TOPOLOGY AND ITS APPLICATION
(to appear). Preprint 1998

ACHIRALITY OF KNOTS AND LINKS

Boju Jiang[1], Xiao-Song Lin[2], Shicheng Wang[3] and Ying-Qing Wu[4]

Abstract. We will develop various methods, some are of geometric nature and some are of algebraic nature, to detect the various achiralities of knots and links in S^3. For example, we show that the twisted Whitehead double of a knot is achiral if and only if the double is the unknot or the figure eight knot, and we show that all non-trivial links with ≤ 9 crossings are not achiral except the Borromean rings. A simple procedure for calculating the η-function is given in terms of a crossing change formula and its initial values.

§1. Introduction

Topological chirality of compact polyhedra in the 3-space is an important notion in physics and chemistry. Nevertheless, it seems that there are not many general theorems about this notion. One such general theorem appeared recently in [JW], where it is proved that a compact polyhedron X has an achiral embedding into S^3, in the sense that its image is contained in the fixed point set of an orientation-reversing diffeomorphism, if and only if X is abstractly planar, that is, it can be embedded into S^2. The related question of which embedding of X has its image contained in the fixed point set of an orientation-reversing diffeomorphism of S^3 is, however, much more complicated. A special case, if we are allowed to abuse the notation by not distinguishing X and its image under an embedding, is when X is a link L in S^3. An oriented link L is *achiral* (or amphicheiral) if there is an orientation-reversing diffeomorphism $g : S^3 \to S^3$ such that $g|_L = id$. Equivalently, L is achiral if it is isotopic to its mirror image, preserving the order and orientation of its components. When L is a knot (one component), our definition of achirality coincides with that in [BZ].

More generally, given $\epsilon = (\epsilon_1, ..., \epsilon_n)$ with $\epsilon_i = \pm 1$, we say that an oriented knot or link $L = K_1 \cup ... \cup K_n$ in S^3 is *ϵ-achiral*, or achiral of type ϵ, if there is an orientation-reversing diffeomorphism g of S^3 which sends each component K_i to itself, with its orientation preserved if $\epsilon_i = +1$ and reversed if $\epsilon_i = -1$; otherwise it is *ϵ-chiral*. When all $\epsilon_i = 1$, we say that L is *positive achiral*, or simply *achiral*, and when all $\epsilon_i = -1$ we say that L is *negative achiral*. A link L in S^3 is *absolutely chiral* if it is ϵ-chiral for all ϵ. Finally, an (un-)oriented link L is *(absolutely) set-wise chiral,* if there is no orientation reversing homeomorphism h such that $h(L) = L$ as unordered links. It should be noticed that when L is an oriented link, our definition is different from that in the usual sense, because usually L is considered chiral if

[1] Partially supported by NSFC grant
[2] Partially supported by NSF grant #DMS 9796130
[3] Partially supported by Outstanding Youth Fellowship of NSFC grant
[4] Partially supported by NSF grant #DMS 9802558

L is not isotopic to its mirror image as *unordered*, oriented links, which is set-wise chiral in our definition.

There are some well known link invariants which detect various chiralities of $L = K_1 \cup ... \cup K_n$. For example, if L is achiral, then

(1) the signature $\sigma(L') = 0$ for any sublink L' of L;

(2) the linking number $\mathrm{lk}(K_i, K_j) = 0$ for all $i \neq j$;

(3) Milnor $\bar{\mu}$-invariants of even length all vanish;

(4) the Jones polynomial of any sublink L' of L is symmetric, i.e. $V_{L'}(t) = V_{L'}(t^{-1})$; and more generally

(5) the HOMFLY polynomial $P_{L'}(l, m)$ as defined in [LM] is symmetric with respect to l, i.e $P_{L'}(l, m) = P_{L'}(l^{-1}, m)$.

Also, Vassiliev knot invariants of odd order can be used to detect the chirality of a knot [Va].

Note that these invariants are either not very effective or hard to calculate in principle, so that it is hard to draw any general conclusion about chirality from them. For example, let's think of the figure eight knot as a twisted Whitehead double of the unknot. It is achiral and has zero signature. For any non-trivial knot K, the same twisted Whitehead double of K always has zero signature. Is it achiral? There seems to be no way to answer this question in general by calculating the Jones polynomial or the HOMFLY polynomial. Also, it is not practical trying to use Vassiliev knot invariants to answer such a general geometric question. Nevertheless, for a specific knot or link, if it is not too complicated, the first thing one should try to detect its chirality is probably to use some invariants. For example, let us see what kind of ϵ-achiralities the Borromean rings have. Although the Jones polynomial and the HOMFLY polynomial of the Borromean rings satisfy conditions in (3) and (4) above respectively so they are not useful here, considering Milnor's invariant $\mu(123)$ leads immediately to the conclusion that the Borromean rings are ϵ-achiral for $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ only if $\epsilon_1\epsilon_2\epsilon_3 = 1$.

We will see that except the figure eight knot, all nontrivial (twisted) double knots are absolutely chiral (Corollary 3.2.(3)), and the Borromean rings are ϵ-achiral if and only if $\epsilon_1\epsilon_2\epsilon_3 = 1$ (Example 3.5. (3)). In fact, to understand the chirality of satellite knots and links in general is one of the purposes of this paper. In Section 2 we study the achirality of links in solid tori. In Section 3 we will show that under some mild restriction, a satellite link $L(J)$ is achiral of some type if and only if both J and L are achiral of certain related types. Combining with results of Section 2, it will be shown that many satellite knots are chiral.

The other purpose of this paper is to apply the η-function of Kojima and Yamasaki to the study of chirality of links. The linking number provides the first obstruction to the achirality of a two component link. We observe that if the η-function of a two component link L with zero linking number is not zero, then L is absolutely chiral. Moreover if $\eta_1 \neq \eta_2$ then L is absolutely set-wise chiral. See Theorem 4.2. A crossing change formula for the η-function (Theorem 4.5) , due originally to G.T. Jin [J1], together with the exact determination of its initial values in terms of the Conway polynomials (Theorem 5.5), will allow us to calculate the η-function effectively. As applications, we will use our calculation of the η-function to show that all prime links with more than one component and up to 9 crossings are chiral, except the Borromean rings (Theorem 4.8).

Notations and conventions. All links oriented, with a fixed order on the components. If L is a link in a 3-manifold M, denote by $|L|$ the underlying unoriented link, with the same order on the components. Given two links $L = K_1 \cup ... \cup K_n$ and $L' = K_1' \cup ... \cup K_n'$, $|L| = |L'|$ means that $|K_i|$ and $|K_i'|$ are the same subspace of M for each i. We do not allow permutation on the components. Similarly, $L = L'$ means that K_i and K_i' are exactly the same oriented knot for all i. No isotopy is allowed here. Two links L and L' are *equivalent*, denoted by $L \cong L'$, if they are isotopic as oriented, ordered links. Similarly for $|L| \cong |L'|$. We say that L and L' are *weakly equivalent* in M, denoted by $L \sim L'$, if there is a (possibly orientation-reversing) homeomorphism f of M, such that $f(|L|) = |L'|$. In other words, two links L, L' in S^3 are weakly equivalent if and only if $|L|$ is isotopic, as an unoriented ordered link, to either $|L'|$ or its mirror image. Given $\epsilon = (\epsilon_1, ..., \epsilon_n)$, denote by ϵL the link $\epsilon_1 K_1 \cup ... \cup \epsilon_n K_n$. Thus a link L in S^3 is ϵ-achiral if and only if there is an orientation-reversing homeomorphism $f : S^3 \to S^3$ such that $f(L) = \epsilon L$. The map f is called an *ϵ-achiral map* (for L).

If $J = K_1 \cup ... \cup K_n$ is a link in a 3-manifold M, denote by $E_M(J) = M - \mathrm{Int}N(J)$ the exterior of J in M. When $M = S^3$, simply write it as $E(J)$.

Suppose M is an oriented 3-manifold satisfying $H_2(M, Z) = 0$. For example, M may be a rational homology sphere or the infinite cyclic cover of the complement of a knot in S^3. Let $L = L' \cup K$ be an oriented link in M such that L' is null-homologous. Let F be a compact, orientable surface bounded by L', with orientation induced by that of L'. Deform F so that it meets K transversely. The orientations of F and K then determine the orientation of $F \cap K$, i.e. a sign for each point x in $F \cap K$. More precisely, define $\mathrm{sign}(x) = 1$ if the product orientation of F and K at x gives the orientation of M at x, and -1 otherwise. The linking number $\mathrm{lk}(L', K)$ is defined as the algebraic intersection number of F and K, i.e. it is the sum of $\mathrm{sign}(x)$ over all $x \in F \cap K$. Since $H_2(M, \mathbb{Z}) = 0$, this intersection number is well defined. When M is R^3 or S^3 and L' is also a knot, the above definition of linking number is equivalent to any of the eight definitions in [R]. Denote by $-K_i$ the knot K_i with orientation reversed. We have the following basic property of $\mathrm{lk}(L', K)$, which will be used in the paper repeatedly, in particular in §4 to investigate the role played by the Kojima-Yamasaki η-function in detecting the chirality of the link $K_1 \cup K_2$ when $\mathrm{lk}(K_1, K_2)$ vanishes.

Lemma 1.1. *Let M, L', K be as above. If g is an orientation-reversing homeomorphism of M, then* $\mathrm{lk}(g(L'), g(K)) = -\mathrm{lk}(L', K)$*; in particular, if g preserves or reverses the orientation of $L = L' \cup K$, then* $\mathrm{lk}(L', K) = 0$.

§2. Achirality of links in solid tori

Fix a trivially embedded solid torus $V = S^1 \times D^2$ in S^3 and fix an orientation of its core λ'. Let (μ, λ) be a preferred meridian-longitude pair on ∂V, i.e, μ is a meridian of λ' on ∂V such that $\mathrm{lk}(\mu, \lambda') = 1$, and λ is a longitude oriented in the same way as λ', and is null homologous in $S^3 - \mathrm{Int}V$. A link $L = K_1 \cup ... \cup K_n$ in V is *essential* if no 3-ball in V intersects L in a nonempty sublink of L, and (V, L) is not homeomorphic to $(D^2 \times S^1, \mathrm{points} \times S^1)$. The link L is *atoroidal* if $V - \mathrm{Int}N(L)$ is an atoroidal 3-manifold. Given $\epsilon = (\epsilon_0, ..., \epsilon_n)$, where $\epsilon_i = \pm 1$, the pair (V, L) is *ϵ-achiral* if there is an orientation-reversing homeomorphism $g : V \to V$, such that $g(\lambda) = \epsilon_0 \lambda$, and $g(K_i) = \epsilon_i K_i$ for all i. A pair (V, L) is *absolutely chiral* if it is not ϵ-achiral for any ϵ.

