THE CLASSIFICATION OF DEHN FILLINGS ON THE OUTER TORUS OF A 1-BRIDGE BRAID EXTERIOR WHICH PRODUCE SOLID TORI

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Abstract. Let $K = K(w, b, t)$ be a 1-bridge braid in a solid torus V, and let γ be a (p, q) curve on the torus $T = \partial V$ of the exterior M_K of K. It will be shown that Dehn filling on T along γ produces a solid torus if and only if p and q satisfy one of four conditions determined by the parameters (w, b, t) of the knot K. This solves the classification problem raised by Menasco and Zhang for such Dehn fillings.

1. Introduction

A knot K in a 3-manifold M is a 0-bridge knot if it is isotopic to a simple closed curve on ∂M, and it is a *1-bridge knot* if it is not 0-bridge, and is isotopic to a curve $\alpha \cup \beta$, where $\alpha \subset \partial M$, and β is a trivial arc in M in the sense that it is rel ∂ isotopic to an arc on ∂M. These knots have been related to many examples of exceptional Dehn surgery on hyperbolic knots, and have been studied quite extensively, see for example [Ga1, Ga2, Be, Eu, MZ, Wu1]. In particular, it was proved by Gabai [Ga1, Ga2] that if some surgery on a knot in a solid torus yields a solid torus then the knot must be a 1-bridge braid. Berge completely classified all such surgeries in [Be]. Menasco and Zhang [MZ] studied Dehn fillings on the outer torus of 1-bridge braids, showing that some of those Dehn fillings produce solid tori. The main purpose of this paper is to solve a problem raised in their paper [MZ, Problem 7], which asked for a complete classification of all such Dehn fillings.

Let B_w be the braid group on w strands, and let σ_i be the standard generators of B_w. See [Bi] for definitions. A braid σ is represented by a set of w strings in $D^2 \times I$. Thus when gluing $D^2 \times 0$ to $D^2 \times 1$, we obtain the closure of σ, which is a knot or link in the solid torus $V = D^2 \times S^1$. A *1-bridge braid* is a knot $K = K(w, b, t)$ in V which is the closure of the braid

$$\sigma(w, b, t) = \sigma_b \cdots \sigma_2 \sigma_1 (\sigma_{w-1} \cdots \sigma_2 \sigma_1)^t.$$

See Figure 1.1 for the braid $\sigma(7, 4, 2)$. Note that not all 1-bridge knots in V are 1-bridge braids. Note also that if $b = 0$ or $w-1$ then $K(w, b, t)$ is isotopic to a curve on $T = \partial V$, hence is a 0-bridge knot, in which case the exterior of K is a cable space. Dehn filling on cable spaces are well understood, see [Go]. Thus we will restrict our attention to $K(w, b, t)$ with $1 \le b \le w-2$. Up to homeomorphism of V obtained by twisting along meridional disks, we may also assume that $1 \le t \le w-1$.

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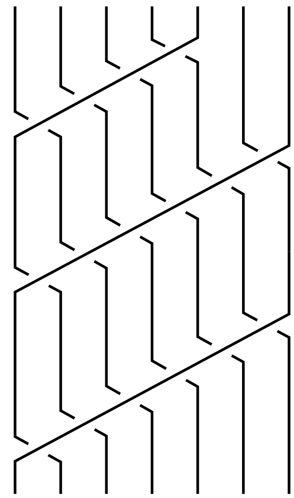

Figure 1.1

Let γ be a (p, q) curve on T with respect to the standard longitude-meridian pair of V. Let $\alpha \cup \beta$ be a 1-bridge presentation of a 1-bridge knot K in V. Thus $\alpha \subset T$, and β is a trivial arc in V. Up to isotopy we may assume that α is disjoint from γ. It can be shown (Lemma 2.5) that the manifold $M_K(\gamma)$ obtained by performing a Dehn filling on the exterior M_K of K along the curve γ is homeomorphic to the manifold $X[\gamma]$ obtained by attaching a 2-handle to the genus 2 handlebody $X = V - \mathrm{Int}N(\beta)$ along the curve γ. Thus the problem of determining which $M_K(\gamma)$ is a solid torus is the same as to determine which $X[\gamma]$ is a solid torus.

A trivial arc β in a solid torus is *γ-trivial* if it is isotopic rel ∂ to an arc β' on T intersecting γ always in the same direction, in which case the minimal intersection number between all such β' and γ is called the *jumping number* of β with respect to γ. See Definition 2.1 for more details. The main theorem of Section 2 states that $X[\gamma]$ is a solid torus if and only if β is γ-trivial, and the jumping number of β with respect to γ is either 1 or $\min(q, p - q)$. See Theorem 2.2. The result will be used in [Wu2] to classify completely tubing compressible tangles.

Because of the relation between $M_K(\gamma)$ and $X[\gamma]$, the above theorem gives a necessary and sufficient condition for $M_K(\gamma)$ to be a solid torus. See Corollary 2.6. It will also be shown (Theorem 2.8) that $M_K(\gamma)$ is reducible if and only if K is a cable of a knot parallel to γ. We remark that in general $M(\gamma)$ may not be a solid torus for any γ since we are performing Dehn filling on the outer torus. Because of that, the powerful Reducible Surgery Theorem of Scharlemann [Sch, Theorem 6.1] does not apply to this situation.

The results in Section 2 will be used in Section 3 to classify all 1-bridge knots in V which admits a Dehn filling on the outer torus producing a solid torus. We will also determine all such Dehn filling slopes for a given 1-bridge braid $K(w, b, t)$. It will be shown that $M_K(\gamma)$ is a solid torus if and only if p and q satisfy one of four conditions determined by the parameters of the 1-bridge braid $K = K(w, b, t)$. See Theorem 3.6. This solves the classification problem raised by Menasco and Zhang for such Dehn fillings [MZ, Problem 7]. Some computational results based on these theorems will be given at the end of that section.

