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THE CLASSIFICATION OF EXCEPTIONAL DEHN SURGERIES ON 2-BRIDGE KNOTS

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Abstract. We will classify all exceptional Dehn surgeries on 2-bridge knots according to whether they produce reducible, toroidal, or Seifert fibered manifolds.

1. Introduction

A nontrivial Dehn surgery on a hyperbolic knot K in S^3 is *exceptional* if the resulting manifold is either reducible, or toroidal, or a Seifert fibered manifold whose orbifold is a sphere with at most three exceptional fibers, called a *small Seifert fibered space*. Thus an exceptional Dehn surgery is non-hyperbolic, and using a version of Thurston's orbifold theorem proved by Boileau and Porti [BP], it can be shown that a non-exceptional surgery on a 2-bridge knot is also hyperbolic, see Remark 6.3. There has been many researches in determining how many exceptional surgeries a knot can admit. See the survey articles [Go, Lu] for details.

The purpose of this paper is to classify all exceptional Dehn surgeries on 2-bridge knots. Hatcher and Thurston [HT] have shown that there is no reducible surgery on hyperbolic 2-bridge knots. We will determine all toroidal surgeries and small Seifert fibered surgeries on these knots, which will then complete the classification.

We use $[b_1, \dots, b_n]$ to denote the partial fraction decomposition $1/(b_1 - 1/(b_2 - \dots - 1/b_n)\dots)$. Recall that a 2-bridge knot K is non-hyperbolic if and only if $K = K_{1/q}$ for some q, in which case K is a $(2, q)$ torus knot, and surgery on K is well understood. K is a *twist knot* if it is equivalent to some $K_{p/q}$ with $p/q = [b, \pm 2]$ for some integer b. Since $[b, \pm 2] = [b \mp 1, \mp 2]$, we may assume that b is even. Let $K(\gamma)$ be the manifold obtained by γ surgery on K. We always assume that $\gamma \neq \infty$, that is, the surgery is nontrivial. With respect to the standard meridian-longitude pair on $\partial N(K)$, each slope γ is identified with a rational number, see Rolfsen's book [R]. The following is the main theorem of this paper. Note that part (4) of the theorem is the case of surgery on Figure 8 knot, and is due to Thurston [Th1]. It is included in the theorem for the sake of completeness.

Theorem 1.1. *Let K be a hyperbolic 2-bridge knot.*

(1) If $K \neq K_{[b_1,b_2]}$ for any b_1, b_2, then K admits no exceptional surgery.

(2) If $K = K_{[b_1,b_2]}$ with $|b_1|, |b_2| > 2$, then $K(\gamma)$ is exceptional for exactly one γ, which yields a toroidal manifold. When both b_1 and b_2 are even, $\gamma = 0$. If b_1 is odd and b_2 is even, $\gamma = 2b_2$.

(3) If $K = K_{[2n,\pm 2]}$ and $|n| > 1$, $K(\gamma)$ is exceptional for exactly five γ: $K(\gamma)$ is toroidal for $\gamma = 0, \mp 4$, and is small Seifert fibered for $\gamma = \mp 1, \mp 2, \mp 3$.

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(4) If $K = K_{[2,-2]}$ is the Figure 8 knot, $K(\gamma)$ is exceptional for only nine γ: $K(\gamma)$ is toroidal for $\gamma = 0, 4, -4$, and is Seifert fibered for $\gamma = -1, -2, -3, 1, 2, 3$.

We will use a result of Hatcher and Thurston [HT] to determine all toroidal surgeries on a hyperbolic 2-bridge knot K, see Lemma 2.2 below. In general it is more difficult to determine small Seifert fibered surgeries, due to the fact that there is no essential surfaces in such a manifold, unless its Euler number is 0. Here we will use essential lamination theory developed by Gabai and Oertel [GO]. The readers are referred to [GO] for definitions and basic properties concerning essential branched surfaces and essential laminations, which play a central role in the proof of the theorem. We will use Brittenham's criteria [Br], which says that if M is a small Seifert fibered space containing an essential branched surface $\mathcal{F}$, then each component of $M - \text{Int}N(\mathcal{F})$ is an I-bundle over some compact surface G. The idea of the proof is to construct essential laminations in surgered manifolds whose complementary regions are not I-bundles. We will apply some techniques developed by Delman [De1, De2] and Roberts [Ro] to construct essential laminations in the surgered manifolds.

