

# IMMERSED ESSENTIAL SURFACES AND DEHN SURGERY

YING-QING WU<sup>1</sup>

**ABSTRACT.** It is known that an embedded essential surface  $F$  in a hyperbolic manifold  $M$  remains essential in Dehn filling spaces  $M(\gamma)$  for most slopes  $\gamma$  on a torus boundary component  $T$  of  $M$ . The main theorem of this paper is to generalize this result to immersed surfaces. More explicitly, if an immersed essential surface  $F$  has coannular slopes  $\beta_1, \dots, \beta_n$  on  $T$ , then there is a constant  $K$  such that  $F$  remains essential in  $M(\gamma)$  when  $\Delta(\gamma, \beta_i) > K$  for all  $i$ . It will also be shown that all but finitely many Freedman tubings of a geometrically finite surface in  $M$  are  $\pi_1$ -injective.

## §1. INTRODUCTION

The problem of how many Dehn fillings on a torus boundary component  $T$  of a compact, orientable, irreducible 3-manifold  $M$  will make a closed embedded essential surface  $F$  compressible has been settled. A slope  $\beta$  on  $T$  is a coannular slope if it is homotopic to some curve on  $F$ . As an embedded essential surface,  $F$  can have at most one coannular slope. If  $F$  has a coannular slope  $\beta$  on  $T$ , then by a result of Culler-Gordon-Luecke-Shalen [10, Theorem 2.4.3],  $F$  is incompressible in all  $M(\gamma)$  such that  $\Delta(\beta, \gamma) > 1$ , where  $\Delta(\beta, \gamma)$  denotes the minimal intersection number between the slopes  $\beta$  and  $\gamma$ . If  $F$  has no coannular slopes, then it is incompressible in the Dehn filling space  $M(\gamma)$  for all but at most three  $\gamma$  [28]. There are examples showing that these are the best possible.

While many manifolds do not contain embedded essential surfaces, it has been shown by Cooper, Long and Reid [8] that most bounded 3-manifolds, in particular all hyperbolic manifolds with some toroidal boundaries, contain immersed closed essential surfaces. There has been a lot of interest recently on immersed surfaces, see for example [1, 6, 7, 8, 23, 24]. It is important to understand to which extent the above theorems for embedded surfaces can be generalized to immersed surfaces.

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Let  $S$  be a surface of finite type, i.e. compact surface with finitely many points removed.  $S$  may be disconnected or unorientable. We define a *surface* (of type  $S$ ) in  $M$  to be a continuous piecewise smooth map  $F : S \rightarrow M$  which is an immersion almost everywhere.  $F$  is *hyperbolic* if all components of  $F$  have negative Euler characteristic. A compact 3-manifold  $M$  is hyperbolic if its interior admits a complete hyperbolic structure. We allow the hyperbolic structure to have infinite volume, hence  $M$  may have non-toroidal boundary components.

Let  $T$  be a set of tori on  $\partial M$ . A curve on a surface is *simple* if it has no self-intersection. A *slope*  $\gamma_i$  on  $T$  is the isotopy class of a simple nontrivial curve on  $T$ . A slope  $\gamma$  is a *coannular slope* if some nontrivial multiple of  $\gamma$  is homotopic to a curve on  $F$ , in which case we say that  $F$  is coannular to  $T$ . A *multiple slope*  $\gamma = (\gamma_1, \dots, \gamma_n)$  on  $T$  is a set of slopes  $\gamma_i$ , one for each component  $T_i$  of  $T$ . Denote by  $M(\gamma)$  the *Dehn filling space* along  $\gamma$ , i.e. the manifold obtained by attaching a solid torus  $V_i$  to each  $T_i$  ( $i \leq n$ ) so that  $\gamma_i$  bounds a meridian disk in  $V_i$ . Given two slopes  $\alpha_1, \alpha_2$  on a torus, we use  $\Delta(\alpha_1, \alpha_2)$  to denote the minimal geometric intersection number between  $\alpha_1$  and  $\alpha_2$ . Let  $\beta$  be a finite set of slopes on  $T$ , with possibly none or several slopes on a component  $T_i$  of  $T$ . Let  $\gamma$  be a multiple slope. Define

$$\Delta(\gamma, \beta) = \min \{ \Delta(\gamma_i, \beta_j) \mid \beta_j \subset T_i \}$$

In particular,  $\Delta(\gamma, \beta) > 0$  if and only if  $\gamma_i \notin \beta$  for all  $i$ . If  $\beta = \emptyset$ , define  $\Delta(\gamma, \beta) = \infty$ . The following is our main theorem.

**Theorem 5.3.** *Let  $T$  be a set of tori on the boundary of a compact, orientable, hyperbolic 3-manifold  $W$ . Let  $F$  be a compact essential surface in  $W$  with  $\partial F \subset \partial M - T$ , and let  $\beta$  be the set of coannular slopes of  $F$  on  $T$ . Then there is an integer  $K$  and a finite set of slopes  $\Lambda$  on  $T$ , such that  $F$  is  $\pi_1$ -injective in  $W(\gamma)$  for all multiple slopes  $\gamma = (\gamma_1, \dots, \gamma_n)$  on  $T$  satisfying  $\Delta(\gamma, \beta) \geq K$  and  $\gamma_i \notin \Lambda$ .*

By enlarging the constant  $K$  we may assume that  $\Lambda$  is disjoint from those components of  $T$  which contain some coannular slopes of  $F$ ; in particular, if each component of  $T$  has a coannular slope of  $F$  then we may assume that  $\Lambda = \emptyset$ . On the other hand, if  $F$  is not coannular to  $T$ , then  $\beta = \emptyset$ ,  $F$  survives all surgeries after excluding a finite set of slopes on each component of  $T$ . The result is best possible in the sense that there is no universal bound on the constant  $K$ , see Theorem 6.1. Note that  $F$  has only finitely many coannular slopes on  $T$ , i.e.  $\beta$  is a finite set. See the remark before Theorem 5.2. Thus in certain sense, Theorem 5.3 says that  $F$  survives most Dehn fillings on  $M$ .

The special case that  $T \subset \partial M$  is a single torus and  $F$  is a closed surface is of particular interest. We summarize the result for this special case in the following theorem for future reference.

**Theorem 1.1.** *Let  $M$  be a compact, orientable, hyperbolic 3-manifold, and let  $T$  be a torus component of  $\partial M$ . Let  $F$  be a closed essential surface in  $M$ .*

(1) *If  $F$  is coannular to  $T$  with coannular slopes  $\beta_1, \dots, \beta_n$ , then there is an integer  $K$  such that  $F$  remains  $\pi_1$ -injective in  $M(\gamma)$  for all  $\gamma$  satisfying  $\Delta(\gamma, \beta_i) \geq K$ ,  $i = 1, \dots, n$ .*

(2) *If  $F$  is not coannular to  $T$ , then  $F$  is  $\pi_1$ -injective in  $M(\gamma)$  for all but finitely many  $\gamma$ .*

(3) *There is no universal bound on the constant  $K$  in (1): For any constant  $C$ , there is a closed essential surface  $F$  in a hyperbolic manifold  $M$  with coannular slope  $\beta$  on  $T = \partial M$ , such that  $F$  is not  $\pi_1$ -injective in  $M(\gamma)$  for all  $\gamma$  with  $\Delta(\gamma, \beta) \leq C$ .*

Part (2) is a well-known result. It follows from Theorem 5.3, but there is a very simple direct proof. See for example [3] or [4]. Part (1) follows from Theorem 5.3, noticing that when  $K$  is sufficiently large, the set of slopes  $\gamma$  with  $\Delta(\gamma, \beta_i) < K$  contains any prescribed finite set. Part (3) is Theorem 6.1(ii). Theorem 1.1 can be generalized to arbitrary compact orientable irreducible 3-manifolds. It is not known if there is a universal bound on the number of slopes  $\gamma$  in (2), although there are examples showing that no such universal bound exists if  $M$  is allowed to be non-hyperbolic.

Theorem 5.3, however, is not true for non-hyperbolic compact orientable 3-manifolds. An easy example is when  $T$  is a pair of tori  $T_1, T_2$  coannular to each other. If  $F$  is not  $\pi_1$ -injective in  $M(\gamma_1, \gamma_2)$ , then it is not  $\pi_1$ -injective in  $M(\gamma'_1, \gamma'_2)$  for all  $(\gamma'_1, \gamma'_2)$  obtained by twisting  $(\gamma_1, \gamma_2)$  along an essential annulus with one boundary component on each of  $T_i$ . More complicated examples can be constructed where no two components of  $T$  are coannular. However, the theorem is true if one further excludes all slopes of distance at most one from Seifert fibred slopes on those components of  $T$  which lie in Seifert fibred components in the Jaco-Shalen-Johanson decomposition of  $M$ .

Another interesting topic is to construct immersed essential surfaces in hyperbolic 3-manifolds. See for example [2, 6, 7, 8, 14, 19]. One of the most important method is the Freedman tubing introduced in [14]. Given a proper surface  $F$  in  $M$ , a *Freedman tubing*  $\hat{F}$  of  $F$  is a surface obtained from  $F$  by adding some annuli on  $\partial M$  with boundary on  $\partial F$ . This idea has been used in several important works, see [6, 7, 8, 19]. In particular, it was first proved by Cooper and Long [7] that a Freedman tubing of an *embedded*, geometrically finite surface is essential if the tubes are long enough.

A combinatorial proof has been given by Li [19], which also yields an upper bound of tube length in terms of genus and number of boundary components of  $F$ . They have also shown that the tubed surface survives most Dehn fillings, which, combined with a result of Culler and Shalen [9], implies that all but finitely many Dehn filling spaces of a hyperbolic manifold contain an immersed surface.

Define the *wrapping number*  $\text{wrap}(A)$  of an annulus  $A$  on a torus  $T$  to be the minimum algebraic intersection number between  $A$  and all points of  $T$ . If  $\hat{F}$  is a Freedman tubing of  $F$ , define  $w(\hat{F}, F)$  to be the minimum of  $\text{wrap}(A_i)$  over all components  $A_i$  of  $\hat{F} - F$ . The following theorem generalizes the above result to immersed essential surfaces.

**Theorem 5.7.** *Let  $F$  be a geometrically finite surface in a compact hyperbolic 3-manifold  $W$ . Then there is a constant  $K$  such that if  $\hat{F}$  is a Freedman tubing of  $F$  with  $w(\hat{F}, F) \geq K$ , then  $\hat{F}$  is  $\pi_1$ -injective in  $W$ .*

The assumption that  $F$  be geometrically finite is necessary, otherwise  $F$  would be a virtual fiber, and hence no Freedman tubing of it would be essential. Immersed surfaces are much more abundant than embedded ones. For example, Oertel [23] and Maher [20] showed that in certain manifolds all slopes are realized as boundary slopes of immersed essential surfaces, while Hatcher [16] showed that there are only finitely many boundary slopes of embedded surfaces in these manifolds. More immersed surfaces can be obtained by projecting to  $M$  embedded surfaces in covering spaces of  $M$ . The boundary of such a surface may have several different slopes on the same torus component of  $\partial M$ . Our theorem applies to such surfaces as well, and there is no restriction on the orientability of  $F$  or  $\hat{F}$ . When  $\partial M$  is a set of tori and  $F$  is a proper surface, an essential Freedman tubing is automatically geometrically finite because it has accidental parabolics, hence by Theorem 5.3 it survives most Dehn fillings.

The idea of our proof is to use area estimation of minimal surfaces to show that certain curves in a negatively curved space cannot bound an immersed disk, and hence are nontrivial. In section 2 we will use some results in minimal surface theory to show that if a piecewise geodesic curve  $\alpha$  is trivial in a negatively curved 3-manifold  $M$ , then it bounds a disk whose intersection with the hyperbolic part of  $M$  has area bounded above by the total external angle of  $\alpha$ . In section 3 we give some estimation for areas of surfaces in truncated hyperbolic cusps, using integral of certain differential forms. These result will then be used in section 4 to show that curves in  $M$  satisfying certain conditions do not bound any disk, hence is nontrivial in  $M$ . The essentiality of surfaces in Dehn filling space and the essentiality of tubing surfaces in hyperbolic manifolds

follow from these results by showing that all nontrivial curves on the surface satisfy those conditions. In section 6 we will show that there is no universal upper bounds for the bad fillings, and post several problems arisen in this research.

