KNOTS AND LINKS WITHOUT PARALLEL TANGENTS

Ying-Qing Wu[1]

Abstract. This paper solves a problem of Colin Adams, showing that any link L in $\mathbb{R}^3$ is isotopic to a smooth link $\hat{L}$ which has no parallel or antiparallel tangents.

§1. Introduction

Steinhaus conjectured that every closed oriented C^1-curve K has a pair of anti-parallel tangents, that is, there are two points p_1, p_2 on K, such that the tangent vectors p_1', p_2' at those points satisfy $p_1' = cp_2'$ for some negative number c. The conjecture is false, as Porter [3] showed that there exists an unknotted curve which has no anti-parallel tangents. Colin Adams raised the question of whether there exists a *nontrivial* knot in $\mathbb{R}^3$ which has no parallel or antiparallel tangents. In this paper we will solve this problem, showing that any (smooth or polygonal) link L in $\mathbb{R}^3$ is isotopic to a smooth link $\hat{L}$ which has no parallel or antiparallel tangents.

Theorem 1.1. *Given any tame link L in S^3 and any neighborhood $\eta(L)$ of L, there is a smooth link L' isotopic to L in $\eta(L)$, which has no parallel or antiparallel tangents.*

If $\mathcal{S}(L)$ denotes the set of all smooth links isotopic to L, then the subset $\hat{\mathcal{L}}(L)$ of all $\hat{L}$ which has no parallel or antiparallel tangents is not dense in $\mathcal{S}(L)$ if it is endowed with C^1 or C^∞ topology. The reason is that if two links L_1 and L_2 are very close to each other in the C^1 topology, then their Gauss maps are also close to each other, hence if we choose L_1 so that its Gauss map has a transverse self intersection, then the Gauss map of any L_2 in a small neighborhood of L_1 in the C^1 topology will also have a self intersection point, and hence L_2 has a pair of points with parallel tangents. Nevertheless, from the proof of Theorem 1.1 one can see that the link L' in the theorem is C^0 close to L, so $\hat{\mathcal{L}}(L)$ is dense in $\mathcal{S}(L)$ under the C^0 topology. Theorem 1.1 is useful in studying supercrossing numbers, see the recent work of Adams, Lefever, Othmer, Pahk and Tripp [1].

To present the idea of the proof, let us consider a smooth knot $K : S^1 \to \mathbb{R}^3$. The unit tangent vectors of K defines the Gauss map $\alpha : S^1 \to S^2$ of K, that is, $\alpha(t) = K'(t)/\|K'(t)\|$, where K' is the derivative of the map K. Let $\rho : S^2 \to P^2$ be the standard double covering of the projective plane P^2. Then K has no parallel or anti parallel tangents if and only if α is admissible in the sense that $\rho \circ \alpha : S^1 \to P^2$ is an injective map.

The first step is to approximate K with a generic polygonal knot K_1. The "Gauss map" of K_1 is a non-continuous map with image a set of points, one for each edge

1991 *Mathematics Subject Classification.* Primary 57M25.
Key words and phrases. Knots, links, tangent lines.
[1] Partially supported by NSF grant #DMS 9802558

of K_1. One can bend each edge a little bit, so that $\rho \circ \alpha : S^1 - V \to P^2$ is an injective map, where V is a small neighborhood of the vertices of K_1. The map α can be extended to a smooth map $\alpha_1 : S^1 \to S^2$ such that $\rho \circ \alpha_1 : S^1 \to P^2$ is still an injective map. This can be done for links as well, see Lemma 2.3 below. Our goal is then to find a knot K_1 isotopic to K, such that its Gauss map is equal to α.

The speed function of K is defined as $f(t) = \|K'(t)\|$. Given a smooth function $f : S^1 \to \mathbb{R}_+$, the integral of $f(t)\alpha(t)$, starting at a base point of S^1, is then a map $K_f : I \to \mathbb{R}^3$ with α as its Gauss map. Note that K_f may not be a closed curve. We will define certain knots to be allowable ϵ-approximations of a polygonal knot K_1, and show (Lemmas 2.1 – 2.2) that such a knot is isotopic to K_1. It then suffices to show that f can be chosen so that K_f is an allowable ϵ-approximation of K_1.

Choose f so that its restriction to $S^1 - V$ is the speed function of the bended K_1 above, and is very small on V. Then K_f is a union of bended edges of K_1 and some small arcs near the vertices of K_1. The major technical difficulty is to choose f so that these small arcs near the vertices of K_1 are unknotted in certain sense, which is a key property of allowable ϵ-approximations to assure that K_f is isotopic to K_1. This is done in Lemmas 2.4 – 2.5. One also has to modify f to make sure that K_f is a closed curve, that is, $K_f(0) = K_f(1)$. The proof of the theorem is given at the end of Section 2.

§2. Proof of the main theorem

We refer the readers to [4] for concepts about knots and links. Throughout this paper, we will use I to denote a closed interval on $\mathbb{R}$. Denote by S^2 the unit sphere in $\mathbb{R}^3$, and by S_1 the circle $S^2 \cap \mathbb{R}_{xy}$ on S^2, where $\mathbb{R}_{xy}$ denotes the xy-plane in $\mathbb{R}^3$. Denote by $Z[z_1, z_2]$ the set $\{v = (x, y, z) \in \mathbb{R}^3 \mid z_1 \le z \le z_2\}$. Similarly for $Y[y_1, \infty)$ etc. A curve $\beta : I \to \mathbb{R}^3$ is an *unknotted curve* in $Z[z_1, z_2]$ if (i) β is a properly embedded arc in $Z[z_1, z_2]$, with endpoints on different components of $\partial Z[z_1, z_2]$, and (ii) β is rel ∂ isotopic in $Z[z_1, z_2]$ to a straight arc.

