STANDARD GRAPHS IN LENS SPACES

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We prove a conjecture of Menasco and Zhang that if a tangle is completely tubing compressible then it consists of at most two families of parallel strands. This is related to problems of graphs in 3-manifold. The above conjecture follows from a theorem which says that a 1-vertex graph in M is standard in certain sense if and only if the exteriors of all its nontrivial subgraphs are handlebodies.

1. Introduction

In this paper, a tangle is a pair (W, t), where W is a compact orientable 3-manifold with ∂W a sphere, and $t = \alpha_1 \cup ... \cup \alpha_n$ a set of mutually disjoint properly embedded arcs in W, called the strands. Denote by $N(t)$ a regular neighborhood of t, and by $\eta(t) = \text{Int}N(t)$ an open neighborhood of t. Let $X = X(t)$ be the tangle space $W - \eta(t)$, and let P be the planar surface $\partial W \cap X = \partial W - \eta(\partial t)$. Denote by A_i the annulus $\partial N(\alpha_i) \cap X$. Thus $\partial X = P \cup (\cup A_i)$. The A_i-*tubing* of P is the surface $F_i = P \cup A_i$. Following Gordon [G], we say that a set of curves $\{c_1, ..., c_k\}$ on the boundary of a handlebody H is *primitive* if there exist disjoint disks $D_1, ..., D_k$ in H such that ∂D_i intersects $\cup c_j$ transversely at a single point lying on c_i. A set of annuli is primitive if their core curves form a primitive set. The surface P is A_i-*tubing compressible* if F_i is compressible, and it is *completely A_i-tubing compressible* if F_i can be compressed until it becomes a set of annuli parallel to $\cup_{j \neq i} A_j$. Equivalently, P is completely A_i-tubing compressible if X is a handlebody, and the set of annuli $\cup_{j \neq i} A_j$ is primitive on ∂X.

A tangle (W, t) is *completely tubing compressible* if the surface P above is completely A_i-tubing compressible for all i. Such tangles arise naturally in the study of reducible surgery on knots. For example, it follows from the proof of Propositions 2.2.1 and 2.3.1 of [CGLS] that if some surgery on a hyperbolic knot K produces a nonprime manifold M, then either the knot complement contains a closed essential surface, or there is a reducing sphere S cutting (M, K') into two non-split completely tubing compressible tangles, where K' is the core of the Dehn filling solid torus.

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Define a *band* in W to be an embedded disk D in W such that $D \cap \partial W$ consists of two arcs on ∂D. A subcollection of strands $t' = \{\alpha_1, ..., \alpha_k\}$ of t is *parallel* if there is a band D such that $D \cap t = t'$.

Define a *core arc* to be an arc α in W such that $W - \eta(\alpha)$ is a solid torus. Because of uniqueness of Heegaard splittings of S^3, $S^2 \times S^1$ and lens spaces, it is easy to see that W has at most two core arcs up to isotopy, and one if W is a punctured S^3, $S^2 \times I$ or $L(p, 1)$. However, a set of core arcs may contain arbitrarily many parallel families. This is the same phenomenon as links in S^3: A link L of n components may have the property that all of its components are trivial knot (so the components are isotopic to each other in S^3), but the components of L are mutually non-parallel in the sense that they do not bound an annulus with interior disjoint from the link. The following theorem proves a conjecture of Menasco and Zhang [**MZ**, Conjecture 5], which shows that this phenomenon will not happen if (W, t) is a completely tubing compressible tangle.

Theorem 1. *If (W, t) is a completely tubing compressible tangle, then t consists of at most two families of parallel core arcs.*

The problem is related to graphs in 3-manifolds. Let $M = \hat{W}$ be the union of W and a 3-ball B, and let $\Gamma = \hat{t}$ be the union of t and the straight arcs in B connecting ∂t to the central point v of B. Thus we have a graph $\hat{t}$ in the closed 3-manifold $\hat{W}$ with one vertex v and n edges $e_1, ..., e_n$ corresponding to the arcs $\alpha_1, ..., \alpha_n$ of t. A graph Γ is *nontrivial* if it contains at least one edge. The *exterior* of a graph Γ in a 3-manifold M is $E(\Gamma) = M - \eta(\Gamma)$. It will be shown (see Lemma 8 below) that the tangle (W, t) is completely tubing compressible if and only if the exterior of any nontrivial subgraph of $\hat{t}$ in $\hat{W}$ is a handlebody. Thus the classification problem for completely tubing compressible tangles becomes a classification problem for 1-vertex graphs Γ in a 3-manifold M which have the property that the exteriors of all its nontrivial subgraphs are handlebodies.