If L is a link in V, denote by $\hat{L}$ the link $C \cup L$ in S^3, where C is the core of the torus $S^3 - \text{Int}V$. Denote by $L(O)$ the link L when considered as a link in S^3.

Lemma 2.1. *The pair (V, L) is (ϵ_0, ϵ')-achiral if and only if $\hat{L}$ is $(-\epsilon_0, \epsilon')$-achiral.*

Proof. An (ϵ_0, ϵ')-achiral map f of (V, L) sends $|\lambda|$ to itself, hence extends to a homeomorphism $f' : S^3 \to S^3$ with $f'(L) = \epsilon' L$ and $f'(|C|) = |C|$, and vise versa. Since $f : V \to V$ is orientation-reversing, it maps a longitude λ of V to $\epsilon_0 \lambda$ if and only if it maps a meridian μ of V to $-\epsilon_0 \mu$. Since μ is a longitude of C, it follows that f is (ϵ_0, ϵ')-achiral if and only if f' is $(-\epsilon_0, \epsilon')$-achiral. □

Suppose $L = K_1 \cup ... \cup K_n$ is a link in a 3-manifold M. An n-tuple $\gamma = (\gamma_1, ..., \gamma_n)$, where each γ_i is a slope on $\partial N(K_i)$, is called a *slope of* L. Denote by $(M, L)(\gamma)$ the manifold obtained by γ_i-Dehn surgery on each K_i. If $M = S^3$, denote $(S^3, L)(\gamma)$ by $L(\gamma)$. When L has a preferred meridian-longitude pair, for example when L is in S^3 or V, each γ_i is represented by a rational number or ∞, see for example [R, Page 259]. The following lemma is useful in determining chirality of links.

Lemma 2.2. *Suppose $L = L' \cup L''$ is a link in S^3. If L is ϵ-achiral for some $\epsilon = (\epsilon', \epsilon'')$, then for any slope γ' of L', there is an orientation-reversing homeomorphism $f : L'(\gamma') \to L'(-\gamma')$ such that $f(L'') = \epsilon'' L''$. In particular, if $(L'(\gamma'), |L''|)$ is not homeomorphic to the pair $(L'(-\gamma'), |L''|)$ for some γ', then L is absolutely chiral.*

Proof. Let (μ_i, λ_i) be the preferred meridian-longitude pair of $K_i \subset L$. By assumption there is an orientation-reversing homeomorphism $g : S^3 \to S^3$ such that $g(|K_i|) = |K_i|$. Now g maps (μ_i, λ_i) to itself, with orientation preserved on one curve and reversed on the other, so it maps a curve of slope γ on $\partial N(K_i)$ to a curve of slope $-\gamma$, hence induces a homeomorphism $f : L'(\gamma') \to L'(-\gamma')$. We have $f(L'') = g(L'') = \epsilon'' L''$. □

Given a link L in V, we may cut V along a meridian, perform r right hand twists, then glue back to get a new link in V, denoted by L^r. If $J = K \cup L$ is a link in S^3 with K an unknotted component, then L is a link in $V = E(K)$. Denote by $\tau_K^n L$ the link $L^n(O)$ in S^3. More explicitly, $\tau_K^n L$ is obtained from L by performing n right hand Dehn twists along a disk bounded by K. There is an orientation-preserving homeomorphism $\varphi^n : (K(-1/n), L) \to (S^3, \tau_K^n L)$.

Corollary 2.3. *(1) If $J = K \cup L \subset S^3$ is (ϵ_0, ϵ')-achiral and K is trivial in S^3, then $\tau_K^n L$ is equivalent to the mirror image of $\tau_K^{-n}(\epsilon' L)$. In particular, $\tau_K^n L \sim \tau_K^{-n} L$.*

(2) If (V, L) is ϵ-achiral for some ϵ, then $L^n(O) \sim L^{-n}(O)$ as links in S^3.

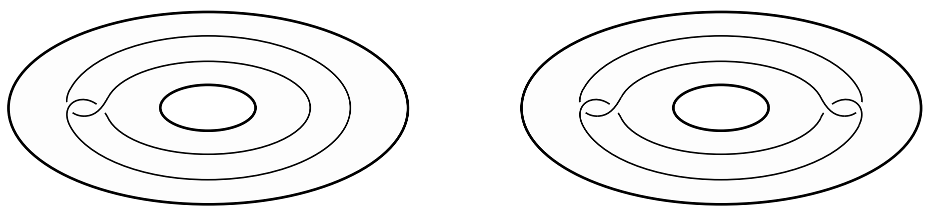
Proof. By Lemma 2.2 and definition, we have the following homeomorphisms:

$$(S^3, \tau_K^n L) \xrightarrow{(\varphi^n)^{-1}} (K(-1/n), L) \xrightarrow{f} (K(1/n), \epsilon' L) \xrightarrow{\varphi^{-n}} (S^3, \tau_K^{-n}(\epsilon' L)),$$

where φ^n and φ^{-n} are defined above, and f is the orientation-reversing homeomorphism given by Lemma 2.2. Composing with a reflection of S^3, we get an orientation-preserving homeomorphism η of S^3 sending $\tau_K^n L$ to the mirror image of $\tau_K^{-n}(\epsilon' L)$, preserving the order and orientation of components. Since any orientation-preserving homeomorphism of S^3 is isotopic to the identity map, the result follows.

(2) By Lemma 2.1, (V, L) is ϵ-achiral for some $\epsilon = (\epsilon_0, \epsilon')$ if and only if $\hat{L} = C \cup L$ is $(-\epsilon_0, \epsilon')$-achiral, where C is the core of $V = S^3 - \mathrm{Int}V$. Since $L^n(O) = \tau_C^n L$ and $L^{-n}(O) \sim \tau_C^{-n} L$, the result follows from (1). □

The knot W and the link B in V shown in Figure 2.1(1) and 2.1(2) are called the *Whitehead knot* and the *Bing link* in V, respectively. These are hyperbolic, hence atoroidal. The knot W^r is called a *twisted Whitehead knot.* A (p, q) *cable knot* in V is a knot isotopic to a curve on ∂V representing $p\lambda + q\mu$ in $H_1(\partial V)$, where (μ, λ) is a preferred meridian-longitude pair on ∂V. We will always assume that $p > 1$. The exterior of a cable knot is a Seifert fiber space with orbifold an annulus with a single cone point, so any simple closed curve on the orbifold is isotopic to a boundary curve; hence cable knots are atoroidal.



(1) (2)

Figure 2.1

Corollary 2.4. *Suppose $L \subset V$ contains either a Whitehead knot or a cable knot $C_{p,q}$ in V. Then (V, L^r) is absolutely chiral for all r.*

Proof. If L contains a knot K such that (V, K^r) is absolutely chiral then (V, L^r) is also absolutely chiral. Hence we may assume without loss of generality that $L = W$ or $C_{p,q}$.

If (V, L^r) were not absolutely chiral, then by Corollary 2.3, the knots $(L^r)^n(O) = L^{r+n}(O)$ and $(L^r)^{-n}(O) = L^{r-n}(O)$ would be weakly equivalent. First suppose $L = W$. When $r = 0$, $L^1(O)$ is the figure 8 knot while $L^{-1}(O)$ is the trefoil knot; when $r \neq 0$, $L^{r+r}(O)$ is a nontrivial twist knot, while $L^{r-r}(O) = L(O)$ is a trivial knot; in either case they are not weakly equivalent. Similarly, if $L = C_{p,q}$, then $L^{r+n}(O)$ is a torus knot $K_{p,q+(r+n)p}$ and $L^{r-n}(O) = K_{p,q+(r-n)p}$, which are not weakly equivalent for any r. □

The following theorem says that the above corollary is almost true for any essential atoroidal link L in V.

Theorem 2.5. *If L is an essential atoroidal link in V, then (V, L^r) is absolutely chiral for all but at most one $r \in \mathbb{Z}$.*

Proof. By assumption $E_V(L)$ is irreducible and atoroidal, hence by Thurston's Geometrization Theorem [Th1], $E_V(L)$ is either hyperbolic or Seifert fibred. Since L is essential, (V, L) is not homeomorphic to a pair $(D^2 \times S^1, \text{points} \times S^1)$. Thus if $E_V(L)$ is Seifert fibred, then L contains a cable knot $C_{p,q}$ for some $p > 1$, so by Corollary 2.4 (V, L^r) is absolutely chiral for all r, and the result follows. Therefore we may assume that $E_V(L)$ is hyperbolic.

Suppose that (V, L^r) and (V, L^s) are not absolutely chiral for some $r \neq s$. Then by Corollary 2.3 we have $L^{r+n}(O) \sim L^{r-n}(O)$ and $L^{s+n}(O) \sim L^{s-n}(O)$. Thus

$$L^m(O) = L^{r+(m-r)}(O) \sim L^{r-(m-r)}(O) = L^{s-(m+s-2r)}(O) \sim L^{s+(m+s-2r)}(O) = L^{m+2(s-r)}(O)$$

for all m. It follows that there are infinitely many $L^m(O)$ weakly equivalent to each other. On the other hand, since $E_V(L) = E(\hat{L})$ is hyperbolic, by Thurston's Hyperbolic Surgery Theorem [Th1], all but finitely many Dehn surgeries on C produce hyperbolic manifolds, where as before, C stands for a core of $S^3 - \text{Int}V$, and $\hat{L} = C \cup L$. Moreover, when n approaches ∞, the volumes of manifolds $(S^3 - L(O), C)(1/n)$ obtained by $1/n$ surgery on C in $S^3 - L(O)$ approach the volume of $S^3 - L(O)$; see [NZ]. Hence there are at most finitely many surgery manifolds having the same volume. Since $(S^3 - L(O), C)(1/n)$ is the complement of $L^{-n}(O)$, it follows that there are at most finitely many $L^m(O)$ weakly equivalent to each other, a contradiction. □

Given $\epsilon = (\epsilon_1, ..., \epsilon_n)$, define $\pi(\epsilon)$ as the product $\epsilon_1 ... \epsilon_n$.