We remark that 1-bridge knots are not the only ones which may admit some solid torus Dehn fillings on the outer tori of their exteriors. Given a knot K in a solid torus V and a slope γ on ∂V, let K' be the core of the Dehn filling solid torus in $V(\gamma)$. Then $M_K(\gamma)$ is a solid torus if and only if the link $L = K \cup K'$ is a generalized Brunnian link in $V(\gamma)$ in the sense that the complement of each component of L is a solid torus; in particular, if $L = K \cup K'$ is a Brunnian link in S^3 and $V = S^3 - \mathrm{Int}N(K')$ then Dehn filling along the meridian slope of K' on M_K is a solid torus. However, if M_K admits two solid torus Dehn fillings on the outer torus, then by Gabai's theorem [Ga1] the core of such a Dehn filling solid torus is a 1-bridge braid, in which case it is easy to show that K must be a 1-bridge knot, and hence a 1-bridge braid by Corollary 2.6.

We work in the smooth or piecewise linear category. Denote by $N(Y)$ a closed regular neighborhood of a subset Y in a 3-manifold M, and by $|Y|$ the number of components of Y. Given a knot K in a solid torus V, let $M_K = V - \mathrm{Int}N(K)$. Let γ be a (p,q)-curve on T disjoint from $\partial\beta$, i.e., γ represents $p[l] + q[m]$ in $H_1(T)$, where $l = x \times S^1$ and $m = \partial D^2 \times y$ for some $x \in \partial D^2$ and $y \in S^1$. We use both $M_K(\gamma)$ and $M_K(p/q)$ to denote the manifold obtained from M_K by Dehn filling on T along the slope γ.

2. A criterion

Consider a proper arc β in a solid torus $V = D^2 \times S^1$, which is *trivial* in the sense that it is rel $\partial\beta$ isotopic to an arc β' on $T = \partial V$. Then $X = V - \mathrm{Int}N(\beta)$ is a genus 2 handlebody. Let γ be a (p,q) curve on T. We are interested in the question of when the manifold $X[\gamma]$ obtained by attaching a 2-handle to X along γ is a solid torus. Note that $X[\gamma]$ can also be obtained by removing a regular neighborhood of the arc β from the punctured lens space $V[\gamma]$, i.e., $(V - \mathrm{Int}N(\beta))[\gamma] = V[\gamma] - \mathrm{Int}N(\beta)$.

Since β is a trivial arc in V, there is a meridian disk D of V containing β. Since γ is a (p,q)-curve, there is also a meridian disk D' of V such that $\partial D'$ intersects γ in exactly p points in the same direction. However, in general one cannot choose D and D' to be the same disk; in other words, it may not be possible to find an isotopy relative to γ which deforms β to an arc lying on a disk D' whose boundary intersects γ at p points. For example, let K be an arbitrary 1-bridge braid with 1-bridge presentation $\alpha \cup \beta$, which is not a 0-bridge braid. Let γ be a longitude on T, and let D be a meridian of V intersecting γ at a single point. Up to isotopy we may assume $\alpha \cap \gamma = \emptyset$. Then β cannot be rel γ isotopic to an arc on D, as otherwise one can show that β would be rel ∂ isotopic to an arc β' on T which intersects γ at a single point, so $\alpha \cup \beta'$ would be a simple closed curve on T, contradicting the fact that K is not a 0-bridge knot.

Definition 2.1. Let γ be a (p,q)-curve on $T = \partial V$. A properly embedded arc β in V is γ-*trivial* if it lies on a meridian disk D of V such that ∂D intersects γ at p points. In this case $\partial\beta$ divides ∂D into two arcs β' and β''. The smaller of the intersection numbers $|\beta' \cap \gamma|$ and $|\beta'' \cap \gamma|$, is called the *jumping number* of β with respect to γ, denoted by $u = u(\beta, \gamma)$.

The following theorem characterizes trivial arcs β in V such that $V[\gamma] - \mathrm{Int}N(\beta)$ is a solid torus. This should be compared with [MZ, Proposition 3], where it does not seem to have been realized that a trivial arc in V may not be γ-trivial. Because of this, the proof to [MZ, Corollary 4] is not complete. The following theorem will fill the gap.

Theorem 2.2. *Let β be a trivial arc in a solid torus V, and let γ be a (p,q)-curve on $T = \partial V$ disjoint from β. Let $X = V - \text{Int}N(\beta)$. Then $X[\gamma]$ is a solid torus if and only if (i) β is γ-trivial, and (ii) the jumping number $u(\beta,\gamma)$ equals* 1 *or* $\min(q, p-q)$.

We need a result which determines when the boundary of a handlebody with a curve removed is incompressible. The following lemma is due to Starr [St]. An alternative proof can be found in [Wu1, Theorem 1.2 and Corollary 1.3].

Lemma 2.3. *Let H be a handlebody, and let γ be a simple closed curve on ∂H. Then $\partial H - \gamma$ is incompressible if and only if there is a set of essential disks $D_1, ..., D_k$ in H cutting ∂H into a set of twice punctured disks $P_1, ..., P_r$, such that for each i, we have (1) each component of $\gamma \cap P_i$ is an essential arc on P_i, and (2) there is at least one component of $\gamma \cap P_i$ connecting each pair of boundary components of P_i.*

Recall that a manifold M is ∂-irreducible if ∂M is incompressible in M. The following lemma proves the necessity of (1) in Theorem 2.2.

Lemma 2.4. *Let β, V, γ, X be as in Theorem 2.2. If β is not γ-trivial, then $X[\gamma]$ is irreducible and ∂-irreducible.*

Proof. Let D_0 be a meridian disk of V containing β, and let D_1 be another meridian disk of V disjoint from D_0. We may assume that γ has been deformed by an isotopy of V *relative to* β so that γ intersects ∂D_i minimally. Let A be the annulus obtained by cutting $T = \partial V$ along the meridian curve ∂D_1. Then $\gamma \cap A$ is a set of arcs $\gamma_1, ..., \gamma_n$. If all γ_i are essential arcs in A, then by an isotopy we may assume that γ_i are straight arcs from one boundary component of A to another. But in this case β would be rel ∂ isotopic to a straight arc β' in A intersecting γ always in the same direction, which would imply that β is γ-trivial, contradicting the assumption.