Given a curve $\alpha : I = [a, b] \to S^2$ and a positive function $f : I \to \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, we use $\beta = \beta(f, \alpha, t_0, v_0)$ to denote the integral curve of $f\alpha$ with $\beta(t_0) = v_0$, where $t_0 \in I$. More explicitly,

$$\beta(t) = \beta(f, \alpha, t_0, v_0)(t) = v_0 + \int_{t_0}^{t} f(t)\alpha(t)\, dt.$$

When $t_0 = a$ and $v_0 = 0$, simply denote it by $\beta(f, \alpha)$.

If $\gamma : [a, b] \to \mathbb{R}^3$ is a map and $[c, d]$ is a subinterval of $[a, b]$, denote by $\gamma[c, d]$ the restriction of γ on $[c, d]$. If u, v are points in $\mathbb{R}^3$, denote by $e(u, v)$ the line segment with endpoints at u and v, oriented from u to v. Denote by $d(u, v)$ the distance between u and v. Denote by $\|e\|$ the length of e if e is a line segment or a vector in $\mathbb{R}^3$. Thus $d(u, v) = \|e(u, v)\| = \|u - v\|$.

Given n points $v_1, ..., v_n$ in $\mathbb{R}^3$, let $e_i = e(v_i, v_{i+1})$. The subscripts are always mod n numbers. Thus $e_n = e(v_n, v_1)$. In generic case, the union of these edges forms a knot, denoted by $K(v_1, ..., v_n)$. To avoid trivial case, we will always assume $n \ge 4$.

Lemma 2.1. *Let $K = K(v_1, ..., v_n)$ be a polygonal knot, and let N be a regular neighborhood of K. Then there is a number $r > 0$ such that*

(i) N contains the r-neighborhood $N(K)$ of K;

(ii) $K' = K(v_1', v_1, ..., v_n', v_n)$ *is isotopic to* K *in* N *if* $d(v_i, v_i') < r$*; and*
(iii) $K'' = K(v_1', ..., v_n')$ *is isotopic to* K *in* N *if* $d(v_i, v_i') < r$*.*

Proof. Choose $r > 0$ to satisfy (i) and $r < d/4$, where d is the minimal distance between non-adjacent edges of K. The knot K' is contained in $N(K)$. Let D_i be the meridian disk of the r-neighborhood $N(e_i)$ of the i-th edge $e_i = e(v_i, v_{i+1})$ of K, intersecting e_i perpendicularly at its middle point m_i. It is easy to check that the distance from m_i to any edge e_j $(j \neq i)$ is at least $d/2 = 2r$, hence D_i is a meridian disk of $N(K)$. The edge $e(v_i', v_i)$ is contained in an r-neighborhood of v_i, hence is disjoint from all D_j. Thus D_i intersects K' at a single point on the edge $e(v_i, v_{i+1}')$, so the disks $D_1, ..., D_n$ cut $N(K)$ into balls $B_1, ..., B_n$, each intersecting K' in an arc consisting of three edges, hence unknotted. Therefore K' is isotopic to K in $N(K)$. This proves (ii).

By (ii), both K and $K(v_1', ..., v_n')$ are isotopic to $K(v_1, v_1', ..., v_n, v_n')$ in N. Therefore, they are isotopic to each other, and (iii) follows. □

Denote by $C(\theta, u, v)$ the solid cone based at u (the vertex of the cone), open in the direction of v, with angle θ. More explicitly, if we set up the coordinate system with u the origin and v in the direction of $(0, 0, 1)$, then

$$C(\theta, u, v) = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq \cot\theta\sqrt{x^2 + y^2}\}.$$

A smooth curve $\beta : [a, b] \to B$ in a ball B is θ-*allowable* if (i) β is properly embedded and unknotted in B, (ii) the cones $C_a = C(\theta, \beta(a), -\beta'(a))$ and $C_b = C(\theta, \beta(b), \beta'(b))$ are mutually disjoint, each intersecting B only at its cone point.

A smooth arc $\beta : [a, b] \to \mathbb{R}^3$ is called an ϵ-*suspension* if it is an embedding into an isosceles triangle Δ in $\mathbb{R}^3$ with base the line segment $e = e(\beta(a), \beta(b))$ and height ϵ. It is called a *round* ϵ-suspension if furthermore it is a subarc of a round circle in $\mathbb{R}^3$, and $\|\beta'(t)\|$ is a constant function. The line segment e is called the *base arc* of β, and the disk bounded by β and e is called the *suspension disk*. Put $\theta = 2\epsilon/\|e\|$. Then the two angles of Δ adjacent to e are at most $\arctan(2\epsilon/\|e\|) < \theta$. Therefore Δ, hence the curve β, is contained in the cones $C(\theta, \beta(a), \beta'(a))$ and $C(\theta, \beta(b), -\beta'(b))$.

