DEPTH OF PLEATED SURFACES IN TOROIDAL CUSPS OF HYPERBOLIC 3-MANIFOLDS

Ying-Qing Wu[1]

Abstract. Let F be a closed essential surface in a hyperbolic 3-manifold M with a toroidal cusp N. The depth of F in N is the maximal distance from points of F in N to the boundary of N. It will be shown that if F is an essential pleated surface which is not coannular to the boundary torus of N then the depth of F in N is bounded above by a constant depending only on the genus of F. The result is used to show that an immersed closed essential surface in M which is not coannular to the torus boundary components of M will remain essential in the Dehn filling manifold $M(\gamma)$ after excluding C_g curves from each torus boundary component of M, where C_g is a constant depending only on the genus g of the surface.

1. Introduction

In this paper a *surface* F in a 3-manifold M is a pair $F = (S, \varphi)$, where S is a connected, possibly nonorientable surface, and $\varphi : S \to M$ is a continuous map which is an immersion almost everywhere. Concepts such as points on F and π_1-injectivity of F are defined in the usual way. See Section 2 for more details.

Let N be a toroidal cusp in a hyperbolic 3-manifold M. The *depth* of a surface F in N, denoted by $d_N(F)$, is defined as the maximal distance from all points of $F \cap N$ to the boundary torus T of N. When F is disjoint from N, the depth of F in N is defined as zero. The depth of a closed surface F in N can be arbitrarily large, as one can deform F via homotopy to push F deep into N. On the other hand, if F is closed, totally geodesic and has a bounded genus, then the depth of F in N is bounded above by a constant which is independent of N and M. See for example [Ba1, Ba2]. Here it is important to assume that the genus of F is bounded, as the depth of closed totally geodesic surfaces in a cusp of the figure 8 knot complement is unbounded. (This follows from a theorem of Leininger [Le] and the proof of Theorem 3.1 below.)

We would like to generalize this result to pleated surfaces, which were invented by Thurston in [Th2]. Roughly speaking, a pleated surface is a surface in a hyperbolic 3-manifold M which is obtained from a hyperbolic surface by bending along a geodesic lamination. Because of that, it shares some properties of totally geodesic surfaces; for example, it has the same area as the underlying hyperbolic surface. However, many important properties of totally geodesic surfaces are not shared by pleated surfaces; for example, a totally geodesic surface in M is always π_1-injective, but that is not true for pleated surfaces.

1991 *Mathematics Subject Classification.* Primary 57N10.

Key words and phrases. Pleated surfaces, hyperbolic manifolds, immersed surfaces, Dehn surgery.

[1] Partially supported by NSF grant #DMS 0203394

The following theorem gives an estimation of the depth of closed essential pleated surfaces F in a toroidal cusp N of a hyperbolic manifold M. It shows that if F is not coannular to the boundary torus of a toroidal cusp N then its depth in N is bounded above by a constant which depends only on the genus g of F.

Theorem 1.1. *Let M be a complete hyperbolic 3-manifold, and let N be a toroidal cusp of M. Let F be an essential closed pleated surface of genus g in M which is not coannular to the torus $T = \partial N$ in M. Then $d_N(F) < \ln(2\pi g)$.*

The proof of this theorem will be given in Section 2. One is referred to that section for definitions of cusps and pleated surfaces.

We use the depth estimation of pleated surfaces to study the question of how many Dehn fillings on a toroidal boundary component T_0 of a compact hyperbolic manifold M_0 will preserve the π_1-injectivity of an immersed essential surface F in M_0. A surface $F = (S, \varphi)$ in M_0 is *incompressible* if it is π_1-injective, i.e., $\varphi_* : \pi_1(S) \to \pi_1(M_0)$ is injective. F is *essential* if it is incompressible and is not homotopic to a surface on ∂M_0. F is *coannular to* T_0 if there is an incompressible annulus A in M_0 with one boundary component on each of F and T_0, in which case the slope of $A \cap T_0$ is called a coannular slope of F on T_0.

The answer to this question is well understood if F is embedded. Denote by $M_0(\gamma)$ the manifold obtained from M_0 by Dehn filling on T_0 along a slope γ. It is known that if F is not coannular to T_0, then F remains essential in $M_0(\gamma)$ for all but at most three slopes γ [Wu1], and if F is coannular to T_0 with coannular slope γ_0 then F remains essential in $M_0(\gamma)$ unless $\Delta(\gamma, \gamma_0) \leq 1$, where $\Delta(\alpha, \beta)$ denotes the geometric intersection number between the two slopes [CGLS, Theorem 2.4.3]. Thus in the latter case F remains essential in $M_0(\gamma)$ for all but at most three lines of slopes γ. Similar results hold for essential laminations and essential branched surfaces. See [Wu2].

For immersed essential surfaces, the above theorems can be generalized in a weaker sense. Suppose F is an immersed essential surface in a compact hyperbolic manifold M_0 with $\partial M_0 = T_0$. If F is not coannular to T_0, then there is a constant K such that F remains essential in $M_0(\gamma)$ for all but at most K slopes γ [AR, Ba1]. If F is coannular to T_0 with coannular slopes $\alpha_1, ..., \alpha_n$, then there is a constant K such that F remains essential in $M_0(\gamma)$ when $\Delta(\gamma, \alpha_i) > K$ for all i [Wu3, Theorem 5.3].

