Exceptional Dehn surgery on large arborescent knots

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Abstract

A Dehn surgery on a knot K in S^3 is exceptional if it produces a reducible, toroidal or Seifert fibred manifold. It is known that a large arborescent knot admits no such surgery unless it is a type II arborescent knot. The main theorem of this paper shows that up to isotopy there are exactly three large arborescent knots admitting exceptional surgery, each of which admits exactly one exceptional surgery, producing a toroidal manifold.

1 Introduction

A Conway sphere for a knot K in S^3 is a sphere S which intersects K at 4 points, such that punctured sphere $S - K$ is incompressible in $S^3 - K$. In this case the sphere S cuts (S^3, K) into two non-splittable tangles (B_1, τ_1) and (B_2, τ_2), where B_i is a 3-ball, and t_i is a pair of properly embedded arcs in B_i. An arborescent knot is obtained by gluing rational tangles together in various ways. See for example [Wu2] or [Ga]. An arborescent knot K is *large* if it has a Conway sphere. It is known [HT, Oe, Wu2] that K is large if and only if its complement is large in the sense that it contains an embedded closed essential surface.

A nontrivial Dehn surgery on a hyperbolic knot K in S^3 is *exceptional* if the resulting manifold is either reducible, toroidal, or a small Seifert fiber space. By the Geometrization Conjecture proved by Perelman, non-exceptional surgeries yield hyperbolic manifolds. Thurston [Th] showed that a hyperbolic knot admits only finitely many exceptional surgeries.

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All large arborescent knots are hyperbolic. It is known that most large arborescent knots admit no exceptional surgery. Define $T(r,s)$ to be a Montesinos tangle which is the sum of two rational tangles associated to rational numbers r, s respectively. See Section 2 for more details. A knot K is an arborescent knot of type II if it has a Conway sphere cutting it into two Montesinos tangles of type $T(r_i,\, 1/2)$. It was shown in [Wu2, Theorem 3.6] that if a large arborescent knot K is not of type II then all nontrivial surgeries on K are Haken and hyperbolic, so there is no exceptional surgery on K. When K is an arborescent knot of type II, all non-integral surgeries are Haken and hyperbolic, and all integral surgeries are laminar in the sense that the resulting manifolds contain essential laminations; in particular, it is irreducible. It remains to determine which type II knots admit integer surgeries producing toroidal or Seifert fibred manifolds. The following is our main theorem, which determines all such knots and the exceptional surgeries on them. The knot K_1 in the theorem is given in Figure 2.2(b). K_3 is actually the mirror image of K_1, so there are essentially only two knots up to homeomorphism of (S^3, K). K_2 is obtained from K_1 by changing crossings on the right half of the diagram of K_1 in Figure 2.2(b).

Theorem 1.1 *Let K_1, K_2, K_3 be the three knots in Definition 2.3. Let K be a large arborescent knot in S^3, and let δ be a non-meridional slope on $\partial N(K)$. Then $K(\delta)$ is an exceptional surgery if and only if (K,δ) is isotopic to $(K_1, 3)$, $(K_2, 0)$ or $(K_3, -3)$, in which case $K(\delta)$ is toroidal.*

The essential punctured torus F constructed in the exterior of the knots K has four boundary components, and that is minimal; see Lemma 6.2 and Proposition 5.8 below. The referee pointed out that this is an interesting fact as there are few examples of classes of knots with such property. Some examples were given by Eudave-Muñoz [Eu] and others more recently by Teragaito [Te]. Those were constructed via double branched covers and the punctured torus is not explicitly described in the knot exterior. The examples constructed in this paper are perhaps the simplest known knots whose exterior contains such a punctured torus F, and F is explicitly constructed. I would like to thank the referee for this as well as some other helpful comments.

The paper is organized as follows. In Section 2 we will define some special disks in rational tangle spaces. These are the pieces which will be used to build the toroidal surfaces in the exteriors of the knots in Theorem 1.1. Section 3 defines an index $i(G,Q)$ for a surface G relative to Q, and proves its additivity and some other properties. Now let K be a type II knot, which is the union of two Montesinos tangles $T_i = T(r_i, 1/2)$. Let F be a punctured essential torus in the exterior of K with integer boundary slope, and let F_i be the intersection of F with the tangle space of T_i. The important fact is that F can be chosen so that each component G of F_i must have zero index relative to the tube around the unknotted string of T_i. It will then be shown in Section 4 that G must be a special surface in the sense that it is the union of special disks in the two rational tangle spaces. This quickly leads to a proof that $r_i \equiv \pm 1/3 \bmod 1$. In Section 5 we define the relative framing for a surface in a tangle space. They can be used to calculate the boundary slope of the surface $F = F_1 \cup F_2$, and show that if F has integer boundary slope then its intersection with the punctured Conway sphere must have a very special configuration, which completely determines the gluing map between the two tangles. It will follow that if $K(\delta)$ is toroidal then K must be one of the three knots in the theorem, and there is only one possible choice of F. In Section 6 it will be shown that the surface F constructed is incompressible and ∂-incompressible, and it gives rise to an essential torus in the surgered manifold. This, together with some known results about surgery on type II knots, will complete the proof of Theorem 1.1.

Unless otherwise stated, a surface in a 3-manifold M is assumed to be either on the boundary of M or properly embedded in M. Given a set X in a surface or 3-manifold, denote by $N(X)$ a regular neighborhood of X, and by $|X|$ the number of components in X. If P is a surface in a 3-manifold M, denote by $M|P$ the manifold obtained by cutting M along P.

2 Special disks in tangle spaces

A *tangle* T is a pair (B, τ), where B is a 3-ball with four specified points on ∂B, and $\tau = \tau_1 \cup \tau_2$ is a pair of arcs in B connecting these points. We will identify B with either a pillow case with $\partial\tau$ the four corners, or the one point compactification of the lower half space, i.e. $B = \hat{R}^3_- = \{(x, y, z) \,|\, z \leq 0\} \cup \{\infty\}$, with $\partial\tau$ identified to the four points $(\pm 1, \pm 1)$ and the front side of the pillow case identified to the square $[-1, 1] \times [-1, 1]$ in $\hat{R}^2 = R^2 \cup \{\infty\}$, which is identified to $(R^2 \times \{0\}) \cup \{\infty\} = \partial\hat{R}^3_-$.

Given a tangle $T = (B, \tau)$, let $E(T)$ be the closure of $B - N(\tau)$, called the tangle space of T. Let $S(T) = \partial B = \hat{R}^2$, and let $P(T)$ be the 4-punctured sphere $\partial B \cap E(T)$. Let $S_+(T)$ (resp. $S_-(T)$ be the closure of the right (resp. left) half plane of $\hat{R}^2 = \partial B$. Similarly, define $P_\pm(T) = S_\pm(T) \cap E(T)$, which is a twice punctured disk. Denote by $U(T) = U_+(T) \cup U_-(T)$ the two annuli

$\partial E(T) - \text{Int}P(T)$, with $U_+(T)$ the one containing the upper right component of ∂P on $\hat{R}^2$. We now have a decomposition of the boundary of $E(T)$ as follows.

$$\partial E(T) = P_-(T) \cup P_+(T) \cup U_-(T) \cup U_+(T).$$

We refer the readers to [Co] or [Wu1] for the definition of rational tangles. Roughly speaking, the strings τ of a rational tangle of slope $r = p/q$ is obtained by pushing into the interior of B the interior of two arcs of slope r on the boundary of the pillow case B connecting the four corners of B. *Throughout this paper we will always assume that* $q \geq 2$.

Given two tangles $T_i = (B_i, \tau_i)$, we may construct a new tangle $T_1 + T_2$ by identifying the disk $S_+(T_1) \subset \hat{R}^2$ with $S_-(T_2) \subset \hat{R}^2$ using the map $(x, y) \to (-x, y)$ and then identifying $B_1 \cup B_2$ to $B = \hat{R}^3_-$ so that a boundary point of B_1 or B_2 on $\partial(B_1 \cup B_2)$ is mapped to the point with the same coordinates on $\partial B = \hat{R}^2$. Denote by $T(r_1, r_2)$ the Montesinos tangle $T(r_1) + T(r_2)$.

Two tangles T_1, T_2 are *weakly equivalent* if T_1 can be deformed to T_2 by an isotopy φ_t of B. They are *P-equivalent* if the isotopy φ_t above is rel $\partial P_+(T_1)$, and *equivalent* if φ_t is rel $S(T)$. Thus for example, $T(r)$ is P-equivalent to $T(r+k)$ for any integer k, and $T(1/3, -1/2)$ is P-equivalent to $T(1/3, 1/2)$. Two tangles are considered the same if they are equivalent.

A surface F in $E(T)$ is *tight* if $\partial F \neq \emptyset$, and it intersects each of $P_\pm$ and $U_\pm$ in essential arcs or essential circles. In this case ∂F is a set of essential loops on $\partial E(T)$. Thus a tight disk is an essential disk in $E(T)$ (i.e. a compressing disk of $\partial E(T)$), and any essential disk in $E(T)$ is isotopic to a tight disk.