Let K be a polygonal knot, with edges $e_1, ..., e_n$. A smooth curve $\beta : S^1 \to \mathbb{R}^3$ is an *allowable* ϵ-*approximation* of K if it is a union of arcs $\beta_1, ..., \beta_{2n}$, such that

(i) each β_{2k} is an ϵ-allowable arc in some ball B_k of radius at most ϵ;

(ii) each β_{2k-1} is an ϵ-suspension, such that its base arc E_k is parallel to e_k, and the difference between the lengths of e_k and E_k is at most ϵ.

Lemma 2.2. *Given any polygonal knot* $K = K(v_1, ..., v_n)$ *and a regular neighborhood* N *of* K*, there is an* $\epsilon > 0$ *such that any allowable* ϵ*-approximation* γ *of* K *with the same initial point is a knot, which is isotopic to* K *in* N*.*

Proof. Rescaling $\mathbb{R}^3$ if necessary, we may assume that the length of each edge of K is at least 3. Let $e_i = e(v_i, v_{i+1})$. Denote by m the minimum distance between nonadjacent edges, and by r the number given in Lemma 2.1.

Let ϵ be a very small positive number (for example, $\epsilon < \min(1, m/10n, r/10n)$). Let $\beta_1, ..., \beta_{2n}$ be the arcs of γ, and B_i the ball containing β_{2i}, as in the definition of allowable ϵ-approximation. Let Δ_i be the isosceles triangles containing β_{2i-1}, as in the definition of ϵ-suspension arcs. Denote by v_i'', v_{i+1}' the initial and ending points of β_{2i-1}, respectively. Consider the union of all Δ_i and B_i.

Sublemma. *The triangles Δ_i are mutually disjoint, the balls B_i are mutually disjoint, and Δ_i intersects B_j only if $j = i$ or $i-1$ mod n, in which case they intersects at a single point.*

Since the base arc $E_i = e(v_i'', v_{i+1}')$ of β_{2i-1} is parallel to e_i with length difference at most ϵ, and since $d(v_i', v_i'')$ is at most 2ϵ (the upper bound of diameters of B_i), one can show by induction that $d(v_i, v_i') \leq (3i-2)\epsilon$, and $d(v_i, v_i'') \leq 3i\epsilon$. Put $\delta = 4n\epsilon < m/2$. Then β_{2i-1} is in the δ-neighborhood of e_i, and B_i is in the δ-neighborhood of v_{i+1}. Since the distance between two vertices or nonadjacent edges of K is bounded below by m, it follows that the balls B_i are mutually disjoint, Δ_i is disjoint from Δ_j when i and j are not adjacent mod n, and disjoint from B_j if j is not equal or adjacent to i mod n. Since $\|E_i\| > 3 - \epsilon$ and the height of Δ_i is at most ϵ, the two angles of Δ_i adjacent to E_i is at most $2\epsilon/(3-\epsilon) < \epsilon$. Thus for each endpoint v of E_i, Δ_i is contained in a cone of angle ϵ based at v in the direction of the tangent or negative tangent of β at v. Since β_{2i} is an ϵ-allowable arc, it follows from definition that Δ_i is disjoint from Δ_{i+1}, and they each intersects B_i only at a single point. This completes the proof of the sublemma.

Since each β_i is an embedding, it follows from the sublemma that $\gamma : S^1 \to \mathbb{R}^3$ is an embedding, hence is a knot. We can deform β_{2i-1} to the edge E_i by an isotopy through the suspension disk. Since β_{2i} is unknotted in B_i, it can be deformed via an isotopy rel ∂ to a straight arc E_i' in B_i. By the sublemma these isotopies form an isotopy of γ to the polygonal knot $K_2 = E_1 \cup E_1' \cup ... \cup E_n \cup E_n' = K(v_1', v_1'', ..., v_n', v_n'')$. Since $d(v_i', v_i'')$ is very small, by Lemma 2.1(ii) K_2 is isotopic to the knot $K(v_1'', ..., v_n'')$, which is isotopic to K by Lemma 2.1(iii). □

Let A be a compact 1-manifold. A smooth map $\alpha : A \to S^2$ is *admissible* if (i) α is an embedding, and (ii) it has no antipodal points, i.e., $\alpha(t) \neq -\alpha(s)$ for all $t \neq s$. Denote by $\lambda : S^2 \to S^2$ the antipodal map, and by $\rho : S^2 \to P^2$ the standard double covering map onto the projective plane P^2. Then α is admissible if and only if $\rho \circ \alpha : A \to P^2$ is a smooth embedding.

Lemma 2.3. *Suppose Y is the disjoint union of finitely many circles, and suppose A is a compact submanifold of Y. Let $\alpha : A \to S^2$ be an admissible map such that each circle component of $\alpha(A)$ bounds a disk Δ with interior disjoint from $\alpha(A)$ and $\lambda(\Delta)$. Then α extends to an admissible map $\hat{\alpha} : Y \to S^2$.*

Proof. Let I be the closures of components of $Y - A$. We need to extend α to an admissible map $\hat{\alpha} : A \cup I \to S^2$ which still satisfies the assumption of the lemma. The result would then follow by induction. If I is a circle, define $\hat{\alpha} : I \to S^2$ to be a smooth map embedding I into a small disk D of S^2 such that D, $\rho(D)$ and $\alpha(A)$ are mutually disjoint. So suppose I is an interval with endpoints u_1, u_2 on a component Y_0 of Y. Denote by $\tilde{\alpha} = \rho \circ \alpha$.