The major difference between the embedded and immersed cases is that in the immersed case there is in general no upper bound for the constant K above. It was shown in [Wu3] that for any constant K there is an essential immersed surface F in a hyperbolic manifold M_0 with a coannular slope γ_0 on T_0 such that F is compressible in $M_0(\gamma)$ for all γ satisfying $\Delta(\gamma, \gamma_0) \leq K$. For non-coannular case, Leininger showed in [Le] that for any K there is an essential surface in a hyperbolic manifold M_0 which is totally geodesic (and hence is not coannular to T_0), and is compressible in $M_0(\alpha)$ and $M_0(\beta)$ with $\Delta(\alpha, \beta) > K$. Thus there is no universal upper bound on the geometric intersection number between "killing slopes" for F. It is still an open question whether there is a universal bound for the number of killing slopes for such an F in a hyperbolic manifold M_0, although we will show that no such upper bound exists if M_0 is not assumed hyperbolic. See Theorem 3.7 below.

We use the result about depth of pleated surface in hyperbolic cusps to study the question of whether there exists an upper bound for the constant K which depends only on the genus of the surface F. The following is a simplified version of Corollary 3.2 in the special case that the Dehn filling is performed on a single torus component T_0 of ∂M_0.

Theorem 1.2. *Let M_0 be a compact orientable hyperbolic 3-manifold with T_0 a torus boundary component, and let F be an immersed essential surface in M_0 which is not coannular to ∂M_0. Then there is a constant $C_1(g)$ ($< 3200g^2$) depending only on the genus g of F, such that F remains essential in $M_0(\gamma)$ for all but at most $C_1(g)$ slopes γ on T_0.*

The theorem shows that an upper bound for the constant K above depending only on the genus of F does exist when M_0 is hyperbolic and F is not coannular to T_0. This should be compared with Theorem 3.5, which says that similar bound does not exist in the case that F is coannular to T_0. It will also be shown (Theorem 3.7) that the condition that M_0 be hyperbolic is crucial and cannot be removed from the above theorem. A general version of Theorem 1.2 (for Dehn fillings on multiple boundary components of M_0) and its variations are given in Theorem 3.1, Corollaries 3.2 and 3.3.

The constant $3200g^2$ in Theorem 1.2 is certainly not the best possible. A smaller upper bound in the special case that F is totally geodesic can been found in [Ba1, Ba2].

2. Proof of Theorem 1.1

Let M_0 be a compact, connected, orientable 3-manifold with T a set of tori on ∂M_0, such that $M = M_0 - \partial M_0$ admits a complete hyperbolic structure. Thus the universal covering space of M is the hyperbolic space $\mathbb{H}^3$. We use the upper half space model to identify $\mathbb{H}^3$ with

$$\mathbb{H}^3 = \{(x, y, z) \mid z > 0\}$$

which is endowed with a riemannian metric $ds^2 = (1/z^2)dr^2$, where dr^2 is the Euclidean metric of $\mathbb{R}^3$.

Let $\rho : \mathbb{H}^3 \to M$ be the covering map. Recall that a *horoball* of $\mathbb{H}^3$ is a subset isometric to the set $\{(x, y, z) \mid z \geq 1\}$. The boundary of a horoball is called a *horosphere.* A *cusp* N of M at a torus boundary component T_0 of M_0 is a regular neighborhood of T_0 in M_0 with T_0 removed, such that $\rho^{-1}(N)$ is a set of horoballs in $\mathbb{H}^3$ with disjoint interiors. Each component of ∂M_0 has a unique maximal cusp in M, which has the property that there are a pair of horoballs covering it which intersect each other at some point on their boundary. Given a cusp N, we may assume that $\rho : \mathbb{H}^3 \to M$ has been chosen so that $\tilde{N} = \{(x, y, z) \mid z \geq 1\}$ is a horoball covering N. Let $T = \partial N$, and let $\tilde{T} = \partial \tilde{N}$.

Recall that a surface F in M is a pair $F = (S, \varphi)$, where S is a connected, possibly nonorientable surface, and $\varphi : S \to M$ is a continuous map which is an immersion almost everywhere. A surface F in M is incompressible if it is π_1-injective, i.e., $\varphi_* : \pi_1(S) \to \pi_1(M)$ is injective, in which case we define the fundamental group of F to be $\pi_1 F = \varphi_*(\pi_1(S))$. This is well defined up to conjugacy. We can define some other notions for F in the obvious way. For example a point on F is a pair (x, φ) with $x \in S$, and a loop on F is the composition of φ with a loop on S. Thus if F is transverse to a surface T in M then $F \cap T$ is a set of loops in F. When there is no confusion, we will also refer to the set $F = \varphi(S)$ as a surface.

A *piecewise geodesic arc* in a Riemannian manifold M is an arc α which contains a nowhere dense closed subset X such that the closure of each component of $\alpha - X$ is a geodesic in M. We use the definition given by Thurston in [Th2] for pleated surfaces: A surface $F = (S, \varphi)$ in a hyperbolic manifold M is a *pleated surface* if S is a complete hyperbolic surface, and $\varphi : S \to M$ is a continuous map , such that

(1) φ is an isometry in the sense that every geodesic segment in S is taken to a piecewise geodesic arc in M which has the same length, and
(2) for each point $x \in S$, there is at least one open geodesic segment l_x through x which is mapped to a geodesic segment in M.