Theorem 2.6. *Let B be the Bing link B in V. Then (V, B) is ϵ-achiral if and only if $\pi(\epsilon) = -1$.*

Proof. Let $B = L_1 \cup L_2$ be the Bing link in V, as shown in Figure 2.1(2), and let $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2)$. One can find a reflection ρ_1 along some plane intersecting V in two meridian disks which is a $(-1, -1, -1)$-achiral map, a reflection ρ_2 along some annulus A in V containing L_1 which is a $(1, 1, -1)$-achiral map, and similarly a $(1, -1, 1)$-achiral map ρ_3. Now $\rho_4 = \rho_1\rho_2\rho_3$ is a $(-1, 1, 1)$-achiral map. We need to show that there is no other types of achiral maps.

Assuming otherwise, and let $g : (V, B) \to (V, B)$ be an ϵ-achiral map for some ϵ with $\pi(\epsilon) = 1$. After composing with some of the ρ_i's above, we may assume that $\epsilon = (1, 1, 1)$, so $g(\lambda) = \lambda$, and $g(L_i) = L_i$ for $i = 1, 2$. Consider the universal covering $\tilde{V} = D^2 \times \mathbb{R}$ of V. Then B lifts to a link

$$\tilde{B} = ... \cup \tilde{L}_{-1} \cup \tilde{L}_0 \cup \tilde{L}_1 \cup \tilde{L}_2 \cup ...,$$

where $\tilde{L}_i$ covers L_1 if and only if i is odd. See Figure 2.2. Notice that $\text{lk}(\tilde{L}_i, \tilde{L}_{i+1}) = (-1)^i$.

Let $\tilde{g} : \tilde{V} \to \tilde{V}$ be a lifting of g such that the restriction of $\tilde{g}$ to $\tilde{L}_0$ is the identity map. Since g is orientation-reversing, so is $\tilde{g}$. Hence for any two components $\tilde{L}', \tilde{L}''$ of $\tilde{B}$, we have

$$\text{lk}(\tilde{g}(\tilde{L}'), \tilde{g}(\tilde{L}'')) = -\text{lk}(\tilde{L}', \tilde{L}'').$$

In particular, $\text{lk}(\tilde{L}_0, \tilde{g}(\tilde{L}_1)) = \text{lk}(\tilde{g}(\tilde{L}_0), \tilde{g}(\tilde{L}_1)) = -\text{lk}(\tilde{L}_0, \tilde{L}_1) = -1$, so we must have $\tilde{g}(\tilde{L}_1) = \tilde{L}_{-1}$ because $\tilde{L}_{-1}$ is the only component of $\tilde{B}$ whose linking number with $\tilde{L}_0$ is -1. By induction one can show that $\tilde{g}(\tilde{L}_n) = \tilde{L}_{-n}$. On the other hand, since g preserves the orientation of a longitude of V, there is a homotopy h_t, deforming g to the identity map. Since V is compact, the length of the trace of any point $x \in V$ under the homotopy h_t is bounded above by some number N. The lifting of h_t to $\tilde{V}$ is a homotopy of $\tilde{g}$ to id with the same upper bound N on the trace of points $\tilde{x} \in \tilde{V}$. But since the distance between $\tilde{L}_n$ and $\tilde{g}(\tilde{L}_n) = \tilde{L}_{-n}$ approaches ∞ as n approaches ∞, this is impossible. □

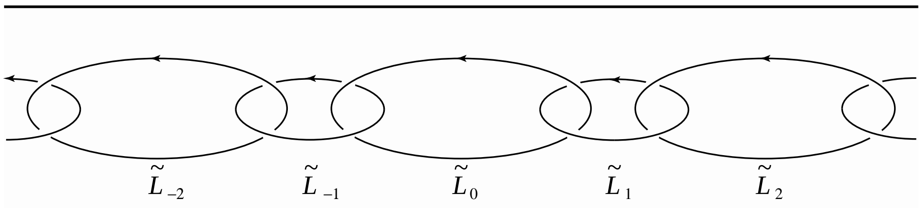


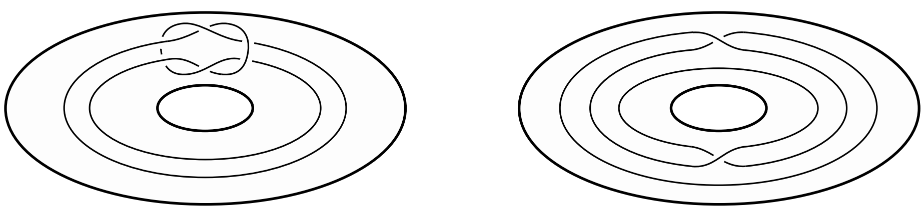
Figure 2.2

Theorem 2.7. *Suppose $L = K_1 \cup ... \cup K_n$ is a link in V, and suppose (V, L) is $(\epsilon_0, ..., \epsilon_n)$-achiral. If $\epsilon_0\epsilon_i = -1$, then the winding number of K_i in V is 0.*

Proof. By Lemma 2.1 L is $(\epsilon_0, ...\epsilon_n)$-achiral if and only if $\hat{L} = C \cup L$ is $(-\epsilon_0, \epsilon_1, ..., \epsilon_n)$-achiral, where C is the core of $S^3 - \text{Int}V$, with the same orientation as the meridian μ of V. Let $f : S^3 \to S^3$ be an achiral map of this type. Since f is orientation-reversing, we have

$$\text{lk}(C, K_i) = -\text{lk}(f(C), f(K_i)) = -\text{lk}(-\epsilon_0 C, \epsilon_i K_i) = \epsilon_0\epsilon_i \text{lk}(C, K_i).$$

Hence if $\epsilon_0\epsilon_i = -1$, then $\text{lk}(C, K_i) = 0$. Since this linking number is exactly the winding number of K_i in V, the result follows. □



(1) (2)

Figure 2.3

Example 2.8. (1) The knot K in V shown in Figure 2.3(1) is $(1, -1)$-achiral and $(-1, 1)$-achiral. One may obtain a $(1, -1)$-achiral map by reflecting along an annulus in V perpendicular to the paper. Composing with a rotation along the vertical axis on the paper, one gets a $(-1, 1)$-achiral map. By Theorem 2.7 such a knot K must have winding number 0 in V.

(2) The pair (V, K) in Figure 2.3(2) is ϵ-achiral for $\epsilon = (1, 1)$ and $(-1, -1)$. Since the winding number of K in V is nonzero, by Theorem 2.7 it cannot be ϵ-achiral for $\epsilon = (-1, 1)$ or $(1, -1)$.

For a braid B, write B as a word in standard generators of the braid group and denote by $e(B)$ the sum of exponents. If $L = \hat{B}$ is a closed braid in V, then $e(L) = e(B)$ is well defined as an isotopy invariant of (V, L), since $L' = \hat{B}'$ is isotopic to L in V if and only if B' is conjugate to B.

By Theorem 2.5, all nontrivial twists K^r $(r \neq 0)$ of the above knots are absolutely chiral. The knot K in Figure 2.3(2) is a closed braid in V with $e(K) = 0$. The following result shows that this is not a coincidence.

Theorem 2.9. *If a link L is a closed braid in V such that this exponent sum $e(L)$ is nonzero, then (V, L) is absolutely chiral.*

Proof. Let $f : (V, L) \to (V, L)$ be an ϵ-achiral map. Since each component K_i has winding number nonzero in V, by Theorem 2.7 f either reverses or preserves the orientation of both λ and K_i. Thus f is either positive achiral or negative achiral.

Since f maps $|\lambda|$ to itself, f is isotopic to a homeomorphism g which is either a reflection along an annulus A containing λ, or a reflection along a plane in $\mathbb{R}^3$ intersecting V in two meridian disks. In either case $e(g(L)) = -e(L)$. Since the exponent sum is an isotopy invariant of links in $V = D^2 \times S^1$, we have $e(L) = e(f(L)) = e(g(L)) = -e(L)$, hence $e(L) = 0$. □

§3. Achirality of satellite knots and links

In this section we will use the results of the last section to investigate achirality of satellite knots and links in S^3. In particular, we will discuss the relationship between the achirality of a knot and that of its Whitehead double and Bing double, respectively.

Given an ordered, oriented link $J = K_1 \cup ... \cup K_n \subset S^3$, there is an orientation-preserving homeomorphism $h : V \to N(K_1)$, unique up to isotopy, sending the preferred meridian-longitude pair (μ, λ) of ∂V to that of $\partial N(K_1)$. Suppose L is an essential link in V. Denote the link $h(L) \cup K_2 \cup ... \cup K_n$ in S^3 by $L(J)$ or $L(K_1) \cup ... \cup K_n$, called an *L-satellite* of J. Notice that only the first component of J has been changed. The original embedding of $L \subset V \subset S^3$, denoted by $L(O)$ before, is $L(J)$ with $J = O$ the trivial knot. This justifies the notation.

Recall that $E(J)$ denotes the exterior of J in S^3. A component K of a link J in S^3 is *J-nontrivial* if it does not bound a disk with interior disjoint from J.

Theorem 3.1. *Suppose L is an essential atoroidal link in a trivial solid torus $V \subset S^3$, and suppose $J = K \cup J' = K \cup K'_1 \cup ... \cup K'_n$ is a link in S^3 such that K is J-nontrivial. The following are equivalent.*

(1) $L(J) = L(K) \cup J'$ is (ϵ'', ϵ')-achiral, where ϵ' has n components;

(2) J is (ϵ_0, ϵ')-achiral and (V, L) is (ϵ_0, ϵ'')-achiral for some $\epsilon_0 = \pm 1$;

(3) J is (ϵ_0, ϵ')-achiral and $\hat{L}$ is $(-\epsilon_0, \epsilon'')$-achiral for some $\epsilon_0 = \pm 1$.

Proof. The equivalence of (2) and (3) follows from Lemma 2.1. Clearly (2) implies (1). Hence we need only prove that (1) implies (2).

Let $f : S^3 \to S^3$ be a (ϵ'', ϵ')-achiral map. Let $h : V \to N(K)$ be the orientation-preserving homeomorphism sending L to $L(K)$. To simplify notations, we will identify (V, L) with $(N(K), h(L))$ below. Put $T = \partial N(K)$. Since K is J-nontrivial, T is incompressible in $E(J)$; since L in essential in V, ∂V is also incompressible in $E_V(L)$. Hence T is incompressible in $E(L(J)) = E(J) \cup_T E_V(L)$.