If $J = Y_0 - \text{Int}I$ is connected, then by assumption $\tilde{\alpha}$ is an embedding, so there is a small disk neighborhood D of $\tilde{\alpha}(J)$ which is disjoint from $\tilde{\alpha}(A - J)$. Let D_1 be the component of $\rho^{-1}(D)$ containing $\alpha(J)$, and extend α to a smooth embedding $\hat{\alpha} : A \cup I \to S^2$ so that $\hat{\alpha}(I) \subset D$.

Now suppose J is disconnected. Let J_1, J_2 be the components of J containing u_1, u_2 respectively. Let $K_1, ..., K_r$ be the circle components of A, and let D_i be the disk on P^2 bounded by K_i. By assumption $\tilde{\alpha}(J_i)$ are in $P^2 - \cup D_i$, so there are two non-homotopic arcs $\tilde{\gamma}_1, \tilde{\gamma}_2 : I \to P^2$ such that $\tilde{\gamma}_i \cup \tilde{\alpha} : I \cup A \to P^2$ is a smooth embedding. One of the $\tilde{\gamma}_i$ lifts to a path $\gamma : I \to S^2$ connecting u_1 to u_2. It follows that $\gamma \cup \alpha : I \cup A \to S^2$ is the required extension. □

Lemma 2.4. *Suppose $\alpha = (\alpha_1, \alpha_2, \alpha_3) : I = [a, b] \to S^2$ is an admissible curve intersecting S_1 transversely at two points in the interior, and $\alpha_3(a) > 0$. Then there is a function $f : I \to \mathbb{R}_+$ such that (i) $f(t) = 1$ in a neighborhood of ∂I, and (ii) the integral curve $\beta = (\beta_1, \beta_2, \beta_3)$ is unknotted in $Z[z_1, z_2]$, where $z_1 = \beta_3(a)$ and $z_2 = \beta_3(b)$.*

Proof. By assumption α_3 has exactly two zeros $u, v \in I$, $(u < v)$, so $a_3(t) < 0$ if and only if $t \in (u, v)$. Since $\alpha_3(u) = \alpha_3(v) = 0$, the two points $\alpha(u)$ and $\alpha(v)$ lie on the unit circle S_1 on the xy-plane, hence by a rotation along the z-axis if necessary we may assume that they have the same x-coordinate, i.e., $\alpha_1(u) = \alpha_1(v)$. Since α is admissible, $\alpha(u) \neq \pm\alpha(v)$, so we may assume that $\alpha_1(u), \alpha_1(v) > 0$. Notice that in this case $\alpha_2(u), \alpha_2(v)$ must have different signs. Without loss of generality we may assume that $\alpha_1(t), \alpha_2(t) > 0$ when t is in an ϵ-neighborhood of u, and $\alpha_1(t) > 0$, $\alpha_2(t) < 0$ when t is in an ϵ-neighborhood of v, where $0 < \epsilon < \min(u - a, b - v)$.

We start with the constant function $f(t) = 1$ on I, and proceed to modify $f(t)$ so that $f(t)$ and the integral curve $\beta = \beta(f, \alpha, t_0, v_0)$ satisfy the conclusion of the lemma. Put $\beta = (\beta_1(t), \beta_2(t), \beta_3(t))$, and choose the base point v_0 so that $\beta(u) = 0$. Thus

$$\beta_i(t) = \int_u^t f(t)\alpha_i(t)\, dt.$$

Since $\alpha_1(u), \alpha_2(u) > 0$, and $\beta_1(u) = \beta_2(u) = 0$, by enlarging $f(t)$ in a small ϵ-neighborhood of u, we may assume that $\beta_1(t), \beta_2(t) > 0$ for all $t \in (u, v]$. Since $\alpha_2(t) < 0$ in a neighborhood of v, we may then enlarge $f(t)$ near v so that $\beta_2(v) = \beta_2(u) = 0$. This does not affect the fact that $\beta_1(t) > 0$ for $t \in (u, v]$, and $\beta_2(t) > 0$ for $t \in (u, v)$. Geometrically the curve β has the property that its projection to the xy-plane lies in the half plane $\{y \geq 0\}$, with initial point at the origin and ending point on the positive x-axis.

The function β_3 is decreasing in $[u, v]$ because $\alpha_3(t)$ is negative in this interval. Thus $\beta_3(v) < \beta_3(u)$. Since α_3 is positive in $[a, u]$ and $[v, b]$, β_3 is increasing in these intervals. We may now enlarge $f(t)$ in $(u - \epsilon, u)$ and $(v, v + \epsilon)$, so that $z_3 = \beta_3(u-\epsilon) < \beta_3(v)$ and $z_4 = \beta_3(v+\epsilon) > \beta_3(u)$. Thus the curve β on $[u-\epsilon, v+\epsilon]$ is a proper arc in $Z[z_3, z_4]$. We want to show that it is unknotted.