*Definitions and conventions.* All 3-manifolds in this paper are assumed orientable. Let  $F : S \rightarrow M$  be a surface. A point  $p \in S$  is a *regular point* if  $F$  is a local immersion at  $p$ . Otherwise it is a *singular point*. Almost all points of  $S$  are regular points since  $F$  is assumed to be an immersion almost everywhere. We will use  $F$  the same way as we would for embedded surfaces. Thus for example  $\partial F$  denotes the restriction of  $F$  to  $\partial S$ , and if  $N$  is a submanifold of  $M$  then  $F \cap N$  denotes the restriction of  $F$  on  $F^{-1}(N) \subset S$ , which is considered as a subsurface of  $F$  if  $F^{-1}(N)$  is a subsurface of  $S$ . By a *curve* on a surface  $F : S \rightarrow M$  we mean the composition  $F \circ \alpha$ , where  $\alpha : S^1 \rightarrow S$  is a closed curve on  $S$ . Similarly if  $\alpha : I \rightarrow S$  is an arc then  $F \circ \alpha$  is called an *arc* on  $F$ . We say that the arc  $F \circ \alpha$  has endpoints on  $\partial F$  if  $\partial \alpha \subset \partial S$ , in which case it is called a *proper arc*.

Given two arcs or curves  $\alpha, \beta$  on  $F$  or in a manifold  $M$ , we use  $\alpha \sim \beta$  to denote that  $\alpha, \beta$  are homotopic. Homotopy of arcs and curves are different. Two arcs  $\alpha, \beta$  are *homotopic* if they are homotopic rel boundary in the usual sense, while two curves are homotopic if they are freely homotopic. A curve in a space is *trivial* if it is null-homotopic. An arc  $\alpha$  on a surface  $F$  is *essential* if it is not homotopic to an arc on  $\partial F$ .

A surface  $F : S \rightarrow M$  is  $\pi_1$ -*injective* if any nontrivial curve on  $F$  is also nontrivial in  $M$ . Note that  $F$  is  $\pi_1$ -injective if and only if  $F_* : \pi_1 S_i \rightarrow \pi_1 M$  is an injective map for all components  $S_i$  of  $S$ . A compact surface  $F$  is *proper* if  $\partial F \subset \partial M$ .  $F$  is  $\partial$ -*incompressible* if no essential arc of  $F$  is homotopic in  $M$  to an arc on  $\partial M$ ; otherwise it is  $\partial$ -compressible. A proper surface  $F$  in  $M$  is *essential* if it is  $\pi_1$ -injective,  $\partial$ -incompressible, and is not rel  $\partial F$  homotopic to a surface on  $\partial M$ .

We refer the readers to [25] and [21] for basic concepts about hyperbolic 3-manifolds. In different sections below,  $M$  may denote either a complete hyperbolic manifold or a compact manifold with interior a complete hyperbolic manifold. If  $M$  is a complete hyperbolic manifold, the *injective radius* of a point  $x$  in  $M$  is the supremum of radii of all embedded balls in  $M$  centered at  $x$ . Denote by  $M_{(0, \epsilon]}$  the set of points which has injective radius at most  $\epsilon$ , and by  $M_{[\epsilon, \infty)}$  the set with injective radius at least  $\epsilon$ . It is well known (see [21]) that when  $\epsilon$  is sufficiently small,  $M_{(0, \epsilon]}$  is a set of cusps, in which case we use  $N = N_\epsilon$  to denote the toroidal cusp components of  $M_{(0, \epsilon]}$ , and  $T = T_\epsilon$  the boundary tori of  $N$ , which are in the interior of  $M$ .

The hyperbolic structure of  $M$  induces a Euclidean metric on  $T = T_\epsilon$ . If  $\alpha$  is either a curve on  $T$  or an arc in  $M$  which is homotopic to an arc on  $T$ , then  $\alpha$  can be homotoped to a geodesic  $\alpha'$  on  $T$ . Define  $t(\alpha)$  to be the Euclidean length of  $\alpha'$ , and call it the  $T$ -length of  $\alpha$ . Notice that it depends only on  $\epsilon$  and the homotopy class of  $\alpha$ . If  $\gamma$  is another curve or arc on  $T$ , and  $\gamma'$  the geodesic on  $T$  homotopic to  $\gamma$ , then the  $T$ -length of  $\alpha$  relative to  $\gamma$ , denoted by  $t_\gamma(\alpha)$ , is defined as

$$(1-1) \quad t_\gamma(\alpha) = t(\alpha) |\sin \theta|,$$

where  $\theta$  is the angle between  $\alpha'$  and  $\gamma'$ . Geometrically,  $t_\gamma(\alpha)$  is the length of the orthogonal projection of  $\alpha'$  to a line orthogonal to  $\gamma'$ . These notations will be used throughout the paper.

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## §2. MINIMAL SURFACES AND THE PLATEAU PROBLEM

Let  $F : S \rightarrow M$  be a surface of type  $S$  in a Riemannian manifold  $M$ . Recall that  $F$  is assumed piecewise smooth. In this section we will also assume that  $F$  is oriented. If  $\omega$  is a differential 2-form of  $M$ , then by the restriction of  $\omega$  to  $F$  we mean the 2-form  $F^*(\omega)$  on  $S$  defined on all smooth points of  $F$ , and the integral of  $\omega$  on  $F$  is defined as

$$\int_F \omega = \int_S F^*(\omega).$$

Since  $F$  is piecewise smooth, this is well defined.

The Riemannian metric on  $M$  induces a Riemannian metric on the set of regular points of  $F$ , which determines a volume form  $\omega_F$ . More explicitly, if  $(u_1, u_2)$  is a local coordinate system of  $S$  at a regular point  $p$  of  $F$  which is compatible with the orientation of  $S$ , then the tangent vectors  $\partial_i = \frac{\partial}{\partial u_i} \in T_p S$  are mapped to  $F_*(\partial_i)$  in  $T_{F(p)} M$ . The Riemannian metric of  $M$  determines an inner product  $\langle \cdot, \cdot \rangle$  on  $T_{F(p)} M$ . Let  $g_{ij} = \langle F_*(\partial_i), F_*(\partial_j) \rangle$ . Then

$$\omega_F = \sqrt{\det(g_{ij})} \, du_1 \wedge du_2.$$

This is a well-defined 2-form on  $S$ . Given a function  $f(p)$  on  $S$ , which we consider as a function on  $F$ , the integral of  $f$  on  $F$  is defined as

$$\int_F f = \int_S f(p) \, \omega_F.$$

In particular, when  $f = 1$ , this defines the area of  $F$ :

$$\text{Area}(F) = \int_F 1 = \int_S \omega_F.$$

If  $M$  is of dimension two, then it has a volume form  $\omega_M$ , in which case  $\omega_F = \pm F^*(\omega_M)$ , where the sign depends on whether  $F$  is orientation preserving or orientation reversing at that point. Given a 2-form  $\omega$  on  $S$  with local presentation  $\omega = \varphi du \wedge dv$ , where  $(u, v)$  is a local coordinate system compatible with the orientation of  $S$ , we use  $|\omega|$  to denote the 2-form  $|\varphi| du \wedge dv$ . Thus  $\omega_F = |F^*(\omega_M)|$  when  $M$  is a surface.

We refer the readers to [12] for the definitions of curvatures and second fundamental form of submanifolds. Let  $(h_{ij})$  be the second fundamental form of  $F$  at a regular point  $p$ , with respect to a basis  $(v_1, v_2)$  of  $T_p F \subset T_p M$ , then the Gauss formula (cf. [12, p.130]) says

$$K = \overline{K}(v_1, v_2) + \det(h_{ij}) = \overline{K}(v_1, v_2) + h_{11}h_{22} - h_{12}^2$$

where  $K$  is the curvature of  $F$ , and  $\overline{K}$  is the sectional curvature of  $M$ . A continuous map  $F : S \rightarrow M$  is a *minimal surface* if it is smooth in the interior of  $S$ , and its mean curvature  $h_{11} + h_{22}$  is always zero.  $F$  is not required to be smooth on  $\partial S$ . Thus if  $F$  is a minimal surface then  $h_{11}h_{22} \leq 0$ , so from the above we have  $K \leq \overline{K}(v_1, v_2)$ . In particular, if  $M$  is a hyperbolic manifold, which by definition has constant sectional curvature  $\overline{K} = -1$ , then  $K \leq -1$ .

The classical Plateau problem asks if a Jordan curve in  $\mathbb{R}^n$  bounds a surface of disk type with minimal area. A solution to the Plateau problem is necessarily a minimal surface, which is harmonic in the interior of  $D$ , and is continuous on  $D$ . The problem was first solved by Douglas [13], and has been generalized by Morrey [22] to many Riemannian manifolds. The regularity of solutions has also been deeply studied. For our purpose, the following result suffices.

**Lemma 2.1.** *Let  $C$  be a null-homotopic, smooth, embedded circle in a complete, negatively curved 3-manifold  $M$  with hyperbolic ends. Then*

- (i)  $C$  bounds a minimal surface  $F : D^2 \rightarrow M$  of disk type, which minimizes the area of all disk type surfaces bounded by  $C$ ;
- (ii)  $F$  is a smooth map on  $D^2$ ;
- (iii) if  $K$  is the curvature function of  $F$ , and  $\kappa$  the geodesic curvature function of  $C$  in  $M$ , then

$$\int_F K + \int_C \kappa \geq 2\pi.$$

*Proof.* (i) This follows from Morrey's solution of the Plateau problem for Riemannian manifolds [22]. Morrey's result says that if  $M$  is a complete Riemannian manifold which is almost homogeneous, then any null-homotopic curve  $C$  in  $M$  bounds a minimal surface which minimizes the area of all disk type surfaces bounded by  $C$ . Since we have assumed that  $M$  is complete and has hyperbolic ends, Morrey's result applies.

(ii) This follows from Theorem 4 in Chapter 7 of [11], which says that the degree of smoothness of a minimal surface on its boundary  $C$  is at least that of  $C$  and  $M$ . Since both  $C$  and  $M$  are assumed smooth, the result follows.

(iii) We need the following Gauss-Bonnet theorem for minimal surfaces with smooth boundary:

$$\int_F K + \int_{\partial F} \kappa_g = 2\pi + 2\pi \sum_{w \in \sigma'} \nu(w) + \pi \sum_{w \in \sigma''} \nu(w),$$

where  $\kappa_g$  is the geodesic curvature of  $\partial F$  on  $F$ ,  $\sigma'$  and  $\sigma''$  the set of interior and boundary branch points, respectively, and  $\nu(w)$  the branch index of  $w$ , which is nonnegative. For minimal surfaces in  $\mathbb{R}^n$ , this is Theorem 1 in Chapter 7 of [11], and for minimal surfaces in Riemannian manifolds it is proved by Kaul [18]. The proof would be easy if one knows the smoothness of  $F$  on its boundary, which is (ii) above, and the local behavior of  $F$  near its branch points, which was done by Heinz and Hildebrandt [17]. If  $p$  is a regular point of  $F$  on  $\partial F$ ,  $\mathbf{n}$  the principal normal vector of  $\partial F$  at  $p$ ,  $\mathbf{n}'$  the inward normal vector of  $p$  on  $F$ , and  $\theta$  the angle between  $\mathbf{n}$  and  $\mathbf{n}'$ , then  $\kappa_g = (\cos \theta) \kappa$ . Therefore we have  $\kappa_g \leq \kappa$ , and the result follows.  $\square$

If  $C$  is a piecewise geodesic curve, and  $p$  is a corner point of  $C$ , then by going around  $C$  in a certain direction, we get two tangent vectors  $v_1, v_2$  at  $p$ . The *external angle* of  $C$  at  $p$  is the angle between  $v_1$  and  $v_2$ . The *total external angle* of  $C$  is the sum of external angles at all the corner points of  $C$ .

**Proposition 2.2.** *Let  $M$  be a complete negatively curved 3-manifold with hyperbolic ends, and let  $M_h$  be a hyperbolic submanifold of  $M$ . Suppose  $C$  is a null-homotopic piecewise geodesic curve in  $M$  such that  $M$  is hyperbolic near all corners of  $C$ . Let  $\Theta$  be the total external angle of  $C$ . Then  $C$  bounds a surface  $F$  of disk type in  $M$  such that*

$$\text{Area}(F \cap M_h) \leq \Theta - 2\pi.$$



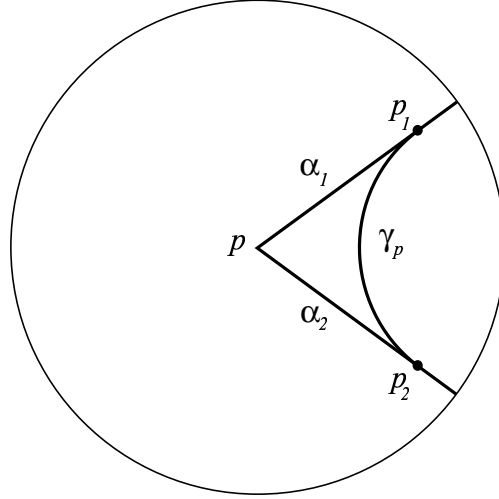


Figure 2.1

*Proof.* At each corner  $p$ , let  $D_p = \exp D_\delta$ , where  $D_\delta$  is a disk of radius  $\delta$  on the plane in  $T_p M$  containing the two tangent vectors of  $C$  at  $p$ , and  $\exp$  the exponential map. Since  $M$  is hyperbolic near  $p$ , by choosing  $\delta$  small enough we may assume that  $D_p$  is an embedded totally geodesic disk in  $M$ . Let  $\alpha'_1, \alpha'_2$  be the two geodesic segment of  $C \cap D_p$ . Choose a point  $p_i$  in the interior of each  $\alpha'_i$ , and let  $\alpha_i$  be the subarc of  $\alpha'_i$  connecting  $p_i$  to  $p$ . Connect  $p_1$  to  $p_2$  by a smooth arc  $\gamma_p$  such that  $(C - \alpha_1 \cup \alpha_2) \cup \gamma_p$  is smooth in  $D_p$ , and  $\gamma_p$  is concave on the region  $\Delta_p$  bounded by  $\alpha = \alpha_1 \cup \alpha_2 \cup \gamma_p$ . See Figure 2.1.