Put $X = E(L(J))$, and denote by $\mathcal{T}$ the Jaco-Shalen-Johannson decomposition tori of X. See [JS]. First assume that T is a component of $\mathcal{T}$. Notice that this is true if $E_V(L)$ is hyperbolic and T is not parallel to a component of ∂X. Since the JSJ-decomposition is unique up to isotopy, by an isotopy we may assume that f maps $\mathcal{T}$ to itself. By assumption $E_V(L)$ is atoroidal, hence it is a component of X cut along $\mathcal{T}$. Since it contains some boundary components of X and since f maps each boundary torus of X to itself, f maps $E_V(L)$ to itself; in particular, it maps T to T, and $N(K)$ to $N(K)$. By an isotopy in $N(K)$ rel boundary, f can be modified to a map f' sending $|K|$ to itself, hence f' is an (ϵ_0, ϵ')-achiral map of J

for some ϵ_0. The restriction of f to $N(K)$ sends a longitude λ_0 of K on $\partial N(K)$ to $\epsilon_0\lambda_0$, hence $f|_{N(K)}$ is the required (ϵ_0, ϵ'')-achiral map of (V, L), and the theorem follows.

We now assume that T is not a component of $\mathcal{T}$. By assumption $E_V(L)$ is irreducible, atoroidal, and is not a product $T \times I$, so it is either hyperbolic or Seifert fibred. Thus T is not a component of $\mathcal{T}$ only if either

(i) T is parallel to a component of ∂X outside of $E_V(L)$, or

(ii) some component M of X cut along $\mathcal{T}$ is a Seifert fiber space containing $E_V(L)$, with T an essential torus in its interior.

In the first case, $E(J)$ is a product $T \times I$, so J is a Hopf link, which is $(-\epsilon', \epsilon')$-achiral. Since T is parallel to a boundary component of X, which is f-invariant, we may assume that T is f-invariant; therefore the restriction of f to $N(K)$ is a $(-\epsilon', \epsilon'')$-achiral map, so (V, L) is $(-\epsilon', \epsilon'')$-achiral, and the result follows.

In case (ii), consider the orbifold Y of M. Since S^3 contains no embedded Klein bottle or non-separating tori, Y is planar, and each component W_i of $S^3 - \text{Int}M$ has boundary a single torus T_i, so W_i is either a solid torus or a nontrivial knot exterior. By assumption $E_V(L)$ is not a product $(D^2 \times S^1, \text{points} \times S^1)$, so it contains a singular fiber of type (p, q) for some $p > 1$, which is a core of V. We may thus assume that K is a singular fiber of M. If W_i is a solid torus, then its meridian cannot be a fiber of M, otherwise S^3 would contain a punctured lens space $L(p, q)$. Therefore the fibration of M extends over W_i. Let $\hat{M}$ be the union of M and all W_i which are solid tori. Then $\hat{M}$ is a Seifert fiber space, and the core of each W_i, in particular each component K_j of L in $V = N(K)$, is a fiber of $\hat{M}$. The homeomorphism f sends each solid torus W_i to some solid torus W_j, so it maps $\hat{M}$ to itself.

First assume that $\partial\hat{M} \neq \emptyset$. By definition $\partial\hat{M}$ is incompressible in $S^3 - \text{Int}\hat{M}$, so it must be compressible in $\hat{M}$. Since $\hat{M}$ is Seifert fibred, it must be a solid torus. The fibration of a solid torus can have at most one singular fiber. Since we already has a singular fiber K in M, each K_j of L is a regular fiber, so it is a cable knot $C_{p,q}$ in $\hat{M}$ for some $p > 1$. Since f maps $(\hat{M}, K_j)$ to itself, this contradicts Corollary 2.4, which says that cable knots in solid tori are absolutely chiral.

Now assume $\partial\hat{M} = \emptyset$, so $\hat{M} = S^3$. In this case $\hat{M}$ has at most two singular fibers. If it has two, then each regular fiber, hence each component K_j of L, is a nontrivial (p, q) torus knot, which has nonzero signature and hence is absolutely chiral, a contradiction. Therefore K, the core of $N(K)$, is the only singular fiber, i.e. $q = \pm 1$. Notice that in this case K is unknotted in S^3. Since K is assumed J-nontrivial, J has another component, say K', which is contained in some component W_i of $S^3 - \text{Int}M$. Since f maps K' to itself, it maps W_i to itself. Since $\hat{M} = S^3$, all W_i are solid tori. The core of W_i is a regular fiber, hence is a trivial knot in S^3. Therefore $V' = S^3 - \text{Int}W_i$ is a fibred solid torus which contains L as a regular fiber, and is mapped to itself by f. As above, this contradicts Corollary 2.4, completing the proof of the theorem. □

Corollary 3.2. *Suppose $J = K_1 \cup ... \cup K_n$ is a link in S^3 such that K_1 is J-nontrivial. Then*

(1) $L^r(J)$ is absolutely chiral for all but at most one r if L is an essential atoroidal link in V.

(2) $L(J)$ is absolutely chiral if $L \subset V$ contains either a (twisted) Whitehead knot, or a cable knot $C_{p,q}$, or a closed braid in V with nonzero exponent sum.

(3) A cable knot $C_{p,q}(K)$ or (twisted) double knot $W^r(K)$ is absolutely chiral, unless it is the trivial knot or the figure 8 knot.

Proof. (1) and (2) follow from Theorems 3.1, 2.5, 2.9 and Corollary 2.4. When K is nontrivial, (3) is a special case of (2). When $K = O$ is trivial, $C_{p,q}(O)$ is either the trivial knot, or has nonzero signature and hence is absolutely chiral. A nontrivial twisted double $K^r = W^r(O)$ of the trivial knot is called a twist knot, which is a 2-bridge knot associated to a rational number with partial fraction decomposition $[\pm 2, r]$ for some integer r. Its mirror image is associated to $[\mp 2, -r]$. By the classification theorem of 2-bridge knots, (see [BZ, p.189]), one can see that K^r is absolutely chiral if and only if it is not the figure 8 knot. □

Corollary 3.3. *Let B be the Bing link in V, and let $J = K_1 \cup ... \cup K_n$ be a link in S^3 such that K_1 is J-nontrivial. Then $B^r(J)$ is $(\epsilon_1, \epsilon_2, \epsilon')$-achiral if and only if (i) $r = 0$, and (ii) J is $(-\epsilon_1\epsilon_2, \epsilon')$-achiral.*

Proof. The sufficiency follows from Theorems 2.6 and 3.1. Now suppose $B^r(J)$ is $(\epsilon_1, \epsilon_2, \epsilon')$-achiral. By Theorem 3.1 this implies that there is an ϵ_0, such that J is (ϵ_0, ϵ')-achiral and (V, B^r) is $(\epsilon_0, \epsilon_1, \epsilon_2)$-achiral. Since (V, B) is ϵ-achiral for some ϵ, and since B is clearly essential and atoroidal in V, by Theorem 2.5 (V, B^r) is absolutely chiral unless $r = 0$; hence (i) follows from Theorem 3.1. By Theorem 2.6, (V, B) is $(\epsilon_0, \epsilon_1, \epsilon_2)$-achiral if and only if $\epsilon_0\epsilon_1\epsilon_2 = -1$, that is $\epsilon_0 = -\epsilon_1\epsilon_2$. Therefore J is $(-\epsilon_1\epsilon_2, \epsilon')$-achiral. □

Remark. Theorem 3.1 is not true if L is allowed to be toroidal. For example, let L be a right hand trefoil in S^3, let V be the exterior of a meridian of L, and let J be a left hand trefoil. Then $L(J)$ is the connected sum of a right hand trefoil and a left hand trefoil, so $L(J)$ is achiral, but both J and (V, L) are absolutely chiral.

Denote by K^* the mirror image of K with induced orientation. The knot $K\#\epsilon K^*$ is called the *ϵ-square of K*. Since $(K\#\epsilon K^*)^* = K^*\#\epsilon K \cong \epsilon(K\#\epsilon K^*)$, we see that the ϵ-square of any knot is ϵ-achiral. Also, if K_1 and K_2 are ϵ-achiral, so is their connected sum. If follows that if K is the connected sum of prime ϵ-achiral knots and ϵ-squares of prime knots, then K is ϵ-achiral. The following theorem shows that the converse is also true. Notice that connected sum of links is not well defined, so the result does not apply to links.

Theorem 3.4. *A knot $K \subset S^3$ is ϵ-achiral if and only if it is the connected sum of prime ϵ-achiral knots and ϵ-squares of prime knots.*

Proof. Suppose the oriented knot K has a decomposition $K \cong m_1K_1\# \dots \#m_nK_n$, where the knots $K_1, \dots, K_n$ are prime and all distinct, and the natural numbers $m_1, \dots, m_n$ are the multiplicities. Since K is ϵ-achiral, we have $K^* \cong \epsilon K$, or

$$m_1K_1^*\# \dots \#m_nK_n^* \cong m_1(\epsilon K_1)\# \dots \#m_n(\epsilon K_n).$$

Note that the knots K_i^*'s are also prime and all distinct, and so are the (ϵK_i)'s. By the uniqueness of prime decomposition of knots, for each index i there is a unique index j such that $K_i^* \cong \epsilon K_j$, and $m_i = m_j$. Clearly this relation is symmetric: $K_i^* \cong \epsilon K_j$ if and only if $K_j^* \cong \epsilon K_i$. Thus, an index i either is self-related, or is paired with another index j. In the former case, K_i is ϵ-achiral. In the latter, the pair $K_i\#K_j$ is an ϵ-square. Hence the theorem. □

Example 3.5. (1) Let L_1, L_2 be the knots in solid tori shown in Figure 2.3. If K is ϵ-achiral, then by Example 2.8 and Theorem 3.1 we see that $L_1(K)$ is $(-\epsilon)$-achiral and $L_2(K)$ is ϵ-achiral. In particular, if K is the figure 8 knot, which is both positive and negative achiral, then $L_i(K)$ is ϵ-achiral for both $\epsilon = 1$ and -1.

(2) Since the figure 8 knot K is (± 1)-achiral, by Corollary 3.3 its Bing double $B(K)$ is ϵ-achiral for all $\epsilon = (\epsilon_1, \epsilon_2)$. By Corollary 3.3, the twisted Bing double $B^r(K)$ of a nontrivial knot K is always absolutely chiral when $r \neq 0$. It is easy to check that this is also true when K is trivial.

(3) Let J be the Hopf link. The Borromean rings are the Bing double $B(J)$. Since J is clearly ϵ-achiral for $\epsilon = (1,-1)$ and $(-1,1)$, it follows from Corollary 3.3 that $B(J)$ is ϵ'-achiral if and only if $\pi(\epsilon') = 1$, i.e., $\epsilon' = (1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$ or $(1,-1,-1)$.