By the above, the curve $\beta[u, v]$ lies in $Z[z_3, z_4] \cap Y[0, \infty)$, with endpoints on the xz-plane. Since β_3 is decreasing on $[u, v]$, β is rel ∂ isotopic in $Z[z_3, z_4] \cap Y[0, \infty)$ to a straight arc $\hat{\beta}[u, v]$ on the xz-plane. Since $\alpha_2(t) > 0$ for $t \in [u - \epsilon, u]$, and $\beta_2(u) = 0$, we have $\beta_2(t) < 0$ for $t \in [u - \epsilon, u]$. Similarly, since $\alpha_2(t) < 0$ near v, we have $\beta_2(t) < 0$ for $t \in [v, v + \epsilon]$. Therefore, the above isotopy is disjoint from the arcs $\beta[u - \epsilon, u]$ and $\beta[v, v + \epsilon]$, hence extends trivially to an isotopy of $\beta[u - \epsilon, v + \epsilon]$, deforming $\beta[u - \epsilon, v + \epsilon]$ to the curve $\hat{\beta} = \beta[u - \epsilon, u] \cup \hat{\beta}[u, v] \cup \beta[v, v + \epsilon]$.

Since $\alpha_1(t)$ is positive near u, v, β_1 is increasing in $[u - \epsilon, u]$ and $[v, v + \epsilon]$. Since $\hat{\beta}$ is a straight arc connecting $\beta(u)$ and $\beta(v)$, and $\beta_1(v) > \beta_1(u)$ by the above, the first coordinate function of $\hat{\beta}$ is also increasing in $[u, v]$. It follows that the first coordinate of $\hat{\beta}$ is increasing in $[u - \epsilon, v + \epsilon]$, therefore, $\hat{\beta}$ is unknotted in $Z[z_3, z_4]$, hence is rel ∂ isotopic to a straight arc $\tilde{\beta}$ in $Z[z_3, z_4]$.

Since $\beta_3(t)$ is increasing on $[a, u-\epsilon] \cup [v+\epsilon, b]$, the above isotopy extends trivially to an isotopy deforming $\beta : I \to \mathbb{R}^3$ to the curve $\beta[a, u - \epsilon] \cup \tilde{\beta} \cup \beta[v + \epsilon, b]$. Since the third coordinate of this curve is always increasing, it is unknotted in $Z[z_1, z_2]$, where $z_1 = \beta_3(a)$ and $z_2 = \beta_3(b)$. Therefore, β is also unknotted in $Z[z_1, z_2]$. $\square$

Given $a \in \mathbb{R}$ and $\delta > 0$, let $\varphi = \varphi[a, \delta](x)$ be a smooth function on $\mathbb{R}^1$ which is symmetric about a, $\varphi(a) = 1$, $\varphi(x) = 0$ for $|x - a| \geq \delta$, and $0 \leq \varphi(x) \leq 1$ for all x. Given $a, b \in \mathbb{R}$ with $a < b$, let $\psi(x) = \psi[a, b](x)$ be a smooth monotonic function such that $\psi(x) = 0$ for $x \leq a$, and $\psi(x) = 1$ for $x \geq b$. Such functions exist, see for example [2, Page 7].

For any point $p \in S^2$, denote by $U(p, \epsilon)$ the ϵ-neighborhood of p on S^2, measured in spherical distance. Thus for any $q \in U(p, \epsilon)$, the angle between p, q (considered as vectors in $\mathbb{R}^3$) is less than ϵ.

Lemma 2.5. *Let $0 < \epsilon < \pi/8$, and let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : I = [a_1, a_2] \to S^2$ be an admissible arc transverse to S_1, such that $\alpha_3(a_i) > \epsilon$. Let $\mu > 0$. Then there is a smooth positive function $f(t)$ such that (i) $f(t) = 1$ near a_i, and (ii) the integral curve $\beta = \beta(f, \alpha, t_0, v_0)$ is an ϵ-allowable arc in a ball of radius μ in $\mathbb{R}^3$.*

Proof. Notice that $U(\alpha(a_i), \epsilon)$ are on the upper half sphere S^2_+. Choose $0.1 > \delta > 0$ sufficiently small, so that $\alpha(t) \in U(\alpha(a_i), \epsilon)$ for t in a δ-neighborhood of a_i, $i = 1, 2$. Choose $c_0 = a_1 + \delta, c_1, ..., c_p = a_2 - \delta$ so that the curve $\alpha(I_j)$ intersects S_1 exactly twice in the interior of $I_j = [c_{j-1}, c_j]$, $j = 1, ..., p$.

By Lemma 2.4 applied to each I_j, we see that there is a function $f_1(t)$ on I, such that $f_1(t) = 1$ near c_i and on $[a_1, c_0] \cup [c_p, a_2]$, and the part $\beta_1[c_0, c_p]$ of the integral curve $\beta_1 = \beta(f_1, \alpha, t_0, v_0)$ is unknotted in $Z[z_0, z_p]$, where $z_i = \beta(c_i)$. Without loss of generality we may choose $t_0 = c_0$ and $v_0 = 0$. Since the curve is compact, the isotopy is within a ball, so there is a disk D in $\mathbb{R}^2$, such that $\beta_1[c_0, c_n]$ is unknotted in $D \times [z_0, z_p]$. Choose N large enough, so that the ball $B(N)$ of radius N centered at the origin contains both $D \times [z_0, z_p]$ and the curve β_1 in its interior. We want to modify $f_1(t)$ on $[a_1, c_0) \cup (c_p, a_2]$ to a function $f_3(t)$, so that $\beta_3 = \beta(f_3, \alpha, c_0, 0)$ is an ϵ-admissible curve in $B(10N)$, and $f_3(t) = 10N/\mu$ near ∂I.