The *pleating locus* of F is the set of points on S which do not have a neighborhood mapped by φ to a totally geodesic disk in M. We refer the readers to [Th2] for some basic properties of pleated surfaces. In particular, it is known that the pleating locus of a pleated surface (S, φ) is a geodesic lamination on S, which has measure 0.

Now let M, N and F be as in Theorem 1.1. We may assume that $F = (S, \varphi)$ intersects N nontrivially as otherwise we would have $d_N(F) = 0$ and hence Theorem 1.1 holds. Fix a point $p_0 = \varphi(s_0) \in F \cap N$ such that the distance from p_0 to T equals $d_N(F)$. By definition and assumption S is a closed hyperbolic surface of genus g, so its universal covering $\tilde{S}$ is the hyperbolic plane $\mathbb{H}^2$. Denote by ρ_1 the covering map $\rho_1 : \tilde{S} \to S$. Let $\tilde{s}_0 \in \tilde{S}$ be a point with $\rho_1(\tilde{s}_0) = s_0$, and let $\tilde{\varphi} : \tilde{S} \to \tilde{M}$ be a lifting of φ such that $\tilde{p}_0 = \tilde{\varphi}(\tilde{s}_0)$ is a point in the horoball $\tilde{N} = \{(x, y, z) \mid z \geq 1\}$, and the distance from $\tilde{p}_0$ to the horosphere $\tilde{T} = \partial \tilde{N}$ equals $d_N(F)$. We have the following commutative diagram.

$$\begin{array}{ccc} (\tilde{S}, \tilde{s}_0) & \xrightarrow{\tilde{\varphi}} & (\mathbb{H}^3, \tilde{p}_0) \\ \rho_1 \downarrow & & \downarrow \rho \\ (S, s_0) & \xrightarrow{\varphi} & (M, p_0) \end{array}$$

Note that $\tilde{F} = (\tilde{S}, \tilde{\varphi})$ is a pleated surface in $\tilde{M}$ with pleating locus $\rho_1^{-1}(\lambda)$, where $\lambda \subset S$ is the pleating locus of F.

Lemma 2.1. *Let $\tilde{T}_r = \{(x, y, z) \in \mathbb{H}^3 \mid z = r\}$ be a horosphere in $\mathbb{H}^3$ at level $r \geq 1$ which intersects $\tilde{F}$ transversely. Then each component of $\tilde{F} \cap \tilde{T}_r$ is a circle bounding a disk D in $\tilde{F}$ lying above $\tilde{T}_r$ in $\mathbb{H}^3$.*

Proof. By assumption $F = (S, \varphi)$ intersects the torus $T_r = \rho(\tilde{T}_r)$ transversely, so $\varphi^{-1}(T_r)$ is a set of circles on S. If some circle component γ is mapped by φ to an essential loop on T_r then since T_r is incompressible in M, γ must also be essential on S, so F would be coannular to T_r, contradicting the assumption of Theorem 1.1. Therefore each component of $\varphi^{-1}(T_r)$ is a circle which is trivial on S, and hence bounds a disk D' on S. It follows that $\tilde{\varphi}^{-1}(\tilde{T}_r)$ is a set of disjoint circles on $\tilde{S}$, each of which bounds a disk D in $\tilde{S}$. We need to show that $\tilde{\varphi}(D)$ lies above the horosphere $\tilde{T}_r$.

If this were not true, then since D is compact, there is an *interior* point u of D such that $\tilde{\varphi}(u)$ has minimal z-coordinate in a neighborhood of $\tilde{u}$ in D. On the other hand, since $\tilde{F} = (\tilde{S}, \tilde{\varphi})$ is a pleated surface, there is a geodesic segment β of $\mathbb{H}^3$ lying on $\tilde{\varphi}(\tilde{S})$, containing the point $\tilde{\varphi}(u)$ in its interior. Since a geodesic in

$\mathbb{H}^3$ is either a euclidean half circle perpendicular to the xy-plane or a vertical line parallel to the z-axis, no point of β can have locally minimal z-coordinate, which is a contradiction. $\square$

Lemma 2.2. *Let $R(t)$ be the region on the $\mathbb{H}^2$ above the horizontal line at height 1 and inside the euclidean half circle of radius $t > 1$ centered at the origin, as shown in Figure 1. Denote by $A(R(t))$ the area of $R(t)$. Then*

$$A(R(t)) = 2(\sqrt{t^2-1} - \arctan\sqrt{t^2-1}) > 2t - 6.$$

Proof. We have

$$\begin{aligned} A(R(t)) &= \iint_{R(t)} \frac{1}{z^2}\,dy\,dz = 2\int_1^t \frac{1}{z^2}\,dz \int_0^{\sqrt{t^2-z^2}} dy \\ &= 2\int_1^t \frac{\sqrt{t^2-z^2}}{z^2}\,dz \\ &= 2\left[\arctan\frac{\sqrt{t^2-z^2}}{z} - \frac{\sqrt{t^2-z^2}}{z}\right]_1^t \\ &= 2(\sqrt{t^2-1} - \arctan\sqrt{t^2-1}) \\ &> 2((t-1) - \pi/2) > 2t - 6. \quad \square \end{aligned}$$