(4) After Bing doubling both components of the Hopf link J, we get a link $B(K_1) \cup B(K_2)$. Since $K_1 \cup B(K_2)$ is not $(-1,1,1)$-achiral by (3), it follows from Corollary 3.3 that $B(K_1) \cup B(K_2)$ is not (positive) achiral. It can be shown that $B(K_1) \cup B(K_2)$ is ϵ-achiral if and only if $\pi(\epsilon) = -1$. More generally, if L is obtained from the Hopf link by Bing doubling n times (along any components), then it is ϵ-achiral if and only if $\pi(\epsilon) = (-1)^{n+1}$.

(5) Let $J = K_1 \cup K_2$ be the Hopf link. Since Whitehead link is of the form $W(K_1) \cup K_2$, by Corollary 3.2(2) it is absolutely chiral.

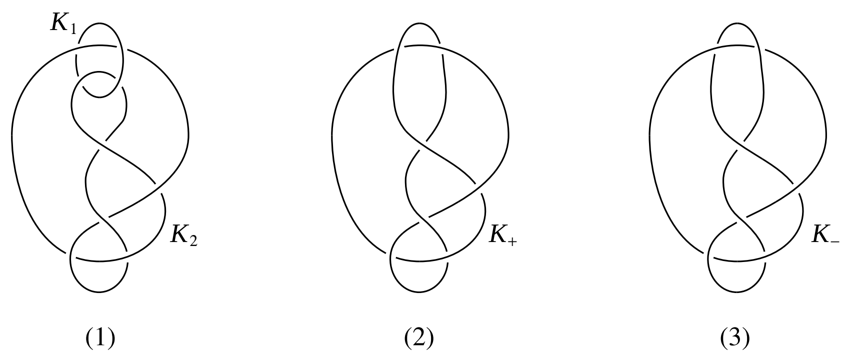


Figure 3.1

(6) Let $J = K_1 \cup K_2$ be the link 8^2_{10} in the table of [R], and let K_1 be the unknotted component. See Figure 3.1 (1). After ± 1 surgery on K_1 the knot K_2 becomes K_+ and K_- in S^3 as shown in Figure 3.1 (2) and (3), which are the knots 6_3 and 7_7 in the knot table of [R]. By Lemma 2.3(1), L is absolutely chiral.

§4. The η-functions

The η-functions $\eta_i(L;t)$ are defined by Kojima and Yamasaki [KY] for two component links $L = K_1 \cup K_2$ with zero linking number. In this section we will show that these functions detect chirality of the link (Theorem 4.2). We will also prove a crossing change formula for these functions, which makes the calculation of $\eta_1(L;t)$ very easy when K_2 is null-homotopic in the complement of K_1. See Theorem 4.5 and Corollary 4.6 below.

In the remaining part of the paper, we always assume that $L = K_1 \cup K_2$ is an oriented link in S^3 with linking number zero. Let $X = S^3 - K_1$, and let $p : \tilde{X} \to X$ be the infinite cyclic covering. Then $H_1(\tilde{X}, \mathbb{Z})$ is a $\mathbb{Z}[t^{\pm 1}]$-module, where t generates the group of deck transformations of the covering $p : \tilde{X} \to X$. It is a $\mathbb{Z}[t^{\pm 1}]$ torsion module since the Alexander polynomial $\Delta_{K_1}(t)$ of K_1 is not zero.

Let l_2 be the preferred longitude of K_2, $\mathrm{lk}(K_2, l_2) = 0$ by definition. Fix a lifting $\tilde{K}_2$ of K_2 in $\tilde{X}$, and let $\tilde{l}_2$ be the lifting of l_2 which is a longitude of $\tilde{K}_2$. There is a Laurent polynomial $f(t)$ such that $[f(t)\tilde{l}_2] = 0 \in H_1(\tilde{X}, \mathbb{Z})$ (e.g., we may choose $f(t) = \Delta_{K_1}(t)$). Let ξ be a 2-chain in $\tilde{X}$ such that $\partial\xi = f(t)\tilde{l}_2$. Define

$$\eta_1(L; t) = \frac{1}{f(t)} \sum_{n=-\infty}^{+\infty} I(\xi, t^n \tilde{K}_2) t^n, \tag{4.1}$$

where $I(\, , \,)$ stands for algebraic intersection number. This is the η-function defined by Kojima and Yamasaki in [KY]. Similarly, we may define $\eta_2(L; t)$ using the liftings of K_1 in the infinite cyclic covering of $S^3 - K_2$. In general, $\eta_1(L; t) \neq \eta_2(L; t)$. The following theorem was proved in [KY].

Theorem 4.1. *(1) $\eta_1(L; t)$ is well-defined (independent of the choice of $\tilde{K}_2$, $f(t)$, and ξ);*
(2) $\eta_1(L; t) = \eta_1(L; t^{-1})$;
(3) $\eta_1(L; 1) = 0$;
(4) $\eta_1(L; t)$ is independent of the orientation of the components of L.

We may use the η-function to detect the achirality of links with zero linking number.

Theorem 4.2. *Suppose $L = K_1 \cup K_2$ is an oriented link with zero linking number. If $\eta_1(L; t) \neq 0$, then L is absolutely chiral. Moreover if $\eta_1(L; t) \neq \eta_2(L; t)$, then L is absolutely set-wise chiral.*

Proof. Suppose that L has an (ϵ_1, ϵ_2)-achiral map $h : S^3 \to S^3$. Up to isotopy we may assume that h maps a regular neighborhood of K to itself, so it maps the preferred longitude ℓ_2 of K_2 to $\epsilon_2 \ell_2 + p m_2$ for some p. By Lemma 1.1, we have $\epsilon_2 p = \mathrm{lk}(\epsilon_2 \ell_2 + p m_2, \epsilon_2 K_2) = \mathrm{lk}(h(\ell_2), h(K_2)) = -\mathrm{lk}(\ell_2, K_2) = 0$. Hence up to isotopy we may assume $h(K_2) = \epsilon_2 K_2$ and $h(\ell_2) = \epsilon_2 \ell_2$.

Let $\tilde{K}_2$, $\tilde{\ell}_2$ be the liftings of K_2 and ℓ_2 in the definition of $\eta_1(L; t)$. Let $\tilde{h} : \tilde{X} \to \tilde{X}$ be the lifting of h such that $\tilde{h}(\tilde{K}_2) = \epsilon_2 \tilde{K}_2$ and $\tilde{h}(\tilde{\ell}_2) = \epsilon_2 \tilde{\ell}_2$.

The deck transformation group of $\tilde{X}$ is naturally isomorphic to the first homology group of the complement of K_1, so the action of $\tilde{h}$ on $\tilde{X}$ is completely determined by the action of h on the meridian μ_1 of K_1. Since h sends μ_1 to $-\epsilon_1 \mu_1$, we have $\tilde{h} \circ t = t^{-\epsilon_1} \circ \tilde{h}$ on $\tilde{X}$.

Choose ξ such that $\partial\xi = f(t)\tilde{\ell}_2$, where $f(t)$ is the Alexander polynomial of K_1. Put $\eta_1(L; t) = \sum a_n t^n$. By the definition of linking number at the end of §1, we have $a_n = I(\xi, t^n \tilde{K}_2) = \mathrm{lk}(\partial\xi, t^n \tilde{K}_2) = \mathrm{lk}(f(t)\tilde{\ell}_2, t^n \tilde{K}_2)$. Also, we have

$$\begin{aligned}
\mathrm{lk}(f(t)\tilde{\ell}_2, t^n \tilde{K}_2) &= -\mathrm{lk}(\tilde{h}(f(t)\tilde{\ell}_2), \tilde{h}(t^n \tilde{K}_2)) \\
&= -\mathrm{lk}(f(t^{-\epsilon_1})\tilde{h}(\tilde{\ell}_2), t^{-\epsilon_1 n}\tilde{h}(\tilde{K}_2)) \\
&= -\mathrm{lk}(\epsilon_2 f(t^{-\epsilon_1})\tilde{\ell}_2, \epsilon_2 t^{-\epsilon_1 n}\tilde{K}_2) \\
&= -\mathrm{lk}(f(t)\tilde{\ell}_2, t^{-\epsilon_1 n}\tilde{K}_2),
\end{aligned}$$

where the first equality follows from Lemma 1.1, the second from $\tilde{h} \circ t = t^{-\epsilon_1} \circ \tilde{h}$, and the fourth from the fact that $f(t) = f(t^{-1})$. Therefore $a_n = -a_{-\epsilon_1 n}$, which is equal to $-a_n$ by Theorem 4.1(2). Thus $a_n = 0$ for all n, hence $\eta_1(L;t) = 0$.

If $\eta_1(L;t) \neq \eta_2(L;t)$, then at least one η-function not zero, hence L is absolutely chiral. Moreover there is no homeomorphism to exchange the two components, so L is absolutely set-wise chiral. □

General calculation of $\eta_1(L,t)$ is a little complicated, and will be discussed in the next section. When K_2 is null-homotopic in $S^3 - K_1$, however, the calculation is much simpler.

Lemma 4.3. *If K_2 is null-homotopic in $X = S^3 - K_1$, then*

$$\eta_1(L;t) = \sum_{n=-\infty}^{+\infty} \mathrm{lk}(\tilde{l}_2, t^n \tilde{K}_2) t^n = \sum_{1}^{\infty} \mathrm{lk}(\tilde{l}_2, t^n \tilde{K}_n)(t^n + t^{-n} - 2).$$

Proof. In this case the lifting $\tilde{K}_2$ of K_2 in the universal abelian cover $\tilde{X}$ of X is also null-homotopic, hence null-homologous, so we can choose $f(t) = 1$, and the intersection number $I(\xi, t^n \tilde{K}_2)$ becomes the linking number $\mathrm{lk}(\tilde{l}_2, t^n \tilde{K}_2)$ in $\tilde{X}$, hence the first equality follows from the definition of the η-function. The second equality follows from Theorem 4.1(2)–(3). □

When K_1 is a trivial knot, it is easy to draw the diagram of the liftings of K_2 in $\tilde{X} = R^1 \times D^2$, from which one can easily read off the η-function $\eta_1(L;t)$.