First, consider the function

$$f_2 = f_1 + (\frac{10N}{\mu} - 1)(1 - \psi[a_1, a_1 + \epsilon_1] + \psi[a_2 - \epsilon_1, a_2]),$$

where ϵ_1 is a very small positive number, say $\epsilon_1 < \min(\delta, \mu/10)$. By the property of the ψ functions, we have $f_2(t) = f_1(t)$ for $t \in [c_0, c_p]$, and $f_2(t) = 10N/\mu$ near a, b. Let $\beta_2 = \beta(f_2, \alpha, z_0, 0)$. Since ϵ_1 is very small, one can show that $\|\beta_2(t)\| < 2N$ for all $t \in [a, b]$. Let b_1, b_2 be positive real numbers. Define

$$f_3(t) = f_2(t) + b_1\, \varphi[a_1 + \frac{\delta}{2}, \frac{\delta}{4}](t) + b_2\, \varphi[a_2 - \frac{\delta}{2}, \frac{\delta}{4}](t).$$

Let β_3 be the integral curve $\beta_3(f_3, \alpha, z_0, 0)$. We have

$$\beta_3(a_2) = \beta_2(a_2) + b_2 \int_{z_p}^{b} \varphi[a_2 - \frac{\delta}{2}, \frac{\delta}{4}](t)\alpha(t)\, dt = \beta_2(a_2) + b_2 v_2.$$

Since $\alpha(t) \in U(\alpha(a_2), \epsilon)$, by the choice of ϵ we have $\alpha_3(t) > 0$ for all $t \in [z_p, b]$, hence the vector v_2 above is nonzero. Since $\|\beta_2(a_2)\| < 2N$, we may choose $b_2 > 0$ so that $\|\beta_3(a_2)\| = 10N$. Similarly, choose $b_1 > 0$ so that $\beta_3(a_1) = 10N$. Now the arc β_3 has both endpoints on the boundary of $B(10N)$. We want to show that it is an unknotted arc properly embedded in this ball.

Consider a point $t \in [c_p, a_2]$ such that $\|\beta_3(t)\| \geq 10N$. Let $\theta(t)$ be the angle between $\beta_3(t)$ and $\beta_3'(t)$. Put $u_0 = \beta_3(c_p)$, and notice that $\|u_0\| < N$. Since $\alpha(t) \in U(\alpha(a_2), \epsilon)$, the curve $\beta_3[c_p, b]$ lies in the cone $C(\epsilon, u_0, \alpha(a_2))$, so the angle between $(\beta_3(t) - u_0)$ and $\alpha(t)$ is at most 2ϵ. We have

$$\begin{aligned}\cos\theta(t) &= \frac{\beta_3(t)\cdot\alpha(t)}{\|\beta_3(t)\|} = \frac{(\beta_3(t)-u_0)\cdot\alpha(t) + u_0\cdot\alpha(t)}{\|\beta_3(t)\|} \\ &\geq \frac{(10N-N)\cos(2\epsilon) - N}{10N} = 0.9\cos(2\epsilon) - 0.1 \\ &> \frac{1}{2}\end{aligned}$$

Therefore, $\theta(t) < \pi/3$. In particular, this implies that the norm of $\beta_3(t)$ is strictly increasing if it is at least $10N$ and $t \in [c_p, a_2]$; but since $\|\beta_3(a_2)\| = 10N$, it follows that $\beta_3(t)$ lies in the interior of $B(10N)$ for $t \in [c_p, a_2)$. Similarly, one can show that this is true for $t \in (a_1, c_0]$. Therefore, β_3 is a proper arc in $B(10N)$. It is unknotted because its third coordinate is increasing on $[a_1, c_0] \cup [c_p, a_2]$ and the curve $\beta_3[c_0, c_p] = \beta_1[c_0, c_p]$ is unknotted in $D \times [z_0, z_p]$, with $\beta_3(c_0)$ on $D \times z_0$.

We need to show that the cone $C(\epsilon, \beta_3(a_2), \beta_3'(a_2))$ intersects $B(10N)$ only at the cone point, but this is true because $\epsilon + \theta(a_2) < \pi/8 + \pi/3 < \pi/2$. Similarly for $C(\epsilon, \beta_3(a_1), -\beta_3'(a_1))$. Also, notice that the cone $C(\epsilon, \beta_3(a_2), \beta_3'(a_2))$ lies above the xy-plane, while $C(\epsilon, \beta_3(a_1), -\beta_3'(a_1))$ lies below the xy-plane, so they are disjoint. It follows that β_3 is an ϵ-allowable curve in $B(10N)$.

Finally, rescale the curve by defining $f(t) = f_3(t)\mu/10N$, and $\beta = \beta(f, \alpha, c_0, 0)$. Then β is an ϵ-allowable curve in a ball of radius μ, and $f(t) = 1$ near ∂I. □

Lemma 2.6. *Suppose the integral curve $\beta = \beta(f, \alpha, a, 0)$ is a round ϵ-suspension. Then for any $k \in [\frac{1}{2}, \frac{3}{2}]$, there is a positive function $g(t)$ such that (i) $g(t) = f(t)$ near a, b, and (ii) the integral curve $\gamma = \beta(g, \alpha, a, 0)$ is a $(k\epsilon)$-suspension with $\gamma(b) - \gamma(a) = k(\beta(b) - \beta(a))$.*