Since the exterior of a regular neighborhood of an edge of Γ is a solid torus, M has a Heegaard splitting of genus 1, hence it must be S^3, $S^2 \times S^1$, or a lens space $L(p, q)$. Since $L(p, q) \cong L(p, -q) \cong L(p, p-q)$ up to (possibly orientation reversing) homeomorphism, we may always assume that $1 \le q \le p/2$. When M is S^3, it follows from [**G**, Theorem 1] that the complement of any subgraph of Γ is a handlebody if and only if Γ is planar, i.e., it is contained in a disk in S^3. Scharlemann and Thompson [**ST**] generalizes this to all abstractly planar graphs in S^3. See also [**W2**] for an alternative proof. For the general case, we need the following definitions.

A *v-disk* D in M is the image of a map $f : D^2 \to M$ such that f is an embedding except that it identifies two boundary points of D^2 to a point v in M. The boundary of D is $\partial D = f(\partial D^2)$. A v-disk D in a solid torus V is *standard* if (i) $D \cap \partial V = v$, and (ii) D is rel v isotopic to a v-disk D'

on ∂V, which is longitudinal in the sense that there is a meridional disk Δ of V such that $D' \cap \Delta$ is a nonseparating arc on D'. We remark that it is important to require that the above isotopy be relative to v as it guarantees that the exterior of D is a handlebody.

A graph with a single vertex is called a *1-vertex graph*. Such a graph is connected, and all of its edges are loops. A 1-vertex graph $\Gamma = e_1 \cup ... \cup e_k$ in V with vertex v is *in standard position* if it is contained in a standard v-disk D in V. In this case we also say that the edges of Γ are parallel.

Let $V_1 \cup V_2$ be a genus one Heegaard splitting of a closed 3-manifold M. Then a 1-vertex graph Γ in M is *in standard position* (relative to the Heegaard splitting) if either (i) M is homeomorphic to S^3, $S^2 \times S^1$ or $L(p,1)$, and Γ is contained in a single standard v-disk in V_1 or V_2, or (ii) M is homeomorphic to $L(p,q)$ with $2 \leq q < p/2$, and Γ is contained in two standard v-disks, one in each V_i. A 1-vertex graph Γ in M is *standard* if it is isotopic to a graph in standard position. Since genus one Heegaard splittings of 3-manifolds are unique up to isotopy [**Wa**, **BO**, **S**], this is independent of the choice of (V_1, V_2). The following theorem characterizes standard graphs in 3-manifolds.

Theorem 2. *A nontrivial 1-vertex graph Γ in a closed orientable 3-manifold M is standard if and only if the exterior of any nontrivial subgraph of Γ is a handlebody.*

It should be noticed that the 3-manifold M in the theorem must be S^3, $S^2 \times S^1$, or a lens space. For if Γ is standard then by definition M has a genus one Heegaard splitting. On the other hand, if the exterior of any nontrivial subgraph of Γ is a handlebody, then in particular the exterior of an edge of Γ is a solid torus, so again M has a genus one Heegaard splitting. Therefore M must be one of the above manifolds.

I would like to thank Menasco and Zhang for raising the problems, and for posting the insightful conjecture [**MZ**, Conjecture 5].

2. Proof of the Theorems

The following lemma proves the easy direction of Theorem 2.

Lemma 3. *If a 1-vertex graph Γ in a 3-manifold M is standard, then the exterior of any nontrivial subgraph Γ' of Γ is a handlebody.*

Proof. Clearly a subgraph of Γ is still standard, hence we need only prove the lemma for $\Gamma' = \Gamma$. Let (V_1, V_2) be a genus one Heegaard splitting of M, and assume that Γ is contained in the union of $D_1 \cup D_2$, where D_i is a standard v-disk in V_i. (The case that Γ is contained in a single standard v-disk is similar and simpler.) Put $\Gamma_1 = \Gamma \cap D_1 = e_1 \cup ... \cup e_{r-1}$ and $\Gamma_2 = \Gamma \cap D_2 = e_r \cup ... \cup e_n$.