Since  $D_p$  is totally geodesic, the curvature  $\kappa$  of  $\gamma_p$  in  $M$  is the same as that in  $D_p$ . Since it is concave as a boundary curve of  $\Delta_p$ , its curvature  $\kappa_g$  as boundary curve of  $\Delta_p$  is  $-\kappa$ . The total external angle of  $\partial\Delta_p$  is  $2\pi + \theta(p)$ , where  $\theta(p)$  is the external angle of  $C$  at  $p$ . Therefore by the Gauss-Bonnet theorem applied to  $\Delta_p$ , we have

$$\int_{\Delta_p} (-1) + \int_{\partial\Delta_p} \kappa_g + (2\pi + \theta(p)) = 2\pi.$$

The first integral is  $-\text{Area}(\Delta_p)$ , and the second equals  $-\int_{\gamma_p} \kappa$ . Hence

$$\text{Area}(\Delta_p) + \int_{\gamma_p} \kappa = \theta(p).$$

Let  $C'$  be the smooth curve obtained from  $C$  by replacing  $\alpha_1 \cup \alpha_2$  with  $\gamma_p$  at each corner  $p$ , and let  $F'$  be the minimal surface bounded by  $C'$  as given in Lemma 2.1.

Then  $F = F' \cup (\cup \Delta_p)$  is a surface bounded by  $C$ . Since the curvature  $K$  of  $F'$  satisfies  $K \leq -1$  in  $M_h$  and  $K < 0$  elsewhere, by Lemma 2.1(3) and the above we have

$$\begin{aligned} \text{Area}(F \cup M_h) &\leq \sum_p \text{Area}(\Delta_p) + \text{Area}(F' \cap M_h) \leq \sum_p \text{Area}(\Delta_p) - \int_{F'} K \\ &\leq \sum_p \text{Area}(\Delta_p) + \int_{\partial F'} \kappa - 2\pi = \sum_p \left[ \text{Area}(\Delta_p) + \int_{\gamma_p} \kappa \right] - 2\pi \\ &= \sum \theta(p) - 2\pi = \Theta - 2\pi. \end{aligned}$$

□

*Remark 2.3.* (1) Charles Frohman pointed out that when  $M$  is hyperbolic, Proposition 2.2 can be proved easily by considering a disk bounded by  $C$  which is a union of totally geodesic triangles. Thus the above proof using minimal surface theory is necessary only if  $M$  is negatively curved but not hyperbolic.

(2) Proposition 2.2 would follow more directly if we had a Gauss-Bonnet type formula for minimal surfaces with boundary a smooth curve with corners. It should look like:

$$\int_F K + \int_{\partial F} \kappa_g + \sum_{p \in \sigma'''} (\pm \theta_p) = 2\pi + 2\pi \sum_{w \in \sigma'} \nu(w) + \pi \sum_{w \in \sigma''} \nu(w)$$

where  $\sigma'''$  is the set of corner points, and  $\theta_p$  the external angle of  $C$  at  $p$ . Note that negative sign could appear before  $\theta_p$  if  $p$  is a branch point. The formula could be proved in the usual way if we know the local behavior of  $F$  near the corner points, which was done in Chapter 8 of [11] in the special case that  $F$  is in Euclidean space. Unfortunately I cannot find a reference for either the formula or the local behavior near corners of a minimal surface  $F$  in a Riemannian manifold.

### 3. AREA ESTIMATION FOR SURFACES IN TRUNCATED HYPERBOLIC CUSPS

Throughout this paper, we will always consider the hyperbolic space  $\mathbb{H}^3$  as in the upper half space model. Denote by  $\mathbb{H}_1^3$  the hyperbolic horoball  $\{(x, y, z) \mid z \geq 1\}$ . For  $b > 1$ , denote by  $\mathbb{H}_{1,b}^3$  the subset of  $\mathbb{H}_1^3$  where  $z \leq b$ . Consider  $\mathbb{H}^2$  as the subset of  $\mathbb{H}^3$  corresponding to the  $yz$ -plane. Define  $\mathbb{H}_1^2 = \mathbb{H}_1^3 \cap \mathbb{H}^2$ , and  $\mathbb{H}_{1,b}^2 = \mathbb{H}_{1,b}^3 \cap \mathbb{H}^2$ . For simplicity, we use  $(y, z)$  to denote the point  $(0, y, z)$  in  $\mathbb{H}^2$ .

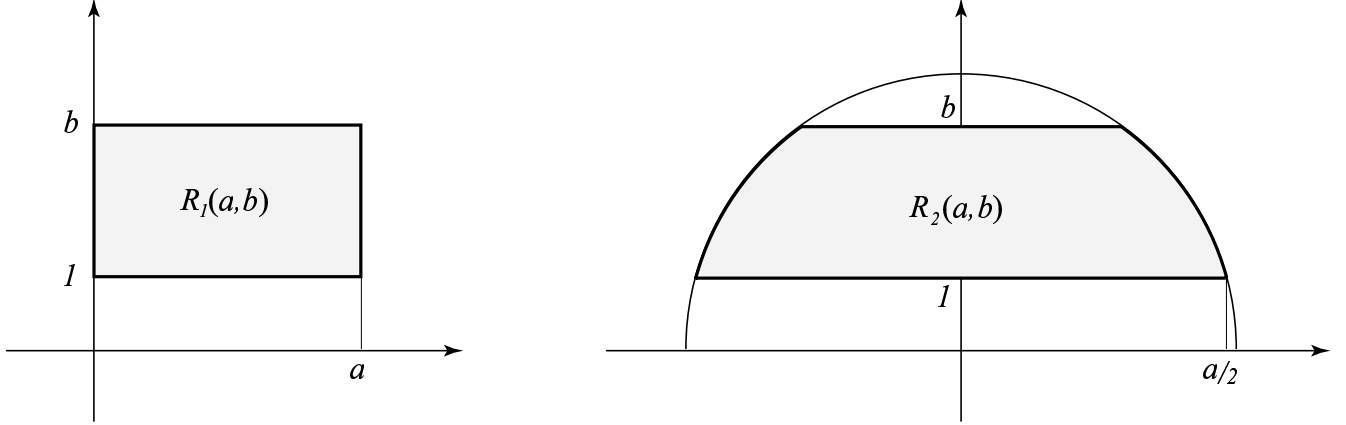


Figure 3.1

Consider the following subset  $R_1(a, b)$  and  $R_2(a, b)$  of  $\mathbb{H}_1^2$  as shown in Figure 3.1, where  $R_1(a, b)$  is a Euclidean rectangle, and  $R_2(a, b)$  is the intersection with  $\mathbb{H}_{1,b}^2$  of a Euclidean disk which is centered at the origin and intersects the horizontal line at  $z = 1$  in an arc of length  $a$ . Thus it has radius  $\sqrt{1 + (a/2)^2}$ . More explicitly, we have

$$R_1(a, b) = \{(y, z) \in \mathbb{H}^2 \mid 0 \leq y \leq a, 1 \leq z \leq b\},$$

$$R_2(a, b) = \{(y, z) \in \mathbb{H}^2 \mid 1 \leq z \leq b, y^2 + z^2 \leq 1 + (\frac{a}{2})^2\}.$$

Define a function  $\eta(x)$  by

$$\eta(x) = x - 2 \arctan \frac{x}{2}.$$

**Lemma 3.1.** (1)  $\text{Area}(R_1(a, b)) = a(1 - 1/b)$ .

$$(2) \text{Area}(R_2(a, b)) = \begin{cases} \eta(a) & b^2 \geq 1 + \frac{a^2}{4} \\ \eta(a) - \eta\left(\frac{\sqrt{1+a^2/4-b^2}}{b}\right) & b^2 \leq 1 + \frac{a^2}{4} \end{cases}$$

(3) If  $a \geq 3\pi$  and  $b \geq 5$ , then  $\text{Area}(R_2(a, b)) > 2\pi$ .

*Proof.* These would follow from the Gauss-Bonnet theorem and the fact that a horizontal line in  $\mathbb{H}^2$  at height  $b$  has curvature  $1/b$ , with normal vector pointing upward. The following is a direct calculation.

$$(1) \text{Area}(R_1(a, b)) = \int_0^a dy \int_1^b \frac{1}{z^2} dz = a(1 - \frac{1}{b}).$$

(2) Let  $r = \sqrt{1 + \frac{a^2}{4}}$ . First assume  $b \geq r$ . Then

$$\begin{aligned} \text{Area}(R_2(a, b)) &= \iint_{R_2(a, b)} \frac{1}{z^2} dy dz = 2 \int_1^r \frac{1}{z^2} dz \int_0^{\sqrt{r^2 - z^2}} dy \\ &= 2 \int_1^r \frac{\sqrt{r^2 - z^2}}{z^2} dz = 2 \left[ \arctan \frac{\sqrt{r^2 - z^2}}{z} - \frac{\sqrt{r^2 - z^2}}{z} \right]_1^r \\ &= 2 \left( \frac{a}{2} - \arctan \frac{a}{2} \right) = \eta(a). \end{aligned}$$

When  $b < r$ ,  $R_2(a, b) = R_2(a, \infty) - R'$ , where  $R'$  is the subregion of  $R_2(a, \infty)$  above the line  $z = b$ . The transformation  $(y, z) \rightarrow (y/b, z/b)$  is a hyperbolic isometry, which maps  $R'$  to the region  $R_2(\frac{\sqrt{r^2 - b^2}}{b}, \infty)$ , so the result follows from the above.

(3) From the definition it is clear that  $\text{Area}(R_2(a, b))$  is an increasing function of both  $a$  and  $b$ . Since  $a \geq 3\pi$  and  $b^2 \geq 5^2 > 1 + (1.5\pi)^2 = 1 + a^2/4$ , by (2) we have

$$\text{Area}(R_2(a, b)) \geq \text{Area}(R_2(3\pi, b)) = 3\pi - 2 \arctan \frac{3\pi}{2} > 2\pi.$$

□

The hyperbolic metric on  $\mathbb{H}^3$  induces a Euclidean metric on the Euclidean plane  $P = \partial\mathbb{H}_1^3$ . Recall from [25] that a hyperbolic cusp  $N$  of toroidal type is isometric to  $\mathbb{H}_1^3/G$  for some Euclidean translation group  $G$  of  $P$  of rank 2. Denote by  $T$  the boundary torus of  $N$ , and by  $N^b$  the truncated cusp  $\mathbb{H}_{1,b}^3/G$ . We allow  $b = \infty$ , in which case  $N^b = N$ .

