COMMENT. MATH. HELV.
(to appear). Preprint 1999.

ANNULAR DEHN FILLINGS

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Abstract. We show that if a simple 3-manifold M has two Dehn fillings at distance $\Delta \geq 4$, each of which contains an essential annulus, then M is one of three specific 2-component link exteriors in S^3. One of these has such a pair of annular fillings with $\Delta = 5$, and the other two have pairs with $\Delta = 4$.

§1. Introduction

Let M be a (compact, connected, orientable) 3-manifold with a torus boundary component T_0. If r is a *slope* (the isotopy class of an essential unoriented simple loop) on T_0, then as usual we denote by $M(r)$ the 3-manifold obtained from M by *r-Dehn filling*, that is, attaching a solid torus J to M along T_0 in such a way that r bounds a meridian disk in J.

We shall say that a compact, connected, orientable 3-manifold M is *simple* if it contains no essential surface of non-negative Euler characteristic, i.e., sphere, disk, annulus or torus. If M has non-empty boundary and is not the 3-ball, then M is simple if and only if M with its boundary tori removed has a complete hyperbolic structure of finite volume with totally geodesic boundary [Th1, Th2]. If M is closed, then the geometrization conjecture asserts that M is simple if and only if M is either hyperbolic or belongs to a certain small class of Seifert fiber spaces [Th1, Th2].

If M is hyperbolic, then Dehn fillings on M are hyperbolic if we exclude finitely many slopes from each torus boundary component [Th1, Th2]. By doubling M along its non-torus boundary components, we see that if M is simple then $M(r)$ is simple for all but finitely many slopes r on any given torus boundary component T_0, and a good deal of attention has been directed towards obtaining a more precise quantification of this statement. Denote by $\Delta(r_1, r_2)$ the distance, or minimal geometric intersection number, between two slopes r_1, r_2 on a torus. Define a 3-manifold to be of type S, D, A or T if it contains an essential sphere, disk, annulus or torus, respectively. For $X_i \in \{S, D, A, T\}$, $i = 1, 2$, define $\Delta(X_1, X_2)$ to be the maximum of $\Delta(r_1, r_2)$, where r_1 and r_2 are Dehn filling slopes of some simple manifold M such that $M(r_i)$ is of type X_i. These numbers $\Delta(X_1, X_2)$ are now known in all ten cases; see [GW2] for more details.

Except when $(X_1, X_2) = (A, A)$, (A, T) or (T, T), it is also known that $\Delta(X_1, X_2)$ is realized by infinitely many simple manifolds M; see [EW]. On the other hand, $\Delta(T, T) = 8$, and there are exactly two simple manifolds M admitting toroidal

1991 *Mathematics Subject Classification.* Primary 57N10.
Key words and phrases. 3-manifolds, Dehn filling, essential annuli.
[1] Partially supported by NSF grant #DMS 9626550.
[2] Partially supported by NSF grant #DMS 9802558.

fillings $M(r_1)$, $M(r_2)$ with $\Delta = \Delta(r_1, r_2) = 8$, exactly one with $\Delta = 7$, exactly one with $\Delta = 6$, and infinitely many with $\Delta = 5$ [Go]. Similarly, $\Delta(A, T) = 5$ [Go, GW1], and there is exactly one simple manifold M having an annular filling $M(r_1)$ and a toroidal filling $M(r_2)$ with $\Delta = 5$, exactly two with $\Delta = 4$, and infinitely many with $\Delta = 3$ [GW1]. In the present paper we complete the picture by dealing with the case (A, A). In this case, $\Delta(A, A) = 5$ [Go, GW1], and there are infinitely many simple manifolds M admitting annular fillings $M(r_1), M(r_2)$ with $\Delta = 3$ [GW1]. Here we show that there is exactly one such manifold M with $\Delta = 5$, and exactly two with $\Delta = 4$. More precisely, we have the following theorem.

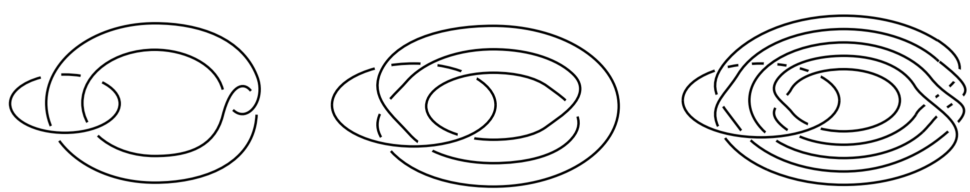
Theorem 1.1. *Suppose M is a compact, connected, orientable, irreducible, ∂-irreducible, anannular 3-manifold which admits two annular Dehn fillings $M(r_1)$, $M(r_2)$ with $\Delta = \Delta(r_1, r_2) \geq 4$. Then one of the following holds.*

(1) M is the exterior of the Whitehead link, and $\Delta = 4$.

(2) M is the exterior of the 2-bridge link associated to the rational number $3/10$, and $\Delta = 4$.

(3) M is the exterior of the $(-2, 3, 8)$ pretzel link, and $\Delta = 5$.

The three manifolds listed in the theorem are the exteriors of the links in S^3 shown in Figure 1.1.



(1) (2) (3)

Figure 1.1

That each of these link exteriors does have a pair of annular fillings with $\Delta = 4$, 4 and 5 respectively is proved in [GW1]. The fillings in question are also toroidal [GW1], so in fact these are exactly the same manifolds which admit an annular and a toroidal filling with $\Delta \geq 4$ [GW1, Theorem 1.1]. Using [GW1, Theorem 1.1], Qiu has independently proved Theorem 1.1 in the special case where M is the exterior of a knot in a solid torus [Q].

According to the proof of [GW1, Theorem 7.5], the annular fillings on the three manifolds listed in Theorem 1.1 are non Seifert fibered graph manifolds. If M admits some Seifert fibered surgery, then ∂M consists of tori, in which case M is hyperbolic if and only if it is simple. Hence the following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2. *Suppose M is a compact orientable hyperbolic 3-manifold with at least two torus boundary components, and suppose $M(r_1), M(r_2)$ are Seifert fibered manifolds. Then $\Delta(r_1, r_2) \leq 3$.*

The condition that M has at least two boundary components cannot be removed. For example, if M is the figure 8 knot complement, then $M(3)$ and $M(-3)$ are

Seifert fibered, and $\Delta(-3,3) = 6$. It is not known whether the bound 3 in the corollary is the best possible.

The proof of Theorem 1.1 proceeds as follows. For $\alpha = 1, 2$, let A_α be an essential annulus in $M(r_\alpha)$, meeting the Dehn filling solid torus J_α in n_α meridian disks, with n_α minimal over all choices of A_α. This gives rise to a punctured annulus $F_\alpha = A_\alpha \cap M$ in M, such that the boundary components of F_α which lie on T_0 have slope r_α, $\alpha = 1, 2$. The arcs of intersection of F_1 and F_2 then define labeled graphs G_α in A_α with n_α vertices, $\alpha = 1, 2$. We assume that $\Delta = \Delta(r_1, r_2) = 4$ or 5, and show by a detailed analysis that there are only three such pairs of graphs, corresponding to the three examples listed in the theorem.

The paper is organized as follows. In Section 2 we give some definitions and establish some basic properties of the graphs G_α. In Section 3 we show that any graph in an annulus with no trivial loops or parallel edges must satisfy one of four possibilities; if the reduced graph $\widehat{G}_\alpha$ is of the fourth type we say that G_α is *special*. Section 4 is devoted to showing that if one of the graphs G_1, G_2 is special then they both are, and (up to relabeling) $n_1 = 1$, $n_2 = 2$. Section 5 considers the generic case, $n_1, n_2 > 2$. This is shown to be impossible, by eliminating in turn the first three possibilities of Section 3 for the reduced graphs $\widehat{G}_\alpha$. Section 6 shows that the case $n_1 = 2$, $n_2 > 2$ is also impossible. In Section 7 we show that if G_1 and G_2 are special, so $n_1 = 1$ and $n_2 = 2$, then there is exactly one possible pair G_1, G_2, with $\Delta = 4$, corresponding to case (1) of Theorem 1.1. Finally in Section 8, we show that if G_1, G_2 are not special and $n_1, n_2 \le 2$, then there are exactly two possible pairs G_1, G_2, one with $\Delta = 4$, $n_1 = n_2 = 2$, and one with $\Delta = 5$, $n_1 = n_2 = 2$, corresponding to cases (2) and (3) of Theorem 1.1.

We would like to thank the referee for his/her careful reading and helpful comments.

§2. Preliminary Lemmas

Throughout this paper, we will always assume that M is a compact, connected, irreducible, ∂-irreducible, anannular 3-manifold, with a torus boundary component T_0. We use α, β to denote the numbers 1 or 2, with the convention that if they both appear, then $\{\alpha, \beta\} = \{1, 2\}$. Let r_1, r_2 be slopes on T_0 such that $M(r_1), M(r_2)$ are annular, and let A_α be an essential annulus in $M(r_\alpha)$ such that n_α, the number of components of intersection of A_α with the Dehn filling solid torus J_α, is minimal among all essential annuli in $M(r_\alpha)$, $\alpha = 1, 2$. Denote by F_α the punctured annulus $A_\alpha \cap M$. Denote by $\Delta = \Delta(r_1, r_2)$ the minimal geometric intersection number between r_1 and r_2. By [Go, Theorem 1.3] we have $\Delta \le 5$. Throughout this paper, we will always assume $\Delta = 4$ or 5, unless otherwise stated.

Minimizing the number of components of $F_1 \cap F_2$ by an isotopy, we may assume that $F_1 \cap F_2$ consists of arcs and circles which are essential on both F_α. Let $u_1, \dots, u_{n_\alpha}$ be the disks that are the components of $A_\alpha \cap J_\alpha$, labeled successively when traveling along J_α. Similarly let $v_1, \dots, v_{n_\beta}$ be the disks in $A_\beta \cap J_\beta$. Let G_α be the graph on A_α with the u_i's as (fat) vertices, and the arc components of $F_1 \cap F_2$ with at least one endpoint on T_0 as edges. Note that we do not regard an edge endpoint on the boundary of the annulus as a vertex, so we are abusing terminology somewhat in that our graphs may have edge endpoints that do not lie on vertices. The minimality of the number of components in $F_1 \cap F_2$ and the minimality of n_α imply that G_α has no trivial loops, and that each disk face of G_α

in A_α has interior disjoint from F_β.