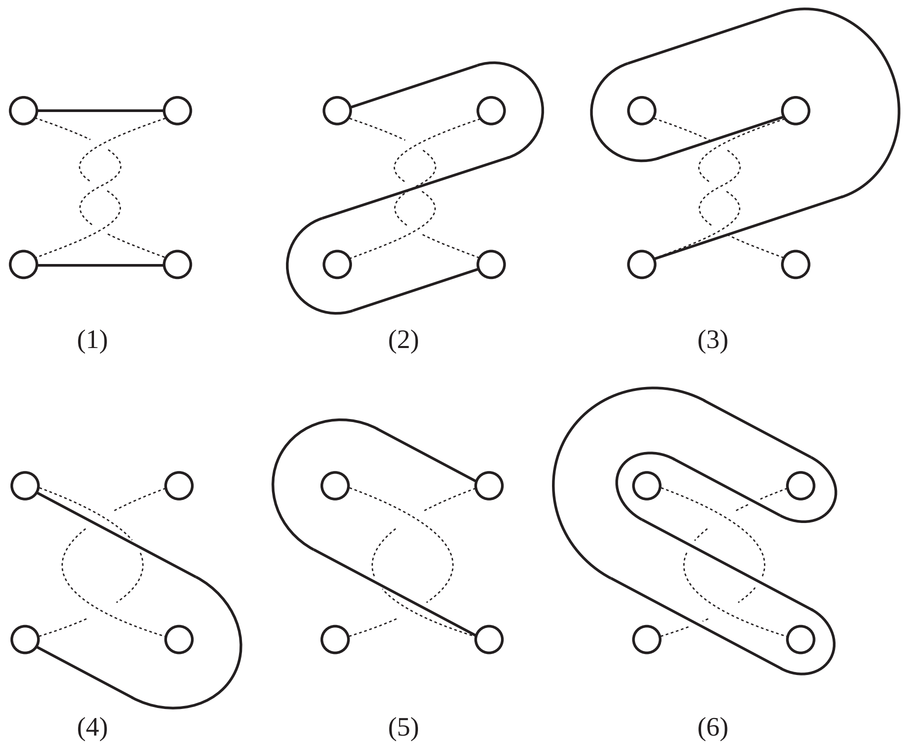


Figure 2.1

The strings τ in a rational tangle $T(p/q) = (B, \tau)$ are rel $\partial\tau$ isotopic to a pair of arcs $\tau' = \tau'_1 \cup \tau'_2$ on ∂B. To be specific, let τ'_2 be the one with an endpoint at $(1,1) \in \hat{R}^2$. Let τ'_0 be a pair of horizontal arcs on R^2 connecting $\partial\tau$. Let $c_i = \tau'_i \cap P(T)$ for $i = 0, 1, 2$, and let c_3 be the curve on P that separates c_1 and c_2, which is unique up to isotopy. Thus for example, for $T(1/3)$ the curves c_0, c_1, c_2 are shown in Figure 2.1(1)–(3), and for $T(-1/2)$ the curves c_1, c_2, c_3 are shown in Figure 2.1(4)–(6), respectively. A disk D in $E(T)$ is a special disk if its intersection with $P(T)$ is one of these curves. It is further required that $q \leq 3$ if $D \cap P(T) = c_1, c_2$, and $q = 2$ if $D \cap P(T) = c_3$. More explicitly, we have the following definition. Note that the curve in Figure 2.1(k) is the boundary curve of a type (k) special disk.

Definition 2.1 Let $T = T(\pm 1/q)$. A disk D in $E(T)$ is a *special disk* if it is one of the following type.

Type (1). $T = T(\pm 1/q)$, q odd, and $D \cap P(T) = c_0$.

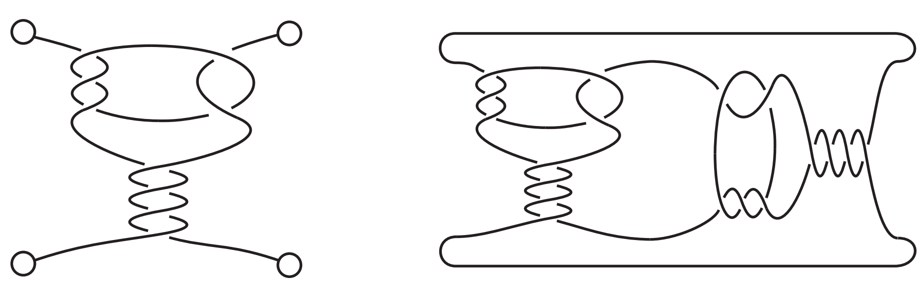
Type (2). $T = T(\pm 1/3)$, and $D \cap P(T) = c_1$.

Type (3). $T = T(\pm 1/3)$, and $D \cap P(T) = c_2$.

Type (4). $T = T(\pm 1/2)$, and $D \cap P(T) = c_1$.
Type (5). $T = T(\pm 1/2)$, and $D \cap P(T) = c_2$.
Type (6). $T = T(\pm 1/2)$, and $D \cap P(T) = c_3$.

Lemma 2.2 *Let $T = T(p/q)$ with $q = 2$ or odd. Let $Q = P_+(T)$ if $q \geq 3$, and $Q = P_-(T) \cup U_+(T)$ if $q = 2$. Suppose D is a tight disk in $T(p/q)$ such that $|D \cap Q| \leq 2$. Then T is P-equivalent to $T(\pm 1/q)$, and D is a special disk. In particular, Q and $Q' = \partial E(T) - \text{Int}Q$ are incompressible, and there is no disk in $E(T)$ intersecting each of Q and Q' at a single essential arc.*

Proof. This is essentially [Wu1, Lemmas 2.1 and 2.2]. Let $P = P_+(T)$ if $q \geq 3$, and $P = P_-(T)$ if $q = 2$. Clearly D must intersect P in a nonempty set of arcs as otherwise we would have $T = T(1/0)$. Since $|D \cap P| \leq 2$, D is a monogon or bigon as defined in [Wu1], which have been classified in Lemmas 2.1 and 2.2 of [Wu1]. When it is a monogon D is a special disk of type (5). Bigons appear when T is a torus tangle or wrapping tangle (see [Wu1] for definition), or a twist tangle; but since T is rational and $q = 2$ or odd, the first two cases does not happen. Thus from the proof of [Wu1, Lemma 2.2] we see that if D is a bigon then it is one of the types (1), (2), (3), (4) or (6) in Definition 2.1, or T is P-equivalent to $T(1/4)$ and D intersects P in two arcs with boundary on the outer component of ∂P, but since we have assumed that $q = 2$ or odd, the last case is impossible. $\square$



(a) (b)

Figure 2.2

Denote by $T(r_1, r_2; n)$ the tangle obtained from $T(r_1, r_2) = (B_3, \tau)$ by twisting the two lower endpoints of τ by n left hand half twists. See Figure 2.2(a) for the tangle $T(1/3, -1/2; 4)$.

Definition 2.3 Let $\eta : \hat{R}^2 \to \hat{R}^2$ be the map which is a $\pi/2$ counterclockwise rotation about the origin followed by a reflection along the y-axis. Define three knots K_1, K_2, K_3 by

$(S^3, K_1) = T(1/3, -1/2;\ 4) \cup_\eta T(1/3, -1/2;\ 4)$,
$(S^3, K_2) = T(1/3, -1/2;\ 4) \cup_\eta T(-1/3, 1/2; -4)$, and
$(S^3, K_3) = T(-1/3, 1/2; -4) \cup_\eta T(-1/3, 1/2; -4)$.

Alternatively, K_i can be obtained by shifting the first tangle to the left, the second tangle to the right, rotating the second tangle counterclockwise by an angle of degree $\pi/2$, and then connecting the endpoints of the tangles by arcs on R^2 which are horizontal except near the endpoints of the strings of the tangles, as shown in Figure 2.2(b) for K_1. Note that K_3 is the mirror image of K_1, and K_2 is obtained from K_1 by taking mirror image the right half of K_1. Theorem 1.1 shows that these are the only large arborescent knots which admit exceptional surgery, and each of them admits exactly one such surgery.

3 Index of essential surfaces

Let Q and F be surfaces in M, intersecting in general position. Denote by $a(F, Q)$ the number of arc components of $F \cap Q$. The *index of* F in M relative to Q is defined as

$$i(F, Q) = \chi(F) - \frac{1}{2}a(F, Q)$$

where $\chi(F)$ is the Euler characteristic of F. This is the same as the cusped Euler characteristic defined in [Wu3] for sutured manifold, only that now Q is not required to be "cusps", which by definition is a set of annuli and tori on the boundary of a 3-manifold.

Lemma 3.1 *Let Q be a surface on ∂M, and Q' an essential surface properly embedded in M and disjoint from Q. Let $M' = M|Q'$, and let Q'_1, Q'_2 be the two copies of Q' on $\partial M'$. Let F be a surface in M, and let $F' = F|Q'$ be the corresponding surface in M'. Then*

$$i(F', Q'_1 \cup Q'_2 \cup Q) = i(F, Q).$$

Proof. Put $k = a(F, Q')$. Note that Q'_1, Q'_2, Q are mutually disjoint compact surfaces on $\partial M'$, hence $a(F', Q'_1 \cup Q'_2 \cup Q) = 2k + a(F, Q)$. Since Q'

intersects F in k arcs and possibly some circle components, after cutting F along $F \cap Q'$ we have $\chi(F') = \chi(F) + k$. It follows that

$$\begin{aligned} i(F,Q) &= \chi(F) - \frac{1}{2}a(F,Q) = \chi(F') - k - \frac{1}{2}a(F,Q) \\ &= \chi(F') - \frac{1}{2}a(F', Q_1' \cup Q_2' \cup Q) \\ &= i(F', Q_1' \cup Q_2' \cup Q). \quad \square \end{aligned}$$

The lemma shows that index is invariant when cutting along a surface disjoint from Q. We remark that it is important to assume that Q is a compact surface and is disjoint from Q', otherwise the lemma may not be true.

The most useful case is when Q' is a separating surface, cutting M into M_1 and M_2. The following additivity lemma follows immediately from Lemma 3.1.