If  $\gamma$  is a nontrivial closed curve on  $T$ , then there is a totally geodesic annulus  $A_\gamma$  in  $N^b$  perpendicular to the boundary, such that  $A_\gamma \cap T$  is homotopic to  $\gamma$ . More explicitly, up to rotation and translation of  $\mathbb{H}_1^3$  we may assume that  $\gamma$  lifts to an arc on  $P = \partial\mathbb{H}_1^3$  with both endpoints on the  $y$ -axis. Let  $A'_\gamma$  be the annulus obtained from  $R_1(t(\gamma), b)$  by identifying the two vertical lines. Then the quotient map  $q$  from  $\mathbb{H}_{1,b}^3$  to  $N^b$  induces a map on  $A'_\gamma$ , which we define as the surface  $A_\gamma$  in  $N^b$ . Notice that if  $\gamma = k\beta$  in  $H_1(T)$ , then  $A_\gamma$  is a  $k$ -fold cover of  $A_\beta$ . By Lemma 3.1(1) we have

$$\text{Area}(A_\gamma) = \text{Area}(R_1(t(\gamma), b)) = t(\gamma) \left(1 - \frac{1}{b}\right).$$

Let  $F$  be a surface of type  $S$  in  $N$  or  $\mathbb{H}^3$ . We would like to estimate the area of  $F$ . Consider the 2-form

$$\omega = \frac{1}{z^2} dy \wedge dz$$

on  $\mathbb{H}^3$ . Notice that its restriction to  $\mathbb{H}^2$  is the standard volume form  $\omega_{\mathbb{H}^2}$  of  $\mathbb{H}^2$ , and if we denote by  $p : \mathbb{H}^3 \rightarrow \mathbb{H}^2$  the Euclidean orthogonal projection  $p(x, y, z) = (y, z)$ , then  $\omega = p^*(\omega_{\mathbb{H}^2})$ . Therefore if  $F : S \rightarrow \mathbb{H}^3$  is a surface in  $\mathbb{H}^3$  then

$$\text{Area}(p \circ F) = \int_S |(p \circ F)^*(\omega_{\mathbb{H}^2})| = \int_S |F^*(\omega)| = \int_F |\omega| \geq \left| \int_F \omega \right|$$

The map  $p$  is area non-increasing, so the area of  $F$  is at least that of  $p \circ F$ . In fact, more is true. Recall that  $\omega_F$  denote the volume form of  $F$  induced by the Riemannian metric of  $\mathbb{H}^3$ .

**Lemma 3.2.** *Let  $F : S \rightarrow \mathbb{H}^3$  be a surface in  $\mathbb{H}^3$ . Let  $\theta(p)$  be the angle between the normal vector of  $F$  at a regular point  $p$  and the positive  $x$ -axis. Then  $F^*(\omega) = \cos \theta(p) \omega_F$ . In particular, if  $F$  is a Euclidean planar surface in  $\mathbb{H}^3$  (so  $\theta$  is a constant), then*

$$\int_F \omega = (\cos \theta) \text{Area}(F).$$

*Proof.* Let  $(u, v)$  be a local coordinate system at a regular point  $p$ . Then  $\mathbf{n} = F_u \times F_v$  is a normal vector of  $T_p F$ . Put  $F = (x(u, v), y(u, v), z(u, v))$ , and  $\mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}$ . Then  $n_1 = y_u z_v - y_v z_u$  and  $\cos \theta(p) = n_1 / \|\mathbf{n}\|$ . Use  $\mathbf{a} \cdot \mathbf{b}$  to denote the dot product of two vectors in  $\mathbb{R}^3$ . Then  $g_{11} = \langle F_u, F_u \rangle = \frac{1}{z^2} F_u \cdot F_u$ . Similarly for the other  $g_{ij}$ . Thus

$$\det(g_{ij}) = \frac{1}{z^4} [(F_u \cdot F_u)(F_v \cdot F_v) - (F_u \cdot F_v)^2] = \frac{1}{z^4} \|F_u \times F_v\|^2.$$

Hence

$$\omega_F = \sqrt{\det(g_{ij})} du \wedge dv = \frac{1}{z^2} \|F_u \times F_v\| du \wedge dv = \frac{\|\mathbf{n}\|}{z^2} du \wedge dv.$$

On the other hand, we have

$$\begin{aligned} F^*(\omega) &= F^*\left(\frac{1}{z^2} dy \wedge dz\right) = \frac{1}{z^2} (y_u du + y_v dv) \wedge (z_u du + z_v dv) \\ &= \frac{1}{z^2} (y_u z_v - y_v z_u) du \wedge dv = \frac{n_1}{z^2} du \wedge dv = \cos \theta(p) \omega_F. \end{aligned}$$

□

Now consider a hyperbolic cusp  $N = \mathbb{H}_1^3/G$ , with torus boundary  $T = \partial N$ . Clearly  $\omega$  is invariant under Euclidean translations, hence it induces a 2-form  $\omega_N$  on  $N$ . Suppose  $F : S \rightarrow N$  is a surface in  $N$ . Since Lemma 3.2 is a local property, we still have

$$F^*(\omega_N) = \cos \theta(p) \omega_F$$

where  $\theta(p)$  is the angle between the normal vector of  $F$  at  $p$  and a vector in  $T_p N$  whose lifting to  $\mathbb{H}_1^3$  points to the positive  $x$ -axis direction.

**Lemma 3.3.** *Let  $F, F_1, F_2 : S \rightarrow N^b$  be compact, oriented surfaces in  $N^b$  with boundary on  $\partial N^b$ . Then*

$$(1) \text{Area}(F) = \int_S \omega_F \geq \int_F |\omega_N| \geq \left| \int_F \omega_N \right|;$$

$$(2) \text{ If } [F_1] = [F_2] \in H_2(N^b, \partial N^b), \text{ then } \int_{F_1} \omega = \int_{F_2} \omega;$$

(3) *If  $[\partial F \cap T] = [\gamma] \neq 0 \in H_1(T)$  and  $\delta$  is a geodesic arc on  $T$  which lifts to an arc on  $\partial \mathbb{H}_1^3$  parallel to the  $x$ -axis, then  $\left| \int_F \omega_N \right| = t_\delta(\gamma)(1 - \frac{1}{b})$ .*

$$(4) \text{ If } [\partial F \cap T] = [\gamma] \neq 0 \in H_1(T), \text{ then } \text{Area}(F) \geq t(\gamma)(1 - \frac{1}{b}).$$

*Proof.* (1) We have

$$\int_S \omega_F = \int_S |\omega_F| \geq \int_S |\cos \theta(p) \omega_F| = \int_S |F^*(\omega_N)| = \int_F |\omega_N|.$$

(2) Notice that

$$d(\omega) = d\left(\frac{1}{z^2} dy \wedge dz\right) = \frac{-2}{z^3} dz \wedge dy \wedge dz = 0,$$

so  $\omega$  is a closed form. Since  $\omega_N$  is induced from  $\omega$ , it is also a closed form. Denote by  $\overline{F}_2$  the surface  $F_2$  with orientation reversed. The assumption means that there is a surface  $F_3$  on  $\partial N^b$ , such that  $\hat{F} = F_1 \cup \overline{F}_2 \cup F_3$  is a closed oriented surface which is null homologous in  $N^b$ . Therefore there is an oriented 3-manifold  $W$  and a map  $f : W \rightarrow N^b$  with  $f|_{\partial W} = \hat{F}$ . By Stokes theorem, we have

$$\begin{aligned} \int_{\hat{F}} \omega_N &= \int_{\partial W} f^*(\omega_N) = \int_W d(f^*(\omega_N)) \\ &= \int_W f^*(d\omega_N) = \int_W 0 = 0. \end{aligned}$$

Since  $F_3$  lifts to a horizontal planar surface in  $\mathbb{H}^3$ , by Lemma 3.2

$$\int_{F_3} \omega_N = \left(\cos \frac{\pi}{2}\right) \text{Area}(F_3) = 0.$$

Therefore

$$0 = \int_F \omega_N = \int_{F_1} \omega_N + \int_{\overline{F}_2} \omega_N + \int_{F_3} \omega_N = \int_{F_1} \omega_N - \int_{F_2} \omega_N$$

and the result follows.

(3)  $F$  is homologous to the surface  $A_\gamma$  defined above, which lifts to a region on a vertical plane in  $\mathbb{H}_1^3$ . Let  $\theta'$  be the angle between  $\delta$  and  $\gamma$ . Then the acute angle  $\theta$  between the normal vector of  $A_\gamma$  and the  $x$ -axis satisfies  $\theta = |(\pi/2) - \theta'|$ . Hence by the definition of  $t_\delta(\gamma)$  in section 1, we have  $\cos \theta = |\sin \theta'| = t_\delta(\gamma)/t(\gamma)$ . It follows from (2) and Lemma 3.2 that

$$|\int_F \omega_N| = |\int_{A_\gamma} \omega_N| = (\cos \theta) \text{Area}(A_\gamma) = (\cos \theta) t(\gamma)(1 - \frac{1}{b}) = t_\delta(\gamma)(1 - \frac{1}{b}).$$

(4) Choose a coordinate system of  $\mathbb{H}^3$  so that the geodesic  $\gamma'$  homotopic to  $\gamma$  lifts to the  $y$ -axis. Let  $\delta$  be an arc perpendicular to  $\gamma'$ . Then

$$\text{Area}(F) \geq |\int_F \omega_N| = t_\delta(\gamma)(1 - \frac{1}{b}) = t(\gamma)(1 - \frac{1}{b}).$$

□

**Lemma 3.4.** *Let  $\beta$  be an arc on  $T$ , and let  $\alpha$  be the geodesic segment in  $N$  homotopic to  $\beta$ . Let  $F$  be a compact, oriented surface in  $N^b$  such that  $\partial F \cap \text{Int}N^b = \alpha \cap \text{Int}N^b$ . Put  $[\gamma] = [(\partial F \cap T) \cup \beta] \in H_1(T)$ .*

(1) *If  $[\gamma] = 0$ , then  $\text{Area}(F) \geq \text{Area}(R_2(t(\alpha), b))$ .*

(2) *If  $[\gamma] \neq 0$ , then  $\text{Area}(F) \geq t_\delta(\gamma)(1 - \frac{1}{b}) - t_\delta(\alpha)$  for all slopes  $\delta$  on  $T$ .*

*Proof.* Without loss of generality we may assume that  $\beta$  is a geodesic on  $T$ . Choose a coordinate system of  $\mathbb{H}^3$  so that the lifting of  $\beta$  is an arc on the  $y$ -axis, and is symmetric about the  $z$ -axis. Let  $R$  be the image of  $R_2(t(\alpha), b)$  in  $N^b$  under the projection map. Then  $\hat{F} = F \cup R$  is a properly embedded surface in  $N^b$ . Notice that  $(\partial \hat{F}) \cap T = (\partial F \cap T) \cup \beta$ , so it is homologous to  $\gamma$ .

In case (1), the surface  $\hat{F}$  is null-homologous in  $H_2(N^b, \partial N^b) \cong H_1(T)$ , hence by Lemma 3.3(2) we have  $\int_{\hat{F}} \omega_N = 0$ . Thus  $\int_F \omega_N = -\int_R \omega_N$ . Since  $R$  lifts to the region  $R_2(t(\alpha), b)$  on  $\mathbb{H}^2$ , we have

$$\text{Area}(F) \geq |\int_F \omega_N| = |\int_R \omega_N| = \text{Area}(R_2(t(\alpha), b)).$$

In case (2), rechoose the coordinate system so that the geodesic on  $T$  homotopic to  $\delta$  lifts to the  $x$ -axis. Then  $R$  lifts to a surface  $\tilde{R}$  which is a rotation of  $R_2(t(\alpha), b)$  by an angle  $\theta$ . As in the proof of Lemma 3.3(3), we have  $\cos \theta = t_\delta(\alpha)/t(\alpha)$ , so by Lemma 3.2 we have

$$\int_R \omega_N = (\cos \theta) \text{Area}(R) = \frac{t_\delta(\alpha)}{t(\alpha)} \text{Area}(R_2(t(\alpha), b)).$$

By Lemma 3.3(3),  $|\int_{\hat{F}} \omega_N| = t_\delta(\gamma)(1 - 1/b)$ . Therefore,

$$\begin{aligned} \text{Area}(F) &\geq |\int_F \omega_N| \geq |\int_{\hat{F}} \omega_N - \int_R \omega_N| \geq |\int_{\hat{F}} \omega_N| - |\int_R \omega_N| \\ &\geq t_\delta(\gamma)(1 - \frac{1}{b}) - \frac{t_\delta(\alpha)}{t(\alpha)} \text{Area}(R_2(t(\alpha), b)) > t_\delta(\gamma)(1 - \frac{1}{b}) - t_\delta(\alpha). \end{aligned}$$

The last inequality is because by Lemma 3.1(2) we have  $\text{Area}(R_2(t(\alpha), b)) < t(\alpha)$ .  $\square$

#### §4. NONTRIVIAL CURVES IN NEGATIVELY CURVED MANIFOLDS

Let  $M$  be a complete hyperbolic manifold of finite volume. Let  $N$  be a set of mutually disjoint cusps of  $M$ . Let  $M_0 = M - \text{Int}N$ . Put  $T = \partial M_0 = \partial N$ , which is a union of tori. We may choose  $N$  so that  $T$  lifts to a set of horospheres in  $\mathbb{H}^3$ , hence it has a Euclidean metric induced by the hyperbolic metric of  $M$ .

A geodesic arc  $\alpha$  in  $M$  with endpoints on  $T$  is said to be of *type I* if a neighborhood of  $\partial\alpha$  lies in  $M_0$ , and of *type II* if  $\alpha \subset N$ . Notice that a geodesic arc may be neither of type I nor of type II, but we will not consider such arcs.