Therefore we may assume that some component γ' of $\gamma \cap A$ is an inessential arc in A. Since the two boundary components of A contain the same number of points of γ, there must be another component γ'' of $\gamma \cap A$ with both endpoints on the boundary component of A which does not contain the endpoints of γ'. Let Δ' and Δ'' be the disks on A cut off by γ' and γ'', respectively. By the minimality of $|\gamma \cap \partial D_i|$, we see that each of Δ' and Δ'' must contain exactly one endpoint of β. Rechoosing D_0 if necessary, we may assume that the intersection of ∂D_0 with each of Δ' and Δ'' is a single arc. See the first figure on Figure 2.1.

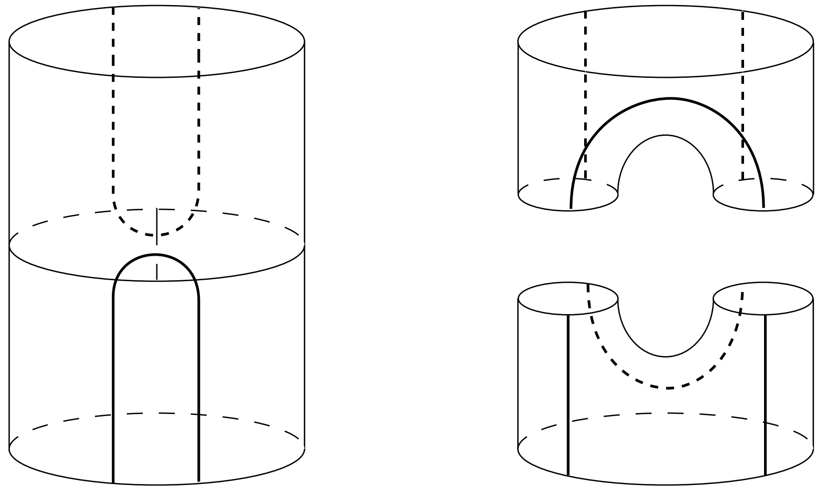

Figure 2.1

Consider the genus 2 handlebody $X = V - \mathrm{Int}N(\beta)$. The disk D_0 containing β intersects X in two disks D_2, D_3. Now the three disks D_1, D_2 and D_3 cut the genus 2 surface ∂X into two pairs of pants P_1 and P_2, as shown in the second figure of Figure 2.1. The minimality of $|\gamma \cap D_i|$, $i = 0, 1$, guarantees that the arcs of $\gamma \cap P_i$ are essential. The two arcs γ' and γ'' on A now give rise to 6 mutually non-parallel essential arcs, three on each of P_1 and P_2. It follows from Lemma 2.3 that $\partial X - \gamma$ is incompressible. Since X is irreducible and ∂X is compressible, by the Handle Addition Lemma [CG], after adding a 2-handle along γ, the manifold $X[\gamma]$ is irreducible and ∂-irreducible. □

Recall that we use $M(\gamma)$ to denote the Dehn filling on a torus boundary component of M along a slope γ. The following lemma relates Dehn filling to 2-handle addition. It is true for a general 3-manifold V, although we will only use it for $V = D^2 \times S^1$.

Lemma 2.5. *Let K be a knot in a 3-manifold V which is isotopic to a curve $\alpha \cup \beta$, where α is an arc on a torus boundary component T of V, and β is a properly embedded arc. Let γ be a simple closed curve on T disjoint from α. Let $M = V - \mathrm{Int}N(K)$, and $X = V - \mathrm{Int}N(\beta)$. Then $M(\gamma) = X[\gamma]$.*

Proof. Consider the manifold $V(\gamma) = V \cup V'$, where V' is the Dehn filling solid torus. Let α' be a proper arc in V' isotopic to α rel $\partial\alpha$. Then K is isotopic to $\alpha' \cup \beta$ in $V(\gamma)$. Let $N(D)$ be a regular neighborhood of a meridian disk D in V' such that $D \cap \alpha' = \emptyset$. Then $V' - \mathrm{Int}N(D)$ is a regular neighborhood of α' in $V(\gamma)$, denoted by $N(\alpha')$. Hence

$$\begin{aligned} M(\gamma) &= V(\gamma) - \mathrm{Int}N(K) \cong V(\gamma) - \mathrm{Int}(N(\beta) \cup N(\alpha')) \\ &= (V(\gamma) - \mathrm{Int}N(\alpha')) - \mathrm{Int}N(\beta) \\ &= V[\gamma] - \mathrm{Int}N(\beta) = X[\gamma] \end{aligned}$$

□

Proof of Theorem 2.2. Let α be an arc in $T-\gamma$ connecting the endpoints of β. By Lemma 2.5 we have $X[\gamma] = M_K(\gamma) = V(\gamma) - \text{Int}N(K)$, where K is a knot in V isotopic to $\alpha \cup \beta$. Note that $V(\gamma)$ is a lens space $L(p,q)$; hence by the uniqueness of Heegaard splittings of lens spaces [BO] we see that $X[\gamma] = V(\gamma) - \text{Int}N(K)$ is a solid torus if and only if K is a core of either V or the attached solid torus V' in $V(\gamma) = L(p,q) = V \cup V'$.