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2. Reducible, toroidal, or Seifert fibered surgeries

Reducible surgeries and toroidal surgeries on 2-bridge knots are completely determined by Lemmas 2.1 and 2.2. Certain surgeries on twist knots are shown to be Seifert fibered in Corollary 2.4.

Lemma 2.1. (Hatcher-Thurston) *Let K be a 2-bridge knot. Then $K(r)$ is reducible if and only if K is a $(2, q)$ torus knot, and $r = 2q$.*

Proof. See [HT, Theorem 2]. □

Lemma 2.2. *Let K be a hyperbolic 2-bridge knot.*

(1) If $K(\gamma)$ is toroidal for some γ, then $K = K_{[b_1,b_2]}$ for some b_1, b_2.

(2) If $|b_i| > 2$ for $i = 1, 2$, there is exactly one such γ. When both b_i are even, $\gamma = 0$. When b_1 is odd and b_2 is even, $\gamma = 2b_2$.

(3) If $K = K_{[2n,2]}$ and $|n| > 1$, $K(\gamma)$ is toroidal if and only if $\gamma = 0$ or -4. For $K = K_{[2n,-2]}$, $\gamma = 0$ or 4.

(4) If $K = K_{[2,-2]}$, then $K(\gamma)$ is toroidal if and only if $\gamma = 0$, 4, or -4.

Proof. We refer the reader to [HT] for notations. If $K(\gamma)$ is toroidal, there is an essential punctured torus T in the knot exterior. By Theorem 1 of [HT], T is carried by some $\Sigma[b_1, \dots, b_k]$, where $[b_1, \dots, b_k]$ is an expansion of p/q. By the proof of Theorem 2 of [HT], we have $0 = 2 - 2g = n(2 - k)$, where g is the genus of T, and n is the intersection number between ∂T and a meridian of K. Therefore $k = 2$. This proves (1). The rest follows by determining all the possible expansions of type $[b_1, b_2]$ for p/q. The boundary slopes of the surfaces can be calculated using Proposition 2 of [HT]. By the proof of [Pr, Corollary 2.1], an incompressible punctured torus T in the exterior of a 2-bridge knot will become an essential torus after surgery along the slope of ∂T. □

Lemma 2.3. *Let $L = k_1 \cup k_2$ be the Whitehead link, which is the 2-bridge link associated to the rational number $p'/q' = [2, 2, -2]$. Let $L(\gamma_1, \gamma_2)$ be the manifold obtained by γ_i surgery on k_i. If $\gamma_1 = -1/n$ and $\gamma_2 = -1$, -2 or -3, then $L(\gamma_1, \gamma_2)$ is a small Seifert fibered space.*

Proof. By definition $L(\infty, \gamma_2)$ is the manifold obtained from S^3 by γ_2 surgery on k_2. After -1 surgery on k_2, the knot k_1 becomes a trefoil knot in $L(\infty, -1) = S^3$. Since the exterior of a torus knot is a Seifert fibered space with orbifold a disk with two cones, it is easy to see that all surgeries but one yield Seifert fibered spaces, each having an orbifold a disk with at most three cone points. For this trefoil, the exceptional surgery has coefficient -6, yielding a reducible manifold. Thus $L(-1/n, -1)$ is a small Seifert fibered space for any n.

After -2 surgery on k_2, the knot k_1 becomes a knot in $\mathbb{R}P^3 = L(\infty, -2)$. The link L is drawn in Figure 1(a), where the curve C is a curve on $\partial N(k_2)$ of slope -2, so it bounds a disk in $L(\infty, -2)$. Thus a band sum of k_1 and C forms a knot k_1' isotopic to k_1 in $L(\infty, -2)$. The link $L' = k_1' \cup k_2$ is shown in Figure 1(b). Using Kirby Calculus one can show that $L(-1/n, -2) = L'(-2 - 1/n, -2)$. The exterior of k_1' in S^3 is a Seifert fibered space with orbifold a disk with two cones, in which k_2 is a singular fiber of index 3. Thus after -2 surgery on k_2, the manifold $L(\infty, -2) - \mathrm{Int}N(k_1')$ is still Seifert fibered, with orbifold a disk with two cones. The fiber slope on $\partial N(k_1')$ is 6. It follows that all but the 6 surgery on k_1' in $L(\infty, -2)$ yield small Seifert fibered manifolds. In particular, $L(-1/n, -2) = L'(-2-1/n, -2)$ are small Seifert fibered manifolds for all n.