**Theorem 4.1.** *Let  $M$  be a complete hyperbolic 3-manifold, and let  $M_0, N$  be as above. If  $\alpha = \alpha_1 \cup \dots \cup \alpha_{2p}$  is a closed curve such that (i) each  $\alpha_{2i+1}$  is a geodesic arc of type I, (ii) each  $\alpha_{2i}$  is a proper arc in  $N$ , and (iii)  $t(\alpha_{2i}) \geq 2\pi$  for  $i < p$ , then  $\alpha$  is nontrivial in  $M$ .*

*Moreover, if each  $\alpha_{2i-1}$  has both endpoints perpendicular to  $T$ , then (iii) can be replaced by (iii')  $t(\alpha_{2i}) \geq \pi$  for  $i < p$ .*

*Proof.* If the theorem were not true, we can choose a curve  $\alpha$  as in the theorem, such that  $\alpha$  is null-homotopic in  $M$ , and  $p$  is minimal among all such curves. By a homotopy we may assume that all  $\alpha_{2i}$  are geodesics in  $N$ , so they are type II arcs. Now  $\alpha$  is a piecewise geodesic curve with  $2p$  corners, so its total external angle is less than  $2p\pi$ . By Proposition 2.2, it bounds a surface  $F : D^2 \rightarrow M$  of disk type, such that  $\text{Area}(F) < \Theta - 2\pi$ , where  $\Theta$  is the total external angle of  $\alpha$ .

By a small perturbation we may assume that  $F$  is transverse to  $T$ . Then  $A = F \cap T$  is a compact 1-manifold in  $F$ . Recall our convention that we will treat  $F$  the same way as an embedded surface. Thus for example  $A$  is really the restriction of  $F$  on the 1-manifold  $F^{-1}(T)$  in  $D$ , and by a disk cut off by a component of  $A$  we really mean the restriction of  $F$  to a disk in  $D$  cut off by the corresponding component of  $F^{-1}(T)$ .

We claim that each arc component of  $A$  is outmost in the sense that it cuts off a disk  $\Delta$  on  $F$  containing no other arc components of  $A$ . (Note that  $\Delta$  could contain



some circle components of  $A$ .) Assuming otherwise, let  $c$  be a component which is not outmost. Now  $c$  is an arc on  $T$ , whose boundary cuts  $\alpha$  into two arcs  $\alpha'$  and  $\alpha''$ . One of  $\alpha'$  and  $\alpha''$ , say  $\alpha'$ , does not contain  $\alpha_{2p}$ , so  $\alpha' \cup c$  satisfies the condition of the theorem with smaller  $p$ , and is null-homotopic in  $M$  because it bounds a subdisk of  $F$ . This contradicts the minimality of  $p$ , completing the proof of the claim.

Now let  $\Delta$  be an outmost disk cut off by an arc component  $c$  of  $A$ . Then  $\Delta \cap \partial D$  is one of the arcs  $\alpha_i$  in  $\alpha$ . We have assumed above that  $\alpha_i$  is a geodesic, so  $\alpha_i$  being homotopic to the arc  $c$  on  $T$  implies that  $\alpha_i$  is in  $N$ , that is,  $i$  is an even number. Hence we can label the outmost disks as  $\Delta_1, \dots, \Delta_p$ , with  $\Delta_i \cap \partial D = \alpha_{2i}$ .

Recall that  $\Delta_i$  may contain some circle components of  $A$ . Let  $Q$  be the component of  $\Delta_i$  cut along  $A$  which contains  $\partial \Delta_i$ . Since  $M$  is hyperbolic,  $N$  is  $\pi_1$ -injective in  $M$ , hence each boundary component of  $Q$  is null-homotopic in  $N$  because it bounds a disk in  $M$ . Let  $\beta$  be an arc on  $T$  homotopic to  $\alpha_{2i}$ . Then  $(\partial Q \cap T) \cup \beta = (\partial Q - \alpha_{2i}) \cup \beta$  is null-homologous on  $T$  because  $\beta$  is homotopic to the arc component of  $\partial Q \cap T$  and the circle components of  $\partial Q \cap T$  are also null-homotopic. Therefore, by Lemma 3.4(1) (with  $b = \infty$ ) and Lemma 3.1(2), for each  $i < p$  we have

$$\text{Area}(\Delta_i) \geq \text{Area}(Q) \geq \text{Area}[R_2(t(\alpha_{2i}), \infty)] = t(\alpha_{2i}) - 2 \arctan \frac{t(\alpha_{2i})}{2}$$

for all  $i < p$ . Notice that  $\arctan(t(\alpha_{2i})/2)$  is the angle between  $\alpha_{2i}$  and  $T$ . Denote by  $\theta_i$  the external angle at the corner between  $\alpha_i$  and  $\alpha_{i+1}$ . Since  $a_j$  are of type I for odd  $j$ , we have  $\theta_{2i-1}, \theta_{2i} \leq \pi - \arctan(t(\alpha_{2i})/2)$ , so the above inequality together with the assumption  $t(\alpha_{2i}) \geq 2\pi$  implies that  $\text{Area}(\Delta_i) \geq \theta_{2i-1} + \theta_{2i}$  for  $i < p$ . Therefore

$$\text{Area}(F) > \sum_{i=1}^{p-1} \text{Area}(\Delta_i) \geq \sum_{j=1}^{2p-2} \theta_j > \Theta - 2\pi.$$

Since  $F$  is chosen to have area less than  $\Theta - 2\pi$ , this is a contradiction.

If all  $\alpha_{2i-1}$  have endpoints perpendicular to  $T$ , then  $\theta_i \leq \frac{\pi}{2} - \arctan(t(\alpha_{2i})/2)$ , so the assumption  $t(\alpha_{2i}) \geq \pi$  suffices to lead to a contradiction.  $\square$

We now consider Dehn fillings on  $M$ . Recall that  $N$  is a set of disjoint cusps, and  $M_0 = M - \text{Int}N$ .

Assume  $t(\gamma_i) > 2\pi + 1$  for each  $i$ . Choose  $b_i$  so that the geodesic curve  $\gamma'_i$  on  $T'_i = \partial N_i^{b_i} - T_i$  isotopic to  $\gamma_i$  in  $N_i$  has length  $2\pi + 1$ . Choose a coordinate for  $\mathbb{H}^3$  so that the geodesic on  $T_i$  homotopic to  $\gamma_i$  lifts to the  $y$ -axis. Then the upper edge of

$R_1(t(\gamma_i), b_i)$  is projected to  $\gamma'_i$ . Since the upper edge has hyperbolic length  $t(\gamma_i)/b_i$ , we have

$$(4-1) \quad b_i = \frac{t(\gamma_i)}{2\pi + 1}.$$

Denote by  $N_i(\gamma_i)$  the manifold obtained by gluing a solid torus  $V_i$  to  $N_i^{b_i}$  along  $T'_i$  so that  $\gamma'_i$  bounds a meridian disk in  $V_i$ . Put  $N^b = \cup N_i^{b_i}$ ,  $N(\gamma) = \cup N_i(\gamma_i)$ , and  $M(\gamma) = M_0 \cup N(\gamma)$ . The manifold  $M(\gamma)$  is the Dehn filling space of  $M$  (or more precisely, of  $M_0$ ) along the multiple slope  $\gamma$ . By the  $2\pi$ -theorem of Gromov-Thurston [15],  $M(\gamma)$  has a negatively curved metric which coincides with the original hyperbolic metric in a neighborhood of  $M_0 \cup N^b$ . We will assume below that  $M(\gamma)$ ,  $N(\gamma)$  and  $V = \cup V_i$  are endowed with such a metric. Let  $C_i$  be the core of  $V_i$ . The identity map on  $M_0 \cup N^b$  extends to a homeomorphism  $M \cong M(\gamma) - \cup C_i$ . We will always (topologically) identify  $M$  with  $M(\gamma) - \cup C_i$  in this way; in particular, each curve  $\alpha$  in  $M$  is also a curve in  $M(\gamma)$ .

**Lemma 4.2.** *Let  $K > 2\pi + 1$  be a constant, and let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a multiple slope on  $T$  such that  $t(\gamma_i) \geq K$  for all  $i$ . Let  $D$  be a surface of disk type in  $M(\gamma)$  such that  $\partial D \subset T$ , and  $D$  is transverse to  $T$ . If  $\partial D$  is nontrivial on  $T$ , then  $\text{Area}(D \cap N^b) \geq K - (2\pi + 1)$ .*

*Proof.* Let  $Q$  be the component of  $D$  cut along  $T$  containing  $\partial D$ . If some component of  $\partial Q - \partial D$  is nontrivial in  $T$ , then by induction the subdisk  $D'$  of  $D$  bounded by this curve has  $\text{Area}(D' \cap N^b) \geq K - (2\pi + 1)$ , and we are done. So assume that all components of  $\partial Q - \partial D$  are trivial on  $T$ . If  $Q$  were in  $M_0$  then the above would imply that  $\partial D$  is null-homotopic in  $M_0$ , contradicting the incompressibility of  $T$  in  $M_0$ . Therefore  $Q$  is contained in  $N_i(\gamma_i)$  for some component  $N_i$  of  $N$ . The above assumption means that each component of  $\partial Q - \partial D$  bounds a disk on  $T_i$ , hence  $\partial D$  is null-homotopic in  $N_i(\gamma_i)$ . Thus  $[\partial Q \cap T_i] = [\partial D] = [k\gamma_i] \in H_1(T_i)$ , and  $k \neq 0$  because  $\partial D$  is assumed nontrivial on  $T_i$ . Hence by Lemma 3.3(4) we have

$$\text{Area}(D \cap N^b) \geq t(k\gamma_i)(1 - \frac{1}{b_i}) = |k|(t(\gamma_i) - \frac{t(\gamma_i)}{b_i}) \geq t(\gamma_i) - (2\pi + 1).$$

The last inequality follows because  $k \neq 0$ , and because by (4-1) we have  $t(\gamma_i) = (2\pi + 1)b_i$ .  $\square$

**Theorem 4.3.** *Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a multiple slope on  $T$  such that  $t(\gamma_i) \geq 12\pi$  for all  $i$ . Let  $\alpha = \alpha' \cup \alpha''$  be a curve in  $M$  such that either  $\alpha''$  is a closed geodesic and*

$\alpha' = \emptyset$ , or  $\alpha''$  is a type I geodesic arc and  $\alpha'$  is an arc in  $N$ . If each component  $\beta$  of  $\alpha'' \cap N_i$  satisfies  $t_\delta(\beta) \leq t_\delta(\gamma_i) - 5\pi$  for some slope  $\delta$  on  $T$ , then  $\alpha$  is nontrivial in  $M(\gamma)$ .

*Proof.* If  $\alpha$  is a geodesic in  $M_0$ , then it remains a geodesic in the negatively curved manifold  $M(\gamma)$ , hence is nontrivial. (This is well known, and also follows from Lemma 2.1(3) because  $K < 0$  and  $\kappa = 0$ .) Therefore by choosing a component of  $\alpha'' \cap N$  as  $\alpha'$  if necessary, we may always assume that  $\alpha''$  is a type I geodesic. Put  $\alpha'' = \alpha_1 \cup \dots \cup \alpha_{2p-1}$ . Then  $\alpha_{2j}$  lie in  $N$ , and  $\alpha_{2j-1}$  are in  $M_0$ . Assume the result is false, and let  $\alpha$  be as in the theorem so that  $\alpha$  is null homotopic in  $M(\gamma)$ , and  $p$  is minimal among all such curves.

Modify  $\alpha$  as follows. For each  $\alpha_{2i}$  which has nontrivial intersection with the Dehn filling solid tori  $V_i$ , homotope  $\alpha_{2i} \cap V_i$  to a geodesic segment  $\alpha'_{2i}$  in  $V_i$ , and denote the resulting arc  $(\alpha_{2i} \cap N^b) \cup \alpha'_{2i}$  by  $\beta_{2i}$ . Since  $b_i = t(\gamma_i)/(2\pi + 1) \geq 12\pi/(2\pi + 1) > 5$ , from Figure 3.1 we see that such modification happens only if

$$t(\alpha_{2i}) > 2\sqrt{b_i^2 - 1} > 2\sqrt{24} > 3\pi.$$

Let  $r$  be the number of arcs which have been modified. Next, deform  $\alpha'$  to a geodesic  $\beta_{2p}$  in  $N(\gamma)$ . For simplicity, write  $\beta_i = \alpha_i$  for the other arcs. The curve  $\beta = \beta_1 \cup \dots \cup \beta_{2p}$  is now a piecewise geodesic in  $M(\gamma)$  with  $2r + 2$  corners, and is homotopic to  $\alpha$ . Note that from the construction all the corners are in the hyperbolic part of  $M(\gamma)$ .