An edge e of a graph G on an annulus A is a *boundary edge* if it has one endpoint on the boundary of A, otherwise it is an *interior edge.* A vertex v of G is a *boundary vertex* if it is incident to a boundary edge, otherwise it is an *interior vertex.* Similarly, a face of G is a *boundary face* if it contains a boundary edge.

If e is an edge of G_α with an endpoint x on a fat vertex u_i, then x is labeled j if x is in $\partial u_i \cap \partial v_j$. When going around the boundary of a vertex in G_α, the labels of the edge endpoints appear as $1, 2, \ldots, n_\beta$ repeated Δ times.

An edge e at a vertex u_i of G_α is called a *j-edge* at u_i if it has an endpoint at u_i labeled j. Dually, a j-edge at u_i is also an i-edge at v_j in G_β. We say that e is an *(i,k)-edge* if it has labels i and k at its two endpoints.

Each vertex of G_α is given a sign according to whether J_α passes A_α from the positive side or negative side at this vertex. Two vertices of G_α are *parallel* if they have the same sign, otherwise they are *antiparallel.* Note that if A_α is a separating surface in $M(r_\alpha)$, then n_α is even, and u_i, u_j are parallel if and only if i, j have the same parity. An interior edge of G_α is a *positive edge* if it connects parallel vertices. Otherwise it is a *negative edge.* We use $\mathrm{val}(v, G)$ to denote the valency of a vertex v in a graph G.

By considering each family of parallel edges of G_α as a single edge E, we get the *reduced graph* $\widehat{G}_\alpha$ on A_α. It has the same vertices as G_α. Denote by $|E|$ the number of edges in G_α represented by E.

A cycle in G_α consisting of positive edges is a *Scharlemann cycle* if it bounds a disk with interior disjoint from the graph, and all the edges in the cycle have the same pair of labels $(i, i+1)$ at their two endpoints. ($i + 1 = 1$ if $i = n_\beta$.) The pair $(i, i+1)$ is called the *label pair* of the Scharlemann cycle. In particular, a pair of adjacent parallel positive edges with the same label pair is a Scharlemann cycle. The boundary of the disk D bounded by a Scharlemann cycle consists of edges of the Scharlemann cycle and some arcs on the annulus C_i on T_0 between ∂v_i and ∂v_{i+1}. When $n_\beta = 2$, the two annuli C_1 and C_2 are still distinct, allowing one to differentiate between a $(1,2)$-Scharlemann cycle and a $(2,1)$-Scharlemann cycle. A pair of edges $\{e_1, e_2\}$ is an *extended Scharlemann cycle* if there is a Scharlemann cycle $\{e_1', e_2'\}$ such that e_i is parallel and adjacent to e_i'.

A subgraph G' of a graph G on a surface F is *essential* if it is not contained in a disk in F.

Lemma 2.1. *(1)* (The Parity Rule) *An edge e is a positive edge in G_1 if and only if it is a negative edge in G_2.*

(2) A pair of edges cannot be parallel on both G_1 and G_2.

(3) If G_α has a set of n_β parallel negative edges, then on G_β they form mutually disjoint essential cycles of equal length.

(4) If G_α has a Scharlemann cycle, then A_β is separating, and n_β is even. Moreover, the edges of the Scharlemann cycle and the vertices at their endpoints form an essential subgraph of G_β.

(5) G_α contains no extended Scharlemann cycle.

Proof. See [GW1, Lemma 2.2], except for (2) in the case that the pair of edges e_1, e_2 are boundary edges. If e_1, e_2 are boundary edges parallel on both G_1, G_2, then they cut off bands B_1, B_2 on the punctured annuli F_1, F_2, which can be glued together to get an annulus in the manifold M, which intersects the Dehn filling torus T_0 in an essential circle. This contradicts the assumption that M is ∂-irreducible and anannular. □

Let E be an edge of $\widehat{G}_\alpha$ representing n_β parallel negative edges on G_α, connecting u_i to u_j. Then E defines a permutation $\varphi : \{1, \dots, n_\beta\} \to \{1, \dots, n_\beta\}$, such that an edge e in E has label k at u_i if and only if it has label $\varphi(k)$ at u_j. Call φ the *permutation associated to E*. Because of the ambiguity in the order of u_i, u_j, the permutation is only well defined up to inverse. An *E-orbit* is an orbit of φ. Such an orbit determines a cycle in G_β consisting of the edges of E with endpoint labels in this orbit, called the cycle of this orbit. Note that all the vertices in a cycle are parallel. Topologically each such cycle is a circle. Lemma 2.1(3) says that these circles are mutually disjoint, mutually parallel, essential circles on the annulus A_β.

Lemma 2.2. *(1) Any two Scharlemann cycles on G_α have the same label pair.*

(2) If E is a positive edge in $\widehat{G}_\alpha$, then $|E| \le n_\beta/2 + 1$. Moreover, if $|E| = n_\beta/2 + 1$, then the corresponding edges of G_α contain a Scharlemann cycle.

(3) Any family of parallel interior edges in G_α contains at most n_β edges.

Proof. See [GW1, Lemma 2.5]. □

Lemma 2.3. *(1) If some vertex of G_α has more than n_β negative edge endpoints, then G_β contains a Scharlemann cycle.*

(2) No vertex of G_α has more than $2n_\beta$ negative edge endpoints.

Proof. For any label i of G_β, let $G_\beta^+(i)$ be the subgraph of G_β consisting of all vertices of G_β and all positive i-edges of G_β. The edges of $G_\beta^+(i)$ correspond to the negative edges of G_α incident to the vertex u_i. Let the number of such edges be k. Then if f denotes the sum of the Euler characteristics of the faces of $G_\beta^+(i)$, we have

$$0 = \chi(A_\beta) = n_\beta - k + f.$$

Therefore, if $k > n_\beta$, $G_\beta^+(i)$ has a disk face D. Then there is a Scharlemann cycle of G_β in D by [HM, Proposition 5.1]. This proves (1).

To prove (2), assume $k > 2n_\beta$. Then by the above we have $f = k - n_\beta > n_\beta$, so $G_\beta^+(i)$ has more than n_β disk faces, and by [HM, Proposition 5.1] each such face contains a Scharlemann cycle of G_β. Hence G_β contains $s > n_\beta$ Scharlemann cycles, all on the same label pair, say $(1, 2)$, by Lemma 2.2(1). Define a graph H in A_β as follows; see [GL, Proof of Theorem 2.3]. The vertices of H consist of the vertices of G_β, together with a vertex v_D in the interior of each disk face of G_β bounded by a Scharlemann cycle. The edges of H are defined by joining each vertex v_D, within D, to the vertices of G_β in ∂D. Thus H has $n_\beta + s$ vertices and at least $2s$ edges. An Euler characteristic argument then shows that H has a disk face E. This disk E contains a 1-cycle of G_β (see [CGLS, p. 279] for definition), and hence a Scharlemann cycle [CGLS, Lemma 2.6.2]. But this contradicts the fact that E is a face of H, because by definition H would have a vertex in the disk bounded by this Scharlemann cycle. □

Let P, Q be two edge endpoints on the boundary of a vertex u in G_α. Let $P_0 = P, P_1, \dots, P_{k-1}, P_k = Q$ be the edge endpoints encountered when traveling along ∂u in the direction induced by the orientation of u. Then the *distance* from P to Q (at the vertex u) is defined as $\rho_u(P, Q) = k$. Notice that since the valency of u

is Δn_β, we have $\rho_u(Q,P) = \Delta n_\beta - \rho_u(P,Q)$. If e_1, e_2 is a pair of edges, each having a single endpoint P_i on the vertex u in G_α, then define $\rho_u(e_1,e_2) = \rho_u(P_1,P_2)$.

A pair of edges e_1, e_2 connecting two vertices u, v in G_α is an *equidistant pair* if $\rho_u(e_1,e_2) = \rho_v(e_2,e_1)$. In particular, one can check that if e_1, e_2 are a pair of parallel edges connecting a pair of parallel vertices in G_α, then e_1, e_2 is an equidistant pair in G_α.

Lemma 2.4. (The Equidistance Lemma.) *Let e_1, e_2 be a pair of edges connecting the same vertices on G_1 and the same vertices on G_2. Then e_1, e_2 is an equidistant pair in G_1 if and only if it is an equidistant pair in G_2.*

Proof. See [GW1, Lemma 2.8]. □

Given two slopes r_1, r_2 on the torus T_0, let l be a curve intersecting r_1 at a single point. Choosing l and the orientations of the curves properly, we may assume that homologically $r_2 = qr_1 + \Delta l$, where $1 \leq q < \Delta/2$. The number q is called the *jumping number* of r_1, r_2. Note that if $\Delta = 4$ then $q = 1$, and if $\Delta = 5$ then $q = 1$ or 2.

Lemma 2.5. *(1) If the jumping number $q = 1$, in particular if $\Delta = 4$, then a pair of j-edges at a vertex u_i in G_α are adjacent among all the j-edges if and only if on G_β they are also adjacent at v_j among all i-edges.*

(2) If $q = 2$, then a pair of j-edges at a vertex u_i in G_α are adjacent among all j-edges if and only if on G_α they are not adjacent among all the i-edges at v_j.

Proof. This is essentially [GW1, Lemma 2.10]. It was shown that if $P_1, ..., P_\Delta$ are the consecutive j-edge endpoints at u_i, then on ∂v_j they appear in the order $P_q, P_{2q}, ..., P_{\Delta q}$, hence the result follows. □

A graph G on an annulus A is *special* if every vertex has at least two nonparallel boundary edges. Note that G is special if and only if the corresponding reduced graph $\widehat{G}$ is special.

Lemma 2.6. *(1) If G is special then every vertex has exactly two boundary edges in $\widehat{G}$, going to distinct boundary components of A.*

(2) If G_α has $2n_\beta$ parallel boundary edges, then G_β is special. G_α cannot have more than $2n_\beta$ parallel boundary edges.