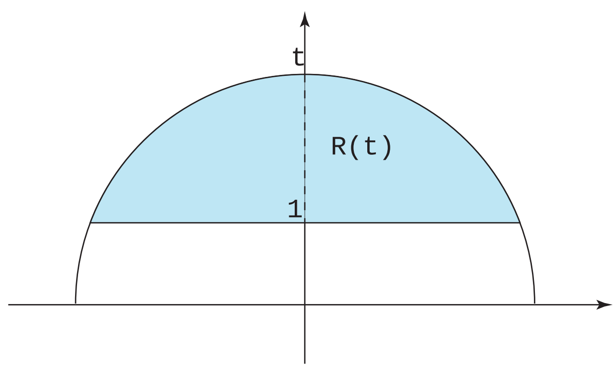

Figure 1

We identify $\mathbb{H}^2$ with the upper half yz-plane in $\mathbb{H}^3 = \mathbb{R}^3_+$, and denote by $\psi : \mathbb{H}^3 \to \mathbb{H}^2$ the euclidean orthogonal projection, i.e.,

$$\psi(x, y, z) = (0, y, z).$$

Denote by $A(F)$ the area of a surface F in $\mathbb{H}^3$. Clearly ψ is area decreasing, i.e., $A(\psi(F)) \le A(F)$ for any surface F in $\mathbb{H}^3$.

Lemma 2.3. *Let D be a disk in $\tilde{N}$ with $\partial D \subset \tilde{T}$. Let α be a proper arc on D such that $\alpha' = \psi(\alpha)$ is a simple arc on $\mathbb{H}^2$. Let β' be a horizontal arc on $\mathbb{H}^2$ joining the two endpoints of α', and let D' be the disk on $\mathbb{H}^2$ bounded by $\alpha' \cup \beta'$. Then $A(D) \geq 2A(D')$.*

Proof. Note that ∂D is mapped into the horizontal line at level $z = 1$ in $\mathbb{H}^2$, hence $\alpha' \cup \beta' = \partial D'$ is a simple closed curve on $\mathbb{H}^2$. The arc α cuts D into D_1 and D_2. Note that ψ maps ∂D_i to a curve on $\mathbb{H}^2$ which is the union of α' and a path on the horizontal line at $z = 1$, hence if u is a point in the interior of D' then the winding number of $\psi(\partial D_i)$ around u is ± 1. It follows that $u \in \psi(D_i)$, in other words, $\psi(D_i)$ contains D'. Since ψ is area decreasing, $A(D') \leq A(\psi(D_i)) \leq A(D_i)$, and the result follows. □

By considering the horosphere with z coordinate $1+\epsilon$ if necessary, we may assume that $\tilde{F}$ is transverse to the horosphere $\tilde{T}$ at level $z = 1$. Let D be the component of $\tilde{F} \cap \tilde{N}$ containing the point $\tilde{p}_0 = (x_0, y_0, z_0)$. By definition of $\tilde{p}_0$, z_0 is maximal among z-coordinates of all points of $\tilde{F}$. By Lemma 2.1 D is a disk.

Lemma 2.4. $A(D) \geq A(R(z_0)) > 2z_0 - 6$.

Proof. Without loss of generality we may assume that $\tilde{p}_0$ is on the z-axis, so $\tilde{p}_0 = (0, 0, z_0)$. Recall that $\tilde{F}$ is a pleated surface. If $\tilde{p}_0$ is on the pleating locus $\tilde{\lambda}$ of $\tilde{F}$, then it is on a geodesic γ of $\mathbb{H}^3$ contained in $\tilde{F}$. If $\tilde{p}_0$ is not on the pleating locus $\tilde{\lambda}$, then it lies in the interior of the closure of a component of $\tilde{F} - \tilde{\lambda}$, which is a totally geodesic surface P in $\mathbb{H}^3$ with boundary a disjoint union of geodesics of $\mathbb{H}^3$. We will call P an ideal polygon, although it may have infinite area.

First assume that $\tilde{p}_0$ is on a geodesic γ of $\mathbb{H}^3$ such that $\gamma \subset \tilde{F}$. Rotating $\mathbb{H}^3$ along the z-axis if necessary, we may assume that γ lies in the upper half yz-plane, which is a hyperbolic plane $\mathbb{H}^2$. Since $\tilde{p}_0 = (0, 0, z_0)$ has maximal z-coordinate on $\tilde{F}$ and $\gamma \subset \tilde{F}$, the curve γ is a euclidean half circle of radius z_0 on $\mathbb{H}^2$ centered at the origin. We now apply Lemma 2.3 to the disk D and the arc $\alpha = \gamma \cap D$. Note that since γ lies on the yz-plane, the projection $\alpha' = \psi(\alpha)$ equals α, which is the part of γ lying above the horizontal line L at level $z = 1$. Thus the disk D' in Lemma 2.3 is exactly the region $R(z_0)$ defined in Lemma 2.2. By Lemma 2.3 we have $A(D) \geq 2A(D') = 2A(R(z_0))$.