Lemma 3.2 (Additivity of index) *Suppose Q is a compact subsurface of ∂M, and Q' is a separating surface in M disjoint from Q, cutting M into M_1 and M_2. Let Q_i' be the copy of Q' on ∂M_i, $Q_i = (Q \cap M_i) \cup Q_i'$, and $F_i = F \cap M_i$. Then*

$$i(F,Q) = i(F_1,Q_1) + i(F_2,Q_2). \quad \square$$

Lemma 3.3 *Let $T = T(p/q)$ with $q = 2$ or odd. Let $Q = P_+(T)$ for $q > 2$, and $Q = P_-(T) \cup U_+(T)$ for $q = 2$. Let F be a tight disk in $E(T)$. Then $i(F,Q) \leq 0$, and equality holds if and only if T is P-equivalent to $T(\pm 1/q)$ and F is a special disk.*

Proof. It is easy to check that each special disk in Definition 2.1 intersects Q in two arcs and hence has $i(F,Q) = 0$. Conversely, if $i(F,Q) \geq 0$ then $|F \cap Q| \leq 2$, so by Lemma 2.2 it is a special disk and hence $i(F,Q) = 0$. $\square$

Lemma 3.4 *Suppose $T = T(p/q, 1/2)$ and F is a tight surface in $E(T)$. Let $U = U_+(T)$. Then $i(F,U) \leq 0$. In particular, $P(T)$ and $\partial E(T) - U$ are incompressible, and there is no disk in $E(T)$ intersecting $U_+(T)$ at a single essential arc.*

Proof. If $i(F,U) > 0$ then F is a disk and $F \cap U$ has at most one arc component. Isotope the decomposition surface P in $E(T) = E(T(p/q)) \cup_P$

$E(T(1/2))$ so that $F \cap P$ is minimal. Then an innermost circle outermost arc argument would lead to a contradiction to Lemma 3.3 because one of the (at least two) disks cut off by outermost arcs will be disjoint from U and hence is isotopic to a tight disk with positive index. $\square$

4 Special surfaces in knot exterior

Throughout this section we assume that K is a type II knot which is the union of two length 2 Montesinos tangles $T_i = (B_i, \tau_i) = T(p_i/q_i, 1/2) = T_{i1} + T_{i2}$, where $T_{i1} = T(p_i/q_i)$ and $T_{i2} = T(1/2)$.

Let $S = \partial B_i$, and let $P = P(T_1) = P(T_2) = S \cap E(K)$, which cuts $E(K)$ into $E(T_1)$ and $E(T_2)$. Let $P_i = E(T_{i1}) \cap E(T_{i2})$ be the twice punctured disk cutting $E(T_i)$ into $E(T_{i1})$ and $E(T_{i2})$. Write $T_{ij} = (B_{ij}, \tau_{ij})$. Thus $E(K)$ is the union of four rational tangle spaces $E(T_{ij})$, $i, j = 1, 2$. Let $U_i = U_+(T_i)$, which is the component of $U(T_i)$ lying in $E(T_{i2})$.

Definition 4.1 A surface F in $E(K)$ or $E(T_i)$ is a *special surface* if it intersects each $E(T_{ij})$ in special disks.

The purpose of this section is to show that if F is an essential punctured torus in $E(K)$ with integer slope then it is a special surface up to isotopy. We will then use this result to show that $q_i = 3$ for $i = 1, 2$.

If F is a compact surface in $E(T)$ intersecting $U(T)$ in arcs on ∂F, then $F - U(T)$ is F with the arcs $F \cap U(T) \subset \partial F$ removed, so it is non-compact. An arc β in $F - U(T)$ is considered to be *essential* if it does not cut off a compact disk from $F - U(T)$. Thus for example, if F is a disk intersecting $U(T)$ in two arcs on ∂F then there is exactly one essential arc on $F - U(T)$ up to isotopy.

A *P-compressing disk* of a surface F in $E(T)$ is a disk D in $E(T)$ such that $\partial D = \alpha \cup \beta$, where α is an arc on P and $\beta = D \cap (F - U(T))$ is an arc on $F - U(T)$ which is essential in the above sense. If such a disk exists then F is *P-compressible*, otherwise it is *P-incompressible*.

Lemma 4.2 *Let F be a punctured essential torus in $E(K)$ such that ∂F has integer slope on $\partial E(K)$, and the complexity $(|F \cap \partial P|, |F \cap P|)$ is minimal in lexicographic order. Let F_j be a component of $F \cap E(T_i)$. Then (i) F_j is tight, (ii) $i(F_j, U_i) = 0$, and (iii) F_j is incompressible and P-incompressible.*

Proof. (i) Since $|F \cap \partial P|$ is minimal, F intersects each of the four annuli $\partial N(K)|\partial P$ in essential arcs. By [Wu1, Lemma 3.3] each $E(T_i)$ is a handlebody and hence irreducible, so if $F \cap P$ contains a trivial loop on P then an innermost circle argument would show that one could isotope F to reduce $|F \cap P|$ without increasing $|F \cap \partial P|$, which is a contradiction. If $F \cap P$ has a trivial arc on P then an outermost one would cut off a ∂-compressing disk for F, which is impossible because F is essential.

(ii) Since P is incompressible [Wu1, Lemma 3.3], each circle component of $F \cap P$ is also essential on F. It follows that each boundary component of $F \cap E(T_i)$ is a nontrivial loop on $\partial E(T_i)$, so each disk component of $F \cap E(T_i)$ is an essential disk in $E(T_i)$.

Let $\hat{F}$ be the torus obtained from F by capping off each boundary component of F with a disk. Define a graph Γ on $\hat{F}$ with the attached disks as fat vertices and $P \cap F$ as edges. A component of $P \cap F$ is either a *circle edge* or an *arc edge* of Γ, depending on whether it is a circle or arc. Note that a circle edge is not incident to any vertex, so Γ is actually a graph in which some edges are loops without vertices.

Since ∂F has integer boundary slope and P has 4 boundary components, each vertex v of Γ has valence 4. Denote by C the number of corners at all vertices of Γ which lie in either U_1 or U_2, where $U_i = U_+(T_i)$, which is the component of $\partial N(K) \cap E(T_i)$ lying in the $E(T_{i2}) = E(T(1/2))$ part. Denote by V and E the numbers of vertices and *arc* edges of Γ, respectively. Since the boundary of each vertex of Γ travels through each of U_1 and U_2 once, we have $C = 2V$. Since each vertex is incident to 4 arc edge endpoints, we also have $E = 2V$. Let F_j be the faces of Γ, and assume it lies in $E(T_i)$. The Euler characteristic formula, which one can check is still valid for graphs with circle edges, gives

$$\begin{aligned} 0 &= \chi(\hat{F}) = V - E + \sum \chi(F_j) = -V + \sum \chi(F_j) = -\frac{1}{2}C + \sum \chi(F_j) \\ &= \sum (\chi(F_j) - \frac{1}{2}|F_j \cap U_i|) = \sum i(F_j, U_i). \end{aligned}$$

By (i) and Lemma 3.4 we have $i(F_j, U_i) \leq 0$ for each component F_j of $F \cap E(T_i)$. Since these are exactly the faces of Γ, it follows from the above that $i(F_j, U_i) = 0$ for all F_j.

(iii) Since F is incompressible and $|F \cap P|$ is minimal, it is easy to see that F_j is also incompressible. By (ii) F_j is either a disk intersecting U_i twice or an annulus disjoint from U_i. Suppose D is a P-compressing disk for F_j as in the definition, and let $\alpha = D \cap P$. Since $\beta = |D \cap F_j|$ is essential on $F_j - U_i$, the two points $\partial\alpha = \partial\beta$ lie on distinct components of $F_j \cap P$, hence an isotopy of F via D would create a surface F' which has the same or smaller complexity than F and yet one of the faces F'_j deformed from F_j has $i(F'_j, U_i) > 0$, which is a contradiction to (ii). $\square$

Lemma 4.3 *If F is a special surface in $E(K)$ then it is not a punctured sphere.*

Proof. This follows from the proof of Lemma 4.2(ii). If $\hat{F}$ is a sphere then $2 = \chi(\hat{F}) = \sum i(F_j, U_i)$, which is a contradiction since $i(F_j, U_i) \leq 0$ for all i, j. $\square$

Lemma 4.4 *Let F be an essential punctured torus in $E(K)$ with integer boundary slope. Then F, P_1, P_2 can be isotoped so that F is a special surface.*

Proof. We may assume that $(|F \cap \partial P|, |F \cap P|)$ is minimal up to isotopy, so Lemma 4.2 applies, and we have $i(F_k, U_i) = 0$ for any component F_k of $F \cap E(T_i)$. We will show below that P_i can be isotoped so that each component D of $F \cap E(T_{ij})$ is a tight disk. In that case by Lemma 3.3 we will have $i(D, U_i) \leq 0$; since F_k is a union of such disks, by the Additivity Lemma 3.2 we will have $i(D, U_i) = 0$ for all D, and hence by Lemma 3.3 D is a special disk in $E(T_{ij})$, which will complete the proof.

Let $Q_i = F \cap E(T_i)$. Isotope P_i so that $(|Q_i \cap \partial P_i|, |Q_i \cap P_i|)$ is minimal. This implies that for $R = P_-(T_{i1})$, $P_+(T_{i2})$ or a component of $U(T_{ij})$, each arc component of $Q_i \cap R$ is essential on R. By Lemma 4.2 Q_i is incompressible and P-incompressible, so the above minimality and the ∂-incompressibility of F imply that each component of $Q_i \cap P_i$ is also essential on $P_i = P_+(T_{i1}) = P_-(T_{i2})$. Therefore all components of $F \cap E(T_{ij})$ are tight. It remains to show that each component of $F \cap E(T_{ij})$ is a disk.

By Lemma 2.2 P_i is incompressible and ∂-incompressible in $E(T_i) - U_i$, so no component of $P_i \cap Q_i$ is a trivial loop on Q_i, or an arc which is trivial on $Q_i - U_i$ in the sense that it cuts off a disk on Q_i disjoint from U_i. Thus if a component F_k of Q_i is a disk then P_i cuts F_k into disks. Similarly if F_k is an annulus then no component of $F_k \cap P_i$ is an inessential arc or inessential circle on F_k. It now suffices to show that $\partial F_k \cap P_i \neq \emptyset$ because then $F_k \cap P_i$ is a nonempty set of essential arcs on F_k, cutting it into disks.