First assume that $X[\gamma]$ is a solid torus. By Lemma 2.4, β is γ-trivial, so it remains to show that $u = u(\beta,\gamma) = 1$ or $\min(q, p-q)$. By definition β is isotopic rel ∂ to an arc β' on T intersecting γ transversely at $u(\beta,\gamma)$ points in the same direction. Thus if we choose the core curve of the attached solid torus V' as a generator of $\pi_1 L(p,q) = \mathbb{Z}_p$, then the curve $K = \alpha \cup \beta$ represents the number $u(\beta,\gamma)$ in $\mathbb{Z}_p$. By the above, K is a core of V' or V, which represents 1 or q in $\pi_1 L(p,q) = \mathbb{Z}_p$, respectively. Hence $u(\beta,\gamma) = 1$ or $\min(q,p-q)$. This completes the proof of necessity.

Now assume that β is γ-trivial, and has jumping number 1 or $\min(q,p-q)$. By definition there is a meridian disk D of V containing β, with ∂D intersection γ at p points in the same direction. If $u(\beta,\gamma) = 1$ then β is rel ∂ isotopic to an arc β' on T intersecting γ at a single point. Note that in this case $\alpha \cup \beta'$ is a simple closed curve on T and is isotopic to a core of V'. If $u(\beta,\gamma) = \min(q,p-q)$ then since γ is a (p,q) curve, there is an arc α' on $T-\gamma$ with $\partial\alpha' = \partial\beta$, running along the longitudinal direction of V only once in the sense that its union with an arc on ∂D is a longitude of V. Since $T-\gamma$ is a meridional annulus on the Dehn filling solid torus V', the two arcs α and α' are isotopic rel ∂ in V'; hence $K \cong \beta \cup \alpha'$ is isotopic to a core of V. In either case $X[\gamma] = V(\gamma) - \text{Int}N(K)$ is a solid torus. □

Corollary 2.6. *Suppose K is a 1-bridge knot in a solid torus V with 1-bridge presentation $\alpha \cup \beta$. Let γ be a (p,q) curve on $T = \partial V$ disjoint from α. Let $M_K = V - \text{Int}N(K)$. Then $M_K(\gamma)$ is a solid torus if and only if (i) β is γ-trivial, and (ii) the jumping number $u(\beta,\gamma) = 1$ or $\min(q,p-q)$.*

In particular, if $M_K(\gamma)$ is a solid torus then the 1-bridge knot K must be a 1-bridge braid.

Proof. Since $M(\gamma) = V[\gamma]$, this follows immediately from Theorem 2.2. Note that if K is isotopic to $\alpha \cup \beta$ such that α disjoint from γ and β is γ-trivial, then K is a 1-bridge braid. □

Corollary 2.7. *Let K be a 1-bridge braid in a solid torus V, and let $\varphi : V \to S^3$ be an embedding. Then $\varphi(K)$ is a nontrivial knot in S^3.*

Proof. This is obvious if $\varphi(V)$ is a nontrivial torus. So assume $\varphi(V)$ is trivial in S^3, and let γ be a longitude of V such that $\varphi(\gamma)$ bounds a meridional disk in $S^3 - \text{Int}\varphi(V)$. Then $V(\gamma) = S^3$, so $\varphi(K)$ is trivial in S^3 if and only if $M_K(\gamma)$ is a solid torus. By Corollary 2.6, this implies that K has a 1-bridge presentation $\alpha \cup \beta$ such that α is disjoint from γ and β is γ-trivial. Since γ is a longitude, the jumping number of β must be 0. It is now easy to see that K is isotopic to a simple closed curve on $T = \partial V$, so it is a 0-bridge knot. Since these have been excluded from 1-bridge braids, the result follows. □

The following theorem characterizes reducible Dehn fillings on the outer torus of 1-bridge knots in solid tori.

Theorem 2.8. *Let K be a 1-bridge knot in a solid torus V, and let $M = V - \text{Int}N(K)$. Let γ be a (p,q) curve on $T = \partial V$. Then $M(\gamma)$ is reducible if and only if (i) $p > 1$, and (ii) K is a cable of a knot K' in V parallel to γ.*

Proof. If K is an (r,s) cable of a (p,q) knot in V, $p > 1$, then M is the union of a (p,q)-cable space $C_{p,q}$ and an (r,s)-cable space $C_{r,s}$ along a boundary component. Since γ is a fiber of the Seifert fibration of $C_{p,q}$, the Dehn filling space $C_{p,q}(\gamma)$ is reducible; hence so is $M(\gamma) = C_{r,s} \cup C_{p,q}(\gamma)$.

Now assume $M(\gamma)$ is reducible. Then $p > 1$, for otherwise K would be a knot in S^3, or a knot in $S^2 \times S^1$ with nontrivial winding number, so $M(\gamma) = V(\gamma) - \text{Int}N(K)$ would be irreducible. This proves (i).

Now let $\alpha \cup \beta$ be a 1-bridge presentation of K such that $\alpha \cap \gamma = \emptyset$. By Lemma 2.5 we have $M(\gamma) = X[\gamma]$, and by Lemma 2.4 the arc β is γ-trivial in V. Since $M(\gamma) = V(\gamma) - \text{Int}N(K)$ and $V(\gamma)$ is a lens space, $M(\gamma)$ is reducible if and only if K is a knot in a ball B in $V(\gamma)$. Since K represents $u(\beta,\gamma)$ in $\pi_1(V(\gamma))$, the jumping number $u(\beta,\gamma) = 0$, which implies that K is isotopic to a cable of a knot K' parallel to γ. □

3. The classification

Let $\alpha \cup \beta$ be a 1-bridge presentation of a 1-bridge braid $K = K(w,b,t)$ in V. Orient K so that the winding number of K in V is $w > 0$. This induces an orientation on α and β. Cutting the solid torus $V = D^2 \times S^1$ along a meridian disk D which is disjoint from β and intersects α minimally, the torus $T = \partial V$ becomes an annulus A. Choose an arc C on A which is disjoint from α, such that A cut along C is a rectangle R as shown in Figure 3.1. The intersection of α with R is a set of arcs $\alpha_0, ..., \alpha_w$, ordered from left to right on R. Let $\alpha(0)$ and $\alpha(1)$ be the initial and ending points of α. We may assume that the orientation of α points downward, and the arc C has been chosen so that the arc α' among the α_i containing $\alpha(0)$ is to the left of the arc α'' containing $\alpha(1)$. See Figure 3.1 for an example, where the α_i are drawn as vertical lines, and $(w,b,t) = (7,4,2)$.