From definition one can see that the manifold $V_i - \eta(D_i)$ is a product $F_i \times I$, where F_i is a once punctured torus. Therefore $X = M - \eta(D_1 \cup D_2)$ is still a product of I and a once punctured torus, which is a handlebody. One can choose a regular neighborhood $N(\Gamma)$ of Γ in M so that it is contained in $N(D_1 \cup D_2)$, and the closure of each component of $N(D_1 \cup D_2) - N(\Gamma)$ is a 3-ball H_i intersecting $\partial N(D_1 \cup D_2)$ at two disks. Now $M - \eta(\Gamma)$ is the union of X and the H_i. Since each H_i can be considered as a 1-handle attached to X, it follows that $M - \eta(\Gamma)$ is a handlebody. □

The following lemma proves the other direction of Theorem 2 under an extra assumption, which by [**MZ**, Lemma 1] implies that $M = S^3$ or $S^2 \times S^1$.

Lemma 4. *Let* Γ *be a 1-vertex graph in a closed orientable 3-manifold such that the exterior of any nontrivial subgraph of* Γ *is a handlebody. Let* $W = M - \eta(v)$*, and* $X = M - \eta(\Gamma)$*. If* $P = \partial W \cap X$ *is compressible, then* Γ *is standard.*

Proof. Let D be a compressing disk of P. First assume that D is separating in W, cutting W into W_1 and W_2. Let Γ_i be the subgraph of Γ consisting of edges whose intersection with W is contained in W_i. Each Γ_i is nontrivial as otherwise ∂D would be trivial on P, contradicting the fact that it is a compressing disk. Now W_i is contained in the exterior of Γ_j $(j \neq i)$, which by assumption is a handlebody. Since $\partial W_i = S^2$ and handlebodies are irreducible, it follows that W_i are 3-balls, hence W is also a 3-ball, so $M = S^3$. In this case by [**G**, Theorem 1] or [**ST**], the graph Γ is planar in S^3, which is easily seen to be equivalent to the condition that it is standard.

Now assume the D is non-separating in W. In this case W cannot be a 3-ball or punctured lens space, so it must be a punctured $S^2 \times S^1$, and D cuts W into $W' = S^2 \times I$. The manifold $X' = W' - \eta(t)$ is obtained from X by cutting along a nonseparating disk D, so it is a handlebody of genus $n-1$, and attaching 2-handles to any proper subset of $\cup A_i$ yields a handlebody. By [**G**, Theorem 2], the set $\cup A_i$ is standard on $\partial X'$, which implies that there is a band $D' = C \times I$ in $W' = S^2 \times I$ containing $t = \Gamma \cap W$. It is clear that such a band D extends to a standard v-disk D'' in $M = S^2 \times S^1$ containing Γ. □

A *trivial arc* in a solid torus V is one which is rel ∂ isotopic to an arc on ∂V. Let γ be a (p,q)-curve on $T = \partial V$. A properly embedded arc α in V is *γ-trivial* if it lies on a meridian disk D of V such that ∂D intersects γ at p points. One can show that α is γ-trivial if and only if it is isotopic to an arc α' on T which intersects γ always in the same direction. When α is γ-trivial, the *jumping number* of α with respect to γ, denoted by $u = u(\alpha, \gamma)$, is defined to be the smallest intersection number between α' and γ for $\alpha' \subset T$ isotopic to α. It should be noticed that not all trivial arcs in V are γ-trivial. Put $X = V - \mathrm{Int} N(\alpha)$, and denote by $X[\gamma]$ the manifold obtained by attaching a

2-handle to X along γ. The following theorem is essentially [**W3**, Theorem 2.2]. It characterizes trivial arcs α in V such that $X[\gamma]$ is a solid torus.