Proof. Without loss of generality we may assume $[a, b] = [-1, 1]$. Set up the coordinate system so that β lies in the triangle with vertices $\beta(a) = (0, 0, 0)$, $\beta(b) = (2u, 0, 0)$ and $(u, \epsilon, 0)$, where $2u = \|\beta(b) - \beta(a)\|$. Put $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Then $\alpha_3(t) = 0$, and $\alpha_2(-t) = -\alpha_2(t)$. Consider the smooth function $\phi = \psi[-1 + \delta, -1 + 2\delta] - \psi[1 - 2\delta, 1 - \delta]$. It is an even function with $\phi(t) = 1$ when $|t| \leq 1 - 2\delta$, and $\phi(t) = 0$ when $|t| \geq 1 - \delta$. Let $g(t) = c + p\phi(t)$, where $p > -c$ is a constant. Since $|\phi(t)| \leq 1$, $g(t)$ is a positive function. We have

$$\gamma(1) = \int_{-1}^{1} g(t)\alpha(t)\,dt = \beta(1) + p\int_{-1}^{1}\phi(t)\alpha(t)\,dt.$$

Since $\phi(t)$ is even and α_2 is odd, $\gamma_2(1) = \gamma_3(1) = 0$. When δ approaches 0, the integral

$$c\int_{-1}^{1}\phi(t)\alpha_1(t)\,dt$$

approaches $\beta_1(1) = 2u$. Hence for any $s \in [u, 3u]$, we may choose δ small and $p \in (-c, c)$ so that $\gamma(1) = (s, 0, 0)$. Note that $\gamma_1'(t) = (c - p)\alpha_1(t) > 0$, so γ is an embedding.

Consider γ and β as curves on the xy-plane. Then The tangent of γ at t is given by

$$\frac{\gamma_2'(t)}{\gamma_1'(t)} = \frac{g(t)\alpha_2(t)}{g(t)\alpha_1(t)} = \frac{\alpha_2(t)}{\alpha_1(t)},$$

which is the same as the tangent slope of β at t, and hence is bounded above by ϵ. Thus γ is below the line $y = \epsilon x$ on the xy-plane. Similarly, it is below the line $y = -\epsilon(x - \gamma_1(1))$. It follows that γ is a $k\epsilon$-suspension, where $k = \gamma_1(1)/\beta_1(1) = s/2u \in [\frac{1}{2}, \frac{3}{2}]$. □

Proof of Theorem 1.1. Without loss of generality we may assume that $L = K_1 \cup ... \cup K_r$ is an oriented polygonal link in general position, with oriented edges $e_1, ..., e_m$, which are also considered as vectors in $\mathbb{R}^3$. Let d be the minimum distance between nonadjacent edges. We may assume each K_i has at least four edges, so d is also an upper bound on the length of e_i. For any $\epsilon_1 \in (0, d/3)$ the connected components of the ϵ_1-neighborhood of L, denoted by $N(K_i)$, are mutually disjoint. Choosing ϵ_1 small enough, we may assume that $N(K_i)$ are contained in $\eta(L)$. By Lemma 2.2 there is an $\epsilon > 0$, such that any allowable ϵ-approximation of K_i is contained in $N(K_i)$ and is isotopic to K_i in $N(K_i)$. Note that $\epsilon \leq d/3$. We will construct such an approximation $\hat{K}_i$ for each K_i, with the property that $\hat{L} = \hat{K}_1 \cup ... \cup \hat{K}_r$ has no parallel or antiparallel tangents. Since $N(K_i)$ are mutually disjoint, the union of the isotopies from $\hat{K}_i$ to K_i will be an isotopy from $\hat{L}$ to L in $N(L)$.

Consider the unit tangent vector of e_i as a point p_i on S^2, which projects to $\hat{p}_i$ on P^2. Since L is in general position, $\hat{p}_1, ..., \hat{p}_n$ are mutually distinct, so by choosing ϵ smaller if necessary we may assume that they have mutually disjoint ϵ-neighborhoods $\hat{D}_1, ..., \hat{D}_n$, which then lifts to ϵ-neighborhoods $D_1, ..., D_n$ of $p_1, ..., p_n$. Adding some edges near vertices of L if necessary, we may assume that the angle between the unit tangent vectors of two adjacent edges e_i, e_{i+1} of L (i.e. the spherical distance between p_i and p_{i+1}), is small (say $< \pi/2$).

Bend each edge e_j a little bit to obtain a round $(\epsilon/2)$-suspension $\hat{e}_j : I_j \to \mathbb{R}^3$ with $\|\hat{e}_j'(t)\| = 1$ (so the length of I_j equals the length of the curve $\hat{e}_j$). Then its derivative $\hat{e}_j'$ is a map $I_j \to S^2$ with image in D_j because D_j has radius ϵ. Let $Y = \cup S^1_i$ be a disjoint union of r copies of S^1, and let $A = \cup I_j$ be the disjoint union of I_j. Embed A into Y by a map ξ according to the order of e_i in L. More precisely, if e_j and e_k are edges of L such that the ending point of e_j equals the initial point of e_k then the ending point of $\xi(I_j)$ and the initial point of $\xi(I_k)$ cobounds a component of $Y - \xi(A)$. The union of the maps $\hat{e}_j' \circ \xi^{-1}$ defines a map $\xi(A) \to S^2$, which is admissible because the disks $\hat{D}_j$ on P^2 are mutually disjoint. By Lemma 2.3, it extends to an admissible map $\hat{\alpha} : Y \to S^2$. It now suffices to show that each K_i has an allowable ϵ-approximation $\hat{K}_i : S^1_i \to \mathbb{R}^3$, with $\hat{\alpha}|_{S^1_i}$ as its unit tangent map.