By Proposition 2.2,  $\beta$  bounds a surface  $D$  of disk type in  $M(\gamma)$ , such that

$$(4-2) \quad \text{Area}(D \cap N^b) < \text{Area}(D \cap (M_0 \cup N^b)) \leq (2r + 2)\pi - 2\pi = 2r\pi.$$

After a small perturbation rel  $\partial$  we may assume that  $D$  is transverse to  $T$ . Let  $A = D \cap T$ . Since  $\partial A = \partial D \cap T = \cup \partial \beta_i$ ,  $A$  has exactly  $p$  arc components. As in the proof of Theorem 4.1, the minimality of  $p$  implies that each arc  $a_i$  of  $A$  is outmost on  $D$  in the sense that it cuts off a disk  $\Delta_i$  with interior containing no arc components of  $A$ . We can label  $a_i$  and  $\Delta_i$  such that either  $\partial \Delta_i = a_i \cup \beta_{2i-1}$  for all  $i$ , or  $\partial \Delta_i = a_i \cup \beta_{2i}$  for all  $i$ .

If  $\Delta_i \cap \partial D = \beta_{2i-1}$  for all  $i$ , then since the geodesic arc  $\beta_{2i-1}$  in  $M_0$  cannot be homotopic in  $M$  to the arc  $a_i$  on  $T$ , there must be some circle component  $\mu_i$  of  $A$  in  $\text{Int} \Delta_i$  which is nontrivial on  $T$ . Applying Lemma 4.3 to the disks  $B_i$  in  $\Delta_i$  bounded by  $\mu_i$ , we get

$$\text{Area}(D \cap N^b) \geq \sum \text{Area}(B_i \cap N^b) \geq p(12\pi - (2\pi + 1)) > 2r\pi$$

which is a contradiction to (4-2).

Now assume  $\partial\Delta_i = a_i \cup \beta_{2i}$  for all  $i$ . Consider a  $\Delta_i$  such that  $\beta_{2i} \neq \alpha_{2i}$ ,  $i < p$ . Recall from the definition of  $\beta_i$  that there are exactly  $r$  such arcs. We have shown that in this case  $t(\alpha_{2i}) > 3\pi$  and  $b_i > 5$ , and we want to show that  $\text{Area}(\Delta_i \cap N^b) \geq 2\pi$ . This follows from Lemma 4.2 if some circle component of  $A$  in  $\Delta_i$  is nontrivial on  $T$ . Hence assume that all circle components of  $A$  in  $\Delta$  are trivial on  $T$ . In particular, if we denote by  $Q$  the component of  $\Delta_i$  cut along  $A$  which contains  $\partial\Delta_i$ , then  $\partial Q - \partial\Delta_i$  is null-homotopic on  $T_i$ , so  $\partial\Delta_i = \alpha_i \cup \beta_{2i}$  is also null-homotopic in  $N_i(\gamma_i)$ . Let  $\beta'$  be an arc on  $T_i$  homotopic to  $\beta_{2i}$  in  $N_i$ . Then  $\beta' \cup a_i$  is null-homotopic in  $N_i(\gamma_i)$ , hence  $[(\partial Q \cap T_i) \cup \beta'] = [a_i \cup \beta'] = k[\gamma_i] \in H_1(T_i)$  for some  $k$ . We can now apply Lemma 3.4 to the surface  $Q = \Delta_i \cap N_i^b$ : If  $k = 0$  then

$$\text{Area}(Q) > \text{Area}(R_2(t(\beta'), b_i)) > 2\pi.$$

The last inequality follows from Lemma 3.1(3) because we have shown that  $t(\beta') = t(\alpha_{2i}) > 3\pi$  and  $b_i > 5$ . If  $k \neq 0$ , choose a slope  $\delta$  as in the statement of the theorem. Then by (4-1) and Lemma 4.3 we have

$$\begin{aligned} \text{Area}(Q) &\geq t_\delta(k\gamma_i)\left(1 - \frac{1}{b}\right) - t_\delta(\alpha) \geq t_\delta(\gamma_i) - t_\delta(\alpha) - \frac{t_\delta(\gamma_i)}{b} \\ &\geq t_\delta(\gamma_i) - t_\delta(\alpha) - \frac{t(\gamma_i)}{b} \geq 5\pi - (2\pi + 1) > 2\pi. \end{aligned}$$

In either case,  $\text{Area}(\Delta_i \cap N^b) \geq \text{Area}(Q) > 2\pi$ . Since there are exactly  $r$  outmost disks  $\Delta_i$  with  $\beta_{2i} \neq \alpha_{2i}$ , it follows that  $\text{Area}(D \cap N^b) \geq 2r\pi$ , which is again a contradiction to (4-2).  $\square$

## 5. DEHN SURGERY AND FREEDMAN TUBING OF IMMersed SURFACES

Let  $M$  be a complete hyperbolic 3-manifold. For  $\mu$  a small positive number, let  $N = N_\mu$  be the toroidal cusp components of  $M_{(0, \mu]}$ , and  $T = T_\mu = \partial N_\mu$ . Let  $M_0 = M - \text{Int}N$ . Then  $M = N \cup_T M_0$ .

A  $\pi_1$ -injective surface  $F : S \rightarrow M$  is *geometrically finite* if  $F_*(\pi_1 S_i)$  is a geometrically finite subgroup of  $\pi_1 M \subset PSL_2(\mathbb{C})$  for each component  $S_i$  of  $S$ . We need some basic facts about geometrically finite surface groups. One is referred to [21] for more details.

Assume that  $S$  is connected, and  $F : S \rightarrow M$  is a hyperbolic, geometrically finite surface in a complete hyperbolic manifold  $M$ . Consider the covering  $p : X = X_\Gamma \rightarrow M$

corresponding to the subgroup  $\Gamma = F_*(\pi_1 S)$  of  $\pi_1(M)$ . Then  $X$  is a geometrically finite complete hyperbolic manifold. Denote by  $C(F) = C(X)$  the convex core of  $X$ , which by definition is the quotient  $C_\Gamma/\Gamma$ , where  $C_\Gamma$  is the convex hull of the limit set of  $\Gamma$ , and the action of  $\Gamma$  on  $C_\Gamma$  is induced by the action of  $\Gamma$  on its limit set. Since  $\Gamma$  contain no  $\mathbb{Z}^2$  subgroup, the following is a special case of Lemma 6.5 and Theorem 6.6 of [21].

**Lemma 5.1.** *There is an  $\epsilon_0 > 0$  such that if  $0 < \epsilon \leq \epsilon_0$ , then (i)  $C(X) \cap X_{[\epsilon, \infty)}$  is compact, (ii)  $X_{(0, \epsilon]}$  has only finitely many components, and (iii) each component of  $X_{(0, \epsilon]}$  is a  $\mathbb{Z}$ -cusp, which intersects  $C(X)$  in a set isometric to*

$$\{(x, y, z) \in \mathbb{H}^3 \mid z \geq 1 \text{ and } A_1 \leq y \leq A_2\}/(g),$$

where  $g$  is a translation in the  $x$ -direction, and  $A_1, A_2$  are constant depending on the cusp.  $\square$

The lifting of  $N = N_\mu$  to  $X$  is a set of horoballs and  $\mathbb{Z}$ -cusps. Denote by  $\tilde{N}$  the  $\mathbb{Z}$ -cusp components of  $p^{-1}(N)$ , and let  $\tilde{T} = \partial\tilde{N}$ . When  $\mu$  is small enough, each component of  $\tilde{N}$  is a component of  $X_{(0, \epsilon]}$  for some  $\epsilon \leq \epsilon_0$ , so we can define  $\mu(F)$  to be the maximum  $\mu$  such that this property holds. Below we will always assume that  $N = N_\mu$  and  $T = T_\mu$  has been chosen such that  $\mu = \mu(F)$ . Note that we usually assume that  $F$  has boundary on  $T$ . When we rechoose  $T = T_{\mu(F)}$ , we extend  $F$  by adding some collars to  $\partial F$  so that  $\partial F$  still lies in  $T$ . Since  $\mu(F)$  depends only on the group  $\Gamma = F_*(\pi_1 S)$ , this will not cause a logic problem.

Let  $P = \tilde{T} \cap C(X)$ . By Lemma 5.1,  $P$  is a finite set of compact annuli, one for each component of  $\tilde{T}$ . The *width* of a component  $P_i$  of  $P$  is defined as  $w(P_i) = A_2 - A_1$ , where  $A_i$  are as in Lemma 5.1. Define  $w(F)$  to be the maximum of  $w(P_i)$  over all component  $P_i$  of  $P$ . (If  $F$  is disconnected, take the maximum over all  $P$  corresponding to all components of  $F$ .)

The core of  $P_i$  projects to a curve  $\alpha'_i$  on  $T$ , which is a nontrivial multiple of some slope  $\alpha_i$  on  $T$ , usually called a parabolic slope of  $F$ . Since  $\pi_1 X = \pi_1 F$ ,  $\alpha'_i$  is homotopic to a nontrivial curve on  $F$ , hence a parabolic slope is a coannular slope. The reverse is also true: If a nontrivial curve  $\alpha'_i$  on  $T$  is homotopic to a curve on  $F$ , then it represents a parabolic element of  $\pi_1 F$ , so its lifting on  $X$  is homotopic into some  $\mathbb{Z}$ -cusp, hence homotopic to some nontrivial curve on some  $P_i$ . Therefore, the set of parabolic slope of  $F$  are the same as the set of coannular slopes of  $F$  on  $T$ . By Lemma 5.1,  $T$  has only finitely many coannular slopes of  $F$ . The following theorem says that if the Dehn

filling slope is far away from all coannular slopes of  $F$ , then  $F$  remains  $\pi_1$ -injective after Dehn filling.

**Theorem 5.2.** *Let  $F$  be a hyperbolic, geometrically finite surface in  $M$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a multiple slope on  $T$  such that  $t(\gamma_i) \geq 12\pi$  and  $t_\beta(\gamma_i) \geq w(F) + 5\pi$  for all coannular slopes  $\beta$  of  $F$ . Then  $F$  is  $\pi_1$ -injective in  $M(\gamma)$ .*

*Proof.* We need to show that if  $\alpha$  is a nontrivial curve on  $F$ , then it is also a nontrivial curve in  $M(\gamma)$ . Let  $\tilde{\alpha}$  be its lifting to  $X = X_F$ . Then  $\tilde{\alpha}$  is homotopic to a geodesic  $\tilde{\alpha}'$  in the convex hull  $C(X)$ . The intersection of  $\tilde{\alpha}$  with  $\tilde{T}$  cuts  $\tilde{\alpha}$  into arcs  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{2n}$ , where  $\tilde{\alpha}_{2i-1}$  lies in  $X_{[\epsilon, \infty)}$ , and  $\tilde{\alpha}_{2i}$  on the cusps. By the choice of  $T = T_{\mu(F)}$ , the image of  $C(X) \cap X_{[\epsilon, \infty)}$  is disjoint from the interior of  $N$ , hence the projection of  $\tilde{\alpha}_i$  gives a decomposition  $\alpha = \alpha_1 \cup \dots \cup \alpha_{2n}$ , with  $\alpha_{2i}$  the components of  $\alpha \cap N$ . Each  $\tilde{\alpha}_{2i}$  is homotopic to an arc lying on a strip of width at most  $w(F)$  bounded by geodesics homotopic to the lifting of some coannular slope  $\beta$  of  $T$ , hence  $t_\beta(\alpha_{2i}) \leq w(F)$ . By assumption we have  $t_\beta(\gamma_i) - t_\beta(\alpha_{2i}) \geq t_\beta(\gamma_i) - w(F) \geq 5\pi$ . Therefore by Theorem 4.3  $\alpha$  is a nontrivial curve in  $M(\gamma)$ .  $\square$

The most interesting case is when  $F$  is a closed essential surface in a compact hyperbolic manifold  $W$ . The following theorem says that when finitely many strips centered at coannular slopes and finitely many other slopes are excluded from the space of Dehn filling slopes, then  $F$  survives surgery. Note that  $F$  is not assumed to be geometrically finite.