(3) If some edge E of $\widehat{G}_\alpha$ represents n_β negative edges, and if G_α has some positive edges, then G_α has at most n_β parallel boundary edges, and each vertex of $\widehat{G}_\beta$ has at most one boundary edge.

Proof. (1) Otherwise there would be a pair of edges of $\widehat{G}$ at some vertex v going to the same boundary component of A. By looking at an outermost such pair one can see that some vertex u of $\widehat{G}$ has a single boundary edge in $\widehat{G}$, contradicting the definition of a special graph.

(2) If G_α has $2n_\beta$ parallel boundary edges, then for any label i it has two parallel i-edges. Since no two edges are parallel on both graphs, these two edges are non-parallel on G_β, hence G_β is special. If G_α has more than $2n_\beta$ parallel boundary edges, then there is a label i such that G_α has three parallel boundary i-edges. Since by (1) the vertex v_i in $\widehat{G}_\beta$ has only two boundary edges, two of these edges would be parallel on both graphs, contradicting Lemma 2.1(2).

(3) Since G_α has some positive edges, the vertices of G_β cannot all be parallel, so there are at least two E-orbits, which form parallel essential cycles on G_β. Hence

all boundary edges at a vertex of G_β must be parallel to each other. If G_α has more than n_β parallel boundary edges then two of them would be parallel on both graphs, contradicting Lemma 2.1(2). □

Lemma 2.7. *Suppose all vertices of $\widehat{G}_\alpha$ are boundary vertices, and suppose there are two boundary edges E_1, E_2 of $\widehat{G}_\alpha$ incident to the same vertex v and going to the same boundary component of A_α. Then $\widehat{G}_\alpha$ has a vertex v' of valency at most 3 which is incident to a single boundary edge, and G_β is special.*

Proof. Let D be the disk on A_α cut off by $E_1 \cup E_2$. Since E_1, E_2 are nonparallel, D contains a vertex $v_1 \neq v$, hence by adding an edge if necessary we may assume that there is an edge incident to v other than E_1, E_2. Let $\widetilde{D}$ be the double of D along $E_1 \cup E_2$, and let $\widetilde{G}$ be the double of $\widehat{G}_\alpha \cap D$. Then each vertex of $\widetilde{G}$ has a boundary edge. By [CGLS, Lemma 2.6.5] $\widetilde{G}$ has a vertex v' of valency at most 3 and incident to at most one boundary edge. Since v has valency at least 4 in $\widetilde{G}$, $v' \neq v$. Hence $\mathrm{val}(v', \widehat{G}_\alpha) = \mathrm{val}(v', \widetilde{G}) \leq 3$. By Lemma 2.2(3) each interior edge of $\widetilde{G}_\alpha$ represents at most n_β edges. Since $\Delta \geq 4$, this implies that the unique boundary edge at v' represents at least $2n_\beta$ edges. By Lemma 2.6(2) in this case G_β is special. □

§3. Reduced graphs on annuli

By a *reduced graph* on a surface we mean one with no trivial loops or parallel edges; in other words, no faces of the graph are monogons or bigons.

Definition 3.1. Let G be a reduced graph on an annulus A. Then G is said to be *triangular* if

(i) every vertex has at most one boundary edge;
(ii) every interior vertex has valency 6;
(iii) every boundary vertex has valency 5;
(iv) every face of G is a disk with three edges.

We remark that the only properties of a triangular graph that we will use are (i), (iii), and the fact that the graph has at least one boundary vertex (which follows from (iv)).

Proposition 3.1. *Let G be a reduced graph in an annulus A. Then either*

(1) G contains an interior vertex of valency at most 5; or
(2) G contains a boundary vertex of valency at most 4 with exactly one boundary edge; or
(3) G is triangular; or
(4) G is special.

Proof. Let G_1 be a graph obtained from G by adding extra edges so as to make each face of G_1 a disk with three edges. In particular, in G_1, each boundary face has three edges, and if some vertex v has two boundary edges e_1, e_2, then v has an edge on each side of $e_1 \cup e_2$, so it has valency at least 4.

Let G_2 be the union of G_1 and ∂A, with the obvious graph structure. Thus the points of $G_1 \cap \partial A$ are now considered vertices, and the segments of ∂A cut by these vertices are considered edges of G_2. Note that $\mathrm{val}(v, G_2) = 3$ for all vertices v on ∂A, and each boundary face now has four edges. Let G_3 be obtained from G_2 by adding a diagonal edge in each boundary face of G_2, all sloping in the same direction; in other words, no two edges added have a common vertex on ∂A. We have $\text{val}(v, G_3) = 4$ if $v \in \partial A$. One can see that if we remove all edges and vertices on ∂A then we get a graph that is obtained from G_1 by adding an extra copy of each boundary edge. Hence if $\text{val}(v, G_1) = p$ and v has q boundary edges in G_1, then $\text{val}(v, G_3) = p + q$. In particular, if v has two boundary edges in G_1, then its valency in G_3 is at least $4 + 2 = 6$. Each face of G_3 is now a triangle.

The double of A along ∂A is a torus T, and the corresponding double of G_3 is a reduced graph $\widetilde{G}_3$ on T with triangular faces. By an Euler characteristic argument, one can show that the number of edges in $\widetilde{G}_3$ is three times the number of vertices of $\widetilde{G}_3$. Thus either (i) some vertex v of $\widetilde{G}_3$ has valency at most 5, or (ii) all vertices of $\widetilde{G}_3$ have valency 6. All vertices on ∂A have valency 4 in G_3, hence valency 6 in $\widetilde{G}_3$, and we have shown that if v has two boundary edges in G_1 then it has valency at least 6 in G_3; therefore (i) implies that either v is an interior vertex of G with valency at most 5, or it is a boundary vertex of G with valency at most $5 - q \leq 4$ and incident to at most one boundary edge, so the graph is of type (1) or (2) in the proposition. Hence we may assume that all vertices of G_3 in the interior of A have valency 6.

If no vertex of G_1 has two boundary edges then each boundary vertex of G_1 has valency $6 - q = 6 - 1 = 5$. Since each interior vertex of G_1 has valency 6, it follows that G_1 is triangular. Since G is a subgraph of G_1 with the same vertices, either $G = G_1$ and hence G is of type (3), or G has a vertex v with $\text{val}(v, G) < \text{val}(v, G_1)$, in which case G is of type (1) or (2).

Now assume some vertex v of G_1 has two boundary edges e_1, e_2 going to different boundary components. Then the valency of v in G_1 is at most $6 - 2 = 4$. Since each face of G_1 has three edges, there is exactly one interior edge e' on each side of $e_1 \cup e_2$. Let v' be the other endpoint of e'. Since each face has three edges, v' must also have two boundary edges going to different boundary components of A. Repeating this process, we see that G_1 is a special graph such that each vertex has valency 4. Since G is a subgraph of G_1, either it is special, hence of type (4), or it has a vertex of valency at most 3 and incident to at most one boundary edge, in which case it is of type (1) or (2).

Finally, assume G_1 has a vertex v which has two boundary edges going to the same boundary component. Then they cut off a disk D from the annulus, which we may assume to be outermost. However, arguing as in the previous paragraph, we see that the vertex on the other end of an edge e' in D incident to v must have two boundary edges, which is a contradiction. Therefore this case does not happen. □

§4. Special graphs

Recall that a graph G on an annulus A is special if every vertex has two non-parallel boundary edges. By Lemma 2.6(1) this implies that every vertex of G has exactly two boundary edges in $\widehat{G}$, going to different boundary components of A.

To simplify notation, denote n_β by n.

Lemma 4.1. *If G_α is a special graph, then G_β is also special.*

Proof. First notice that since each vertex u_i of G_α is incident to at most two families of interior edges and each such family contains at most n edges (Lemma 2.2(3)), there are at most two interior j-edges at u_i for any j. Hence there are at least $\Delta - 2$ boundary j-edges at u_i. Since this is true for all i, j, we see that each vertex v_j of G_β has at least $2n_\alpha$ ($3n_\alpha$ if $\Delta = 5$) boundary edges. Since each parallel family contains at most $2n_\alpha$ edges (Lemma 2.6(2)), the lemma follows immediately when $\Delta = 5$.

Now assume $\Delta = 4$, and assume G_β is not special. Then it has a vertex v_i such that all boundary edges are parallel. By Lemma 2.6(2) and the above, v_i has exactly $2n_\alpha$ boundary edges, all parallel to each other. In particular, there are only two boundary 1-edges e_1, e_1' at v_i. Dually this means that e_1, e_1' are the only boundary i-edges at u_1. Since they are parallel on G_β, they cannot be parallel on G_α, so they belong to different families of boundary edges. Since these two edges are adjacent among all 1-edges at v_i, by Lemma 2.5(1) they must also be adjacent among all i-edges at u_1. This implies that the two interior i-edges at u_1 are on the same side of $e_1 \cup e_1'$, so they belong to the same edge E in $\widehat{G}_\alpha$ because there is only one interior edge of $\widehat{G}_\alpha$ on each side of $e_1 \cup e_1'$. Since by Lemma 2.2(3) E contains at most n edges, this is impossible. □

In the remainder of this section we will assume that both G_1 and G_2 are special.

The sign of a vertex u in G_α induces an orientation on ∂u, called its *preferred orientation*. Thus the preferred orientations of the ∂u's are all in the same direction on T_0. Let e_1, e_2 be a pair of adjacent boundary edges at some vertex u of G_α. When traveling on ∂u along the preferred orientation, the labels at the endpoints of e_1, e_2 appear as $i, i+1$ for some i ($i+1 = 1$ if $i = n$). They cut off a band B on the surface F_α, called an *i-band* at u (of G_α). Note that the label i is determined by the pair e_1, e_2 even if $n = 2$. The edge labeled i at u is called the *initial edge* of B, the other the *terminal edge*. Two i-bands of G_α are of different *types* if their initial edges are nonparallel on G_β; otherwise they are of the same type.

If $e_1, ..., e_k$ are all the edges of a parallel family E at a vertex u, appearing in this order when traveling along the preferred orientation of ∂u, then e_k is called the *ending edge* of E, and the label of e_k at u is called the *ending label* of E. Note that if a boundary i-edge e is not an ending edge, then it is the initial edge of an i-band.