The construction of $\hat{K}_i$ is independent of the other components, so for simplicity we may assume that $L = K(v_1, ..., v_n)$ is a knot, with edges $e_i = e_i(v_i, v_{i+1})$. Since L is in general position, the three unit vectors p_1, p_2, p_3 of the edges e_1, e_2, e_3 are linearly independent, so there is a positive number $\delta < \epsilon$, such that the ball of radius δ centered at the origin is contained in the set $\{\sum u_i p_i \mid \epsilon > |u_i|\}$.

We may assume that the intervals $I_j = [a_j, b_j]$ are sub-intervals of $I = [0, a_{n+1}]$, with $a_1 = 0$ and $b_j < a_{j+1}$. Put $\hat{I}_j = [b_j, a_{j+1}]$. Without loss of generality we may assume that the function $\xi : A \to S^1$ defined above is the restriction of the function $\xi : I \to S^1$ defined by $\xi(t) = \exp(2\pi i t/a_{2n})$. Put $\alpha = \hat{\alpha} \circ \xi : I \to S^2$. Thus $\alpha(t) = \hat{e}'_j(t)$ on I_j.

Consider the restriction of α on $\hat{I}_j = [b_j, a_{j+1}]$. We have assumed that the spherical distance between p_j and p_{j+1} is at most $\pi/2$. Since $\alpha(b_j) \in D_j$ and $\alpha(b_{j+1}) \in D_{j+1}$, the spherical distance between $\alpha(b_j)$ and $\alpha(a_{j+1})$ is at most $\pi/2 + 2\epsilon$. As ϵ is very small, we may choose a coordinate system for $\mathbb{R}^3$ so that the third coordinate α_3 of α is greater than ϵ at b_j and a_{j+1}, and by transversality theorem we may further assume that α is transverse to the circle $S_1 = S^2 \cap \mathbb{R}_{xy}$ in this coordinate system. Now we can apply Lemma 2.5 to get a function $f_j(t)$ on $\hat{I}_j$ such that $f_j(t) = 1$ near $\partial \hat{I}_j$, and the integral curve $\gamma_j = \beta(f_j, \alpha|_{\hat{I}_j})$ is an ϵ-allowable curve in a ball of radius $\delta/2n$. Extend these f_j to a smooth map on I by defining $f(t) = 1$ on I_j.

Consider the integral curve $\beta = \beta(f, \alpha)$. It is the union of $2n$ curves $\beta_i, \hat{\beta}_i$ defined on I_i and $\hat{I}_i$, where $\beta_i = \beta(f|_{I_i}, \alpha|_{I_i}, a_i, \beta(a_i))$ is a translation of $\hat{e}_j$ because $f|_{I_i} = 1$; and $\hat{\beta}_i = \beta(f|_{\hat{I}_i}, \alpha|_{\hat{I}_i}, b_i, \beta(b_i))$ is a translation of γ_i because $f|_{\hat{I}_i} = f_i$. We have

$$\begin{aligned}\|\beta(a_{n+1}) - \beta(a_0)\| &\leq \sum_1^n \|\beta(a_{j+1}) - \beta(b_j)\| + \|\sum_1^n (\beta(b_j) - \beta(a_j))\| \\ &\leq \sum 2(\delta/2n) + \|\sum e_j\| = \delta.\end{aligned}$$

By the definition of δ, there are numbers $u_i \in [-\epsilon, \epsilon]$, such that $\beta(2n) - \beta(0) = \sum_{i=1}^3 u_i p_i$. Notice that $|u_i| < \epsilon < \|e_j\|/2$, so by Lemma 2.6, we can modify $f(t)$ on $[a_j + \epsilon_1, \beta_j - \epsilon_1]$ for $j = 1, 2, 3$ and some $\epsilon_1 > 0$, to a function $g(t)$, so that the integral curve $\gamma = \beta(g, \alpha)$ on I_j is an ϵ-suspension with base arc the vector $e_j + u_j p_j$. Now we have $\gamma(a_{n+1}) = \gamma(0)$, so γ is a closed curve. Since $\gamma'(t) = \alpha(t)$ near 0 and $2n$ and α induces a smooth map $\hat{\alpha} : S^1 \to S^2$, it follows that γ induces a smooth map $\hat{\gamma} : S^1 \to \mathbb{R}^3$.

From the definition we see that $\hat{\gamma}$ is an allowable ϵ-approximation of K. This completes the proof of the theorem. □

Acknowledgement. I would like to thank Colin Adams for raising the problem, and for some helpful conversations. Thanks also to the referee for some helpful comments, which have made the paper more readable.

References

[1] C. Adams, C. Lefever, J. Othmer, S. Pahk and J. Tripp, *The Supercrossing Number of Knots and the Crossing Map*, Preprint.

[2] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, 1974.

[3] J. Porter, *A note on regular closed curves in E^3*, Bull. Acad. Polon. Sci. Sir. Sci. Math. Astronom. Phys. **18** (1970), 209–212.

[4] D. Rolfsen, *Knots and Links*, Publish or Perish, 1990.

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: wu@math.uiowa.edu