Now assume that $\tilde{p}_0$ is in an ideal polygon P of $\mathbb{H}^3$ contained in $\tilde{F}$. Let q be an ideal point of P, and let γ be a geodesic in P passing through $\tilde{p}_0$ and having one ideal endpoint at q. Note that at least one component of $\gamma - \tilde{p}_0$ lies in P, and γ is either disjoint from ∂P (and hence contained in P), or intersects ∂P at a single point. Up to a rotation of $\mathbb{H}^3$ along the z-axis we may assume that γ lies on the yz-plane and the ideal point q is on the negative y-axis. Let R' be the region on $\mathbb{H}^2$ lying on the left of the z-axis, below γ and above the horizontal line $z = 1$. Note that $A(R') = \frac{1}{2}A(R(z_0))$.

If $\gamma \cap \partial P = \emptyset$ then the proof follows as before. So assume γ intersects ∂P at a point u. Then u lies on a geodesic β on ∂P, as shown in Figure 2. Let β_1 and β_2 be the two components of $\beta - u$, and let α_i be the arc $(\gamma \cup \beta_i) \cap D$. From Figure 2 it is clear that at least one of the projection of α_i on $\mathbb{H}^2$ is an arc whose union with an arc on the horizontal line at $z = 1$ bounds a disk D' containing the region R' above. By Lemma 2.3 it follows that $A(D) \geq 2A(D') \geq 2A(R') = A(R(z_0)) > 2z_0 - 6$. □

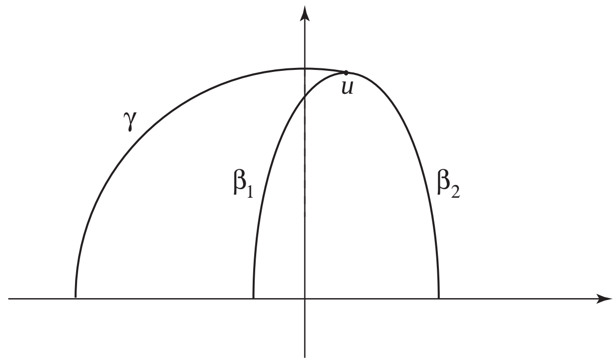

Figure 2

Proof of Theorem 1.1. By Lemma 2.1 each component of $\varphi^{-1}(N)$ is a disk on the hyperbolic surface S. Since $\rho_1 : \tilde{S} \to S$ is a universal covering map, each component of $\rho_1^{-1}(\varphi^{-1}(N))$ is a disk D which maps homeomorphically by ρ_1 to a component D' of $\varphi^{-1}(N)$; therefore

$$A(D) = A(D') \leq A(S) = 2\pi|\chi(S)| = 2\pi(2g-2).$$

Since $\tilde{\varphi} : \tilde{S} \to \mathbb{H}^3$ is a pleated surface, the area of $\tilde{\varphi}(D)$ equals the area of D. Thus by Lemma 2.4 we have

$$2\pi(2g-2) \geq A(D') = A(D) > 2z_0 - 6,$$

where z_0 is the maximum z-coordinate of points on $\tilde{\varphi}(\tilde{S})$. Thus $z_0 < 2\pi g$. Now the hyperbolic distance from a point $p = (x, y, z) \in \mathbb{H}^3$ to the horosphere $\tilde{T}$ at level $z = 1$ is $|\ln z|$. Therefore the depth of F in N satisfies $d_N(F) = \ln z_0 < \ln(2\pi g)$. □

3. Essential surfaces and Dehn filling

Let M_0 be a compact orientable hyperbolic 3-manifold with ∂M_0 a set of tori, and let $M = M_0 - \partial M_0$. A *maximal set of cusps* in M is a set of cusps N in M, one for each end of M, such that (i) the interiors of the cusps are disjoint, and (ii) N is maximal subject to (i). Note that N may not be unique if M_0 has more than one boundary components.

Let N be a set of cusps in M, one for each end of M. Let $T = \partial N$, which is a set of tori in M, and is parallel to ∂M_0 in M_0. The hyperbolic structure of M induces a Euclidean metric on T. The *T-length* $l_T(\gamma)$ of a curve γ on ∂M_0 is defined to be the length of a geodesic on T homotopic to γ in M_0. It is known that if N is maximal and $T = \partial N$ then $l_T(\gamma) \geq 1$ for any nontrivial curve γ on ∂N. See [BH].

A *multiple slope* is a set of essential simple closed curves $\gamma = (\gamma_1, ..., \gamma_k)$, such that there is at most one γ_i for each boundary component of M_0. Define $M_0(\gamma)$ to be the manifold obtained by attaching k solid tori to M_0 such that γ_i bounds a meridian disk in the i-th attached solid torus.

Theorem 3.1. *Let $F = (S, \varphi)$ be a closed incompressible surface of genus g in a compact, orientable, hyperbolic 3-manifold M_0 with ∂M_0 a set of tori, and assume that F is not coannular to ∂M_0. Then there is a constant $C_0(g) < 4\pi^2 g$ depending* *only on g, such that if N is a maximal set of cusps of $M = M_0 - \partial M_0$, and $\gamma = \gamma_1 \cup ... \cup \gamma_k$ is a multiple slope on ∂M_0 such that $l_T(\gamma_i) > C_0(g)$ for all i, then F remains incompressible in $M_0(\gamma)$.*

Proof. This follows from Theorem 1.1 and a theorem of Bart [Ba1, Theorem 1.1], which says F remains essential in $M(\gamma)$ if one can choose a set of cusps disjoint from the projection of the convex hull of the limit set of $\pi_1(F)$, and the length of each γ_i is greater than 2π. We give a proof for the convenience of the readers.