**Theorem 5.3.** *Let  $T$  be a set of tori on the boundary of a compact, orientable, hyperbolic 3-manifold  $W$ . Let  $F$  be a compact essential surface in  $W$  with  $\partial F \subset \partial M - T$ , and let  $\beta$  be the set of coannular slopes of  $F$  on  $T$ . Then there is an integer  $K$  and a finite set of slopes  $\Lambda$  on  $T$ , such that  $F$  is  $\pi_1$ -injective in  $W(\gamma)$  for all multiple slopes  $\gamma = (\gamma_1, \dots, \gamma_n)$  on  $T$  satisfying  $\Delta(\gamma, \beta) \geq K$  and  $\gamma_i \notin \Lambda$ .*

*Proof.* We first assume that  $\partial W$  is a set of tori. Since  $W$  is hyperbolic and  $F$  is essential, no component of  $F$  is an annulus or torus, hence  $F$  is hyperbolic. Let  $M$  be the interior of  $W$ , which by definition has a complete hyperbolic structure. Since  $F$  is disjoint from  $T$ , it cannot be a virtual fiber, hence according to Bonahon and Thurston [5, 25] it is automatically geometrically finite. More explicitly, assume that  $F$  is geometrically infinite and let  $X_F$  be the covering of  $M$  corresponding to the subgroup  $\pi_1(F)$ . Then Bonahon [5] showed that every end of  $X_F$  relative to the cusp neighborhoods is geometrically tame, while Thurston [25, Theorem 9.2.1]

showed that every end of  $X_F$  relative to cusp neighborhoods which is geometrically tame and geometrically infinite must either correspond to a virtual fiber for  $M$  or project to a geometrically tame and geometrically infinite end of  $M$  modulo cusp neighborhoods. Since we have assumed that  $\partial W$  is a set of tori,  $M$  has no geometrically infinite end modulo cusp neighborhoods, therefore  $F$  must be a virtual fiber, which is a contradiction.

Identify the manifold  $M_0$  above with  $W$ , so  $\partial W = T = T_{\mu(F)}$ . Let  $\Lambda$  be the set of slopes  $\lambda$  on  $T$  such that  $t(\lambda) < 12\pi$ . For each  $\beta_i$  on some  $T_j$ , define  $u(\beta_i) = \text{Area}(T_j)/t(\beta_i)$ . Then  $t_{\beta_i}(\gamma_j) = \Delta(\beta_i, \gamma_j)u(\beta_i)$ . Choose  $K$  so that  $K > (w(F) + 5\pi)/u(\beta_i)$  for all  $i$ . The result then follows from Theorem 5.2.

Now assume that  $W$  has some higher genus boundary components. If  $\partial W$  is compressible, then by an innermost circle outermost arc argument one can show that  $F$  can be homotoped to be disjoint from a maximal set of compressing disks  $D$ . Let  $W'$  be  $W$  cut along  $D$ . ( $W' = W$  if  $D = \emptyset$ .) Then  $F$  is essential in  $W'$  except that it is possibly homotopic to a surface in a non-torus component of  $\partial W'$ . Let  $\hat{W}'$  be the double of  $W'$  along the non-torus components of  $\partial W'$ . Denote by  $\hat{F}, \hat{T}, \hat{\beta}, \hat{\gamma}$  the double of  $F, T, \beta, \gamma$  in  $\hat{W}'$ , respectively. By an innermost circle outermost arc argument one can show that  $\hat{F}$  is  $\pi_1$ -injective in  $\hat{W}$ . Let  $q : \hat{W}' \rightarrow W'$  be the obvious quotient map. If  $A$  is an annulus in  $\hat{W}'$  with one boundary component on each of  $\hat{F}$  and  $\hat{T}$ , then  $q(A)$  is an annulus in  $W'$  with one boundary component on each of  $F$  and  $T$ . Hence  $\hat{\beta}$  is the set of all coannular slopes of  $\hat{F}$  in  $\hat{W}'$ . By the above, there is a number  $K$  and a set of slopes  $\Lambda'$  such that  $\hat{F}$  is  $\pi_1$ -injective in  $\hat{W}'(\hat{\gamma})$  when  $\Delta(\hat{\gamma}, \hat{\beta}) \geq K$  and  $\hat{\gamma}_i \notin \Lambda'$ . Since  $F$  is  $\pi_1$ -injective in  $\hat{F}$ , we see that  $F$  is  $\pi_1$ -injective in  $W'(\gamma)$ . Since  $W(\gamma)$  is obtained from  $W'(\gamma)$  by adding some 1-handles,  $F$  is also  $\pi_1$ -injective in  $W(\gamma)$ . Let  $\Lambda = q(\Lambda')$ . Then the result follows.  $\square$

We now consider Freedman tubings of essential surfaces. Let  $\hat{S}$  be a surface containing  $S$ , such that  $\hat{S} - S$  is a set of annuli. Then a surface  $\hat{F} : \hat{S} \rightarrow M_0$  is called a *Freedman tubing* of  $F$  if  $\hat{F}|_S = F$ , and  $\hat{F}(\hat{S} - S) \subset T$ . We will use  $A = \hat{F} - \text{Int}F$  to denote the restriction of  $\hat{F}$  to  $\hat{S} - \text{Int}S$ , and call a component  $A_i$  of  $A$  a *tubing annulus*. Let  $\delta_i$  be a component of  $\partial A_i$ . Then the *length* of a tube  $A_i$  is defined as

$$\ell(A_i) = \min \{ t_{\delta_i}(\alpha) \mid \alpha \text{ an essential arc on } A_i \}$$

Denote by  $\ell(A) = \min \ell(A_i)$ . Clearly,  $\ell(A_i)$  would become very large when  $A_i$  wraps around the torus many times. For example, if  $A_i \subset T_j$  is immersed and contains a

sub-annulus  $A'_i$  with both boundary components on the same geodesic curve of  $T_j$ , and  $A'_i$  wraps  $k$  times around  $T_j$ , then  $\ell(A_i) \geq k \text{Area}(T_j)/t(\delta_i)$ .

Theorem 5.7 below says that a Freedman tubing of a geometrically finite surface is essential if the tubes are long enough. This generalizes a result of Freedman-Freedman [14] and Cooper-Long [7] (see also [19]), where the above result is proved for embedded surfaces. In most cases, one can apply Theorem 5.2 to show that it remains essential after most Dehn fillings. The assumption that  $F$  is geometrically finite is necessary: if  $F$  is geometrically infinite, then  $F$  is a virtual fiber, hence all Freedman tubings of  $F$  are inessential.

A boundary component  $\delta_i$  of  $F$  can be pushed around  $T$  many times. We need a number to measure how far  $\delta_i$  is away from a standard position. We would consider  $F$  to be in a standard position if its lifting to  $X$  lies in the convex core  $C(X)$ . Let  $\tilde{\delta}_i$  be the component of  $\partial\tilde{F}$  which projects to  $\delta_i$ . Each  $\tilde{\delta}_i$  is on some component  $\tilde{T}_i$  of  $\tilde{T}$ , which contains a component  $P_i$  of  $P$ . Define a number  $\rho(\delta_i)$  to be the minimum nonnegative number such that  $\tilde{\delta}_i$  lies in a  $\rho(\delta_i)$  neighborhood of  $P_i$  on  $\tilde{T}_i$ . Since  $\tilde{\delta}_i$  is compact on  $\tilde{T}_i$ , such a number exists. If  $F$  is a (possibly disconnected) geometrically finite surface in  $M$  with some boundary components on  $T = T_{\mu(F)}$ , define

$$\rho(F) = \max \rho(\delta_i)$$

where the maximum is taken over all boundary components of  $F$  which is to be tubed.

**Lemma 5.4.** *Let  $\alpha$  be an arc on  $\tilde{T}$  with one endpoint  $p_1$  on  $\tilde{\delta}_i$  and the other endpoint on  $P_i$ . Then*

$$t_{\tilde{\delta}_i}(\alpha) \leq w(F) + \rho(F).$$

*Proof.* Homotope  $\alpha$  to  $\alpha_1 \cdot \alpha_2$ , where  $\alpha_1$  is a shortest arc from  $p_1$  to some point in  $P_i$ , and  $\alpha_2$  an arc in  $P_i$ . Since  $P_i$  is a strip bounded by geodesics of  $\tilde{T}_i$  parallel to  $\tilde{\delta}_i$ , by definition we have  $t_{\tilde{\delta}_i}(\alpha_1) \leq w(F)$ , and  $t_{\tilde{\delta}_i}(\alpha_2) \leq \rho(F)$ .  $\square$

Two arcs  $\alpha_1, \alpha_2$  in  $X$  with  $\partial\alpha_i \subset \tilde{T}$  are  $\tilde{T}$ -homotopic if there are arcs  $\beta', \beta''$  on  $\tilde{T}$  such that  $\alpha_1 \sim \beta' \cdot \alpha_2 \cdot \beta''$ . Clearly this is an equivalence relation. An arc  $\alpha$  in  $X$  is of type I if it projects to a type I arc in  $M$ .

**Lemma 5.5.** *Any proper essential arc  $\alpha$  of  $\tilde{F}$  is  $\tilde{T}$ -homotopic to a type I arc of  $X$  with endpoints on  $P$ .*

*Proof.* First deform  $\alpha$  by a  $\tilde{T}$ -homotopy to an arc  $\alpha_1$  with  $\partial\alpha_1 \subset P$ . This is possible because each component of  $\tilde{T}$  contains a component of  $P$ . Now homotope  $\alpha_1$  (rel  $\partial$ )



to a geodesic  $\alpha_2$  in  $X$ . Since  $C(X)$  is a convex set,  $\alpha_2 \subset C(X)$ , so  $\alpha_2 = \beta_1 \cdot \alpha_3 \cdot \beta_2$ , where  $\alpha_3$  is a geodesic of type I with endpoints in  $P$ , and  $\beta_1, \beta_2$  are (possibly empty) arcs in  $C(X) \cap X_{(0, \epsilon]}$ , which can be pushed into  $\tilde{T}$ , hence  $\alpha_2$  is  $\tilde{T}$ -homotopic to  $\alpha_3$ .  $\square$

**Lemma 5.6.** *Let  $F$  be a geometrically finite surface in  $M$ . Let  $\alpha$  be an essential arc of  $F$  with endpoints on boundary components  $\delta_0, \delta_1$  of  $F$  which lie on  $T = T_{\mu(F)}$ . Then  $\alpha$  is homotopic to  $\beta_0 \cdot \alpha' \cdot \beta_1$ , where  $\alpha'$  is an arc of type I, and  $\beta_i$  are arcs on  $T$  with  $t_{\delta_i}(\beta_i) \leq \rho(F) + w(F)$ .*

*Proof.* Consider the lifting  $\tilde{\alpha}$  of  $\alpha$  on  $\tilde{F} \subset X$ . By Lemma 5.5,  $\tilde{\alpha}$  is homotopic to  $\tilde{\beta}_0 \cdot \tilde{\alpha}' \cdot \tilde{\beta}_1$ , where  $\tilde{\alpha}'$  is of type I, and  $\tilde{\beta}_i$  is an arc on some component  $\tilde{T}_i$  of  $\tilde{T}$  with one endpoint on each of  $\tilde{\epsilon}_i$  and  $P_i$ . Projecting these curves into  $M$ , we get  $\alpha \sim \beta_0 \cdot \alpha' \cdot \beta_1$ . By Lemma 5.4, we have  $t_{\delta_i}(\beta_i) = t_{\tilde{\delta}_i}(\tilde{\beta}_i) \leq \rho(F) + w(F)$ .  $\square$

Recall that the wrapping number of an annulus  $A$  on a torus  $T$  is defined as

$$\text{wrap}(A) = \{ |A \cdot p| \mid p \in T \}$$

where  $A \cdot p$  denotes the algebraic intersection number between  $A$  and  $p$ , which is well defined for all points  $p \notin \partial A$ .

**Theorem 5.7.** *Let  $F$  be a geometrically finite surface in a compact hyperbolic 3-manifold  $W$ . Then there is a constant  $K$  such that if  $\hat{F}$  is a Freedman tubing of  $F$  with  $\text{wrap}(\hat{F}, F) \geq K$ , then  $\hat{F}$  is  $\pi_1$ -injective in  $W$ .*

*Proof.* Let  $M$  be the interior of  $W$ . By assumption  $M$  is a complete hyperbolic manifold. Identify  $M_0$  above with  $W$ , possibly with some higher genus boundary components removed. Let  $T = \partial M_0$ . Clearly  $\ell(\hat{F} - \text{Int}F)$  goes to infinity when  $\text{wrap}(\hat{F}, F)$  approaches infinity. Choose  $K$  large enough such that when  $\text{wrap}(\hat{F}, F) > K$ , we have  $\ell(\hat{F} - \text{Int}F) > 2(\rho(F) + w(F) + \pi)$ .