Example 4.4. For the Whitehead link L in Figure 4.1(1), we have

$$\eta_1(L;t) = \eta_2(L;t) = 2 - t - t^{-1}.$$

We did not draw the longitude of K_2, but one can check that $\mathrm{lk}(\tilde{l}_2, \tilde{K}_2) = 2$, which also follows from the fact that $\eta_1(L;1) = 0$.

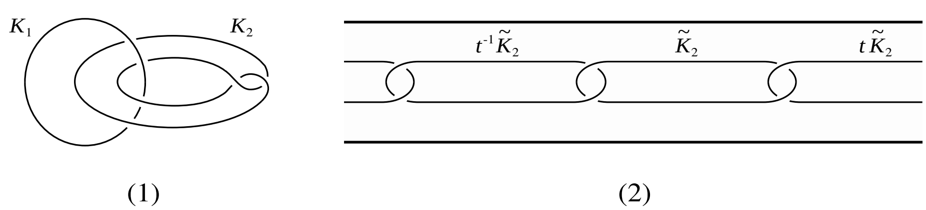


Figure 4.1

The following theorem gives a crossing change formula for $\eta_1(L;t)$, which is due originally to G.T. Jin [J1] but is never published.

Let c be a crossing of K_2, and let K^+, K^- and K^0 be three diagrams which differ only at the crossing c shown in Figure 4.2. (Thus $K_2 = K^+$ or K^-.) We say that K^0 is obtained from K_2 by smoothing the crossing c. Note that K^0 has two components. Let $n = n(c)$ be the absolute value of the linking number between K_1 and one component of K^0. Denote by $\mathrm{sign}(c)$ the sign of the crossing c.

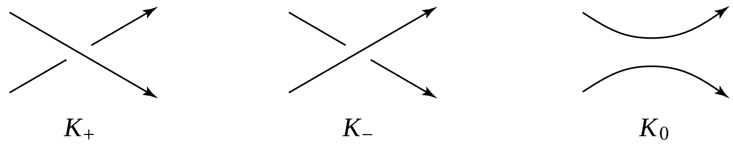


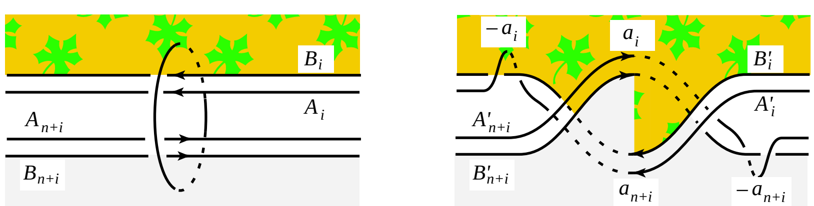
Figure 4.2

Theorem 4.5. *Suppose K' is obtained from K_2 by switching a crossing c. Then*

$$\eta_1(K_1 \cup K_2; t) - \eta_1(K_1 \cup K'; t) = \operatorname{sign}(c)(t^{n(c)} + t^{-n(c)} - 2). \tag{4.1}$$

Proof. Let $p : \tilde{X} \to X = S^3 - K_1$ be the infinite cyclic covering. As before, denote by l_2 the preferred longitude of K_2. Let $\tilde{K}_2$ and $\tilde{l}_2$ be fixed liftings of K_2 and l_2 respectively, such that $\tilde{l}_2$ is a longitude of $\tilde{K}_2$. For simplicity put $A_i = t^i \tilde{K}_2$, and $B_i = t^i \tilde{l}_2$.

Let γ be a small circle around the crossing c such that $\operatorname{lk}(\gamma, K_2) = 0$. Then K' is obtained from K_2 by an $\epsilon = -\operatorname{sign}(c)$ surgery on γ. Note that the preferred longitude l_2 of K_2 becomes the preferred longitude l_2' of K'. Fixing a component of $p^{-1}(\gamma)$ as $\tilde{\gamma}_0$, and put $\tilde{\gamma}_i = t^i \tilde{\gamma}_0$. The liftings of K' and l_2', denoted by A_i' and B_i' respectively, can be obtained from A_i and B_i by performing an ϵ surgery on each $\tilde{\gamma}_j$. Since a component of the link K^0 obtained by smoothing K_2 at c has linking number n with K_1, it lifts to a path connecting a base point x on A_0 to $t^n(x)$ on A_n, so c "lifts" to crossings $\tilde{c}_i$ between A_i and A_{n+i}. Choose $\tilde{\gamma}_0$ to be the one around $\tilde{c}_0$. Then the two components linked with $\tilde{\gamma}_i$ are A_i and A_{n+i}. See Figure 4.3 (1).



(1) (2)

Figure 4.3

Assume $f(t) = \sum a_i t^i$. As in the definition of the η-function, let ξ be a 2-chain with $\partial \xi = \sum a_i t^i \tilde{l}_2 = \sum a_i B_i$. We may choose ξ such that in a neighborhood of B_i the 2-chain ξ consists of a_i copies of annuli. (If $a_i < 0$, the annuli have opposite orientation to that induced by B_i.)

We first analyze the effect of the ϵ surgery on a single $\tilde{\gamma}_i$. After an ϵ surgery on $\tilde{\gamma}_i$, A_j and B_j become A_j' and B_j' respectively. The 2-chain ξ can be modified locally to a 2-chain ξ' such that $\partial \xi' = \sum a_i B_i'$. See Figure 4.3 (2) for the case $\epsilon = 1$.

One can see that

$$I(\xi', A_i') = I(\xi, A_i) + \epsilon(-a_i + a_{n+i});$$
$$I(\xi', A_{n+i}') = I(\xi, A_{n+i}) + \epsilon(a_i - a_{n+i});$$
$$I(\xi', A_j') = I(\xi, A_j), \qquad j \neq i, n+i.$$

Therefore,

$$\sum_{-\infty}^{\infty} I(\xi', A_j')t^j = \sum_{-\infty}^{\infty} I(\xi, A_j)t^j + \epsilon(-a_i + a_{n+i})t^i + \epsilon(a_i - a_{n+i})t^{n+i}.$$

After performing ϵ-surgery on all $\tilde{\gamma}_i$, we get

$$\begin{aligned}\sum_{-\infty}^{\infty} I(\xi, A_j)t^j - \sum_{-\infty}^{\infty} I(\xi', A_j')t^j &= -\epsilon\sum_{-\infty}^{\infty}[(-a_i + a_{n+i})t^i + (a_i - a_{n+i})t^{n+i}]\\ &= \mathrm{sign}(c)[-\sum_{-\infty}^{\infty} a_i t^i + (\sum_{-\infty}^{\infty} a_i t^i)t^{-n} + (\sum_{-\infty}^{\infty} a_i t_i)t^n - \sum_{-\infty}^{\infty} a_i t^i]\\ &= \mathrm{sign}(c)(t^n + t^{-n} - 2)f(t).\end{aligned}$$

The result now follows by dividing both sides by $f(t)$. □

The η-function is trivial if K_2 is separated from K_1, that is, it is contained in a ball disjoint from K_1. The following corollary follows from Theorem 4.5 by induction.

Corollary 4.6. *Suppose there is a sequence of crossings $c_1, ..., c_m$ of K_2 such that after switching these crossings K_2 can be separated from K_1. Then*

$$\eta_1(L;t) = \sum_{i=1}^{m} \mathrm{sign}(c_i)(t^{n(c_i)} + t^{-n(c_i)} - 2).$$

Example 4.7. Let L be the link 9^2_{37} shown in Figure 4.4. Then K_2 can be separated from K_1 by switching the crossings c_1, c_2 in the figure. We have $n(c_1) = n(c_2) = 1$, and both c_1 and c_2 are negative crossings. Therefore by Corollary 4.6 we have

$$\eta_1(L;t) = \mathrm{sign}(c_1)(t + t^{-1} - 2) + \mathrm{sign}(c_2)(t + t^{-1} - 2) = 4 - 2t - 2t^{-1}.$$

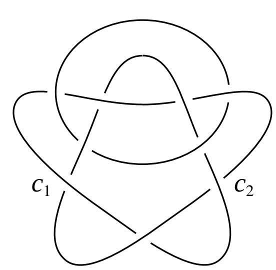


Figure 4.4

Theorem 4.8. *All non-trivial prime links L with at least two components and at most 9 crossings are positive and negative chiral, except the Borromean rings.*

Proof. All links up to nine crossings are listed on pages 416–429 of [R]. Since a prime knot with 3 or 5 crossings is a twisted Whitehead double of the trivial knot and is not the figure eight knot, by Corollary 3.2 it is absolutely chiral. By Example 3.5(5) or 4.4, the Whitehead link is absolutely chiral. Notice that a link containing a chiral sublink is chiral. Thus, if L contains a sublink which is either a Whitehead link or is a knot with 3 or 5 crossings, then L is absolutely chiral; if L contains a sublink which has nonzero linking number, by Lemma 1.1 it is positive and negative chiral. Excluding all these cases and the Borromean rings 6^3_2, the remaining links in the table are: 7^2_3, 8^2_{10}, 8^2_{13}, 8^2_{15}, 9^2_4, 9^2_5, 9^2_9, 9^2_{10}, 9^2_{32}, and 9^2_{37}. Every link in this list has an unknotted component K_1. Thus $\eta_1(t)$ is a polynomial.

To simplify the notation, let us denote by $\eta_1^+(t)$ the sum of the terms of $\eta_1(t)$ with positive t power. Theorem 4.1(2)–(3) shows that $\eta_1(t)$ is completely determined by $\eta_1^+(t)$. One can check that $\eta_1^+(L;t) \neq 0$ for all links in the list except 8^2_{10}. More explicitly, the polynomial $\eta_1^+(t)$ is $-2t$ for 7^2_3, 0 for 8^2_{10}, $-t$ for 8^2_{13}, t for 8^2_{15}, 9^2_5 and 9^2_{32}, $-t-t^2$ for 9^2_4, $-t+t^2$ for 9^2_9, $-3t$ for 9^2_{10}, and $-2t$ for 9^2_{37}. By Theorem 4.2, all links in this list are absolutely chiral except 8^2_{10}. But the link 8^2_{10} is also absolutely chiral by Example 3.5(6). This completes the proof of the theorem. □

§5. Calculation of the η-functions

Because of Theorem 4.2, it becomes important that we have a procedure for effective calculation of the η-function from a link diagram.