Lemma 5. *Let α be a trivial arc in a solid torus V, and let γ be a (p,q)-curve on $T = \partial V$ disjoint from α, $0 < q \leq p/2$. Let $X = V - \text{Int}N(\alpha)$. Then $X[\gamma]$ is a solid torus if and only if (i) α is γ-trivial, and (ii) the jumping number $u(\alpha, \gamma)$ equals 1 or q.* □

Lemma 6. *Theorem 2 is true if $M = L(p,q)$ and Γ has at most two edges.*

Proof. If Γ has only one edge e_1, then $V_1 = N(e_1)$ and $V_2 = M - \text{Int}V_1$ form a genus one Heegaard splitting of $L(p,q)$. By an isotopy we may deform e_1 to standard position in V_1, and the result follows.

We now assume that $\Gamma = e_1 \cup e_2$. Let $V_1 = N(e_1)$, and $V_2 = M - \text{Int}V_1$, which by assumption is a solid torus. Since e_2 intersects e_1 at the vertex v of Γ, we may assume that $e_2 \cap V_1$ is an unknotted arc lying on a meridional disk D' of V_1. Let D be another meridional disk of V_1 disjoint from D', and let γ be the curve ∂D on $T = \partial V_i$. Since M is a lens space $L(p,q)$, γ is a (p,q) curve on T with respect to some longitude-meridian pair of V_2. Let α be the embedded arc $e_2 \cap V_2$ in V_2. The boundary of α lies on the curve $\gamma' = \partial D'$, which is a parallel copy of γ.

Note that $V_2 - \eta(\alpha) = M - \eta(\Gamma)$, so by assumption it is a handlebody, denoted by H. The frontier of $N(\alpha)$ (i.e. $N(\alpha) \cap H$) is an annulus A which must be primitive on H because when attaching the 2-handle $N(\alpha)$ to H along A we obtain the solid torus V_2. It follows that the core curve α of the attached 2-handle $N(\alpha)$ is a trivial arc in V_2.

Let β be an arc on γ' connecting the two endpoints of α. Then β is isotopic to the arc $e_2 \cap V_1$ on the disk D', hence the curve $\alpha \cup \beta$ is isotopic to e_2, which by assumption has exterior a solid torus in $L(p,q)$. Therefore by Lemma 5 α is γ-trivial, and the jumping number $j(\alpha, \gamma)$ is either 1 or q. By definition α is isotopic rel ∂ to an arc α' on T intersecting γ transversely at $j(\alpha, \gamma)$ points in the same direction.

First assume that $j(\alpha, \gamma) = 1$. Then $e_2' = \alpha' \cup \beta$ is a simple closed curve on T intersecting the meridian curve γ of V_1 transversely at a single point, hence it is a longitude of V_1. Since β lies on $\partial D'$ and $e_2 \cap V_1$ is an arc on D', there is an isotopy of $\Gamma \cap V_1$ in V_1 such that $e_2 \cap V_1$ is deformed to the arc β, and e_1 to a loop e_1' in standard position in V_1. The isotopy deforms Γ to the graph $\Gamma' = e_1' \cup e_2'$, with a single vertex v' on T. Since e_2' is a longitude on ∂V_1 and e_1' is in standard position, $e_1' \cup e_2'$ bounds a v'-disk Δ in V_1. Pushing $\Delta - v'$ to the interior of V_1 deforms Γ' to a graph in standard position, hence the result follows.

Now assume that $j(\alpha, \gamma) = q > 1$. Choose a meridional disk D_2 of V_2 containing α', intersecting γ at p points. Since γ' is a (p,q) curve, and the jumping number of α is q, we can choose the arc β on γ' with $\partial\beta = \partial\alpha$ so that the interior of β is disjoint from ∂D_2, hence $e_2'' = \alpha' \cup \beta$ is a longitude of V_2. By an isotopy of $\Gamma \cap V_1$ we can deform $e_2 \cap V_1$ to β, and e_1 to a loop e_1' in standard position in V_1. Let $v' = e_1' \cap e_2''$. By an isotopy rel v' we can deform e_2'' to an edge e_2' in V_2, which by definition is in standard position in V_2 because e_2'' is a longitude of V_2. It follows that Γ is isotopic to the graph $\Gamma' = e_1' \cup e_2'$ in standard position, hence Γ is standard. □

Suppose F is a surface on the boundary of a 3-manifold X, and c a simple closed curve in F. Denote by X_c the manifold obtained from X by attaching a 2-handle to X along c, and by F_c the corresponding surface in X_c. More explicitly, $X_c = X \cup_\varphi (D^2 \times I)$, where φ identifies $\partial D^2 \times I$ to a regular neighborhood A of c in F, and $F_c = (F - A) \cup (D^2 \times \partial I)$. We need the following version of the handle addition lemma.