Lemma 4.2. *There is a label i such that all i-bands of G_α are of the same type.*

Proof. Assuming otherwise, then there are two i-bands B_i^1, B_i^2 of different types for each i. Since the graph G_β is special, there are only two families of parallel boundary edges for each vertex v_i in G_β, so each family contains the initial edge of some B_i^j. Therefore, the terminal edge of each B_i^j is parallel to the initial edge of some B_{i+1}^k, so there is a band D_i^j on F_β connecting these two edges. Note that D_i^j degenerates to a single edge if these two edges coincide.

Consider the 2-complex $Q = \cup(B_i^j \cup D_i^j)$. Then $Q \cap T_0 = \cup(e_i^j \cup d_i^j)$ is a graph G on T_0, where $e_i^j = B_i^j \cap T_0$ and $d_i^j = D_i^j \cap T_0$. We have $Q \cong G \times I$. Shrinking each d_i^j to a point, and orienting e_i^j so that its endpoint is on d_i^j, we get an oriented graph G' in which each vertex d_i^j is the tail of some edge e_{i+1}^k. Hence G' contains an embedded oriented cycle. The corresponding cycle C in G is then an embedded loop in T_0. Let γ be a parallel copy of some boundary component of F_β on T_0, intersecting some e_i^j in C transversely at a single point. The definition of B_i^j and the orientation of e_i^j implies that C intersects γ always in the same direction; hence C is an essential curve. Thus $A = C \times I \subset Q$ is an annulus properly embedded in M intersecting T_0 in the essential curve C, which contradicts the assumption that M is ∂-irreducible and anannular. □

Lemma 4.3. *Each family of boundary edges in G_α contains at least n edges.*

Proof. Let $E_1, ..., E_4$ be the four edges of $\widehat{G}_\alpha$ at u_1, with E_1, E_3 the boundary edges. If $|E_1| < n$ then there is a label i which does not appear at the endpoints of edges in E_1. If $\Delta = 5$ then we would have $|E_3| = 5n - |E_1| - |E_2| - |E_4| \geq 2n + 1$, contradicting Lemma 2.6(2). Hence $\Delta = 4$. Since $|E_3| \leq 2n$, E_3 contains at most two i-edges, so each of E_2, E_4 contains one i-edge. Let e_1, e_2 be the i-edges of E_2, E_4 at u_1, and let e_3, e_4 be the i-edges of E_3. Since e_3, e_4 are adjacent i-edges at u_1, by Lemma 2.5(1) they are adjacent 1-edges at v_i. On G_β the two edges e_3, e_4 belong to different families of boundary edges at v_i, because they cannot be parallel on both graphs. Therefore the two edges e_1, e_2 belong to the same family of interior edges. Since they both have label 1 at v_i, this would imply that the interior family containing them has at least $n_\alpha + 1$ edges, contradicting Lemma 2.2(3). □

Lemma 4.4. *The jumping number $q = 1$.*

Proof. This is automatically true if $\Delta = 4$. Hence assume $\Delta = 5$. First assume that there is a vertex u_j of G_α which has two interior i-edges e_1, e_2 for some i. Since each interior family contains at most n edges, e_1, e_2 are nonparallel on G_α. By Lemma 4.3 each boundary family contains an i-edge, hence e_1, e_2 are non adjacent among the i-edges at u_j. Dually on G_β these are j-edges at the vertex v_i. For the same reason, they are non adjacent among all j-edges at v_i. Therefore by Lemma 2.5 the jumping number $q = 1$.

Now assume that u_j has at most one interior i-edge for all i. Then it has at most n interior edge endpoints. On the other hand, since each boundary family contains at most $2n$ edges and the valency of u_j is $\Delta n = 5n$, we see that it cannot have less than n interior edge endpoints; therefore it has exactly n interior edge endpoints, and each boundary family contains exactly $2n$ edges. If u_j has two interior families, so each family contains less than n edges, then the two boundary families have different ending labels. In this case for each label i there are three i-bands, which cannot all be of the same type because each boundary family of v_i has at most two j-edges. This contradicts Lemma 4.2. Therefore u_j has only one family of interior edges, which contains n edges. For the same reason, each vertex of G_β has only one family of interior edges, containing n_α edges. By the parity rule one of these families is negative, and by Lemma 2.1(3) they form cycles on the other graph, so each vertex of that graph would then have two families of interior edges, contradicting the above conclusion. □

Lemma 4.5. *Suppose all i-bands at a vertex u_j of G_α are of the same type. Then*

(1) there are n parallel interior edges at u_j, and

(2) each family of n parallel interior edges at u_j has i as its ending label.

Proof. Let $E_1, ..., E_4$ be the edges at u_j of $\widehat{G}_1$, appearing in this order around ∂u_j along its preferred orientation, with E_1, E_3 the boundary edges. Let $e_1, ..., e_4$ be four i-edges at u_j, appearing successively along the preferred orientation of ∂u_j.

First assume that all e_i are boundary edges. Then we may assume that $e_1, e_2 \in E_1$, and $e_3, e_4 \in E_3$. Thus e_1, e_3 are not ending edges, so they are initial edges of some i-bands B_1, B_3. Since the jumping number $q = 1$ (Lemma 4.4), and since e_1, e_3 are non adjacent among i-edges at u_j, by Lemma 2.5 they are non adjacent among 1-edges at v_i in G_β, hence they are non parallel boundary edges on G_β. Therefore B_1, B_3 are of different type.

Now assume that E_2 contains an i-edge e_2, say. Since each of E_1, E_3 contains at least n edges, we must have $e_1 \in E_1$ and $e_3 \in E_3$. Assume that either e_2 is not the ending edge of E_2 or $|E_2| < n$. Then e_1 is not an ending edge of E_1, and there is an i-band B_1 with e_1 as the initial edge. If e_3 is not an ending edge either, then there is an i-band B_3 with e_3 as initial edge. For the same reason as above, B_1, B_3 are of different type, and we are done. So assume that e_3 is the ending edge of E_3. Now we must have $|E_4| < n$ as otherwise E_4 would have n edges and have the i-edge e_4 as its ending edge, contradicting the assumption. Hence e_4 is in E_1, and so there is an i-band with e_4 as an initial edge. Since e_1, e_4 are parallel on G_α, they are nonparallel on G_β, so again B_1, B_4 are of different type. This proves (1). To prove (2), notice that if $|E_2| = n$ but e_2 is not the ending edge, then e_1, e_3 are not ending edges of E_1, E_3, so from the above the two i-bands B_1, B_3 are of different type. □

Lemma 4.6. *Suppose $n > 2$. Then each positive edge of $\widehat{G}_\alpha$ represents at most $n/2$ edges.*

Proof. When $n > 2$, the special graph $\widehat{G}_\beta$ has at most one edge connecting any two vertices. If G_α has $n/2 + 1$ parallel positive edges, then it has a Scharlemann cycle $e_1 \cup e_2$ with label pair $(1, 2)$, say. So the two edges e_1, e_2 would be parallel on G_β, contradicting Lemma 2.1(4). □

Proposition 4.7. *If G_α is special then up to relabeling we have $n_1 = 1$, $n_2 = 2$, and G_1 has exactly two interior edges.*

Proof. First assume $n_\alpha \geq 2$ for $\alpha = 1, 2$. By Lemma 4.2 for each graph G_α there is a label i such that all i-bands of G_α are of the same type. Let u_j be a vertex of G_α. By Lemma 4.5(1), it has a set of n parallel interior edges E with i as its ending label at u_j. Let u_k be the vertex on the other endpoint of E, then by Lemma 4.5(2), E also has ending label i at u_k. If E is negative, then the ending edge e of E at u_j is the same as that at u_k, so e would have the same label i on its two endpoints, and hence is a loop on G_β. Since $n \geq 2$ and G_β is special, this is absurd. If E is positive, then the two ending edges would give rise to two negative edges at v_i in G_β, which must be nonparallel because they cannot be parallel on both graphs. Thus both families of interior edges at v_i are negative. Replacing u_j by v_i in the above argument, we get a contradiction because now E must be negative. Therefore up to relabeling we must have $n_1 = 1$.

By Lemma 4.5(1) the only vertex u_1 of G_1 has n_2 parallel interior edges, which by the parity rule must be negative edges on G_2, hence $n_2 \geq 2$. If $n_2 > 2$, then by Lemma 4.6 G_1 has at most $n_2/2$ interior edges, which is a contradiction. Hence the result follows. □

§5. The generic case

In this section except for Lemma 5.1, we assume $n_\alpha, n_\beta > 2$. By Proposition 4.7, G_α and G_β are not special. Again denote n_β by n.

Lemma 5.1. *Suppose $n_\alpha \geq 2$, $n > 2$, and suppose $\widehat{G}_\alpha$ has a negative edge E with $|E| = n$, and a positive edge E' with $|E'| = n/2 + 1$. Let $(1, 2)$ be the label pair of the Scharlemann cycle in E'. Then*

(1) G_β has at most $n/2$ boundary vertices;

(2) when $n = 4$, the two vertices v_3, v_4 of G_β cannot both be boundary vertices; and

(3) when $n = 4$, G_α cannot have both a $(1,4)$-edge and a $(2,3)$-edge.

Proof. Let k be the number of E-orbits. Since E' contains more than $n/2$ edges, hence contains a Scharlemann cycle, the annulus A_β is separating, so G_β has the same number of positive and negative vertices. Each E-orbit contains the same number (n/k) of vertices, all of the same sign, so the number of orbits containing positive vertices is the same as the number of those containing negative ones, and hence k must be even. Recall that each E-orbit forms an essential cycle on G_β, so only the vertices on the two cycles adjacent to the two boundary components of A_β could be boundary vertices. Hence the number of boundary vertices is at most $2(n/k)$, and since k is even, (1) follows unless $k = 2$.

Assume $k = 2$. Let C_1, C_2 be the two cycles of E-orbits on G_β, and let e_1, e_2 be the edges of the Scharlemann cycle in E'. By Lemma 2.1(4) $e_1 \cup e_2$ is an essential cycle on G_β. The two vertices v_1, v_2 of $e_1 \cup e_2$ are on different C_1, C_2 because they are antiparallel, so the cycle $e_1 \cup e_2$ lies between C_1 and C_2, separating the vertex v_3 on the first orbit from the vertex v_n on the second. On the other hand, since E' contains more than two edges, there is an edge adjacent to the Scharlemann cycle which is a $(3, n)$-edge, so on G_β there would be an edge connecting v_3 to v_n. This is a contradiction, showing that $k = 2$ is impossible. In particular, this proves (1).