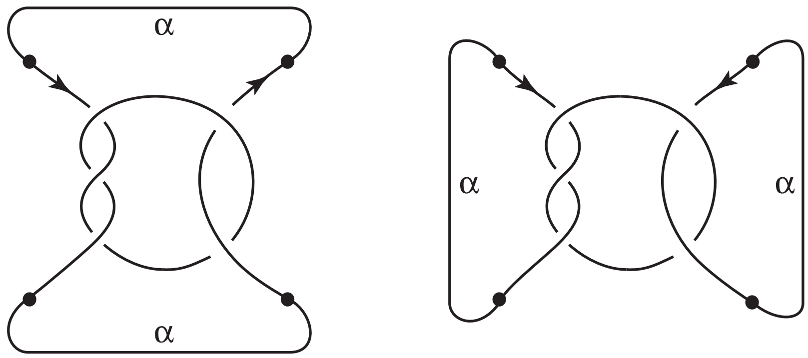
Let F_k be an annulus component of Q_i with ∂F_k disjoint from P_i. This implies that $F_k \cap U_-(T) = \emptyset$ because each component of $F_k \cap U_-(T_i)$ is a component of $F \cap U_-(T_i)$, which intersects P_i at two points. Since $i(F_k, U_i) = 0$, we also have $F_k \cap U_+(T_i) = \emptyset$. Therefore $\partial F_k \subset P$. Note that no component C' of ∂F_k is parallel to a component C of ∂P as otherwise the arc components of $F \cap P$ with endpoints on C would lie in the annulus between C and C' and hence would be trivial arcs, and there are $|\partial F|/2 > 0$ of them, which can be used to ∂-compress F, a contradiction to the fact that F is essential. Therefore the two components of ∂F_k are both parallel to the circle $P_i \cap P$ and hence bound an annulus A on P. By Lemma 4.2(iii) F_k is incompressible, and by [Wu1, Lemma 3.3] $E(T_i)$ is a handlebody, so $F_k \cup A$ must bound a solid torus V in $E(T_i)$. Now A cannot be meridional on V because P is incompressible, and A cannot run more than once along the longitude of V as otherwise the union of V and a regular neighborhood of a disk on ∂B_i bounded by a component of ∂F_k would be a punctured lens space in the 3-ball B_i, which is absurd. It follows that A is longitudinal on ∂V and hence F_k is P-compressible, which contradicts Lemma 4.2(iii). □

Proposition 4.5 *Suppose K is an arborescent knot of type II, and suppose $E(K)$ contains an essential punctured torus with integer boundary slope. Then K is the union of T_1, T_2, each of which is weakly equivalent to $T(1/3, 1/2)$ or $T(-1/3, 1/2)$.*

Proof. By definition K is the union of T_1, T_2 with $T_i = T(p_i/q_i, 1/2) = T_{i1} + T_{i2}$. By Lemma 4.4 we may assume that F intersects each $E(T_{ij})$ in special disks. If some $q_i > 3$ then the only special disk in $E(T_{i1})$ is of type (1), hence each component of $F \cap P_i$ is an arc with only one endpoint on the circle $P_i \cap P$, where P_i is the twice punctured disk $E(T_{i1}) \cap E(T_{i2})$. From the definition we see that the only special disks in $E(T_{i2}) = E(T(1/2))$ intersecting P_i in such arcs are those of type (4), which are disjoint from $U_+(T_2)$. Therefore $F \cap U_+(T_i) = \emptyset$, which is a contradiction because F must intersect each $U_\pm(T_i)$ in exactly $|\partial F| > 0$ arcs. □

5 Boundary slopes of special surfaces

Let $T = (B^3, \tau)$ be a tangle, with B^3 identified to $\hat{R}^3_-$. We assume that τ is oriented. Let α be a pair of arcs on $R^2 - \mathrm{Int} I^2$, shown in Figure 5.1(1) when the orientations of τ are opposite near the two upper endpoints, or in Figure 5.1(2) otherwise. Then $\hat{\tau} = \tau \cup \alpha$ is a link in R^3 with orientation induced from that of τ.



(1) (2)

Figure 5.1

Let γ be a set of essential arcs on $U(T)$, and assume that $n = |\gamma \cap U_-(T)| = |\gamma \cap U_+(T)|$. Each component of γ is isotopic in $N(\tau)$ to a component of τ, so the orientation of τ induces an orientation on γ. Let $p : R^3_- \to R^2 = \partial R^3$ be the standard projection. We always assume that τ and γ are in *regular position* in the sense that

(i) $p(\tau) \subset I^2$, where $I = [-1, 1]$;

(ii) $p : \tau \cup \gamma \to R^2$ is an immersion with only double crossings; and

(iii) $p(\gamma) \cap \alpha = \emptyset$.

Condition (iii) is to guarantee that all crossings between $p(\gamma)$ and $p(\tau \cup \alpha)$ appear inside of the square I^2, so if we close up γ by arcs parallel to α to obtain $\hat{\gamma}$ then the linking number between $\hat{\gamma}$ and $\hat{\tau} = \tau \cup \alpha$ can be calculated using crossings between τ and γ.

Since τ and γ are oriented, each crossing is assigned a sign according to the right hand rule, as given in Rolfsen's book [Ro, p.132].

Definition 5.1 (1) The *relative linking number* between τ and γ, denoted by $lk(\tau, \gamma)$, is the sum of the signs of crossings at which γ passes below τ.

(2) Let F be a surface in $E(T)$ such that $F \cap U(T)$ is in regular position. Then the *relative framing* of F in $E(T)$ is defined as $\theta(F) = lk(\tau, F \cap U(T))$.

An isotopy of $\tau \cup \gamma$ is a *regular isotopy* if $\tau \cup \gamma$ is in regular position at any time during the isotopy. Similarly an isotopy of a surface F in $E(T)$ is a regular isotopy if its restriction to $F \cap U(T)$ is a regular isotopy.

Consider the four disks $\cup D_i = R^2 - \text{Int}P(T)$. Since γ has n endpoints on each ∂D_i, we can connect $\partial\gamma$ by $2n$ arcs γ' on $\partial N(\hat{\tau})$ which lies in the upper half space R^3_+. Define $\hat\gamma = \gamma \cup \gamma'$, with orientation induced by that of γ. Since $\hat\tau \cup \hat\gamma$ is an oriented link, the linking number $lk(\hat\tau, \hat\gamma)$ is well defined.

Lemma 5.2 *(1) $lk(\tau, \gamma) = lk(\hat\tau, \hat\gamma)$.*

(2) $lk(\tau, \gamma)$ and $\theta(F)$ are regular isotopy invariants .

(3) Let ψ be a rotation of R^3_- along the z-axis by an angle of $\pi/2$, deforming τ to τ' and a surface F to F'. Then $\theta(F) = \theta(F')$.

(4) Suppose F_j are the components of F in $E(T)$. Then $\theta(F) = \sum \theta(F_j)$.

Proof. (1) It is well known that $lk(\hat\tau, \hat\gamma)$ can be calculated as the sum of the signs of crossings at which $\hat\gamma$ passes below $\hat\tau$. By definition γ does not pass below α, and γ' does not pass below $\hat\gamma$ because it lies in R^3_+ while $\hat\gamma$ lies in R^3_-. Therefore the crossings at which $\hat\gamma$ passes below $\hat\tau$ are exactly where γ passes below τ, and the result follows.

(2) A regular isotopy does not change the relative position of $\partial\tau$ to α, hence it extends to an isotopy of $\hat\tau \cup \hat\gamma$, and the result follows from (1) because the linking number of a link is an isotopy invariant.

(3) This follows from the definition, because ψ gives a sign preserving one to one correspondence between the crossings.

(4) This follows from definition. □

Let $T = T_1 + T_2$, where $T_1 = T(1/3)$ and $T_2 = T(-1/2)$. Let F be a special surface in $E(T)$ intersecting each $U_\pm(T)$ in n arcs. By definition each component of $F \cap E(T_i)$ is a special disk. We may assume that F is a union of a_i copies of A_i for $i = 1, ..., 6$. From Figure 2.1 we see that $F \cap P(T_i)$ is as shown in Figure 5.2(i), $i = 1, 2$, where an arc with label a_j indicates a_j parallel copies of that arc.

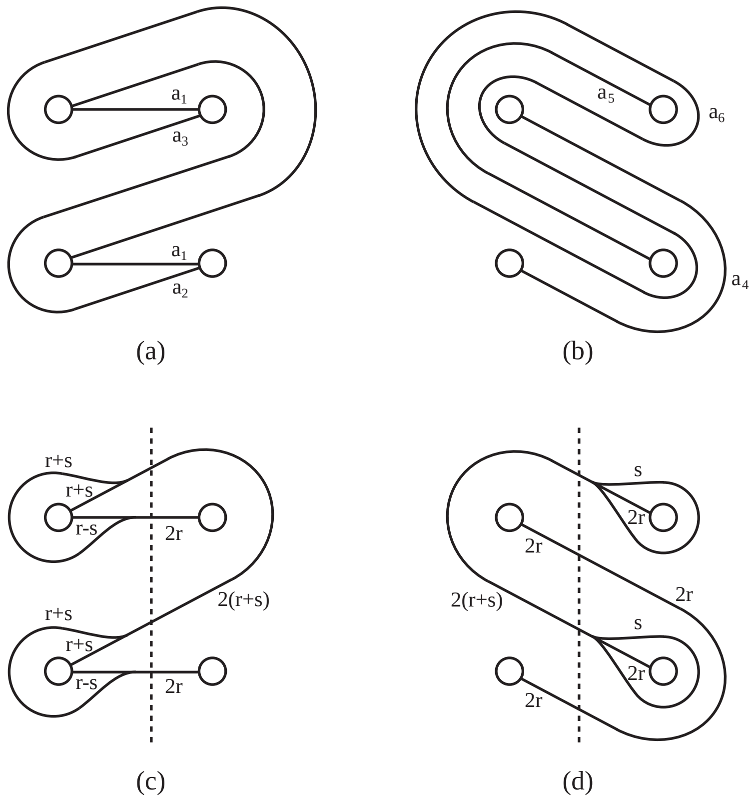


Figure 5.2

Lemma 5.3 *Let $r = n/2$, and $s = a_6$. Then there is a a special surface F in $E(T)$ consisting of a_k copies of disks of type (k) and intersecting each $U_i(T)$ at n arcs, if and only if the non-negative integers a_k satisfy the following equations.*

(1) $n = 2r$ is even;

(2) $a_1 = r - s$;

(3) $a_2 = a_3 = r + s$;

(4) $a_4 = a_5 = a_1 + a_2 = a_1 + a_3 = n$;

(5) $a_6 = s$.