By assumption $M = M_0 - \partial M_0$ admits a complete hyperbolic structure of finite volume. Recall that the group of the surface $F = (S, \varphi)$ is defined to be the subgroup $G = \varphi_*(\pi_1(S))$ of $\pi_1(M)$. By Thurston [Th1], either F is a virtual fiber, or G is geometrically finite; but since F is closed while M is noncompact, F cannot be a virtual fiber. Hence it must be geometrically finite.

Let $H(G)$ be the convex hull of the limit set of G in $\mathbb{H}^3$. Consider the covering space M_1 of M corresponding to the subgroup G. Let $\rho' : \mathbb{H}^3 \to M_1$ and $\rho'' : M_1 \to M$ be the covering maps. Then $\rho'(H(G))$ is a compact set in M_1 homeomorphic to $S \times I$ (or S if G is Fuchsian), and hence $\rho(H(G)) = \rho''(\rho'(H(G)))$ is compact in M. See [Mg]. The boundary of $\rho'(H(G))$ consists of two pleated surfaces F_1', F_2' in M_1 ($F_1' = F_2'$ if G is Fuchsian), which map to pleated surfaces F_1, F_2 in M by ρ''. Since $\rho : \mathbb{H}^3 \to M$ is a covering map, its restriction on $H(G)$ is an immersion; hence the frontier of $\rho(H(G))$ is contained in $\rho(\partial H(G)) = F_1 \cup F_2$. By Theorem 1.1, the depth of F_i in a cusp of M is bounded by $\ln(2\pi g)$, so the depth of $\rho(H(G))$ in N is also bounded by $\ln(2\pi g)$.

For simplicity of notations let us assume $k = 1$. The proof for the general case is similar. Thus N is a cusp adjacent to a torus component T_0 of ∂M_0 containing the Dehn filling slope γ, and the length of γ on $T = \partial N$ is greater than $4g\pi^2$. Let T' be the euclidean torus in the cusp N at depth $\ln(2\pi g)$. By lifting to the universal covering space $\mathbb{H}^3$ one can see that the length of a geodesic curve γ' on T' isotopic to γ in N is greater than $(4g\pi^2)/(2\pi g) = 2\pi$.

Denote by M' the manifold obtained from M by cutting off the cusp bounded by T'. By the 2π theorem of Gromov-Thurston [GT, BH], if γ is a slope on T' with geodesic length greater then 2π then the hyperbolic structure on M' can be extended over the Dehn filling solid torus to obtain a negatively curved metric on the Dehn filled manifold $M'(\gamma)$, which is clearly homeomorphic to $M_0(\gamma)$ by a homeomorphism rel $\rho(H(G))$. Up to homotopy we may assume that F is in $\rho(H(G))$. If α is an essential closed curve on S, then $\varphi(\alpha)$ is homotopic to a geodesic α' in M, which lifts to a geodesic in the convex hull $H(G)$, and hence α' lies in $\rho(H(G))$, which is a subset of M'. It follows that α' remains a geodesic in $M'(\gamma)$, which must be essential because $M'(\gamma)$ is negatively curved. Therefore $F = (S, \varphi)$ is π_1-injective in $M'(\gamma)$, and hence in $M_0(\gamma)$. □

Given a euclidean torus T, there are only finitely many curves of length at most t. In fact, there is a constant $C(t)$, independent of T, such that if the minimum length of closed geodesics on T is 1 then there are at most $C(t)$ simple closed geodesics γ on T with $l_T(\gamma) \le t$. Using the idea in the proof of [BH, Lemma 12] one can show that

$$C(t) \le \frac{\frac{\pi}{2}(t + \frac{1}{2})^2}{\frac{\sqrt{3}}{2}} + 1,$$

so for $t \ge 4\pi^2$ we have $C(t) < 2t^2$. The following result is now an immediate corollary of Theorem 3.1, as it is known that the minimal length of a closed geodesic on the boundary of a maximal set of cusps is at least 1 [BH].

Corollary 3.2. *Let M_0 be a compact orientable hyperbolic 3-manifold with toroidal boundary, and let F be an immersed essential surface of genus g in M_0 which is not coannular to ∂M_0. Then after excluding at most $C(4\pi^2 g)$ ($< 3200g^2$) curves on each component of ∂M_0, F remains essential in all $M(\gamma)$.*

Denote by $\Delta(\gamma_1, \gamma_2)$ the minimal geometric intersection number between two slopes γ_1 and γ_2 on a torus T. There is another constant $C'(t)$ such that if (i) T is a euclidean torus such that the minimal length of closed geodesics on T is at least 1, and (ii) γ_1 and γ_2 are closed geodesics of length at most t on T, then $\Delta(\gamma_1, \gamma_2) < C'(t)$. Using the idea in the proof of [BH, Theorem 14] one can show that

$$C'(t) < \frac{2}{\sqrt{3}} t^2 < 2t^2.$$

Corollary 3.3. *Let M_0 be a compact orientable hyperbolic 3-manifold with toroidal boundary, and let F be an immersed essential surface of genus g in M_0 which is not coannular to ∂M_0. Let γ_1, γ_2 be slopes on a torus component T_0 of ∂M_0. If F is compressible in both $M_0(\gamma_1)$ and $M_0(\gamma_2)$, then $\Delta(\gamma_1, \gamma_2) < C'(4\pi^2 g) < 3200g^2$.*

This should be compared with Leininger's result [Le], which says that there is no universal bound for $\Delta(\gamma_1, \gamma_2)$ if there is no constraint about the genus of F.