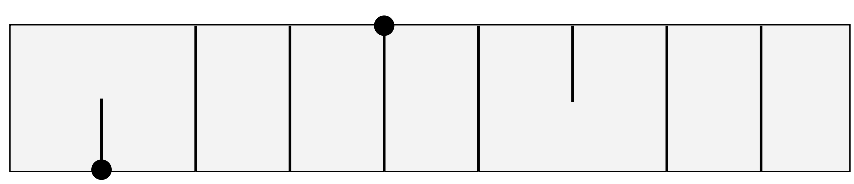

Figure 3.1

The parameters of $K(w,b,t)$ appear in Figure 3.1 as follows. The winding number w indicates the number of points of α on the top or bottom of R, labeled successively by $v_1, ..., v_w$ and $v'_1, ..., v'_w$, respectively. The number t is such that on the torus T a point v'_i at the bottom is identified to v_{i+t} on the top, where the subscripts are integers mod w. The bridge number b in $K(w,b,t)$ indicates the number of arcs between the two ending arcs α' and α''. Note that β is isotopic rel ∂ to an arc in R with interior intersecting α at exactly b points in the same direction.

Now let γ be a (p,q) curve on T such that $M_K(\gamma)$ is a solid torus. By Corollary 2.6, β is γ-trivial, so up to isotopy we may assume that γ is disjoint from α, and intersects the meridian disk D above at p points. Thus $\gamma \cap A$ is a set of p vertical arcs. Cutting A along a component of $\gamma \cap A$, we obtain a rectangle R on which γ becomes a set of $p+1$ vertical arcs $\gamma_0, ..., \gamma_p$, with γ_p identified to γ_0 on A. These arcs cut R further into a set of rectangles $R_0, ..., R_{p-1}$. Since α is disjoint from γ, each arc α_i lies in one of the R_j. We may arrange so that the initial point of α lies in R_0, as shown in Figure 3.2, where $(p,q) = (3,1)$. Note that the bottom of R_i is identified to the top of R_{i+q} on T in such a way that the endpoints of the arcs match each other.

Figure 3.2

Definition 3.1. A 5-tuple (p,q,k,x,ϵ) of integers is *allowable* if

(1) $p > q > 0$, and $\gcd(p,q) = 1$;
(2) $k \geq 0$;
(3) $p > x \geq 0$;
(4) $\epsilon = 1$ if $k = 0$, and $\epsilon \in \{1, -1\}$ if $k > 0$.

Fix a meridian disk D of V. Assume that α is an arc on T disjoint from a (p,q) curve γ on T, such that $\partial\alpha \cap \partial D = \emptyset$, γ intersects ∂D minimally up to isotopy, and α intersects ∂D minimally up to isotopy rel $\gamma \cup \partial\alpha$. Define an allowable 5-tuple $(p,q,k(\alpha),x(\alpha),\epsilon(\alpha))$ as follows.

Let R_i be the rectangles defined above. Up to relabeling we may assume that $\alpha(0) \in R_0$. Let $k(\alpha)$ be the number of endpoints of $\alpha \cap R$ which lie on the top of R_0 (hence $\alpha \cap R_0$ has $k+1$ components, including α'.) Let $x = x(\alpha)$ be the number such that the arc α'' is contained in R_x.

An arc α_i is a *left arc* (resp. *right arc*) if it is adjacent to the left (resp. right) edge of the rectangle R_j containing it. Define $\epsilon(\alpha) = 1$ if α' is a left arc, and $\epsilon(\alpha) = -1$ otherwise. When $k = 0$, α is both a left arc and a right arc, in which case by definition we have $\epsilon(\alpha) = 1$. Note that one of the α' and α'' is a left arc, and the other is a right arc. For if this were false, assuming that neither of α' or α'' is a left arc, say, then the union of the left arcs in R would form a closed circle component of α on T, which would contradict the fact that α is an arc on T.

We have thus associated an allowable 5-tuple $(p,q,k(\alpha),x(\alpha),\epsilon(\alpha))$ to each arc α as above. Conversely, given a 5-tuple (p,q,k,x,ϵ) one can construct such an arc $\alpha = \alpha(p,q,k,x,\epsilon)$ with $(p,q,k(\alpha),x(\alpha),\epsilon(\alpha)) = (p,q,k,x,\epsilon)$, which is unique up to homeomorphism of $(V, \gamma \cup \partial D)$. Let $K = K(p,q,k,x,\epsilon)$ be the knot isotopic to $\alpha(p,q,k,x,\epsilon) \cup \beta$, where β is a trivial arc in $V - D$. Note that $K(p,q,k,x,\epsilon)$ is a well-defined 1-bridge braid $K(w,b,t)$ for any allowable 5-tuple. The parameters (w,b,t) will be calculated explicitly in Lemma 3.5. Note that two different allowable 5-tuples may give rise to the same 1-bridge braid.

The following theorem classifies, for a fixed pair (p, q), all 1-bridge braids K such that $M_K(\gamma)$ is a solid torus.