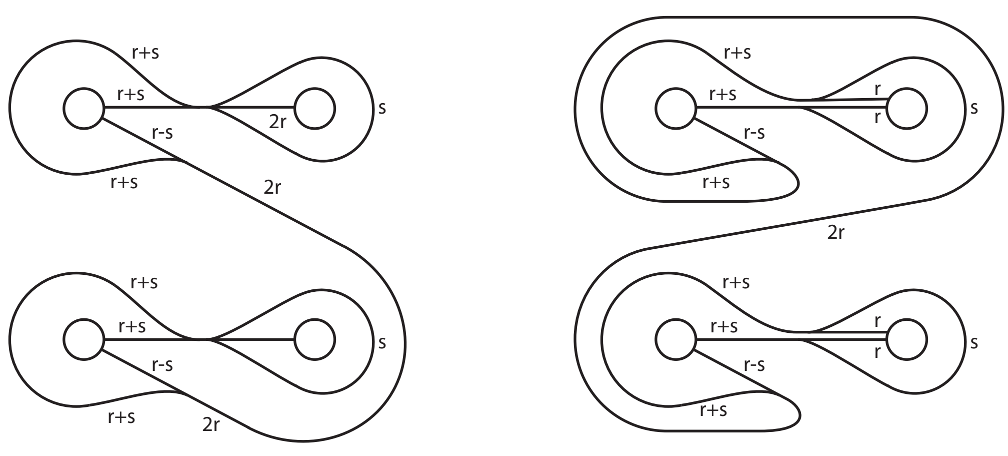
Proof. Suppose F is such a surface. Then (5) is from definition, and (4) follows from the fact that F intersects each boundary component of P at n points. By (4) we have $a_2 = a_3$. Let $P' = P_+(T_1) = P_-(T_2)$ be the twice punctured disk cutting $E(T)$ into $E(T_1)$ and $E(T_2)$. Then on $\partial E(T_1)$ there are $a_2 + a_3$ arcs of $F \cap P'$ with both endpoints on the circle $P' \cap P(T)$ while there are $a_5 + 2a_6$ such arcs on $\partial E(T_2)$. Hence $2a_2 = a_2 + a_3 = a_5 + 2a_6 = n + 2s$, which shows that $n = 2r$ is even, and $a_2 = a_3 = r + s$. (2) follows from this and the equation $a_1 + a_2 = n = 2r$, as shown in Figure 5.2(a).

Conversely, given a set of non-negative numbers a_k satisfying the above equations, let E_k be a_k copies of disks of type (k). Then one can check that the arcs $(E_1 \cup E_2 \cup E_3) \cap P'$ are isotopic to the arcs $(E_4 \cup E_5 \cup E_6) \cap P'$ on P', hence one can glue these together to form a surface F in $E(T)$. $\square$

The curves in Figures 5.2(a) and 5.2(b) can be represented by the weighted train tracks shown in Figure 5.2(c) and 5.2(d), respectively, where an arc of the train track with weight x represents x parallel copies of that arc. The weights on the train tracks are calculated using the above lemma.

Recall that $T(1/3, -1/2) = T_1 + T_2$ is formed by gluing $P_+(T_1)$ to $P_-(T_2)$ using a reflection map along the y axis on R^2. Since the train track on these two half planes match each other under this reflection, after gluing we obtain a special surface F in $E(T)$ with $F \cap P$ represented by the train track γ in Figure 5.3(a).

To simplify the diagram, we perform a counterclockwise full twist on both the top two tangle endpoints and the bottom two tangle endpoints. Note that this is equivalent to twisting the two lower endpoints by four left hand half twists, so by definition it deforms the tangle $T(1/3, -1/2)$ to the tangle $T(1/3, -1/2; 4)$ defined in Section 2. The train track, after splitting along two edges, becomes that in Figure 5.3(b).



(a) (b)

Figure 5.3

After a further splitting and isotopy, we obtain the train track γ in Figure 5.4(a). Note that up to isotopy we can move the end points of γ around ∂P, but so far we have not done that. By moving some endpoints of train track around ∂P, we obtain the one in Figure 5.4(b).

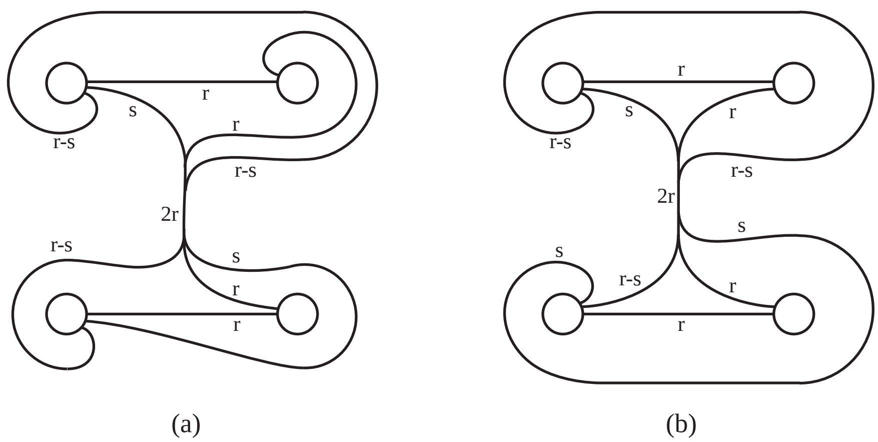


Figure 5.4

The two strings of $T = T(\pm 1/q)$ is said to be *consistently oriented* if they both run from the upper endpoints to the lower endpoints or both from the lower endpoints to the upper endpoints. For $T(1/3)$ we will always assume that its two strings are consistently oriented. For $T(-1/2)$, we introduce a new variable ϵ and set $\epsilon = 1$ if the two strings of $T(-1/2)$ are oriented consistently, and $\epsilon = -1$ otherwise. Recall that a surface F in $E(T)$ is regular if $F \cap U(T)$ is a set of regular curves on $U(T)$.

Lemma 5.4 *Let A_k be a regular special disk such that $A_k \cap P(T)$ is the curve in Figure 2.1(k).*

(1) $\theta(A_1) = 6$, $\theta(A_2 \cup A_3) = 4$, $\theta(A_4 \cup A_5) = -2\epsilon$, and $\theta(A_6) = 0$.

(2) A special surface F in $T(1/3, -1/2)$ is isotopic to a regular surface F' such that $\partial F' \cap P$ is represented by the weighted train track in Figure 5.3(a), and $\theta(F') = (5 - 2\epsilon)n - 2s$, where $s = a_6$ is the number of type (6) special disks in F.

(3) A special surface F in $T(1/3, -1/2; 4)$ is isotopic to a regular surface F' such that $F' \cap P$ is carried by the weighted train track in Figure 5.4(b), and $\theta(F) = (6 - 4\epsilon)n - 2s$.

Proof. (1) follows by drawing the curves γ of ∂A_i on $U(T)$ and counting the signed crossings where γ passes below τ. We omit the details.

(2) By definition F is the union of a_k copies of A_k, which can be put in regular position. We leave it to the reader to check that the regular surfaces in $E(T(1/3))$ and $E(T(-1/2))$ can be combined together to create the surface F in $E(T(1/3,-1/2))$ without creating new crossings between $F \cap U(T)$ and τ. Since $a_2 = a_3$ and $a_4 = a_5$, we have $\theta(F) = a_1\,\theta(A_1) + a_2\,\theta(A_2 \cup A_3) + a_4\,\theta(A_4 \cup A_5) + a_6\,\theta(A_6)$. It follows from (1) and Lemma 5.3 that

$$\begin{aligned}\theta(F) &= 6\,a_1 + 4\,a_2 - 2\,\epsilon\,a_4 \\ &= 6(r-s) + 4(r+s) - 2\,\epsilon\,n = (5-2\epsilon)n - 2s.\end{aligned}$$

(3) By definition $T = T(1/3,-1/2;\,4)$ is obtained from $T(1/3,-1/2)$ by two counterclockwise full twists of the two lower endpoints of the tangle, which deforms the surface F'' in (2) to a new surface F' in $E(T)$ with boundary curve represented by the train track shown in Figure 5.4(a). Note that after the twists each arc component of $\alpha' = F' \cap \partial N(\tau)$ passes below τ four more times than $\alpha'' = F'' \cap \partial N(\tau)$ does, two of which in the positive direction, and the other two in positive direction if and only if $\epsilon = -1$. (Note that the two strings twisted have the same orientation if and only if the two strings in the tangle $T(-1/2)$ have opposite orientation.) Hence $\theta(F') = \theta(F'') + (2-2\epsilon)n$, so by (2) we have $\theta(F') = (7-4\epsilon)n - 2s$.

We now perform an isotopy of F' to obtain the surface F whose boundary curve on P is carried by the train track in Figure 5.4(b). To do this one needs to turn $2r = n$ endpoints of $F \cap P$ on ∂P clockwise for an angle of almost 2π. The isotopy on each endpoints creates one more crossing at which α' passes below τ, and it is in the negative direction. Therefore we have $\theta(F) = \theta(F') - n = (6-4\epsilon)n - 2s$. $\square$

The following lemma can be used to calculate the boundary slope of a surface in $E(K)$.