In Theorem 3.1 we assumed that the manifold M_0 is hyperbolic, and the surface F is not coannular to ∂M_0. Neither condition can be removed. The following theorem is in [Wu3, Theorem 1.1].

Theorem 3.4. *Let M be a compact, orientable, hyperbolic 3-manifold, and let T be a torus component of ∂M. Let F be a closed essential surface in M.*

(1) If F is coannular to T with coannular slopes $\gamma_1, ..., \gamma_n$, then there is an integer K such that F remains π_1-injective in $M(\gamma)$ for all γ satisfying $\Delta(\gamma, \gamma_i) \geq K$, $i = 1, ..., n$.

(2) There is no universal bound on the constant K in (1): For any constant C, there is a closed essential surface F in a hyperbolic manifold M with coannular slope β on $T = \partial M$, such that F is not π_1-injective in $M(\gamma)$ for all γ with $\Delta(\gamma, \beta) \leq C$.

Part (1) of the theorem is a finiteness theorem, while part (2) shows there is no upper bound in general for the number of lines of surgeries which will kill the essential surface. The genera of the surfaces used in the proof of part (2) increase as C increases. Because of Theorem 3.1, one may wonder if a similar theorem would be true for surfaces coannular to ∂M. More precisely, does there exist a constant $C(g)$ depending only on the genus of F, such that F remains essential in $M(\gamma)$ for all but at most $C(g)$ lines of slopes γ? The following theorem shows that no such constant $C(g)$ exist. Recall that with respect to the standard meridian-longitude pair (m, l) of a knot K in S^3, a slope $\gamma = pm + ql$ on $\partial N(K)$ is represented by a rational number p/q or $1/0$. See [Ro]. Thus $M(\gamma) = M(p/q)$.

Theorem 3.5. *Let K be a non-fibred hyperbolic knot in S^3 with genus g, and let $M = S^3 - \text{Int}N(K)$ be the knot exterior. Then for any constant C there is an immersed essential surface F of genus $2g$ with the longitude of K as a coannular slope, such that F is compressible in $M(\gamma)$ for all $\gamma = p/q$ satisfying $0 < p \leq C$.*

Proof. Let S_1 be a minimal Seifert surface of K in M. Let S be the double of S_1, which is an abstract surface obtained by taking two copies of S_1 joined by an annulus A. Let $\varphi_k : S \to M$ be the map which sends the two copies of S_1 to S_1, and A to an annulus wrapping k times around the torus $T = \partial M$. The surface $F_k = (S, \varphi_k)$ is called a Freedman tubing of S_1 (with wrapping number k). See [FF, CL, Wu3] for more details. Since K is hyperbolic and not fibred, by a theorem of Freedman-Cooper-Long [FF, CL, Li], F_k is incompressible in M when k is sufficiently large. See [Wu3] for an alternative proof.

Lemma 3.6. *If k is a multiple of p then the surface F_k is compressible in $M(p/q)$.*

Proof. Put $k = pr$. Let α be an essential arc on the annulus $\varphi_k(A)$ which wraps k times around the meridian m and 0 times along the longitude l. Then $\alpha \cdot (l^{qr})$ represents $km + (qr)l = r(pm + ql)$ in $\pi_1(T)$, and hence is null-homotopic in the Dehn filling solid torus V. Thus α is homotopic to l^{-qr} in V, so the annulus $\varphi_k(A)$ is rel ∂A homotopic to a degenerate annulus with image on the boundary of the Seifert surface S_1. Therefore the immersed surface F_k is homotopic in $M(p/q)$ to an embedding of S in M with image $\partial N(S_1)$, the boundary of a regular neighborhood of S_1. Since $N(S_1)$ is a handlebody, $\partial N(S_1)$ is compressible in M, hence φ_k is also compressible in $M(p/q)$. □

Now given any positive integer C, let k be a positive integer which is a multiple of all p with $0 < p \le C$. We may assume that C is sufficiently large, so the surface F_k is essential by the Freedman-Cooper-Long Theorem. By Lemma 3.6, the surface F_k is compressible in $M(p/q)$ for all p/q with $0 < p \le C$. This completes the proof of Theorem 3.5. □

The following theorem shows that Theorem 3.1 is false if the manifold M is not assumed to be hyperbolic.

Theorem 3.7. *There exists a non-hyperbolic compact orientable irreducible 3-manifold M with $\partial M = T$ a torus, a fixed closed connected orientable surface S of positive genus, and an immersion $\varphi_{n(k)} : S \to M$ for any positive integer k, such that the essential surfaces $F_k = (S, \varphi_{n(k)})$ satisfy the following conditions.*

(i) F_k is incompressible;

(ii) F_k is not coannular to T;

(iii) There are k distinct slopes $\gamma_1, ..., \gamma_k$ on T such that F_k is compressible in $M(\gamma_i)$ for $i = 1, ..., k$.