Theorem 3.2. *Let K be a 1-bridge knot in V, and let p, q be coprime integers such that $0 < q < p$. Then $M_K(p/q)$ is a solid torus if and only if $K = K(p, q, k, x, \epsilon)$ for some allowable 5-tuple (p, q, k, x, ϵ) such that $x = 1$, q, $p - q$ or $p - 1$.*

Proof. Let γ be a (p, q) curve on $T = \partial V$. Fix a meridian disk D of V which intersects γ at p points. If $M_K(\gamma)$ is a solid torus then by Corollary 2.6 the knot K is a 1-bridge braid $\alpha \cup \beta$, such that β is γ trivial, and the jumping number $u(\beta, \gamma)$ is 1 or q. The first condition implies that up to isotopy one may assume that $\alpha = \alpha(p, q, k, x, \epsilon)$ for an allowable 5-tuple (p, q, k, x, ϵ), and β is a γ-trivial arc disjoint from D. The second condition implies that $x = 1$, q, $p - q$, or $p - 1$. Conversely, if $K = K(p, q, k, x, \epsilon)$ for some allowable 5-tuple (p, q, k, x, ϵ) and $x \in \{1, q, p-q, p-1\}$ then one can show that (1) and (2) in Corollary 2.6 hold, hence $M_K(\gamma)$ is a solid torus. □

Corollary 3.3. *Given $p > q > 0$ with $p \geq 3$, there are infinitely many 1-bridge braids K in V such that $M_K(p/q)$ is a solid torus.* □

Given a 1-bridge braid $K = K(w, b, t)$, we would like to find all (p, q) such that $M_K(p/q)$ is a solid torus. By Theorem 3.2 it suffices to find all allowable 5-tuples (p, q, k, x, ϵ) such that $K(w, b, t) = K(p, q, k, x, \epsilon)$, and $x = 1, q, p - q$ or $p - 1$. We need to find the relationship between the parameters (w, b, t) and (p, q, k, x, ϵ).

Denote by Z_p the set of integers $\{0, 1, ..., p - 1\}$, and by $[a] \in Z_p$ the mod p residue of an integer a. (We will also use $[x]$ later to denote the integer part of a real number x. It will be clear from the context what the notation stands for.) For any $x \in Z_p$, let $\bar{q}_x \in Z_p$ be such that $q\bar{q}_x \equiv x \bmod p$. Define S_x as the subset of Z_p given by

$$S_x = \{[q], [2q], ..., [\bar{q}_x q]\}.$$

Define $\varphi(x, y)$ to be the number of elements $z \in S_x$ which lie in the open interval $(0, y)$.

Let $\bar{q}, \bar{p}$ be the numbers in Z_p such that $\bar{q}q = \bar{p}p + 1$. Since $\gcd(p, q) = 1$, $\bar{q}$ and $\bar{p}$ are well defined. The following lemma calculates the value of $\varphi(x, s)$ for certain values of x and s.

Lemma 3.4. *With the above notation, the values of $\varphi(x, x)$ and $\varphi(x, q)$ for $x \in \{1, q, p - q, p - 1\}$ are given as follows.*

(1) $\varphi(1, 1) = 0$; $\quad \varphi(1, q) = \bar{p}$;

(2) $\varphi(q, q) = 0$;

(3) $\varphi(p - q, p - q) = p - q - 1$; $\quad \varphi(p - q, q) = q - 1$;

(4) $\varphi(p - 1, p - 1) = p - \bar{q} - 1$; $\quad \varphi(p - 1, q) = q - \bar{p} - 1$.

Proof. (1) In this case we have $x = 1$. By definition $\varphi(1, 1) = 0$. We need to calculate $\varphi(1, q)$. The formula is obvious when $q = 1$, so we assume $q > 1$. Note that $\bar{q}_x = \bar{q}$.

By definition $\varphi(1, q)$ is the number of points in the set $\{[q], [2q], ..., [\bar{q}q]\}$ which are contained in the interval $(0, q)$. Since $\gcd(p, q) = 1$ and $\bar{q} < p$, $[iq] \neq 0$ for $i = 1, ..., \bar{q}$. Hence we need only find the number of $i \in [1, \bar{q}]$ such that $[iq] \in [0, q)$. Note that $[iq] \in [0, q)$ if and only if $iq \in [jp, jp + q)$ for some j.

The set of integers $\{q, 2q, ..., \bar{q}q\}$ distributes evenly on the interval $[0, \bar{q}q]$, with distance q between adjacent numbers. For any $j = 1, 2, ..., \bar{p} - 1$, the interval

$[jp, jp+q)$ is contained in $[0, \bar{q}q)$, hence it contains exactly one iq. Also, the interval $[\bar{p}p, \bar{p}p+q)$ contains the point $\bar{q}q$ because $\bar{q}q = \bar{p}p+1$ and $q > 1$. When $j > \bar{p}$, the interval $[jp, jp+q)$ is disjoint from $[0, \bar{q}q)$, hence it contains no iq in the above set. It follows that exactly $\bar{p}$ of the numbers in $S_1 = \{[q], ..., [\bar{q}q]\}$ are in the interval $(0, q)$. Hence $\varphi(1, q) = \bar{p}$.

(2) When $x = q$, we have $1 \cdot q = x$, so $\bar{q}_x = 1$. Thus S_x has a single element $[q]$. Since $[q] = q \notin (0, q)$, we have $\varphi(x, q) = \varphi(q, q) = 0$.

(3) When $x = p - q$, we have $\bar{q}_x = p - 1$. Hence $S_x = \{[q], [2q], ..., [\bar{q}_x q]\}$ contains all integers mod p except 0. Thus all the numbers in $(0, q)$ are in S_x, so $\varphi(p-q, q) = q - 1$. Similarly we have $\varphi(p-q, p-q) = p - q - 1$.