Lemma 5.5 *Suppose F_i is a regular surface in $E(T_i)$. Let $\eta : \partial B_1 = \hat{R}^2 \to \hat{R}^2 = \partial B_2$ be the reflection along the y axis, such that $\eta(F_1 \cap P(T_1)) = F_2 \cap P(T_2)$. Let $(S^3, K) = (B_1, \tau_1) \cup_\eta (B_2, \tau_2)$ with orientation of τ_i induced by that of K, and let $F = F_1 \cup_\eta F_2$. Suppose F has m boundary components with slope p/q, where p, q are coprime and $q > 0$. Then $mp = \theta(F_1) + \theta(F_2)$. In particular, $q = 1$ if and only if $\theta(F_1) + \theta(F_2) \equiv 0$ mod n, where $n = mq$ is the number of times ∂F intersects each meridian of K.*

Proof. We can shift $T_1 = (B^3, \tau_1)$ to the left and $T_2 = (B^3, \tau_2)$ to the right so that τ_1, τ_2 are separated by the yz-half-plane in $B^3 = R^3_-$. Now $K \subset R^3$ is isotopic to the knot K' obtained by adding four arcs on R^2, each connecting an endpoint p_i of τ_1 to $\eta(p_i)$ on τ_2, with two below the line $y = -1$ and the other two above the line $y = 1$. We may also assume that near $\partial\tau_i$ these arcs match the arcs α in Figure 5.1, so they are disjoint from the projection of $F_i \cap U(T_i)$ because F_i are regular. The isotopy from K to K' extends to an isotopy which deforms F in $E(K)$ to the surface F' in $E(K')$ obtained from $F_1 \cup F_2$ by connecting their boundary on R^2 by bands in the upper half space. More explicitly, let $C = F_1 \cap R^2$ and embed $C \times I$ in $R^3_+ \cap E(K')$ so that $C \times \{-1\} = F_1 \cap R^2$, $C \times \{1\} = F_2 \cap R^2$, and $\partial C \times I$ lies on $\partial N(K')$. Then $F' = F_1 \cup (C \times I) \cup F_2$. Note that $(\partial C) \times I$ does not pass below K', so $lk(K', \partial F')$ is the sum of the signs of crossings where ∂F_i passes below τ_i, hence $mp = lk(K, \partial F) = lk(K', \partial F') = \theta(F_1) + \theta(F_2)$.

If $q = 1$ then $mp \equiv 0 \mod n = mq$. Conversely, if $mp \equiv 0 \mod n = mq$ then p is a multiple of q. Since p, q are coprime and $q > 0$, this is possibly only if $q = 1$. $\square$

We now assume that K is a type II knot and F is an essential punctured torus in $E(K)$ with integer boundary slope. By Proposition 4.5, K is the union of two tangles T_1, T_2, each weakly equivalent to $T(1/3, -1/2)$ or $T(-1/3, 1/2)$. Hence up to weak isotopy we may assume that $T_i = T(1/3, -1/2;\, 4)$ or its mirror image $T(-1/3, 1/2; -4)$. By Lemma 4.4 we may assume that $F_i = F \cap E(T_i)$ is a special surface, so by Lemma 5.4(3) we may assume that F_i is regular, and $F_i \cap P(T_i)$ is represented by the train track γ_i in Figure 5.4(b) if $T_i = T(1/3, -1/2;\, 4)$, and its reflection along the y axis if $T_i = T(-1/3, 1/2; -4)$. Note that the gluing map $\eta : \partial B_2 \to \partial B_1$ could be any orientation reversing map which maps $P(T_2)$ to $P(T_1)$ and $F_1 \cap P(T_1)$ to $F_2 \cap P(T_2)$.

As in Lemma 5.3, let n be the number of times F intersects a meridian of K on $N(K)$, $r = n/2$, and let s_i be the number of type (6) special disks in F_i. There are five possible ways to split γ_i, according to the values of s_i.

(1) $r - s_i > s_i > 0$;
(2) $s_i > r - s_i > 0$;
(3) $s_i = 0$;
(4) $r - s_i = 0$; and
(5) $r - s_i = s_i$.

One can check that for $T_i = T(1/3, -1/2;\, 4)$ the train track in Figure 5.4(b) splits to γ_i in Figure 5.5(1)–(5), respectively, where $s = s_i$. When $T_i = T(-1/3, 1/2; -4)$, γ_i is the reflection of those in the figure along the

y-axis. We say that γ_i is of type (k) if it is the one in Figure 5.5(k).

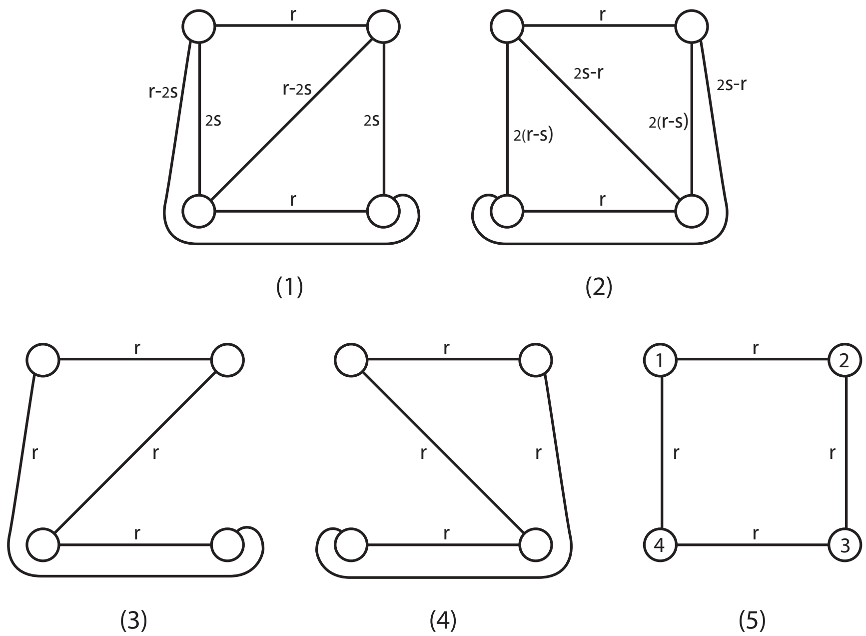


Figure 5.5

Lemma 5.6 *For each i, $s_i \in \{0, r, r/2\}$. Hence γ_i is not of type (1) or (2).*

Proof. We need to show that γ_i cannot be of type (1) or (2). Because of symmetry we may assume without loss of generality that $T_1 = T(1/3, -1/2; 4)$, and γ_1 is of type (1). By Lemma 5.4(3) we have $\theta(F_1) = (6 - 4\epsilon)n - 2s_1$.

Since the graph in Figure 5.5(1) is not homeomorphic to those in Figure 5.5(3)–(5), we see that γ_2 must be of type (1) or (2). Let $\eta : \partial B_2 \to \partial B_1$ be the gluing map, which is orientation reversing. Since the two horizontal edges of γ_i have higher weights than the other edges, η must map horizontal edges to horizontal edges. Without loss of generality we may assume that η maps the upper horizontal line of γ_2 to the upper horizontal line of γ_1 by a reflection along the vertical axis. Note that this completely determine η on γ_2.

First assume that $T_2 = T(1/3, -1/2; 4)$. If γ_2 is the one in Figure 5.5(2) then η is simply a reflection along the vertical line, hence it maps the two right vertices of γ_2 to the two left vertices of γ_1, but since the two left (right) vertices of γ_i belong to the same component of τ_i, K would be a link of two components, which is a contradiction. Therefore γ_2 must be the graph in Figure 5.5(1), which is redrawn in Figure 5.6(1). One can modify the graph by turning the lower horizontal edge clockwise by a half twist to obtain the one in Figure 5.6(2), then isotope some of the edge endpoints at the two lower boundary components of P around to obtain the graph in Figure 5.6(3). The map η is the composition of this isotopy followed by a reflection along a vertical line.

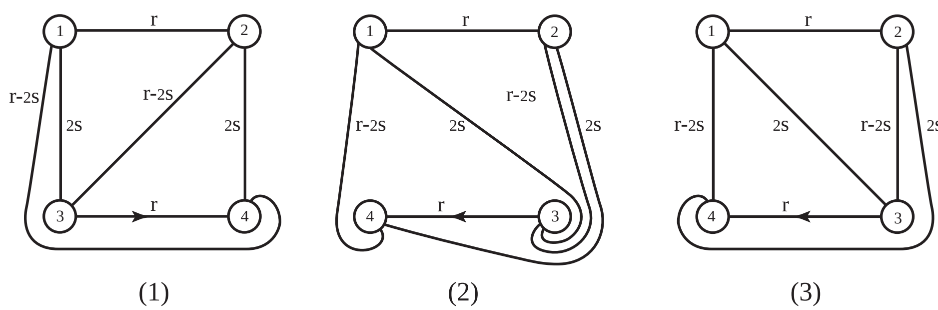


Figure 5.6

Note that the isotopies above are not regular isotopies. They have changed the relative framing of the surface F_2. By Lemma 5.4(3) the framing of F_2 with boundary graph in Figure 5.6(1) is given by $(6-4\epsilon)n - 2s_2$. After twisting the two lower vertices of the graph by a half twist, each boundary arc of F_2 on the tubes $\partial E(T_2) - P$ passes below the part of τ near the vertex 3 in the figure once in the negative direction, but does not pass below the other string of τ, hence the new framing is $(4-4\epsilon)n - 2s_2$. The isotopy from Figure 5.6(2) to Figure 5.6(3) moves r edge endpoints clockwise and another r edge endpoints counterclockwise around vertex 3 and 4 respectively, hence will not change the relative framing. Therefore we have $\theta(F_2) = (4-4\epsilon)n - 2s_2$ for the surface F_2 corresponding to Figure 5.6(3).