The crossing change formula of Theorem 4.5 is useful in calculating the η-functions for those links L such that K_2 is null-homotopic in $S^3 - K_1$. However, Corollary 4.6 fails when K_2 is not null-homotopic in $S^3 - K_1$. By [KY], $\eta_1(L;t)$ can be expressed in terms of Alexander polynomials (see also [J2]), which in turn can be written as the determinant of some matrices. We will look into this procedure of calculating the η-function described in [KY] more specifically and this allows us to determine the η-function *exactly*. Then, once we have reduced the knot K_2 using the crossing change formula in Theorem 4.5 to the unknot, we can express $\eta_1(K_1 \cup K_2;t)$ in terms of the Conway polynomial of K_1 and that of another knot K_1^+, which is obtained from K_1 by performing (+1) surgery on the unknotted K_2. See Theorem 5.5 and the remark which follows.

Finally, in Theorem 5.7, we observe that the crossing change formula in Theorem 4.5 leads to crossing changes formulas for Cochran's derived invariants.

Denote by M_L the 3-manifold obtained from 0-framing surgery on $L = K_1 \cup K_2$. Let $\tilde{M}_L$ be the infinite cyclic covering space of M_L with respect to the meridian of K_1. $H_1(\tilde{M}_L)$ is also a $\mathbb{Z}[t^{\pm 1}]$ module and we denote by $\Delta_1(\tilde{M}_L;t)$ its Alexander polynomial. The following theorem is from [KY].

Theorem 5.1. *We have*

$$\eta_1(L;t) \sim \frac{\Delta_1(\tilde{M}_L;t)}{\Delta_{K_1}(t)}, \tag{5.1}$$

where $\sim$ means that the two sides differ by a factor of a unit in $\mathbb{Z}[t^{\pm 1}]$.

Let us recall the procedure for the calculation of $\Delta_1(\tilde{M}_L;t)$ (and $\Delta_{K_1}(t)$) described in [KY]. Here, we are more specific about orientations than in [KY]. The payoff turns out to be quite pleasant.

The first step is to find a system of oriented unknotting circles for K_1, say $\{T_1, ..., T_m\}$, such that $\mathrm{lk}(T_i, K_1) = 0$ and performing an ϵ_i-surgery on every T_i, where $\epsilon_i = \pm 1$, will change K_1 to an unknot K_1'. Let $\mu_1, ..., \mu_m$ be the meridians of $T_1, ..., T_m$, respectively, oriented so that $\mathrm{lk}(\mu_i, T_i) = 1$. The ϵ_i-surgery on T_i changes μ_i to μ_i'. By the choice of the orientation of μ_i, after the surgery μ_i' is isotopic to $-\epsilon_i T_i$ as oriented knots in $N(T_i)$. See Figure 5.1, in which (1) and (2) illustrate a (-1)-surgery and (3) and (4) illustrate a $(+1)$-surgery. To uniformize the notation, put $T_0 = K_2$, $\mu_0' = l_2$, and $\epsilon_0 = -1$.

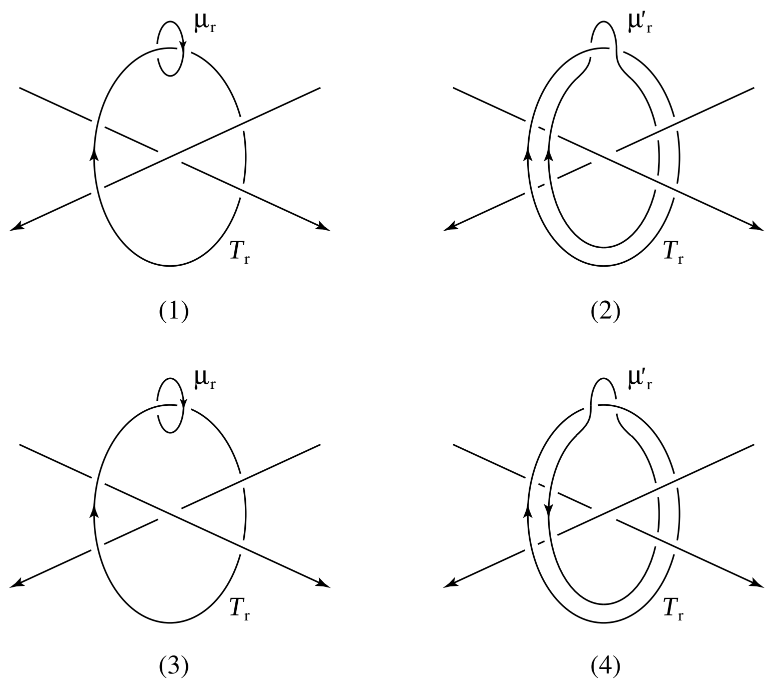


Figure 5.1

The infinite covering of $S^3 - K_1'$ is now homeomorphic to R^3. We fix liftings of $T_0, ..., T_m, \mu_0', ..., \mu_m'$, say $\tilde{T}_0, ..., \tilde{T}_m, \tilde{\mu}_0', ..., \tilde{\mu}_m'$, such that $\tilde{\mu}_r'$ is a parallel of $-\epsilon_r \tilde{T}_r$. Define

$$d_{rs}(t) = \sum_{n=-\infty}^{+\infty} \mathrm{lk}(\tilde{\mu}_r', t^n \tilde{T}_s) t^n, \qquad r, s = 0, \dots, m. \tag{5.2}$$

Clearly we have $d_{sr}(t) = \epsilon_r \epsilon_s d_{rs}(t^{-1})$. It was shown in [KY] that

$$\Delta_1(\tilde{M}_L; t) \sim \det \begin{pmatrix} d_{00} & d_{01} & d_{02} & \dots & d_{0m} \\ d_{10} & d_{11} & d_{12} & \dots & d_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ d_{m0} & d_{m1} & d_{m2} & \dots & d_{mm} \end{pmatrix} \tag{5.3}$$

and

$$\Delta_{K_1}(t) \sim \det \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \dots & d_{mm} \end{pmatrix} \tag{5.4}$$

Lemma 5.2. *We have*

$$\eta_1(L;t) = \frac{\det \tilde{\mathcal{A}}}{\det \mathcal{A}} \tag{5.5}$$

where $\tilde{\mathcal{A}}$ and $\mathcal{A}$ are the matrices in (5.3) and (5.4), respectively.

Proof. This is in fact what the proof of Theorem 5.1 in [KY] actually shows. Essentially, we may choose $\det \mathcal{A}$ to be the Laurent polynomial $f(t)$ used in the definition of $\eta_1(L;t)$ to annihilate the lifting $\tilde{l}_2$. □

The definition (5.2) depends on the orientations of T_i and the liftings $\tilde{T}_i$. But the determinants $\det \mathcal{A}$ and $\det \tilde{\mathcal{A}}$ are independent of these choices. The orientations of the meridians μ_i (relative to that of T_i), which were left unspecified in [KY], may affect the sign of $\det \mathcal{A}$ and $\det \tilde{\mathcal{A}}$ simultaneously. We have specified them explicitly, in order to make $\det \mathcal{A}$ properly normalized.

Recall that the Conway polynomial $\nabla_K(z)$ is given by the substitution $z = t^{1/2} - t^{-1/2}$ in the normalization of the Alexander polynomial satisfying $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$.

Lemma 5.3. *For the matrix $\mathcal{A} = \mathcal{A}(t)$ in (5.4),*

$$\nabla_{K_1}(t^{1/2} - t^{-1/2}) = \det \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \dots & d_{mm} \end{pmatrix}. \tag{5.6}$$

This is a surgery description of the Conway polynomial $\nabla_{K_1}(z)$.

Proof. We have $\det \mathcal{A}(t) = \det \mathcal{A}(t^{-1})$. Also, since $d_{rs}(1) = \delta_{rs}$ for $r, s = 1, 2, \dots, m$, we have $\det \mathcal{A}(1) = 1$. Thus, (5.6) holds. □

We now consider the entries of matrices in (5.3) and (5.4). Let $J \cup J'$ be a link in the complement of the unknot K_1', where both J and J' has zero linking number with K_1', and with liftings $\tilde{J}$ and $\tilde{J}'$, respectively, chosen in the infinite cyclic covering of $S^3 - K_1'$. Define the *linking polynomial*:

$$\Lambda(J, J'; t) = \sum_{n=-\infty}^{+\infty} \mathrm{lk}(\tilde{J}, t^n \tilde{J}') t^n.$$

The entries of the above determinants can be expressed as η-functions and linking polynomials.

Lemma 5.4. *We have*

$$d_{rs}(t) = \begin{cases} \eta_1(K_1' \cup T_0; t), & r = s = 0; \\ 1 - \epsilon_r \eta_1(K_1' \cup T_r; t), & r = s \neq 0; \\ -\epsilon_r \Lambda(T_r, T_s; t), & r \neq s. \end{cases} \tag{5.7}$$

Proof. When $r = s = 0$, this follows from Lemma 4.3 and the definition. When $r = s \neq 0$, let l_r be the preferred longitude of T_r, and let $\tilde{l}_r$ be the lifting of l_r which is a longitude of $\tilde{T}_r$. Since $\mathrm{lk}(\mu'_r, T_r) = 1$, we have $\mathrm{lk}(\tilde{\mu}'_r, \tilde{T}_r) = 1 - \epsilon_r \mathrm{lk}(\tilde{l}_r, \tilde{T}_r)$. Also, when $n \neq 0$, $\mathrm{lk}(\tilde{\mu}'_r, t^n\tilde{T}_r) = -\epsilon_r \mathrm{lk}(\tilde{l}_r, t^n\tilde{T}_r)$. Hence

$$\begin{aligned} d_{rr}(t) &= \sum_{-\infty}^{+\infty} \mathrm{lk}(\tilde{\mu}'_r, t^n\tilde{T}_r)t^n \\ &= 1 - \epsilon_r \sum_{-\infty}^{+\infty} \mathrm{lk}(\tilde{l}_r, t^n\tilde{T}_r)t^n \\ &= 1 - \epsilon_r \eta_1(K'_1 \cup T_r; t). \end{aligned}$$

When $r \neq s$, we have $\mathrm{lk}(\tilde{\mu}'_r, \tilde{T}_s) = -\epsilon_r \mathrm{lk}(\tilde{T}_r, \tilde{T}_s)$, so the result follows from definition. □

Let us now consider the special case when K_2 is the unknot. Let K_1^+ be the knot obtained from K_1 by performing a $(+1)$-surgery on K_2. Recall that $\nabla_K(z)$ denotes the Conway polynomial of K.