(4) The calculation for $x = p-1$ is similar to that for $x = 1$. We have $(p-\bar{q})q \equiv p-1 \mod p$, hence $\bar{q}_x = p - \bar{q}$. From the equation $(p-\bar{q})q = (q-\bar{p}-1)p + (p-1)$, we see that that $q - \bar{p} - 1$ of the numbers in $S_x = \{[q], ..., [(p-\bar{q})q]\}$ are in the open interval $(0, q)$. Hence we have $\varphi(p-1, q) = q - \bar{p} - 1$. Note that all but one of the numbers $\{[q], ..., [(p-\bar{q})q]\}$ are in the interval $(0, p-1)$, (the last number in the set is $p-1$), so we have $\varphi(p-1, p-1) = p - \bar{q} - 1$. $\square$

Lemma 3.5. *Let (p, q, k, x, ϵ) be an allowable 5-tuple. Then $K(p, q, k, x, \epsilon) = K(w, b, t)$, where*

$$w = kp + \bar{q}_x$$
$$t = kq + \varphi(x, q)$$
$$b = k(x + \epsilon) + \varphi(x, x)$$

Proof. Recall that $K(p, q, k, x, \epsilon) = \alpha(p, q, k, x, \epsilon) \cup \beta$. The arc $\alpha = \alpha(p, q, k, x, \epsilon)$ is homotopic to $\hat{\alpha} \cdot \tilde{\alpha}$, where $\hat{\alpha}$ consists of k loops parallel to γ, while $\tilde{\alpha}$ is an arc connecting the two endpoints of α and is disjoint from the top of the rectangle R_0 in Figure 3.2. The arc $\tilde{\alpha}$ intersects each R_i in either 1 or 0 arc. By definition $\bar{q}_x \in Z_p$, and $q\bar{q}_x \equiv x \mod p$. Thus when traveling along $\tilde{\alpha}$ one enters $R_0, R_q, R_{2q}, ..., R_{\bar{q}_x q}$ successively, where the subscripts are integers mod p. Note that $\tilde{\alpha} \cap R_0$ does not have a vertex on the top of R_0, therefore the wrapping number $w = kp + \bar{q}_x$.

Recall that t is the number such that the i-th point of α at the bottom of R is glued to the $(t+i)$-th point of α on the top of R. Since the bottom of R_0 is glued to the top of R_q, the first point of α at the bottom of R_0 is glued to the first point of α on the top of R_q, hence t equals the number of points of α on the top of $R' = R_0 \cup ... \cup R_{q-1}$. As above, the number of points of $\tilde{\alpha}$ on the top of R' is $\varphi(x, q)$, while the number of points of $\hat{\alpha}$ on the top of each R_i is k, hence the second equation follows.

When $\epsilon = 1$, the bridge width b equal the number of arcs of α on $R_0 \cup ... \cup R_x$, not counting the two arcs containing an endpoint of α. Therefore we have $b = k(x+1) + \varphi(x, x)$. When $\epsilon = -1$, the bridge does not pass the edges in R_0 and R_x, hence $b = k(x-1) + \varphi(x, x)$. $\square$

The following theorem gives a simple method to calculate, for a given 1-bridge braid K, all (p, q) such that $M_K(\gamma)$ is a solid torus, where γ is a (p, q) curve. There are four possible solutions for each K; in each case (p, q) can be calculated from w and t, and it has to satisfy an extra condition in terms of b.

Theorem 3.6. *Let $K = K(w, b, t)$ be a 1-bridge braid in a solid torus V such that $1 \le b \le w - 2$, and $0 < t < w$. Denote by $[y]$ the integer part of a real number*

y. Then $M_K(p/q)$ is a solid torus if and only if p, q satisfy one of the following conditions.

(1) $qw - pt = 1$, $p, q \in Z_w$, and $b = 2[w/p]$.

(2) $k = \gcd(w-1, t)$, $p = (w-1)/k$, $q = t/k$, and $b = k(q+\epsilon)$ for some $\epsilon = \pm 1$.

(3) $k = \gcd(w+1, t+1) - 1$, $p = (w+1)/(k+1)$, $q = (t+1)/(k+1)$, and $b = k(p-q+\epsilon) + (p-q-1)$ for some $\epsilon = \pm 1$.

(4) $p(t+1) - qw = 1$, $p, q \in Z_w$, and $b = [w/p](p-2) + (p - \bar{q} - 1)$.

Proof. Let $M = M_K$ be the exterior of K. By Theorem 3.2, $M(\gamma)$ is a solid torus if and only if $K(w, b, t) = K(p, q, k, x, \epsilon)$ for some allowable 5-tuple (p, q, k, x, ϵ), and $x = 1, q, p-q$ or $p-1$. We need to show that these correspond to (1) – (4) in the theorem.

CASE 1: $x = 1$. In this case by Lemma 3.4 the three equations in Lemma 3.5 become

$$\begin{aligned} w &= kp + \bar{q}_1 = kp + \bar{q} \\ t &= kq + \varphi(1, q) = kq + \bar{p} \\ b &= k(1+\epsilon) + \varphi(1,1) = k(1+\epsilon) \end{aligned}$$

If $K(w, b, t) = K(p, q, k, x, \epsilon)$ and $x = 1$, then since $b > 0$, the third equation above gives $\epsilon = 1$, so $b = 2k = 2[w/p]$. We have $qw - pt = q(kp+\bar{q}) - p(kq+\bar{p}) = q\bar{q} - p\bar{p} = 1$. Therefore (1) holds.

Conversely, if $p, q \in Z_w$ satisfy the equation $qw - pt = 1$, let $w = k'p + p'$, and $t = k''q + q'$, where $0 \le p' < p$ and $0 \le q' < q$. Then the equation $qw - pt = 1$ becomes

$$1 = qw - pt = q(k'p + p') - p(k''q + q') = (k' - k'')pq + (qp' - pq').$$

Since $p, q \in Z_w$, this implies that $k' = k''$, and $qp' - pq' = 1$; hence $p' = \bar{q}$ and $q' = \bar{p}$. Hence if we define $k = k' = k''$ and $x = \epsilon = 1$ then (p, q, k, x, ϵ) is an allowable 5-tuple with $x = 1$. By definition we have $w = kp + \bar{q}$ and $t = kq + \bar{p}$. The equation $b = k(1+\epsilon)$ follows from the extra condition $b = 2[w/p] = 2k$ in (1). Therefore $K(w, b, t) = K(p, q, k, x, \epsilon)$.

The arguments for the other cases are similar. We only show the calculations below.