In order to glue F_2 to F_1 by η, the weight of the left vertical edge in Figure 5.6(3) must match the weight of the right vertical edge of Figure 5.5(1), hence we have $r - 2s_2 = 2s_1$. Thus $\theta(F_1) = (6-4\epsilon)n - (r - 2s_2)$, so

$$\begin{aligned} \theta(F_1) + \theta(F_2) &= [(6-4\epsilon)n - (r-2s_2)] + [(4-4\epsilon)n - 2s_2] \\ &= (10-8\epsilon)n - r \equiv r \mod n. \end{aligned}$$

Since $r = n/2$, this is a contradiction to Lemma 5.5 and the assumption that F has integer boundary slope. □

Lemma 5.7 *$s_1 = s_2 = r/2$, so both γ_i are of type (5).*

Proof. Note that if γ_i is of type (3) or (4) then the endpoints of a string of τ_i is separated by those of the other string on γ_i, which is a circle. Hence if both γ_i are of type (3) or (4) then $K = \tau_1 \cup \tau_2$ would be a link of two components, which is a contradiction. Therefore at least one γ_i, say γ_1, is of type (5). If the result is not true then γ_2 must be of type (3) or (4). We assume it is of type (3). The other case is similar.

By the same proof as in that of Lemma 5.6, we can isotope γ_2 by twisting the lower level edge of γ_2 by a half twist, followed by an isotopy which moves some endpoints of γ_2 around ∂P, to change γ_2 to a graph of type (5). As in the proof of Lemma 5.6, this will not change $\theta(F_2)$ mod n. By Lemma 5.4(3) we have $\theta(F_1) + \theta(F_2) \equiv -2s_1 - 2s_2$ mod n. Since γ_2 is of type (3), we have $s_2 = 0$, and since γ_1 is of type (5), we have $2s_1 = r = n/2$. Hence $\theta(F_1) + \theta(F_2) \equiv r$ mod n, which by Lemma 5.5 implies that the boundary slope of the punctured torus F is not an integer slope, a contradiction. □

Proposition 5.8 *Let K be a type II knot, and let F be an essential punctured torus in $E(K)$ with integer boundary slope δ. Then $|\partial F|$ is a multiple of 4, and $(K, \delta) = (K_1, 3)$, $(K_2, 0)$ or $(K_3, -3)$, where K_i are the knots defined in Definition 2.3.*

Proof. By Proposition 4.5 we have $(S^3, K) = T_1 \cup T_2$, where each T_1 is weakly equivalent to $T(1/3, 1/2)$ or $T(-1/3, 1/2)$. Up to weak equivalence we may assume that each T_i is either $T(1/3, -1/2; 4)$ or $T(-1/3, 1/2; -4)$.

Denote by $u_1, ..., u_4$ the four disks $\partial B_i - \mathrm{Int} P(T_i)$, which we will consider as fat vertices. By Lemma 5.7 the train track γ_1 is of type (5), so it is a cycle containing those four vertices, labeled in cyclic order, as shown in Figure 5.5(5). Similarly, γ_2 contains the vertices $v_1, ..., v_4$ in the same order. Orient γ_i clockwise. Let $\eta : \partial B_1 \to \partial B_2$ be the gluing map. Then up to isotopy η is determined by its restriction on γ_1, which is then determined by the image of u_1 and whether $\eta|_{\gamma_1}$ is orientation preserving or not. Note that although η must be orientation reversing on ∂B_1, it may map the disk inside of γ_1 to the disk outside of γ_2, so $\eta|_{\gamma_1}$ could be orientation preserving.

Since K is a knot, the endpoints of a string of τ_2 must be mapped to endpoints of different strings of τ_2, which excludes four possible η. Also, if τ_i is considered as lying in a pillow case B_i, then from Figure 2.2(a) one can see that a π rotation along a horizontal axis will preserve the tangle. One can now easily check that all the four possible choices of η give rise to the same knot, so we may assume that η is obtained by rotation γ_2 counterclockwise by an angle of $\pi/2$ followed by a reflection along a vertical line, as described in Definition 2.3. Therefore K is one of the three knots in the statement.

The surface F is cut into F_i in $E(T_i)$. By Lemma 5.2(3), the $\pi/2$ rotation above will not change $\theta(F_1)$, so by Lemma 5.5 the boundary slope of F is given by $(\theta(F_1) + \theta(F_2))/n$. Examining the orientation of the strings in τ_i we see that they are consistently oriented in the tangle $T(\pm 1/2)$, hence $\epsilon = 1$ for both T_i. Since γ_i are of type (5), by definition we have $2s_i = r$, so by Lemma 5.4(3) we have $\theta(F_i) = (6 - 4\epsilon)n - 2s_i = 2n - r$ if $T_i = T(1/3, -1/2;\, 4)$, and $\theta(F_i) = -2n + r$ if $T_i = T(-1/3, 1/2;\, -4)$. Hence if $K = K_1 = T(1/3, -1/2;\, 4) \cup_\eta T(1/3, -1/2;\, 4)$ then by Lemma 5.5 the boundary slope of T is $[(2n - r) + (2n - r)]/n = 3$. Similarly for the other two cases.

By Lemma 5.3 we have $|\partial F| = n = 2r$, and by Lemma 5.7 we have $r = 2s_i$, hence $|\partial F|$ is a multiple of 4. $\square$

6 Toroidal surgery

Let (K, δ) be one of the three pairs described in Theorem 1.1. In this section we will show that there is a punctured torus F in $E(K)$ with boundary slope δ, and the torus $\hat{F}$ obtained by capping off the boundary components of F with meridional disks in the Dehn filling solid is indeed an essential torus in the surgered manifold $K(\delta)$.

Let $T = T(1/3, 1/2) = T_1 \cup T_2$, where $T_1 = T(1/3)$ and $T_2 = T(1/2)$.

Lemma 6.1 *A special surface Q in $E(T)$ is incompressible and P-incompressible.*

Proof. By considering a component if necessary we may assume that Q is connected. By Lemma 3.3 each special disk has zero index, so by the Additivity Lemma 3.2 Q also has zero index and hence is either an annulus disjoint from $U_+(T)$ or a disk intersecting $U_+(T)$ in two arcs. If Q is a disk then it is automatically incompressible, and a P-compression will produce two disks D_i, each intersecting $U_+(T)$ at a single arc, which contradicts Lemma 3.4.

Now assume Q is an annulus. Let $P' = P_+(T_1) = P_-(T_2)$ be the twice punctured disk cutting $E(T)$ into $E(T_1)$ and $E(T_2)$. By definition P' cuts Q into a set of special disks, each of which is an essential bigon in the sense that it intersects P' in two arcs, and an arc on the bigon with one endpoint on each of these arcs is not rel ∂ homotopic to an arc on P'. Using this and the fact that P' is incompressible one can easily show, by an innermost circle outermost arc argument, that Q is incompressible in $E(T)$. To show it is P-incompressible, one need only show that there is no ∂-compressing disk D of Q in $E(T)$ disjoint from $U_+(T)$. Suppose $\partial D = \alpha \cup \beta$, where α is an essential arc on Q and $\beta \subset \partial E(T) - U_+(T)$. Since P' is incompressible, we may assume $D \cap P'$ has no circle components; since $P' \cap Q$ is a set of essential arcs on Q, α is isotopic to a component of $P' \cap Q$, so by an isotopy of D we may also assume that $\alpha \cap P' = \emptyset$. Hence $D \cap P'$ is a set of arcs with endpoints on β. Choose D so that $|D \cap P'|$ is minimal. If $D \cap P' = \emptyset$ then one can use D to ∂-compress a special disk to produce a pair of disks D_j in some $E(T_i)$ with ∂D_j the union of an essential arc on P' and another arc on $\partial E(T_i) - P' \cup U_+(T)$, which will lead to a contradiction to Lemma 3.3. Now an outermost arc γ cuts off a disk D' from D which lies in one of the $E(T_i)$. Using Lemma 3.3 one can show that $\partial D'$ is trivial on $\partial E(T_i)$, so it cuts off a 3-ball which can be used to reduce $|D \cap P'|$, contradicting its minimality. $\square$

By definition the knot K is the union of two tangles T_1, T_2, each of which is either $T(1/3, -1/2;\ 4)$ or its mirror image $T(-1/3, 1/2;\ -4)$. Let $n = 4$, $s = 1$, and define a_k by $a_1 = a_6 = 1$, $a_2 = a_3 = 3$, and $a_4 = a_5 = 4$. One can check that these numbers satisfy the equations in Lemma 5.3. Since $T(1/3, -1/2; 4)$ is weakly equivalent to $T(1/3, 1/2)$, by Lemma 5.3 there is a special surface F_i in $E(T_i)$ which is the union of a_k copies of special disks of type (k) for $k = 1, ..., 6$. By Lemma 5.4(3) the curves $F_i \cap P(T)$ is represented by the train track in Figure 5.4(b), which splits to the one in Figure 5.5(5) because $r = 2 = 2s$. Similarly for $T_i = T(-1/3, 1/2; -4)$. The graph in Figure 5.5(5) is preserved by the gluing map $\eta : \partial B_1 \to \partial B_2$, which by definition is a $\pi/2$ rotation followed by a reflection along the vertical line. It follows that $F_1 \cup_\eta F_2$ form a surface F in $E(K)$.

Lemma 6.2 *F is a connected essential punctured torus in $E(K)$ with ∂F consisting of 4 circles of slope δ.*

Proof. The boundary slope δ of F is calculated in the proof of Proposition 5.8. In particular, δ is an integer in all three cases of K. Since $n = 4$ in the construction, we see that $|\partial F| = 4$.

Each F_i consists of $\sum a_i = 16$ disks, glued along $4r + 2(r + s) = 14$ edges as shown in Figure 5.2(c)-(d), hence $\chi(F_i) = 16 - 14 = 2$. By Lemma 5.7, $F_i \cap P(T_i)$ is as shown in Figure 5.5(5), hence it consists of $4r = 8$ arcs. Therefore $\chi(F) = \chi(F_1) + \chi(F_2) - 8 = -4$. Denote by $\hat{F}$ the surface obtained from F by capping off each boundary component of F with a disk.