Proof. Let K_1 be a hyperbolic knot in S^3 which is not a fibred knot. Let $K = C_{p,q}(K_1)$ be a (p,q) cable knot of K_1, and let $M = S^3 - \text{Int}N(K)$ be the exterior of K. Denote by $V = N(K_1)$ a regular neighborhood of K_1 which contains K as a cable knot. Let $M_1 = S^3 - \text{Int}V$, and let $X = V - \text{Int}N(K)$. Let $T_1 = \partial M_1$. Thus $M = M_1 \cup_{T_1} X$.

To construct the immersed surfaces F_k, let S_1 be a minimal genus Seifert surface of the hyperbolic knot K_1, and let $\varphi_n : S \to M_1$ be a Freedman tubing of two copies of S_1 with wrapping number n. By the proof of Theorem 3.5 we may choose $n(k)$ so that (a) $F_k = (S, \varphi_{n(k)})$ is incompressible, and (b) F_k is compressible in $M_1(\gamma)$ for all $\gamma = r/s$ satisfying $1 \le r \le kpq + 1$. In particular, F_k satisfies condition (i).

Let $\gamma_i = (ipq + 1)/i$. By a theorem of Gordon [Go, Corollary 7.3], we have

$$M(\gamma_i) = M(\frac{ipq+1}{i}) = M_1(\frac{ipq+1}{iq^2}).$$

By definition, F_k is compressible in $M_1((ipq+1)/iq^2)$ for all i satisfying $1 \le i \le k$. Therefore F_k satisfies condition (iii). It remains to show that F_k is not coannular to T.

First notice that the only coannular slope of F_k in the manifold M_1 is the longitude. This is because F_k lifts to a closed surface in the universal abelian covering of M_1, so any closed curve on F_k lifts, but the longitude is the only primitive closed curve on $T_1 = \partial M_1$ that lifts.

Now suppose A is an essential annulus in M with one boundary component in each of F and T. By a homotopy we may assume that A is transverse to T_1, and the number of components of $A \cap T_1$ is minimal. Since T_1 is incompressible in M, each component of $A \cap M_1$ and $A \cap X$ is an essential annulus in M_1 or X, except the one which has a boundary component on F, denoted by A_1. By the above, $\beta = A_1 \cap T_1$ is a longitude of K_1. On the other hand, the component A_2 of $A \cap X$ which has β as a boundary component is an essential annulus in X, and it is well known that such an annulus is isotopic to a union of fibers [Ja, Theorem VI.34]; in particular, the longitude β of K_1 is a fiber of X. This is a contradiction because after trivial filling along the meridional slope m on T the Seifert fibration on X extends to a Seifert fibration of the solid torus $X(m)$ containing K as a regular fiber, so a fiber on T_1 has slope p/q. $\square$

References

[AR] I. Aitchison and J.H. Rubinstein, *Incompressible surfaces and the topology of 3-dimensional manifolds*, J. Austral. Math. Soc. **55** (1993), 1–22.

[Ba1] A. Bart, *Surface groups in some surgered manifolds*, Topology **40** (2001), 197–211.

[Ba2] ———, *Surface groups in Bianchi groups*, Preprint.

[BH] S. Bleiler and C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), 809–833.

[CL] D. Cooper and D. Long, *Virtually Haken surgery on knots*, J. Diff. Geom. **52** (1999), 173–187.

[CGLS] M. Culler, C. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Ann. Math. **125** (1987), 237–300.

[FF] B. Freedman and M. Freedman, *Kneser-Haken finiteness for bounded* 3*-manifolds locally free groups, and cyclic covers*, Topology **37** (1998), 133–147.

[Go] C. Gordon, *Dehn surgery and satellite knots*, Trans. Amer. Math. Soc. **275** (1983), 687–708.

[GT] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), 1–12.

[Ja] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics 43, 1980.

[Le] C. Leininger, *Compressing totally geodesic surfaces*, Topology Appl. **118** (2002), 309–328.

[Li] T. Li, *Immersed essential surfaces in hyperbolic 3-manifolds*, Comm. Anal. Geom. **10** (2002), 275–290.

[Mg] J. Morgan, *Uniformization theorem for three-dimensional manifolds*, The Smith Conjecture, Academic Press, 1984, pp. 37–125.

[Ro] D. Rolfsen, *Knots and Links*, Publish or Perish, 1990.

[Th1] W. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University, 1978.

[Th2] ———, *Hyperbolic structures on* 3*-manifolds I. Deformation of acylindrical manifolds*, Annals Math. **124** (1986), 203–246.

[Wu1] Y-Q. Wu, *Incompressibility of surfaces in surgered 3-manifolds*, Topology **31** (1992), 271–279.

[Wu2] ———, *Essential laminations in surgered 3-manifolds*, Proc. Amer. Math. Soc. **115** (1992), 245–249.

[Wu3] ______, *Immersed essential surfaces and Dehn surgery*, Topology **43** (2004), 319-343.

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: wu@math.uiowa.edu