Now assume $n = 4$. Since we have shown that k is even and $k \neq 2$, we must have $k = 4$. In this case each vertex v_i of G_β has an essential loop C_i coming from the n parallel negative edges in G_α. These loops and their vertices form essential circles on A_β which are parallel to each other. As above, there is an edge in E' which connects v_3 to v_4. Hence the circles C_3 and C_4 are adjacent to each other, so v_3, v_4 cannot both be boundary vertices. This proves (2). Since the edges in the Scharlemann cycle connect v_1 to v_2, C_1 is adjacent to C_2. Thus either C_3 separates v_4 from v_1, v_2, so there is no edge connecting v_4 to v_1, or C_4 separates v_3 from v_1, v_2, so there is no edge connecting v_3 to v_2. This proves (3). □

Lemma 5.2. *Suppose $E_1, ..., E_5$ are the edges of $\widehat{G}_\alpha$ at a vertex u of valency 5. If E_1, E_2, E_3 are positive, and E_4 is an interior edge, then $|E_5| > n$; in particular, E_5 is a boundary edge.*

Proof. Assume $|E_5| \leq n$. By Lemma 2.2(2) we have $|E_i| \leq n/2 + 1$ for $i \leq 3$, and by Lemma 2.2(3) $|E_4| \leq n$. Since $n > 2$, we must have $\Delta = 4$, and

$$4n = \Delta n = |E_1| + \cdots + |E_5| \leq 3(\frac{n}{2} + 1) + 2n = \frac{7}{2}n + 3,$$

which implies that $n \leq 6$. Moreover, n must be even, otherwise by Lemmas 2.2(2) and 2.1(4) we would have $|E_i| \leq n/2$ for $i = 1, 2, 3$, hence $4n \leq 3(n/2) + 2n$, which is absurd.

If $n = 6$, then all the above inequalities are equalities. In particular, $\Delta = 4$, $|E_4| = |E_5| = 6$, and $|E_i| = 4$ for $i = 1, 2, 3$, so each of E_1, E_2, E_3 contains a Scharlemann cycle, and by Lemma 2.2(1) they all have the same label pair, say $(1, 2)$. But since $|E_4| = |E_5| = n$, these labels also appear in E_4 and E_5. Thus the label 1 appears 5 times, contradicting the fact that $\Delta = 4$.

Now assume $n = 4$. If each of E_1, E_2, E_3 contains a Scharlemann cycle with label pair $(1, 2)$, say, (in particular, if $|E_i| = 3$ for $i = 1, 2, 3$), then again the labels $\{1, 2\}$ appear three times among the endpoints of $E_1 \cup E_2 \cup E_3$ at u. Also, since $E_4 \cup E_5$ has at least $16 - 3 \times 3 = 7$ edge endpoints at u, one of the labels $\{1, 2\}$

appears at least twice among the endpoints of $E_4 \cup E_5$ at u, so it appears 5 times at u, contradicting the fact that $\Delta = 4$. Hence we may assume that $|E_1| = |E_2| = 3$, $|E_3| = 2$, E_3 contains no Scharlemann cycle, and $|E_4| = |E_5| = 4$. Since the two edges of E_3 have labels $3, 4$ at u, they must have label sets $\{1, 4\}$ and $\{2, 3\}$. Since $|E_4| = 4$, the edges in E_4 are negative. This contradicts Lemma 5.1(3), completing the proof of the lemma. □

Lemma 5.3. *$\widehat{G}_\alpha$ has no interior vertex of valency at most* 5.

Proof. Let $E_1, \ldots, E_5$ be the edges of $\widehat{G}_\alpha$ incident to u. Since all these edges are interior edges, by Lemma 5.2 they can have at most two positive edges, say E_1, E_2. By Lemma 2.3(2), u has at most $2n$ negative edges in G_α, hence $E_1 \cup E_2$ represents at least $2n$ positive edges. By Lemma 2.2(2) we have $2n \leq 2(n/2 + 1)$, which contradicts the assumption that $n \geq 3$. □

Lemma 5.4. *$\widehat{G}_\alpha$ cannot have a boundary vertex u of valency at most 4 with a single boundary edge.*

Proof. Let E_0 be the boundary edge, and E_1, E_2, E_3 the interior edges of $\widehat{G}_\alpha$ at u. By Lemma 2.6(2) and Proposition 4.7 we have $|E_0| < 2n$. By Lemma 2.3(2), u can have at most $2n$ negative edges in G_α, so one of the interior edges, say E_1, must be positive, and by Lemma 2.2(2) $|E_1| \leq n/2 + 1 < n$. Since each of E_2, E_3 represents at most n edges, we have $|E_0| > n$.

We claim that *either G_α or G_β contains a Scharlemann cycle.* If u has more than n negative edges, then by Lemma 2.3(1) G_β contains a Scharlemann cycle. So assume u has at most n negative edges. Since u has less than $2n$ boundary edges, it must have more than n positive edges. If at most two of the E_1, E_2, E_3 are positive, then one of them represents more than $n/2$ positive edges; if all the three interior edges at u are positive, then since they represent more than $2n$ edges, again one of them represents more than $n/2$ edges. In either case these parallel edges contain a Scharlemann cycle. This completes the proof of the claim.

Now $|E_0| > n$ implies that some vertex of G_β has two nonparallel boundary edges. In particular, $\widehat{G}_\beta$ cannot be triangular. It follows from Lemma 5.3 and Proposition 3.1 that $\widehat{G}_\beta$ must also have a boundary vertex v of valency at most 4 with a single boundary edge. Since one of G_α and G_β has a Scharlemann cycle, by considering v instead of u if necessary, we may assume without loss of generality that G_α contains a Scharlemann cycle with label pair $(1, 2)$.

We claim that $|E_0| \leq n + 2$. Otherwise each vertex of G_β has a boundary edge, and some vertex v_i other than v_1, v_2 has two such edges e_1, e_2. Since the edges of the Scharlemann cycle form an essential subgraph of G_β (Lemma 2.1(4)), separating the two boundary components of A_β, the edges e_1, e_2 must go to the same boundary component. Applying Lemma 2.7, we see that G_α is special, a contradiction.

Since $|E_0| > n$ and u has some positive edges, by Lemma 2.6(3) the graph G_α cannot have n parallel negative edges. Thus if k of the E_1, E_2, E_3 are positive, then

$$4n \leq (n + 2) + k(\frac{n}{2} + 1) + (3 - k)(n - 1) = (4 - \frac{k}{2})n + (2k - 1)$$

which implies that $n < 4$. But since G_α contains a Scharlemann cycle, n is even. This contradicts the assumption that $n > 2$. □

Lemma 5.5. *If both $\widehat{G}_1, \widehat{G}_2$ are triangular, then each boundary vertex has exactly two positive and two negative edges in $\widehat{G}_\alpha$.*

Proof. Let $E_0, ..., E_4$ be the edges of $\widehat{G}_\alpha$ at a boundary vertex v, with E_0 the boundary edge. Since $\widehat{G}_\beta$ is also triangular, E_0 represents at most n edges. Therefore by Lemma 5.2 at most two of the E_i are positive. On the other hand, by Lemma 2.3(2) v has at most $2n$ negative edges, hence at least n positive edges. Since each E_i represents at most $n/2 + 1 < n$ positive edges, v must have two positive edges in $\widehat{G}_\alpha$. □

Lemma 5.6. *Suppose both $\widehat{G}_1, \widehat{G}_2$ are triangular. Then all vertices of G_1, G_2 are boundary vertices.*

Proof. Let $E_0, ..., E_4$ be the edges of $\widehat{G}_\alpha$ at a boundary vertex v, with E_0 the boundary edge. By Lemma 5.5 we may assume that E_1, E_2 are negative edges, and E_3, E_4 are positive edges.

Suppose G_β has some interior vertices. Then $|E_0| < n$. Since v has at most $2n$ negative edges, it has more than n positive edges, so $E_3 \cup E_4$ contains a Scharlemann cycle with label pair $(1, 2)$, say, and n is even. Also, one of E_1, E_2 must represent n parallel edges, for otherwise $E_0 \cup E_1 \cup E_2$ would contain at most $3n - 3$ edges, so one of E_3, E_4 would contain at least $n/2 + 2$ edges, contradicting Lemma 2.2(2). Now we can apply Lemma 5.1(1) and conclude that G_β has at most $n/2$ boundary vertices. Thus $|E_0| \leq n/2$. We have the inequality

$$4n \leq |E_0| + ... + |E_5| \leq \frac{n}{2} + 2n + 2(\frac{n}{2} + 1) \leq \frac{7}{2}n + 2.$$

Since n is even, this implies $n = 4$, $|E_0| = 2$, $|E_1| = E_2| = 4$, and $|E_3| = |E_4| = 3$. Now each of E_3, E_4 contains a Scharlemann cycle on label pair $(1, 2)$, so these labels appear 4 times among the interior edge endpoints at v. Thus the labels of E_0 must be $3, 4$. This contradicts Lemma 5.1(2). □

Lemma 5.7. *$\widehat{G}_1, \widehat{G}_2$ cannot both be triangular.*

Proof. Assume $\widehat{G}_1, \widehat{G}_2$ are triangular. Then by Lemma 5.6 all vertices of G_1, G_2 are boundary vertices, and by Lemma 5.5 each vertex v of G_α has exactly two positive edges and two negative edges in $\widehat{G}_\alpha$. Since a positive edge in G_α is a negative edge in G_β, it follows that either (i) some vertex v of one of the graphs, say G_1, has more positive edge endpoints than negative ones, or (ii) all vertices of G_1 and G_2 have the same number of positive and negative edge endpoints.