Since $|\partial F| = 4$, we have $\chi(\hat{F}) = 0$. By Lemma 4.3 no component of $\hat{F}$ is a sphere. It follows that each component of $\hat{F}$ is a torus. In particular, no component of F is an annulus.

By Lemma 6.1 each F_i is incompressible and P-incompressible, and by Lemma 3.4 $P = P(T_1) = P(T_2)$ is also incompressible. Thus by an innermost circle outermost argument one can show that $F = F_1 \cup F_2$ is incompressible in $E(K)$. Since no component of F is an annulus, this implies that F is also ∂-incompressible as otherwise two copies of a boundary compression disk and the annulus on $\partial N(K)$ bounded by two components of ∂F would contain a compressing disk of F.

By Proposition 5.8 any essential punctured torus has at least 4 boundary components. Since F has no closed component and $|\partial F| = 4$, it follows that F is connected. $\square$

Lemma 6.3 *Let M be a handlebody of genus 3, let c_1, c_2 be a pair of curves on ∂M, and let M' be the manifold obtained by attaching two 2-handles along c_1 and c_2. If $\partial M - c_i$ is compressible for $i = 1, 2$, and $\partial M - c_1 \cup c_2$ is incompressible, then $\partial M'$ is incompressible.*

Proof. This is a standard application of the Handle Addition Lemma [Ja]. Denote by M_1 the manifold obtained from M by attaching a 2-handle along c_1. By assumption there is a compressing disk D of ∂M which is disjoint from c_1. If D is separating then one component H of $M|D$ is a handlebody disjoint from c_1, so we may re-choose D to be a non-separating disk in H. Thus after attaching a 2-handle to M along c_1, D is still a compressing disk of ∂M_1, so ∂M_1 is compressible. On the other hand, since $\partial M - c_2$ is compressible while $(\partial M - c_2) - c_1$ is incompressible, by the Handle Addition Lemma applied to the pair $(M - c_2, c_1)$ we see that the surface $\partial M_1 - c_2$ is incompressible in M_1. Now since ∂M_1 is compressible while $\partial M_1 - c_2$ is incompressible, we may apply the Handle Addition Lemma again to conclude that after attaching a 2-handle to M_1 along c_2, the boundary of the resulting manifold M' is incompressible. $\square$

Let F_0 be a separating surface in a 3-manifold M with $\partial F_0 = c_1 \cup ... \cup c_n \cup c'_n \cup ... \cup c'_1$, lying successively on a torus component R of ∂M. Let A_1 be the component of $R|\partial F_0$ bounded by $c_1 \cup c'_1$, and let A_k be the annulus on R which is bounded by $c_k \cup c'_k$ and contains A_1. Starting with F_0, one one can construct a sequence of surfaces F'_k and F_k by adding the annulus A_k to F_{k-1} to obtain F'_k and then pushing the A_k part of F'_k off R to obtain F_k. The surfaces F_k are said to be obtained from F by *successively tubing through* A_1. The following lemma is probably due to Gordon.

Lemma 6.4 *Suppose F_0 is a connected separating incompressible surface in a 3-manifold M with ∂F_0 on a torus component R of ∂M. Let M_0', M_0'' be the components of $M|F_0$, and let A_1 be an annulus component of $\partial M_0' - \text{Int} F_0$. If $F_1' = F_0 \cup A_1$ is incompressible in M_0', then the surfaces F_k obtained by successively tubing F through A_1 are all incompressible in M.*

Proof. We use the above notation, and let M_k', M_k'' be the components of $M|F_k$. Note that F_k are all connected and separating, and A_{k+1} is a component of $\partial M_k' - \text{Int} F_k$ or $\partial M_k'' - \text{Int} F_k$. Hence by induction we need only show that (i) F_1 is incompressible, and (ii) if $n > 1$ then F_2' is also incompressible in M_1''.

Since F_1 is obtained by pushing the A_1 part of $F_1' = F_0 \cup A_1$ into the interior of M, the component M_1' of $M|F_1$ contained in M_0' is homeomorphic to M_0', with $F_1 \subset \partial M_1'$ identified to $F_0 \cup A_1 \subset \partial M_0'$, so by assumption F_1 is incompressible in M_1'. When $n = 1$ the other component M_1'' is obtained by attaching a collar $R \times I$ to M_0'' along the annulus $A_1' = R - \text{Int} A_1$. It is clear that A_1' is incompressible in M_1''. If it is also ∂-incompressible then an innermost circle outermost arc argument shows that F_1 is incompressible in M_1'', and if it is ∂-compressible then F_0 must be an annulus which is parallel to A_1' after cutting off some possible summands, so M_1'' is essentially a product $R \times I$ and hence F_1 is also incompressible. When $n > 1$ M_1'' is obtained by attaching $A_2 \times I$ to M_0'' along the two annuli P and P' bounded by $c_1 \cup c_2$ and $c_1' \cup c_2'$, respectively. The incompressibility of F_0 implies that these annuli are incompressible. Also, there is no disk D in M_1'' intersecting $P \cup P'$ at a single essential arc as otherwise the frontier of $N(D \cup P \cup P')$ would contain a compressing disk of F_0. One can now apply an innermost circle outermost arc argument to show that both F_1 and F_2' are incompressible in M_1''. $\square$

Proof of Theorem 1.1. If K is not a type II knot or if δ is not an integer slope then by [Wu2, Theorems 3.6 and 4.4] $K(\delta)$ is Haken and hyperbolic, so we assume that K is a type II knot and δ is an integer slope. Write $(S^3, K) = T_1 \cup T_2$, with $T_i = T(p_i/q_i, 1/2)$. By [Wu2, Theorem 2.3] $E(K)$ contains an essential branched surface $\mathcal{B}$ which remains essential in $K(\delta)$, hence by [GO] $K(\delta)$ is irreducible. Also, by the construction in the proof of [Wu2, Theorem 2.3] the exterior of the $\mathcal{B}$ is the disjoint union of $E(T_1)$ and $E(T_2)$, with vertical surface $U_+(T_i)$ on $\partial E(T_i)$.

We claim that $E(T_i)$ is not an I-bundle with $U_+(T_i)$ as a vertical annulus. If this is false that after attaching a 2-handle to $U_+(T_i)$ the resulting manifold M_i would be an I-bundle over a closed surface, which must be a Klein bottle because ∂M_i is a torus. Since M_i is the exterior of a trefoil knot in S^3, this would imply that there is a Klein bottle embedded in S^3, which is absurd.

By [Br] if a small Seifert fiber space contains an essential lamination then its exterior is an I-bundle, hence the above implies that $K(\delta)$ cannot be a small Seifert fiber space. Therefore $K(\delta)$ is exceptional if and only if it is toroidal. By Proposition 5.8 $K(\delta)$ is toroidal only if (K, δ) is one of the three pairs listed, so we need only show that $K(\delta)$ is indeed a toroidal manifold for each of those pairs.

Let F be the surface constructed before Lemma 6.2. By Lemma 6.2 F is an essential punctured torus with ∂F consisting of four circles of slope δ given in Theorem 1.1.

Let $F_i = F \cap E(T_i)$, $P = P(T_i)$, and $P_i = E(T_{iq}) \cap E(T_{i2})$. Note that the special disks $F \cap E(T_{ij})$ in $E(T_{ij})$ cuts $E(T_{ij})$ into a set of 3-balls B_k. The arcs $P_i \cap F_i$ cut P_i into a set of disks D_r, so the manifold $E(T_i)|F_i$ is obtained by gluing the B_k's along the D_r's, and hence is a set of handlebodies H_k. Also, $F \cap P$ cuts P into a set of disks D'_r, so $E(K)$ is obtained from the H_k's by gluing along the D'_r's, and hence is also a set of handlebodies. Since F is a punctured torus with four boundary components, it is separating in $E(K)$, so $E(K)|F$ has two components M_1, M_2, and ∂M_i is the union of F with two annuli on $\partial N(K)$ and hence is a surface of genus 3. It follows that each M_i is a handlebody of genus 3.

Let A_1, A_2 be the two annuli $\partial M_i - \text{Int}(F)$, and let c_j be the core of A_j. Then by Lemma 6.2 the surface $\partial M_i - c_1 \cup c_2$, which is homotopic to F on ∂M_i, is incompressible. Note that $\partial M_i - c_1$ is homotopic to the surface $F \cup A_2$ obtained from F by tubing along A_2. If this is incompressible then by Lemma 6.4 the closed surface F' obtained from F by successively tubing through A_2 is incompressible. From the construction one can see that F' has coannular slope δ on $\partial N(K)$ in the sense that there is an incompressible annulus A' which has interior disjoint from F', with one boundary component on F' and the other on $\partial N(K)$ with slope δ. It is easy to see that any embedded essential surface in an irreducible 3-manifold has at most one coannular slope on a torus boundary component, hence there is no disk in S^3 with boundary on F' which intersects K exactly once, so F' is K-essential in the sense of [Wu2]. By [Wu2, Lemma 4.7] there is no such closed surface in the exterior of a type II knot, which is a contradiction.

Let $\hat{M}_1, \hat{M}_2$ be the two components of $K(\delta)|\hat{F}$. Then $\hat{M}_i$ is obtained from M_i by attaching two 2-handles along the curves $c_1, c_2 \subset \partial M_i$. We have shown above that M_i is a handlebody of genus 3, $\partial M_i - c_j$ is compressible, and $\partial M_i - c_1 \cup c_2$ is incompressible. Therefore by Lemma 6.3 the surface $\hat{F} = \partial \hat{M}_i$ is incompressible in $\hat{M}_i$. Since this is true for $i = 1, 2$, it follows that $\hat{F}$ is an incompressible torus in $K(\delta)$. $\square$

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