We need to show that any nontrivial curve  $\alpha$  on  $\hat{F}$  is also nontrivial in  $M$ . If  $\alpha$  is homotopic to a curve on  $F$  or  $A = \hat{F} - \text{Int}F$  then  $\alpha$  is nontrivial in  $M$  because  $F$  is  $\pi_1$  injective. So assume  $\alpha = \alpha_1 \cup \dots \cup \alpha_{2n}$ , where  $\alpha_{2i-1} \subset F$  and  $\alpha_{2i} \subset A$  are essential arcs. By Lemma 5.6, we have  $\alpha_{2i-1} \sim \beta_{2i-1} \cdot \alpha'_{2i-1} \cdot \gamma_{2i-1}$ , where  $\alpha'_{2i-1}$  is a type I arc, and  $t_{\delta'}(\beta_{2i-1})$  and  $t_{\delta''}(\gamma_{2i-1}) \leq \rho(F) + w(F)$ , where  $\delta', \delta''$  are boundary components of  $F$  containing the endpoints of  $\alpha_{2i-1}$ . Put  $\alpha'_{2i} = \gamma_{2i-1} \cdot \alpha_{2i} \cdot \beta_{2i+1}$ . Then  $\alpha \sim \alpha'_1 \cdot \alpha'_2 \cdot \dots \cdot \alpha'_{2n}$ . Let  $\delta_i$  be the boundary component of  $F$  containing an endpoint of  $\alpha_{2i}$ . Then

$$\begin{aligned} t(\alpha'_{2i}) &\geq t_{\delta_i}(\alpha'_{2i}) \geq t_{\delta_i}(\alpha_{2i}) - t_{\delta_i}(\gamma_{2i-1}) - t_{\delta_i}(\beta_{2i+1}) \\ &\geq \ell(\hat{F} - \text{Int}F) - 2\rho(F) - 2w(F) > 2\pi \end{aligned}$$

Therefore by Theorem 4.1,  $\alpha$  is a nontrivial curve in  $M$ .  $\square$

## 6. UPPER BOUNDS ON SURGERY DISTANCE AND TUBING LENGTH.

Theorems 5.3 is the best possible in the sense that there is no universal bounds on the number  $K$  in the theorem. Similarly, Theorem 5.7 is the best possible in the sense that there is no universal bound on how many time a surface need to tube around a torus boundary component in order to produce an essential surface. Assume that  $\hat{F}$  is a Freedman tubing of an essential surface  $F$ , with tubes on a torus  $T = \partial M_0$ .

**Theorem 6.1.** (i) *For any constant  $K$ , there is an embedded, geometrically finite surface  $F$  in a hyperbolic manifold  $M$ , such that all Freedman tubing  $\hat{F}$  of  $F$  with  $\text{wrap}(\hat{F}) \leq K$  are inessential.*

(ii) *For any constant  $K$ , there is a closed essential surface  $F$  in a hyperbolic manifold  $M$ , and a slope  $\beta$  on  $T$ , such that  $F$  is not  $\pi_1$ -injective in  $M(\gamma)$  for all  $\gamma$  with  $\Delta(\gamma, \beta) \leq K$ .*

*Proof.* (1) Let  $S$  be a compact orientable surface of genus  $g > K$  with a single boundary component  $c$ . Let  $\alpha_1, \dots, \alpha_g$  be a set of mutually disjoint nonseparating curves cutting  $S$  into a connected planar surface. By Theorem 1.1 of [27], there exists a pseudo-Anosov map  $\varphi : S \rightarrow S$  such that  $\varphi(\alpha_i) = \alpha_{i+1}$  for  $i < g$ . (Note that  $\varphi(\alpha_g) \neq \alpha_1$ , otherwise  $\varphi$  would be reducible.) Let  $W = S \times I - N(\alpha'_1)$ , where  $\alpha'_1$  is the curve  $\alpha_1 \times \frac{1}{2}$  in the interior of  $S \times I$  isotopic to  $\alpha_1$ . Let  $M = W / ((x, 1) \sim (\varphi(x), 0))$ . Since  $\varphi$  is pseudo-Anosov, it is easy to check that  $M$  is irreducible and atoroidal, and it cannot be a Seifert fiber space because  $S = S \times 0$  is an essential hyperbolic surface in  $M$  disjoint from one boundary component of  $M$ . Therefore by Thurston's hyperbolization theorem for Haken manifolds [26],  $M$  is hyperbolic.

Let  $F$  be the disjoint union of two copies of  $S$  with opposite orientation. Then  $F$  is  $\pi_1$  injective, and is not a virtual fiber because it is disjoint from one boundary component of  $M$ . Hence it is geometrically finite. Let  $\hat{F}$  be a Freedman tubing of  $F$  with  $\text{wrap}(\hat{F}) = k \leq K$ . We want to show that  $\hat{F}$  is inessential in  $M$ .

Let  $\tilde{M}$  be the infinite cyclic covering of  $M$  dual to the surface  $S$ . Note that  $\tilde{M}$  can be constructed by taking infinitely many copies of  $W$ , denoted by  $W_i$  ( $i \in \mathbb{Z}$ ), and gluing the surface  $S \times 1$  in  $W_i$  to  $S \times 0$  in  $W_{i+1}$  using the map  $\varphi$ . Let  $X_k$  be the union of  $W_1, \dots, W_k$  in  $\tilde{M}$ . Then  $\hat{F}$  lifts to a surface in  $\tilde{M}$  homotopic to  $\partial X_k$ . Put  $\alpha'_i = \alpha_i \times \frac{1}{2}$ . One can check that when  $k \leq K$ ,  $X_k$  is homeomorphic to the manifold  $(S \times I) - \alpha'_1 \cup \dots \cup \alpha'_k$ . Let  $\beta$  be an essential arc on  $S$  disjoint from all  $\alpha_i$ . Then  $\beta \times I$  is a compressing disk of  $\partial X_k$ . It follows that  $\hat{F}$  is not  $\pi_1$ -injective in  $M$ .

(2) Let  $\hat{F}$  be a Freedman tubing of  $F$  such that  $\hat{F}$  is essential in  $M$ , and the wrapping number  $w$  of  $\hat{F}$  is minimal among all such surface. Since  $F$  is geometrically finite and embedded, the existence of such a surface follows from [7] or [19], or from Theorem 5.7. Let  $\beta$  be the boundary slope of  $F$ , and let  $A$  be the tubing annulus  $\hat{F} - \text{Int}F$ . Assume that  $\Delta = \Delta(\gamma, \beta) \leq K$ . Notice that the annulus  $A$  is rel  $\partial$  homotopic in the Dehn filling solid torus to another annulus  $A'$  on  $\partial M$  with wrapping number  $w' = |w - \Delta|$ , so  $\hat{F}$  is homotopic in  $M(\gamma)$  to a surface  $\hat{F}' = F \cup A'$  which is a Freedman tubing of  $F$  with wrapping number  $w'$ . By the choice of  $w$ , it follows that  $\hat{F}$  is inessential in  $M(\gamma)$  for all  $\gamma$  such that  $\Delta(\gamma, \beta) < 2w$ . By (1) we have  $2w > w > K$ , hence the result follows.  $\square$

Although there is no universal upper bound on the wrapping number of an essential Freedman tubing surface, it has been shown by Li [19] that an upper bound in terms of genus and number of boundary components of  $F$  does exist if  $F$  is an embedded surface. Li showed that if  $F$  is embedded with genus  $g$  and  $b$  boundary components, then a Freedman tubing of  $F$  is essential if its wrapping number is at least  $6g + 2b - 3$ .

**Problem 6.2.** *Find the minimal constant  $C(g, b)$  such that if  $F$  is a geometrically finite embedded surface with genus  $g$  and  $b$  boundary components, then all Freedman tubing of  $F$  with wrapping number at least  $C(g, b)$  is essential.*

Li's result [19] shows that  $C(g, b) \leq 6g + 2b - 3$ , and the proof of Theorem 6.1 shows that  $C(g, b) > g$ .

For immersed surface, no such number would exist if we do not assume that  $F$  is in standard position. The reason is because we can slide one component of  $\partial F$  around the torus many times, so when tubing on the opposite direction, a long part of the tube would just homotope that boundary component of  $F$  back to its original position. However, one can consider the number of tubings which is inessential. For the embedded case, it is at most  $2C(g, b) + 1$ . For simplicity let us consider the case that  $F$  has only two boundary components.

**Conjecture 6.3.** *Let  $F$  be a compact surface with boundary on a torus component of  $\partial M$ . Let  $g$  be the genus and  $b$  the number of boundary components of  $F$ . Then there exists a constant  $C'(g, b)$  depending only on  $g$  and  $b$ , such that all but at most  $C'(g, b)$  of the Freedman tubings of  $F$  are  $\pi_1$ -injective.*

The following result gives an estimation of tubing length when  $F$  is a totally geodesic surface, which leads to an upper bound on wrapping numbers of inessential Freedman

tubing in this special case. Existence of immersed totally geodesic surfaces can be found in [2] and [24].

**Theorem 6.4.** *Let  $T$  be the boundary tori of a set of disjoint cusps  $N$  in  $M$ , let  $F'$  be a totally geodesic surface in  $M$ , and let  $F = F' \cap M_0 = M - \text{Int}N$ . If  $\hat{F}$  is a Freedman tubing of  $F$  with  $\ell(\hat{F} - \text{Int}F) \geq \pi$ , then  $\hat{F}$  is  $\pi_1$ -injective in  $M$ .*

*Proof.* Notice that we do not require  $T = T_{\mu(F)}$ . The intersection of  $F'$  with  $N$  is a set of totally geodesic annuli, hence they are perpendicular to  $T = \partial N$ . A nontrivial curve  $\alpha$  on  $\hat{F}$  can be homotoped on  $\hat{F}$  either to a curve on  $F$  or to a curve  $\alpha_1 \cup \dots \cup \alpha_{2n}$  with  $\alpha_{2i-1}$  a geodesic arc on  $F$  perpendicular to  $T$ , and  $\alpha_{2i}$  an essential arc on  $\hat{F} - F$ . Since  $F$  is totally geodesic,  $\alpha_{2i-1}$  is also a geodesic of  $M$ , hence the result follows from Theorem 4.1.  $\square$

**Corollary 6.5.** *Let  $F$  be as in Theorem 6.4. If  $\hat{F}$  is an inessential Freedman tubing of  $F$ , then  $\text{wrap}(\hat{F}) \leq 2\pi^2(2g + b - 2)/\sqrt{3}$ .*

*Proof.* Let  $A$  be a tubing annulus of  $\hat{F}$ , and let  $\beta$  be the boundary slope of  $A$ . Extend  $F$  to a complete hyperbolic surface  $F'$  by adding a cusp at each of its boundary component. By the Gauss-Bonnet theorem,  $\text{Area}(F') = 2\pi(2g + b - 2)$ . Each cusp with boundary on  $\partial A$  has area =  $t(\beta)$ , and there are two of them, hence  $t(\beta) < \pi(2g + b - 2)$ . Choose  $T$  to be a set of maximal cusps, then each slope of  $T$  has length at least 1, hence the area of each component of  $T$  is at least  $\sqrt{3}/2$ . If  $\gamma$  and  $\beta$  are on the torus  $T_i$ , then

$$\ell(A) \geq \text{wrap}(A)\text{Area}(T_i)/t(\beta) \geq \text{wrap}(\hat{F})\sqrt{3}/2\pi(2g + b - 2).$$

By Theorem 6.4,  $\hat{F}$  is essential if  $\ell(A) \geq \pi$  for all tubes  $A$  of  $\hat{F}$ , which is true if  $\text{wrap}(\hat{F}) \geq 2\pi^2(2g + b - 2)/\sqrt{3}$ .  $\square$

If  $F$  is a closed, embedded, incompressible surface in  $M_0$  which is not coannular to a torus  $T \subset \partial M_0$ , then Theorem 1 of [28] says that  $F$  remains incompressible in  $M(\gamma)$  for all but at most three slopes  $\gamma$  on  $T$ . For immersed essential surfaces  $F$  in  $M$  without coannular slopes (also called accidental parabolics), Theorem 1.1 says that  $F$  remains essential in  $M(\gamma)$  except for finitely many  $\gamma$  on  $T$ . The answer to the following problem is likely to be negative. If  $M$  is not assumed hyperbolic, there are examples showing that no upper bound exists. However, no examples are known for hyperbolic manifolds.

**Problem 6.6.** *Let  $F$  be a closed essential surface in a hyperbolic manifold  $M$ , and assume that  $F$  has no coannular slopes. Does there exist a universal upper bound on the number of slopes  $\gamma$  on a torus boundary component  $T$  of  $M_0$  such that  $F$  is inessential in  $M(\gamma)$ ?*

Many hyperbolic manifolds do not contain closed *embedded* essential surfaces. However, it was proved in [8] that any hyperbolic  $M$  with some toroidal cusps contains a closed essential surface. The surfaces constructed there are Freedman tubings of some surfaces in certain covering spaces of  $M$ , and hence all have coannular slopes. The following seems to be an interesting open problem. The corresponding problem for closed hyperbolic manifold is also open, and is part of the virtual Haken conjecture.

**Conjecture 6.7.** *Every hyperbolic manifold with toroidal cusps contains a closed essential surface without coannular slopes.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IA 52242  
E-mail address: wu@math.uiowa.edu