In case (i), (writing $n = n_2$), v has at most $2(n/2+1) = n+2$ positive edges, at most $n + 1$ negative edges, and at most n boundary edges. From the inequality

$$4n \leq (n + 2) + (n + 1) + n$$

we see that $n \leq 3$. But if $n = 3$ then v has at most $2(n/2) = n$ positive edges, at most $n - 1$ negative edges, and at most n boundary edges, which would lead to the contradiction that $4n \leq n + (n - 1) + n$.

In case (ii), any vertex v of G_α has at most $n+2$ positive edges, the same number of negative edges, and at most n boundary edges; so from $4n \leq (n+2)+(n+2)+n$ we see that $n = 4$, $|E| = 3$ for all positive interior edges of $\widehat{G}_\alpha$, and $|E| = 4$ for all boundary edges of G_α. Each label appears three times among the interior edges endpoints at any vertex v of G_α, but since each of the two families of positive edges at v contains a Scharlemann cycle, which must all have the same label pair $(1,2)$, it follows that these labels appear only once among the negative edge endpoints at v, so the label 3 appears twice among the negative edge endpoints at v. Since this is true for all vertices v in G_α, it means that the vertex v_3 on G_β has $2n_\alpha$ positive edge endpoints, and n_α negative ones, a contradiction. □

Proposition 5.8. *One of the graphs G_α has at most two vertices.*

Proof. Assume $n_1, n_2 \geq 3$. By Proposition 4.7, G_α is not special, by Lemma 5.3 $\widehat{G}_\alpha$ does not have an interior vertex of valency at most 5, and by Lemma 5.4 it cannot have a boundary vertex of valency at most 4 with a single boundary edge. Thus by Proposition 3.1 both $\widehat{G}_1, \widehat{G}_2$ are triangular, which contradicts Lemma 5.7. □

§6. Nonspecial graphs with $n_1 = 2$ and $n_2 > 2$

Throughout this section we will assume that G_1, G_2 are not special graphs. We will show that the case $n_1 = 2$ and $n_2 = n > 2$ does not happen. Together with Propositions 4.7 and 5.8, this shows that n_α must be at most 2 for both $\alpha = 1$ and 2.

Lemma 6.1. *If $n_1 = 2$ then $\widehat{G}_1$ is a subgraph of that shown in Figure 6.1.*

Proof. Since $\widehat{G}_1$ is not a special graph, one of the vertices u_1, u_2 has at most one boundary edge. If either u_1 or u_2 does not have a loop, then one can find a vertex u of valency at most 3 in $\widehat{G}_1$, with at most one boundary edge. Since each interior edge represents at most n edges, u would have at least $2n$ boundary edges, which would imply that G_2 is a special graph, a contradiction. Hence each vertex u_i has a loop. It is now easy to see that $\widehat{G}_1$ must be a subgraph of that in Figure 6.1. □

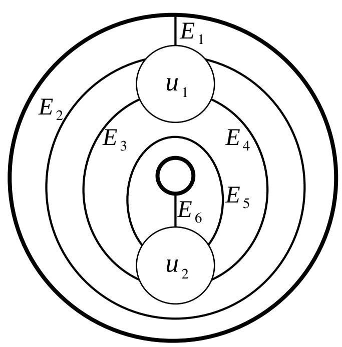


Figure 6.1

Label the edges of $\widehat{G}_1$ as in Figure 6.1. Denote by m the number of non-loop interior edges of G_1, i.e. $m = |E_3| + |E_4|$.

Lemma 6.2. *Suppose $n_1 = 2$, and $n > 2$.*

(1) Either $m = 2n$, or $m = 2n - 2$ and E_2 contains a Scharlemann cycle.

(2) The two vertices of G_1 are antiparallel.

Proof. (1) If no label appears twice among the endpoints of edges in E_2, then from the labeling on ∂u_1 one can see that either $m \geq 2n$ or $|E_1| \geq 2n$. But the second possibility does not occur because then by Lemma 2.6(2) the graph G_2 would be special. Hence in this case we have $m \geq 2n$. Since each of E_3, E_4 represents at most n edges, we conclude that $m = 2n$.

Now assume that some label appears twice among the endpoints of edges in E_2. Then E_2 contains a Scharlemann cycle e_1, e_2, with label pair $(1, 2)$, say. Since $n > 2$, E_2 contains no extended Scharlemann cycle (Lemma 2.1(5)), so one of these two edges, say e_1, must be an outermost edge among those in E_2. Thus the endpoints of e_1 are either adjacent to those in $E_3 \cup E_4$ or to those in E_1. In the first case, the label sequence of $E_3 \cup E_4$ at u_1 is $3, 4, ..., n$, so $m \equiv n - 2 \mod n$. If $m = 2n - 2$ then we are done. If $m \neq 2n-2$, then since $|E_3|, |E_4| \leq n$, we must have $m = n-2$. Thus $|E_1| = \Delta n - m - 2|E_2| \geq 2n$, which by Lemma 2.6(2) would imply that G_2 is special, a contradiction. Therefore e_1 must be adjacent to E_1. As above, we have either $|E_1| = 2n - 2$, or $|E_1| = n - 2$ and $m \geq 2n$. In the second case we have $m = 2n$ because $|E_3|, |E_4| \leq n$. It remains to show that $|E_1| = 2n-2$ is impossible.

Assume $|E_1| = 2n - 2$. Notice that this happens only if E_2 contains a Scharlemann cycle. Moreover, if $(1, 2)$ is the label pair of the Scharlemann cycle then all labels other than $1, 2$ would appear twice among endpoints of edges in E_1. Thus on G_2 each vertex other than v_1, v_2 would have two boundary 1-edges. But since the edges in the Scharlemann cycle and the vertices v_1, v_2 form an essential subgraph of G_2, these two parallel 1-edges must go to the same boundary component of A_2. By looking at an outermost vertex one can see that there is a vertex v_i with $i \neq 1, 2$, at which the two boundary 1-edges are parallel, so they are parallel on both graphs, contradicting Lemma 2.1(2).

(2) If u_1, u_2 are parallel then $2n - 2 \leq |E_3| + |E_4| \leq 2(n/2 + 1)$, implying that $n = 4$ and $|E_3| = |E_4| = 3$. In this case both E_3, E_4 contain Scharlemann cycles, and by Lemma 2.2(1) they must have the same label pair $(1, 2)$ as the one in E_2. But since each of the labels $1, 2$ appears only once among the endpoints at u_1 of edges in $E_3 \cup E_4$, this is impossible. □

Lemma 6.3. *Suppose $n_1 = 2$, and $n > 2$. Then G_1 cannot have $2n$ negative edges.*

Proof. We must have $|E_2| > 0$, otherwise $|E_1| \geq 2n$, so G_2 would be special, contradicting our assumption. Assume that G_1 has $2n$ negative edges. Then $|E_3| = |E_4| = n$, and by Lemma 2.6(3) we have $|E_1| \leq n$, hence $|E_2| \geq n/2$. On the other hand, by Lemma 2.2(2) $|E_2| \leq n/2 + 1$. Hence E_2 contains either $n/2$ or $n/2 + 1$ edges. We want to show that $|E_2| \neq n/2 + 1$. Assuming otherwise, then since E_3 contains n parallel negative edges, by Lemma 5.1 the graph G_1 has at most $n/2$ parallel boundary edges. On the other hand, we have $|E_1| = \Delta n - m - 2|E_2| \geq n-2$, and since E_2 contains a Scharlemann cycle with label pair $(1, 2)$, say, n is even. Therefore we must have $n = 4$. Now in this case the labels of the edges in E_1 are $3, 4$, contradicting Lemma 5.1(2).

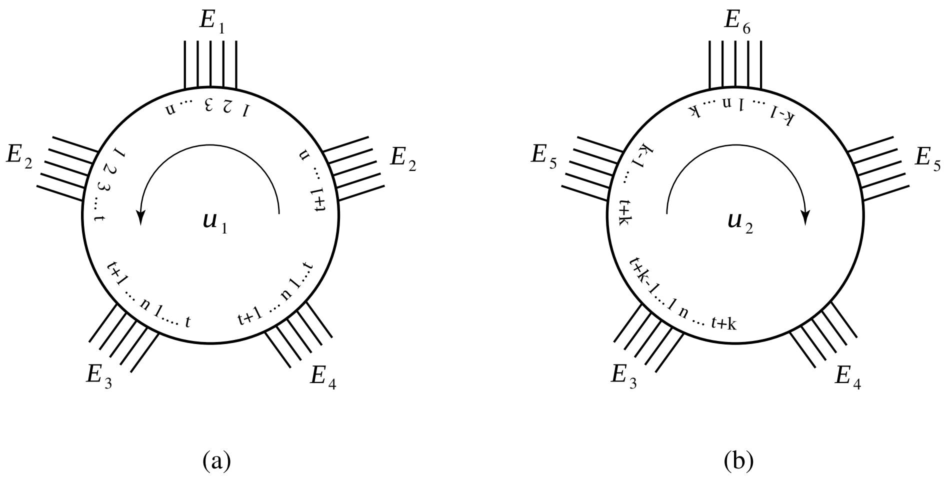


Figure 6.2

We have shown that $|E_2| = n/2$, and $|E_1| = n$. For the same reason, we have $|E_5| = n/2$ and $|E_6| = n$. Without loss of generality we may assume that u_1 is a positive vertex, u_2 is negative, and the edges of E_1 have label sequence $1, ..., n$ at u_1. See Figure 6.2. Let $t = n/2$. Since $|E_2| = n/2$, the label sequence of the endpoints at u_1 of the edges of E_3 is $t+1, ..., n, 1, ..., t$. There is a number k such that the label sequence at the other end of E_3 is $t+k, t+k+1, ..., t+k-1$. The number $k \neq 1$, otherwise these edges would be loops in G_2, so $n > 2$ would imply that some vertex of G_2 does not have a boundary edge, contradicting the fact that $|E_1| = n$. Now from Figure 6.2 we can see that the label sequence of E_6 is $k, ..., n, 1, ..., k-1$, hence the two edges e_3, e_4 in E_6 labeled n and 1 respectively, are adjacent (because $k \neq 1$). Let e_1, e_2 be the edges of E_1 labeled 1 and n respectively. By Lemma 2.6(3), each vertex of G_2 has at most one family of parallel boundary edges, so e_2 is parallel to e_3, and e_4 parallel to e_1 in G_2. Let $B(e_1, e_2)$ be the band on F_1 between e_1 and e_2, and let $B(e_3, e_4)$ be that between e_3 and e_4. Similarly, let $B(e_2, e_3)$ and $B(e_4, e_1)$ be the bands on F_2 between e_2, e_3 and e_4, e_1, respectively. Now we can form an annulus $A = B(e_1, e_2) \cup B(e_2, e_3) \cup B(e_3, e_4) \cup B(e_4, e_1)$ in the manifold M. Since the boundary curve C of A on T_0 intersects the circle ∂v_2 transversely at a single point (on the arc $B(e_1, e_2) \cap \partial u_1$), it is an essential curve. This contradicts the fact that the manifold M is ∂-irreducible and anannular. □