Theorem 5.5. *For the link $L = K_1 \cup K_2$ with zero linking number and the unknotted component K_2, we have*

$$\eta_1(L;t) = \frac{\nabla_{K_1^+}(t^{1/2} - t^{-1/2})}{\nabla_{K_1}(t^{1/2} - t^{-1/2})} - 1. \tag{5.8}$$

Proof. We will keep the notation in the discussion after Theorem 5.1. So, ϵ_r-surgery on T_r for $r = 1, \dots, m$ will change K_1 to the unknot K'_1. And by construction K_1^+ is changed by an ϵ_0-surgery on $K_2 = T_0$, with $\epsilon_0 = -1$, to K_1. Thus, ϵ_r-surgery on T_r for $r = 0, 1, \dots, m$ will change K_1^+ to the unknot K'_1. So, by (5.6),

$$\nabla_{K_1^+}(t^{1/2} - t^{-1/2}) = \det \begin{pmatrix} d_{00}^+ & d_{01} & d_{02} & \dots & d_{0m} \\ d_{10} & d_{11} & d_{12} & \dots & d_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ d_{m0} & d_{m1} & d_{m2} & \dots & d_{mm} \end{pmatrix}$$

where

$$d_{00}^+ = 1 + \eta_1(K'_1 \cup T_0; t).$$

By (5.5), (5.6) and (5.7), we have

$$\frac{\nabla_{K_1^+}(t^{1/2} - t^{-1/2})}{\nabla_{K_1}(t^{1/2} - t^{-1/2})} = 1 + \frac{\det \tilde{\mathcal{A}}}{\det \mathcal{A}} = 1 + \eta_1(L;t).$$

This finishes the proof. □

Remark. We may think of (5.8) as giving us the initial value in the calculation of the η-function using the crossing change formula (4.1). Thus, Theorem 4.5 and Theorem 5.5 combined give us a simple procedure for the calculation of the η-function. See the example at the end of this section for an illustration.

Finally, let us mention the so-called derived invariants. By Theorem 4.1 (1) and (2), we may write $\eta_1(L;t)$ as a power series in $w = 2 - t - t^{-1} = -z^2$:

$$\eta_1(K_1 \cup K_2; t) = \sum_{k=1}^{\infty} \beta_k(K_1, K_2)\, w^k.$$

The sequences of invariants $\beta_k(K_1, K_2)$ are Cochran's *derived invariants* [C]. By Theorem 4.2, if some $\beta_k(K_1, K_2)$ is nontrivial then L is absolutely chiral.

We will call

$$C_1(K_1 \cup K_2; w) = \eta_1(K_1 \cup K_2; t) = \sum_{k=1}^{\infty} \beta_k(K_1, K_2) w^k$$

with $w = -z^2 = 2 - t - t^{-1}$ the *Cochran function.* Theorem 5.5 can be rewritten in the following form.

Corollary 5.6. *For the link $L = K_1 \cup K_2$ with zero linking number and unknotted component K_2, we have*

$$C_1(L; w) = \frac{\nabla_{K_1^+}(z)}{\nabla_{K_1}(z)} - 1$$

with $w = -z^2$.

In general, $\beta_k(K_1, K_2) \neq \beta_k(K_2, K_1)$. But $\beta_1(K_1, K_2) = \beta_1(K_2, K_1)$ and it is known to be equal to the *Sato-Levine invariant* $\beta(K_1, K_2)$. The Sato-Levine invariant is determined by the crossing change formula

$$\beta(K_1, K_2^+) - \beta(K_1, K_2^-) = -n^2$$

where, as before, n is the absolute value of the linking number of a component of K_2^0 with K_1. This was first obtained in [J1] and published in [J3]. The following result generalizes this crossing change formula to all derived invariants and, combined with Corollary 5.6, gives a simple algorithm to calculate the derived invariants.

Theorem 5.7. *Let K^+, K^- be the knots in the complement of K_1 which differ only at a crossing c as shown in Figure 4.2. Assume $\operatorname{lk}(K_1, K^{\pm}) = 0$ and let $n = n(c)$ be the absolute value of the linking number between K_1 and one component of K^0 (also as in Figure 4.2). Then*

$$\beta_k(K_1, K^+) - \beta_k(K_1, K^-) = \frac{(-1)^k n}{k} \binom{n+k-1}{2k-1}.$$

Proof. Write $t^n + t^{-n} - 2 = \sum a_k w^k$. Then by Theorem 4.5 and the definition of $\beta_k(L)$, we have $\beta_k(K_1, K^+) - \beta_k(K_1, K^-) = a_k$. Also,

$$\begin{aligned} a_k &= \left.\frac{t^n + t^{-n} - 2 - (a_1 w + ... + a_{k-1} w^{k-1})}{w^k}\right|_{w=0} \\ &= (-1)^k \left.\frac{t^{n+k} + t^{-n+k} - 2t^k - t^k(a_1 w + ... a_{k-1} w^{k-1})}{(t-1)^{2k}}\right|_{t=1} \\ &= (-1)^k \frac{(n+k)...(n-k+1) + (-n+k)...(-n-k+1)}{(2k)!} \\ &= (-1)^k \frac{(n+k-1)...(n-k+1)((n+k)+(n-k))}{(2k)!} \\ &= (-1)^k \frac{n}{k} \frac{(n+k-1)...(n-k+1)}{(2k-1)!}. \end{aligned}$$

The third equality follows by taking the limit as t approaches 1, and using the l'Hospital's rule $2k$ times. □

We finish this section by giving an example to show how to use Theorems 4.5 and 5.5 (Corollary 5.6 and Theorem 5.7) to calculate the η-function (the derived invariants)

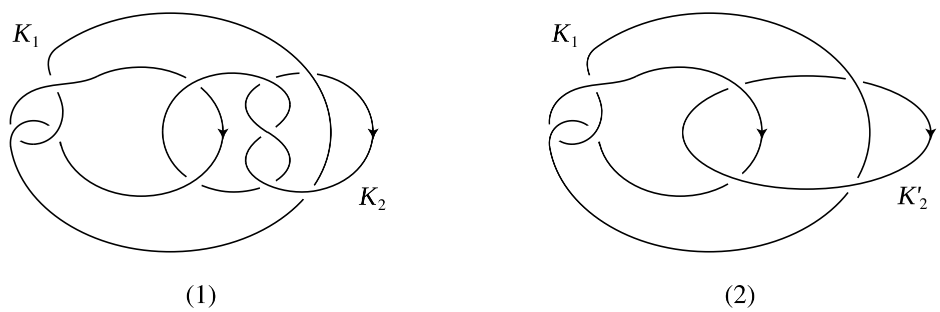


Figure 5.2

Example 5.8. We use the link $L = K_1 \cup K_2$ in Figure 5.2(1) to illustrate the calculation of $\eta_1(L;t)$ and its derived invariants. In Figure 5.2 (1), the knot K_2 is not null-homotopic in $S^3 - K_1$, so one cannot use Corollary 4.6 to calculate $\eta_1(L;t)$. Instead, we proceed as follows.

First, change K_2 by a negative crossing switching to the unknot K_2', as shown in Figure 5.2(2). Secondly, $(+1)$-surgery on K_2' changes K_1 to K_1^+, which is the unknot. The Conway polynomial of K_1 (the trefoil knot) and the unknot are, resepctively, $1 + z^2$ and 1. Thus

$$\eta_1(K_1 \cup K_2'; t) = \frac{1}{1 + (t^{1/2} - t^{-1/2})^2} - 1 = \frac{2 - t - t^{-1}}{t + t^{-1} - 1}.$$

By (4.1), we have

$$\eta_1(L;t) = \eta_1(K_1 \cup K_2'; t) - (t + t^{-1} - 2) = \frac{1}{t + t^{-1} - 1} + 1 - t - t^{-1}.$$

The η-function $\eta_2(L;t)$ can be calculated similarly, and we have $\eta_2(L;t) = \eta_1(L;t)$.

The derived invariants $\beta_k = \beta_k(K_1, K_2)$ can be calculated from $\eta_1(L;t)$. However, it can also be calculated directly using Corollary 5.6 and Theorem 5.7. By Corollary 5.6 we have

$$C_1(K_1, K_2'; w) = \frac{1}{1 - w} - 1 = w + w^2 + w^3 + \dots$$

Since $n = n(c) = 1$, Theorem 5.7 gives

$$\begin{aligned} \beta_k(K_1, K_2) &= \beta_k(K_1, K_2') - \frac{(-1)^k n}{k} \binom{n + k - 1}{2k - 1} \\ &= \begin{cases} 2, & k = 1; \\ 1, & k \neq 1. \end{cases} \end{aligned}$$

References

[BZ] G. Burde and H. Zieschang, *Knots*, de Gruyter Studies in Math. 5, 1985.

[C] T. Cochran, *Geometric invariants of link cobordism*, Comment. Math. Helv. **60** (1985), 291–311.

[JS] W.H. Jaco and P.B. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs of the Amer. Math. Soc. 21, Number 220, 1979.

[JW] B. Jiang and S. Wang, *Achirality and planarity*, preprint.

[J1] G.T. Jin, *Invariants of two-component links*, Ph.D. Thesis, Brandeis University, 1988.

[J2] ———, *On Kojima's η-function*, Differential Topology (Proceedings, Siegen 1987), LNM, vol. 1350, pp. 14–30.

[J3] ———, *The Sato-Levine invariant and Cochran's sequence*, Proceedings of the Topology and Geometry Research Center, Kyungpook National University, Korea, vol. 1, December 1990, pp. 19–36.

[KY] S. Kojima and M. Yamasaki, *Some new invariants of links*, Invent. Math. **54** (1979), 213–228.

[LM] W. Lickorish and K. Millett, *A polynomial invariant of oriented links*, Topology **26** (1987), 107–141.

[NZ] W. Neumann and D. Zagier, *Volume of hyperbolic 3-manifolds*, Topology **24** (1985), 307-332.

[R] D. Rolfsen, *Knots and Links*, Publish or Perish, 1976.

[Th1] W. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University, 1978.

[Th2] ———, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

[Va] V.A. Vassiliev, *Complements of Discriminants of Smooth Maps: Topology and Applications*, Trans. of Math. Mono., vol. 98, Amer. Math. Soc., Providence, 1992..

Department of Mathematics, Peking University, Beijing 100871, P.R. China
E-mail address: `jiangbj@sxx0.math.pku.edu.cn`

Department of Mathematics, University of California Riverside, CA 92521
E-mail address: `xl@math.ucr.edu`

Department of Mathematics, Peking University, Beijing 100871, P.R. China
E-mail address: `swang@sxx0.math.pku.edu.cn`

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: `wu@math.uiowa.edu`