Lemma 7. *Let F be a surface on the boundary of a 3-manifold X, and K a 1-manifold in F with $F - K$ compressible in X. Let c be a simple loop in $F - K$. If F_c has a compressing disk Δ in X_c, then $F - c$ has a compressing disk Δ' in X such that $\partial\Delta' \cap K \subset \partial\Delta \cap K$.*

Proof. This was proved in [**W1**]. Theorem 1 of [**W1**] says that under the assumption of the lemma we have $|\partial\Delta' \cap K| \le |\partial\Delta \cap K|$, but that was proved by showing that $\partial\Delta' \cap K \subset \partial\Delta \cap K$. Note that when $K = \emptyset$, it reduces to Jaco's Handle Addition Lemma [**J**, Lemma 1]. □

Lemma 8. *A tangle (W, t) is completely tubing compressible if and only if the exterior of any nontrivial subgraph of $\hat{t}$ in $\hat{W}$ is a handlebody.*

Proof. Let A_i be the annulus $\partial N(\alpha_i) \cap \partial X$. The exterior of a subgraph Γ' of $\Gamma = \hat{t}$ in $\hat{W}$ is the same as the exterior of the corresponding strands of t in W, which can be obtained from $X = W - \eta(t)$ by attaching 2-handles to those annuli A_i corresponding to the edges e_i in $\Gamma - \Gamma'$. Therefore the condition that the exterior of any nontrivial subgraph of $\hat{t}$ in $\hat{W}$ is a handlebody implies that attaching 2-handles to X along any proper subset of $\cup A_i$ yields a handlebody. By [**G**, Theorem 1] this implies that any proper subset of $\cup A_i$ is a primitive set on ∂X. Hence (W, t) is completely tubing compressible.

On the other hand, if (W, t) is completely tubing compressible, and Γ' is a proper subgraph of Γ which does not contain the edge e_i, say, then the set $\cup_{j \ne i} A_j$ is primitive on ∂X, and since the exterior $E(\Gamma')$ of Γ' can be obtained by attaching 2-handles to X along a subset of primitive set $\cup_{j \ne i} A_j$, it follows that $E(\Gamma')$ is a handlebody. □

Proof of Theorem 2. By Lemma 3 we need only show that if the exterior of any nontrivial subgraph of Γ is a handlebody then Γ is standard. Put $W = M - \eta(v)$, $t = W \cap \Gamma$, $X = M - \eta(\Gamma) = W - \eta(t)$, and $P = \partial W \cap X$. By Lemma 4 we may assume that P is incompressible, so by Lemma 8 and [**MZ**, Lemma 1], the manifold M is a lens space $L(p, q)$. Up to homeomorphism we may assume $1 \le q \le p/2$.

By Lemma 6 we may assume that $n \geq 3$, and by induction we may assume that any nontrivial proper subgraph of Γ is standard. In particular, each e_i is standard in M, so it is isotopic to a core of either V_1 or V_2. Since $n \geq 3$, at least two of the e_i are cores of the same V_j, hence up to relabeling we may assume without loss of generality that e_1 and e_2 are both isotopic to a core of V_2.

Consider the graph $\Gamma' = e_1 \cup ... \cup e_{n-1}$. By induction Γ' is standard, so the edges are contained in two v-disks if $M = L(p, q)$ with $2 \leq q < p/2$, and one v-disk otherwise. Notice that in the first case the core of V_1 is homotopic to q times the core of V_2, so they represents different elements in $\pi_1 M$. Since by assumption e_1 and e_2 are isotopic to the core of V_2, it follows that they are on the same v-disk. In either case there is a v-disk D_1 containing both e_1 and e_2. Taking a subdisk bounded by $e_1 \cup e_2$ and pushing its interior off D_1, we get a v-disk D_2 bounded by $e_1 \cup e_2$ with interior disjoint from Γ'. Note that D_2 may intersect e_n. However, the following lemma says that D_2 can be rechosen to have interior disjoint from e_n as well.