Lemma 6.4. *Suppose $n_1 = 2$, and $n > 2$. Then G_1 cannot have exactly $2n-2$ negative edges.*

Proof. If G_1 has $2n-2$ negative edges, then (up to symmetry) either $|E_3| = |E_4| = n-1$, or $|E_3| = n$ and $|E_4| = n-2$. Looking at the labeling, one can see that the two loops of E_2 near $E_3 \cup E_4$ form a Scharlemann cycle, with label pair $(1, 2)$, say. If $|E_3| = n$ then by Lemma 2.6(3) we have $|E_1| \leq n$, hence $|E_2| \geq n/2 + 1$. Now by Lemma 5.1(1) the graph G_2 has at most $n/2$ boundary vertices, which contradicts the fact that $|E_1| = n$. Therefore we must have $|E_3| = |E_4| = n-1$.

For the same reason, the two loops in E_5 near $E_3 \cup E_4$ form a Scharlemann cycle, which by Lemma 2.2(1) must have the same label pair $(1, 2)$. Now we can see that E_3 has label sequence $3, 4, ..., n, 1$, at u_1, and has label sequence $2, 3, ..., n$ at u_2.

However, in this case E_3 has only one orbit, containing all the labels, so all the vertices of G_2 are parallel to each other, hence all edges of G_1 are negative. But since G_1 contains some loops, this is a contradiction. □

Proposition 6.5. *If $M(r_1), M(r_2)$ are annular, and $\Delta \geq 4$, then $n_\alpha \leq 2$ for $\alpha = 1, 2$.*

Proof. By Proposition 4.7 this is true if one of the G_α is special. By Proposition 5.8 one of the graphs, say G_1, has at most two vertices. Since the two possibilities in Lemma 6.2(1) have been ruled out by Lemmas 6.3 and 6.4, the case $n_1 \leq 2$ and $n_2 > 2$ cannot happen. □

§7. Special graphs with $n_1 = 1$ and $n_2 = 2$

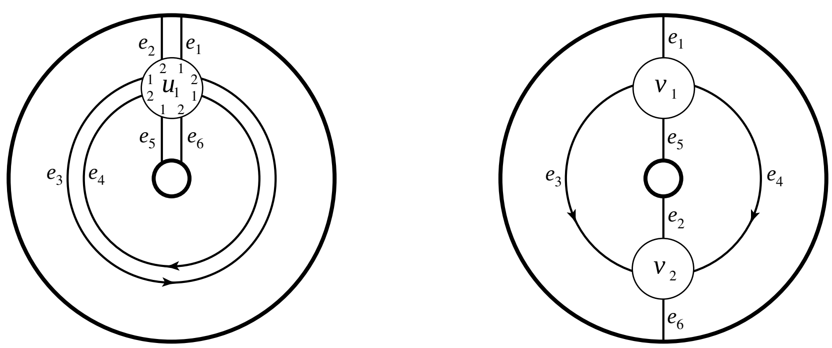
Proposition 7.1. *If G_α is special, then $\Delta = 4$, up to relabeling $n_1 = 1$, $n_2 = 2$, and the manifold M is the exterior of the Whitehead link.*

Proof. By Lemma 4.1, both graphs must be special. By Proposition 4.7, up to relabeling we must have $n_1 = 1$, $n_2 = 2$, and G_1 has exactly two interior edges e_1, e_2.

Assume $\Delta = 5$. By Lemma 4.4 the jumping number $q = 1$. There is a pair of adjacent boundary 1-edges e_1, e_2 at v_1 in G_2, which by Lemma 2.5(1) should also be adjacent at u_1 in G_1 among all 1-edges; but since the two families of boundary edges at u_1 are separated by two interior edges, e_1, e_2 must be in the same family, so they are parallel on both graphs, a contradiction. Therefore we must have $\Delta = 4$.

Now the Whitehead link exterior W does admit two annular Dehn fillings $W(r_1), W(r_2)$ with $\Delta(r_1, r_2) = 4$, $n_1 = 1$, and $n_2 = 2$, see [GW1, Theorem 7.5]. It remains to show that the manifold satisfying these conditions is unique.

Each vertex of G_2 has two boundary edges, which are nonparallel because G_2 is special. Thus the graph G_2 must be as shown in Figure 7.1(b). Similarly, since G_1 is special it has two families of parallel boundary edges. The loops have different labels at their two endpoints, so each family of boundary edges of G_1 contains an even number of edges. Hence G_1 must be as shown in Figure 7.1(a).



(a) (b)

Figure 7.1

Label the six edges of G_1 as in the figure. Orient e_3, e_4 so that on G_1 they have label 1 at their tails. Up to symmetry we may assume that the edge e_1 on G_2 is as shown in Figure 7.1(b). The label 1 endpoints of edges e_1, e_3, e_5, e_4 appear successively on ∂u_1, hence by Lemma 2.5(1) they also appear in this order on ∂v_1 in G_2, so these edges must be as shown in the figure. Similarly by looking at the label 2 endpoints of e_2, e_4, e_6, e_3 one can determine the edges e_4 and e_6. Therefore up to symmetry the graphs G_α are exactly as shown in the figure. We need to show that these graphs uniquely determine the manifold M.

Recall that F_α denotes the punctured annulus $A_\alpha \cap M$. Let $X = N(F_1 \cup T_0)$, and let $Y = N(F_1 \cup F_2 \cup T_0)$, where the regular neighborhoods are taken in M. The frontier of X in M, i.e. $X \cap \overline{M - X}$, is a surface F, which is a four punctured sphere. Note that Y is obtained from X by adding regular neighborhoods of the faces of G_2. Each of the four faces of G_2 is a disk D_i with $\partial D_i = c_i \cup c_i'$, where c_i' is an arc on ∂M, and c_i an arc on $F_1 \cup T_0$. Let $\widetilde{c}_i$ be the arc $D_i \cap F$. Then the frontier of $Y = X \cup (\cup N(D_i))$ in M is a properly embedded surface F', homeomorphic to the surface obtained by cutting F along the arcs $\widetilde{c}_i$. Thus Y and X are homeomorphic, but they are embedded in M differently. Note that Y is uniquely determined by the graphs G_1 and G_2.

It is easy to see that all the $\widetilde{c}_i$ are essential arcs on F. Since each boundary component of F meets $\cup \widetilde{c}_i$ twice, after cutting along all these $\widetilde{c}_i$, the remnant, and hence F', consists of either two disks, or two disks and an annulus. In fact, by examining the graphs, one can see that F' indeed consists of two disks and an annulus. Since M is irreducible and ∂-irreducible, the disk components of F' are boundary parallel. If the annular component A of F' is incompressible in M then A is also boundary parallel because M is anannular and irreducible, so M would be homeomorphic to Y, which in turn is homeomorphic to X. Let C be an essential curve on T_0 disjoint from ∂F_1. Then $C \times I$ in $T_0 \times I$ would be an essential annulus in X, contradicting the fact that M is anannular. Therefore A must be compressible. Let D be a compressing disk of A in M. Then D lies in either Y or $M - \text{Int}Y$. We show that the first case is impossible.

First notice that the surface F_1 cuts X into a manifold $F \times I$, in which both F and the two copies of F_1 are incompressible. By an innermost circle argument one can show that F is incompressible in X. Under the homeomorphism $Y \cong X$, A can be considered as a subsurface of F, hence A is also incompressible unless the core of A is a trivial curve on F. On the other hand, notice that A is a component of $\partial Y - \text{Int}F''$, where $F'' = Y \cap (\partial M - T_0)$ is a neighborhood of $\partial A_1 \cup \partial A_2$ on ∂M, which is connected. Since $\partial A \subset F''$, it follows that the core of A is nonseparating on ∂Y, hence it is nontrivial on F. This completes the proof that A is incompressible in Y.

Hence the compressing disk D of A lies in $M - \text{Int}Y$. Let M' be the union of Y and a regular neighborhood of D. Then the frontier of M' in M is a set of disks, which must be boundary parallel because M is irreducible and ∂-irreducible. Therefore M' is homeomorphic to M. It follows that M is obtained from Y by adding a 2-handle along the core of A, and hence is uniquely determined by the graphs G_1 and G_2. □

§8. Nonspecial graphs with $n_\alpha \leq 2$

First note that if $n_\alpha = 1$ and G_α is not special, then the unique vertex of G_α has valency at most 3 in $\widehat{G}_\alpha$, and hence by Lemma 2.2(3) G_α has at least $2n_\beta$ parallel boundary edges. By Lemma 2.6(2) this implies that G_β, and therefore (by Lemma 4.1) G_α, is special, a contradiction. Hence if G_1, G_2 are not special and $n_1, n_2 \le 2$, we must have $n_1 = n_2 = 2$.

Lemma 8.1. *Suppose that $n_1 = n_2 = 2$ and G_1, G_2 are not special. Then for $\alpha = 1, 2$, the two vertices of G_α are antiparallel, $\widehat{G}_\alpha$ is a subgraph of the graph $\widehat{G}$ in Figure 6.1, and one of the following holds.*

(i) $\Delta = 4$, each interior edge of $\widehat{G}_\alpha$ represents two edges of G_α, and G_α has no boundary edges.

(ii) $\Delta = 5$, each edge of $\widehat{G}_\alpha$ represents two edges of G_α, and the jumping number $q = 2$.