CASE 2: $x = q$. By Lemma 3.4 the equations in Lemma 3.5 become

$$\begin{aligned} w &= kp + \bar{q}_x = kp + 1 \\ t &= kq + \varphi(x, q) = kq \\ b &= k(x+\epsilon) + \varphi(x, x) = k(q+\epsilon) \end{aligned}$$

We can solve the first two equations for k, p, q to get $k = \gcd(w-1, t)$, $p = (w-1)/k$, and $q = t/k$. The third equation gives the extra condition that must be satisfied, i.e., $b = k(q+\epsilon)$ for some $\epsilon = \pm 1$.

CASE 3: $x = p - q$. In this case $\bar{q}_x = p - 1$. We have

$$\begin{aligned} w &= kp + \bar{q}_x = kp + (p-1) \\ t &= kq + \varphi(x, q) = kq + (q-1) \\ b &= k(x+\epsilon) + \varphi(x, x) = k(p-q+\epsilon)) + (p-q-1) \end{aligned}$$

The first two equations give $k = \gcd(w-1, t) - 1$, $p = (w-1)/(k+1)$, and $q = t/(k+1)$. The third equation gives the extra condition that $b = k(p-q+\epsilon)+(p-q+1)$ for some $\epsilon = \pm 1$.

CASE 4: $x = p - 1$. In this case $\bar{q}_x = p - \bar{q}$. We have

$$\begin{aligned} w &= kp + \bar{q}_x = kp + (p - \bar{q}) \\ t &= kq + \varphi(x, q) = kq + (q - \bar{p} - 1) \\ b &= k(x + \epsilon) + \varphi(x, x) = k(p - 1 + \epsilon)) + (p - \bar{q} - 1) \end{aligned}$$

As in Case 1, one can show that the first two equations are equivalent to the condition that $p(t+1) - qw = 1$ for some $p, q \in Z_w$. As usual, we have $k = [w/p]$, hence the last equation is equivalent to $b = [w/p](p - 1 + \epsilon)) + (p - \bar{q} - 1)$. Since $b \neq w - 1$, we must have $\epsilon = -1$. □

Corollary 3.7. *Let $K = K(w, b, t)$ be a 1-bridge braid in a solid torus V such that $1 \le b \le w - 2$, and $0 < t < w$. If $M_K(p/q)$ is a solid torus, then $w + 1 \ge p > q > 0$.* □

4. Computational results

A computer program can be written using Theorems 3.2 and 3.6 to calculate a list, for w up to a given value, of all $K(w, b, t)$ and (p, q) such that (p, q) filling on the outer torus of M_K produces a solid torus. It can also calculate all such (p, q) for any given $K(w, b, t)$. The following are some results from such a calculation.

(1) The exterior of the knot $K(7, 2, 4)$ admits three such fillings, with slopes $(3, 2)$, $(5, 3)$ and $(8, 5)$. Since the dual knot after such a filling is a hyperbolic knot in a solid torus, by a theorem of Berge [Be], up to (possibly orientation reversing) homeomorphism of (V, K) this is the only 1-bridge braid such that M_K admits three solid torus fillings on T. Note that $K(7, 4, 2)$ is equivalent to $K(7, 2, 4)$ by an orientation reversing map of V.

(2) The knot $K(8, 3, 6)$ is the first one (in lexicographic order of parameters) that admits no solid torus filling on T.

(3) As w increases, the percentage of knots $K(w, b, t)$ admitting solid torus fillings on T becomes smaller, as expected, and most of them only admit one such filling, so its dual knot does not admit nontrivial surgery which produces a solid torus. There are 72 knots with $w \le 10$, 60 of which admit a total of 86 solid torus fillings on T. There are exactly 6000 knots $K(w, b, t)$ with $w \le 40$, 2380 of which admit a total of 2692 such fillings.

One can easily show that there is an orientation reversing map of V sending $K = K(w, b, t)$ to $\bar{K} = K(w, w - b - 1, t - b - 1)$, which sends a (p, q) curve on T to a $(p, p - q)$ curve, so $M_K(p/q)$ is a solid torus if and only if $M_{\bar{K}}(p/(p - q))$ is. Thus up to orientation reversing homeomorphism of V we may assume that $b < w/2$, and if $b = (w - 1)/2$ then $t < w/2$. There are 36 such knots for $w \le 10$. The following list shows all such knots and the Dehn filling slopes on T such that $M_K(p/q)$ is a solid torus. As an example, the tuple $(6, 2, 3; 5/3, 7/4)$ indicates that the knot $K(6, 2, 3)$ admits two such Dehn fillings, with slopes 5/3 and 7/4 respectively.

TABLE 1. $(w, b, t; p_1/q_1, p_2/q_2, p_3/q_3)$

$(4,1,2;3/2,5/3)$	$(5,2,1;3/1,4/1)$	$(6,1,2;5/2)$
$(6,1,4;7/5)$	$(6,2,1;5/1)$	$(6,2,3;5/3,7/4)$
$(7,2,1;6/1)$	$(7,2,4;3/2,5/3,8/5)$	$(7,2,5;4/3)$
$(8,1,2;7/2)$	$(8,1,6;9/7)$	$(8,2,1;7/1)$
$(8,2,3;5/2,7/3)$	$(8,3,2;3/1,7/2)$	$(8,3,4;7/4,9/5)$
$(8,3,6;-)$	$(9,2,1;8/1)$	$(9,2,3;8/3)$
$(9,2,5;5/3,7/4)$	$(9,2,6;10/7)$	$(9,2,7;5/4)$
$(9,4,1;-)$	$(9,4,2;4/1)$	$(9,4,3;5/2,8/3)$
$(10,1,2;9/2)$	$(10,1,4;-)$	$(10,1,6;-)$
$(10,1,8;11/9)$	$(10,2,1;9/1)$	$(10,2,7;7/5,11/8)$
$(10,3,2;9/2)$	$(10,3,4;9/4)$	$(10,3,6;3/2,11/7)$
$(10,3,8;-)$	$(10,4,1;-)$	$(10,4,5;9/5,11/6)$

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