Lemma 9. *There is a v-disk D_3 bounded by $e_1 \cup e_2$ with interior disjoint from Γ.*

Proof. Consider the handlebody $X = M - \eta(\Gamma)$. Let c_i be the meridian curve of e_i on $F = \partial X$, and put $C = \{c_1, ..., c_n\}$. Let $K = c_1 \cup ... \cup c_{n-1}$. By Lemma 2, the tangle (W, t) is completely tubing compressible, so K is a primitive set on ∂X, hence $F - K$ is compressible. We now apply Lemma 8 to (X, F, K, c) with $c = c_n$. Note that after attaching a 2-handle to c_n, the manifold $X' = X_{c_n}$ is the same as the exterior of the graph $\Gamma' = e_1 \cup ... \cup e_{n-1}$, and the surface $F_{c_n} = \partial X'$.

Recall that $e_1 \cup e_2$ bounds a v-disk D_2 in M with interior disjoint from Γ', so its restriction to $X' = X_{c_n}$ is a compressing disk Δ of $\partial X' = F_{c_n}$ intersecting each of c_1 and c_2 at a single point, and is disjoint from $c_3, ..., c_{n-1}$. Therefore, by Lemma 8, there is a compressing disk Δ' of $F - c_n$ in X, such that $\partial \Delta'$ intersects each of c_1 and c_2 at most once, and is disjoint from $c_3, ..., c_{n-1}$. Since it is a compressing disk of $F - c_n$, it is also disjoint from c_n.

Now $\partial \Delta'$ cannot be disjoint from C, because we have assumed that the surface P homotopic to $F - C$ is incompressible. Also, $\partial \Delta' \cap C$ cannot be a single point in c_1, say, because then the frontier of a regular neighborhood of $\Delta' \cup c_1$ would be a compressing disk of $F - C$, which is again a contradiction. It follows that $\partial \Delta'$ intersects each of c_1 and c_2 at exactly one point, and is disjoint from the other c_j's. Since Γ is a spine of $N(\Gamma)$, by shrinking $N(\Gamma)$ to Γ, the disk Δ' becomes a v-disk D_3 in M bounded by $e_1 \cup e_2$, with interior disjoint from Γ. This completes the proof of the sublemma. □

We now continue to show that Γ is standard in M. By induction we may assume that $\Gamma'' = e_2 \cup ... \cup e_n$ is in standard position in $M = V_1 \cup V_2$, with e_2

on a v-disk D' in V_2, say, which contains all the edges of Γ'' in V_2. Consider the disk D_3 bounded by $e_1 \cup e_2$ as given by the sublemma. It has interior disjoint from Γ, so by considering $D_3 \cap D'$ and using an innermost circle outermost arc argument one can show that D_3 can be modified so that it intersects D' only along the edge e_2. Pushing the part of D_3 near e_2 slightly off e_2, we get a v-disk D_4 with boundary the union of e_1 and a loop e_1' on D', which is a parallel copy of e_2 intersecting Γ only at v. One can then isotop e_1 via the disk D_4 to the edge e_1', which lies on the v-disk D'. Thus after this isotopy all edges of Γ are now contained in the v-disks which contain Γ''. Therefore Γ is also standard by definition. □

Proof of Theorem 1. Suppose (W, t) is completely tubing compressible. Then by Lemma 8 the corresponding graph $\hat{t}$ in $\hat{W} = W \cup B$ has the property that the exterior of any proper subgraph of $\hat{t}$ is a handlebody. By Theorem 2, $\hat{t}$ is contained in the union of at most two v-disks D_1 and D_2, with D_i in V_i. By an isotopy rel $\hat{t}$ we may assume that $D_i \cap \partial W$ consists of two arcs, hence $D_1 \cap W$ and $D_2 \cap W$ are two disjoint bands in W containing t, and the result follows. □

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