Proof. By Lemma 6.1, G_α is a subgraph of the graph $\widehat{G}$ shown in Figure 6.1. Each vertex v of G_α must have a loop, otherwise some vertex would have valency 3 in $\widehat{G}_\alpha$ with a single boundary edge, so by Lemma 2.6(2) G_β would be special, contradicting the assumption. Since a loop in G_α is a non-loop negative edge of G_β, it follows that each graph G_β has some negative edges, hence the two vertices of G_β must be antiparallel, $\beta = 1, 2$. By Lemma 2.2(3) each interior edge of $\widehat{G}_\alpha$ represents at most two edges of G_α. Similarly, each boundary edge of $\widehat{G}_\alpha$ also represents at most two edges of G_α, by Lemma 2.1(2).

First assume $\Delta = 4$. Notice that a vertex of G_α has either no boundary edge or two boundary edges, for if it has exactly one boundary edge then the loops based at that vertex would have the same label at their two endpoints, which contradicts the parity rule. Since two boundary edges at a vertex of G_α correspond to boundary edges at different vertices of G_β, it follows that either both vertices of G_α have two boundary edges, or they both have no boundary edges. The second possibility gives rise to conclusion (i) in the lemma.

Assume that each vertex of G_α has two boundary edges. Then there are a total of 6 interior edges in each graph. Note that an interior edge is a loop on G_α if and only if it is a non-loop on G_β because of the parity rule, hence one of the graphs, say G_1, has at least three loops. Without loss of generality we may assume that there are two loops e_1, e_2 based at the vertex u_1. Consider their label 1 endpoints. Because there are two boundary edges at u_1, these two endpoints are non adjacent among all label 1 endpoints at u_1. Now look at the graph G_2. By Lemma 2.5(1) e_1, e_2 are non adjacent 1-edges at v_1 among all 1-edges. However, since they are non-loops in G_2, they are contained in the two adjacent families E_3, E_4 in Figure 6.1. Since $E_3 \cup E_4$ contains a total of at most four edges, e_1, e_2 are adjacent among all 1-edges at v_1. This contradiction completes the proof of the lemma for the case $\Delta = 4$.

Now assume $\Delta = 5$. Since each vertex of $\widehat{G}$ has valency 5, and since each edge of $\widehat{G}_\alpha$ represents at most two edges of G_α, $\Delta = 5$ implies that each edge of $\widehat{G}_\alpha$ represents exactly two edges. By the same argument as above one can show that the jumping number q cannot be 1, so we are in case (ii). □

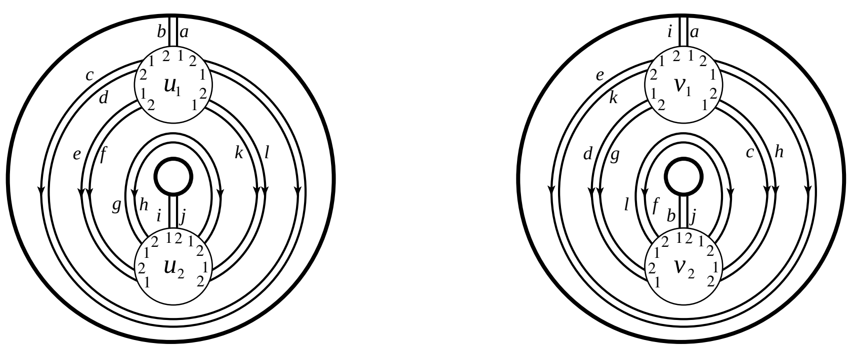
Lemma 8.2. *There is a unique irreducible, ∂-irreducible, anannular manifold M which admits two annular Dehn fillings $M(r_1), M(r_2)$ with $\Delta(r_1, r_2) = 5$.*

Proof. By Lemma 8.1, the graphs must be as shown in Figure 8.1. We first show that the edge correspondence and the labelings of the vertices are unique up to symmetry.

Reflecting the annuli vertically and changing their orientations if necessary, we may assume that the vertices u_1, v_1 are positive, and the labeling of edge endpoints at $\partial u_1, \partial v_1$ are as shown. Any non-loop edge has the same label on its two endpoints, because it is a loop edge on the other graph. Thus the labeling on $\partial u_2, \partial v_2$ is determined by that on $\partial u_1, \partial v_1$, respectively. Orient the edges so that a non-loop edge goes from u_1 to u_2 (resp. v_1 to v_2). Then dually the orientation of a loop edge must go from label 1 to label 2. Label the edges of G_1 as in Figure 8.1(a).

If $P_1, ..., P_5$ are the points of $u_1 \cap v_1$, appearing in this order on ∂u_1 along its orientation, then since the jumping number $q = 2$, they appear in the order P_1, P_3, P_5, P_2, P_4 on ∂v_1 either along or against the orientation of ∂v_1. In other words, along the orientation of ∂v_1 they either appear in this order, or in the order P_1, P_4, P_2, P_5, P_3. In the second case, write $(Q_1, Q_2, ..., Q_5) = (P_1, P_4, P_2, P_5, P_3)$; then $(P_1, ..., P_5) = (Q_1, Q_3, Q_5, Q_2, Q_4)$. Hence by interchanging the roles of G_1 and G_2 if necessary, we may assume that the points appear as $(P_1, P_3, P_5, P_2, P_4)$ on ∂v_1 along the orientation of ∂v_1.

Now we can see that the labeling of the edges on G_2 is completely determined by that of G_1: The 1-edges at u_1 appear in the order a, c, e, k, d in the positive direction, so at v_1 they appear in the order a, e, d, c, k, where a is the unique boundary edge at v_1 labeled 1. The order of the 2-edges at u_1 is b, d, f, l, c, so dually the 1-edges at v_2 are in the order b, f, c, d, l. Similarly by looking at u_2 one can determine the labeling of the remaining edges in G_2. See Figure 8.1(b).



(a) (b)

Figure 8.1

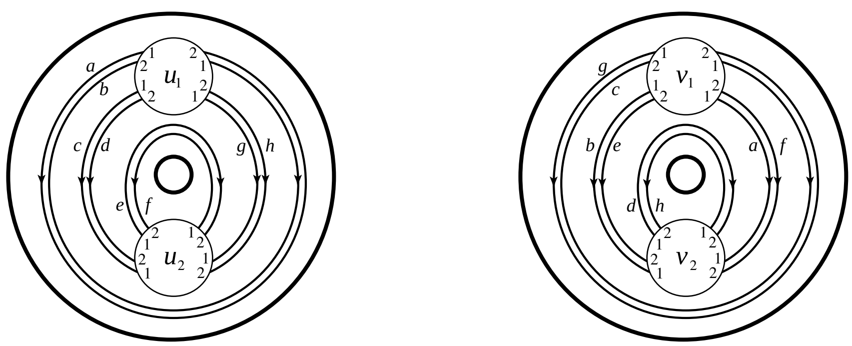
It remains to show that the manifold M is uniquely determined by these graphs. As in the proof of Proposition 7.1, consider the submanifold $X = N(A_1 \cup J_1)$ of $M(r_1)$. Since J_1 intersects A_1 in two meridian disks of opposite sign, the frontier F of X consists of two components F_b, F_w, each being a twice punctured torus, called the *black surface* and the *white surface* respectively. A face of G_2 is black or white according to whether it intersects the black surface or the white surface. Note that each face of G_2 intersects F in a circle or an arc, so it is either black or white, but not both.

Let D_1 be a face of G_2 bounded by a pair of parallel loops, and let D_2 be the triangular interior face of G_2 adjacent to D_1. Since they have an edge in common, they are of different colors, so we may assume that D_1 is black and D_2 is white.

The boundary of D_1 intersects a meridian of J_1 twice in the same direction, hence ∂D_1 is a nonseparating curve on F_b. After adding a neighborhood of D_1 to X, the black frontier is homeomorphic to the surface obtained by 2-surgery on F_b along ∂D_1, hence is an annulus A_b. Since its boundary components are essential curves on ∂M, and since M is ∂-irreducible, A_b is incompressible in M, and hence is boundary parallel in M. Similarly, since the boundary of D_2 intersects a meridian of J_1 three times, ∂D_2 is a nonseparating curve on F_w, so after adding $N(D_2)$ the white frontier becomes an annulus A_w, which for the same reason must be boundary parallel in M. It follows that M is homeomorphic to $N(F_1 \cup T_0 \cup D_1 \cup D_2)$, where F_1 is the punctured annulus $A_1 \cap M$. The boundary curves of D_i are determined by the graphs, which have been determined (up to symmetry) as above. Hence the manifold M is uniquely determined. □

Lemma 8.3. *There is a unique irreducible, ∂-irreducible, anannular manifold M which admits two annular Dehn fillings $M(r_1), M(r_2)$ with $\Delta(r_1, r_2) = 4$ and $n_1 = n_2 = 2$.*

Proof. The proof is similar to that of Lemma 8.2. In this case the jumping number is 1, and one can show that up to symmetry the graphs must be as shown in Figure 8.2. The proof that M is determined by the graphs is the same as in the proof of Lemma 8.2. □



(a) (b)

Figure 8.2

We now prove Theorem 1.1, which we restated here for the reader's convenience.

Theorem 1.1. *Suppose M is a compact, connected, orientable, irreducible, ∂-irreducible, anannular 3-manifold which admits two annular Dehn fillings $M(r_1)$, $M(r_2)$ with $\Delta = \Delta(r_1, r_2) \geq 4$. Then one of the following holds.*

(1) M is the exterior of the Whitehead link, and $\Delta = 4$.

(2) M is the exterior of the 2-bridge link associated to the rational number $3/10$, and $\Delta = 4$.

(3) M is the exterior of the $(-2, 3, 8)$ pretzel link, and $\Delta = 5$.

Proof. By Proposition 6.5, we must have $n_\alpha \leq 2$ for $\alpha = 1, 2$. If G_α is special, then by Proposition 7.1 the manifold M is the exterior of the Whitehead link. If G_α is nonspecial, then by Lemma 8.1 the graphs G_α must be as in Figure 8.1 or 8.2, and by Lemmas 8.2 and 8.3, in each case the manifold M is uniquely determined by the graphs; hence there are at most three manifolds M which may admit two annular Dehn fillings of distance at least 4 apart. On the other hand, it has been shown in [GW1, Theorem 7.5] that each of these manifolds admits two such fillings. Hence the result follows. □

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