The proof for $\gamma_2 = -3$ is similar. One can show that the band sum of k_1 and the curve C of slope -3 on $\partial N(k_2)$ is isotopic to the curve k_1' shown in Figure 1(c), which is a $(3, -2)$ torus knot. By the same argument as above one can show that $L(-1/n, -3) = L(-3 - 1/n, -3)$ are small Seifert fibered manifolds for all n. □

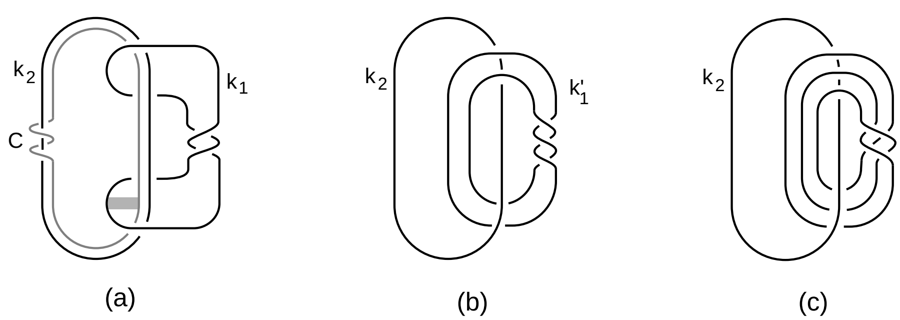

Figure 1

Recall that a 2-bridge knot K is a twist knot if $K = K_{p/q}$, and $p/q = [2n, \pm 2]$ for some n.

Corollary 2.4. *If $K = K_{p/q}$ is a twist knot with $p/q = [2n, \pm 2]$, then $K(\gamma)$ is a small Seifert fibered space for $\gamma = \mp 1$, ∓ 2 and ∓ 3.*

Proof. Consider the case $p/q = [2n, 2]$. The proof for $p/q = [2n, -2]$ is similar. Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p'/q' = [2, 2, -2]$. Notice that after $-1/n$ surgery on k_1, the knot k_2 becomes the knot $K = K_{[2n,2]}$

in $S^3 = L(-1/n, \infty)$. Therefore by Lemma 6, $K(\gamma) = L(-1/n, \gamma)$ are small Seifert fibered spaces for $\gamma = -1$, -2 and -3. □

3. Delman's construction, and the proof of Theorem 1.1(1)

For each rational number p/q, there is associated a diagram $D(p/q)$, which is the minimal subdiagram of the Hatcher-Thurston diagram [HT, Figure 4] that contains all minimal paths from $1/0$ to p/q. See [HT, Figure 5] and [De1]. $D(p/q)$ can be constructed as follows. Let $p/q = [a_1, \dots, a_k]$ be a continued fraction expansion of p/q. To each a_i is associated a "fan" F_{a_i} consisting of a_i simplices, see Figure 2(a) and 2(b) for the fans F_4 and F_{-4}. The edges labeled e_1 are called initial edges, and the ones labeled e_2 are called terminal edges. The diagram $D(p/q)$ can be constructed by gluing the F_{a_i} together in such a way that the terminal edge of F_{a_i} is glued to the initial edge of $F_{a_{i+1}}$. Moreover, if $a_i a_{i+1} < 0$ then F_{a_i} and $F_{a_{i+1}}$ have one edge in common, and if $a_i a_{i+1} > 0$ then they have a 2-simplex in common. See Figure 2(c) for the diagram of $[2, -2, -4, 2]$. Notice that the fans F_{-2} and F_{-4} in the figure share a common triangle.

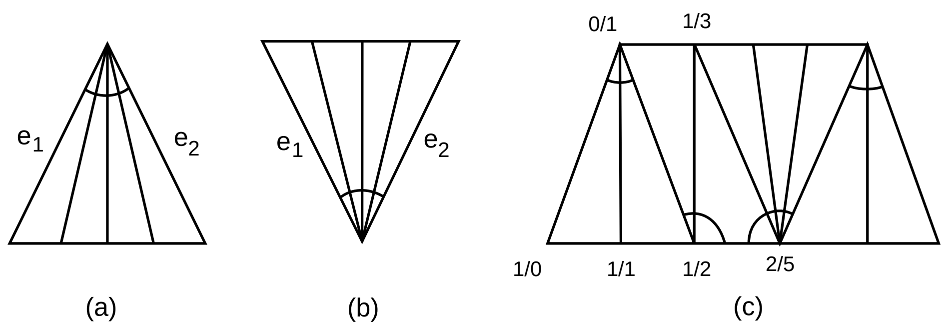

Figure 2

To each vertex v_i of $D(p/q)$ is associated a rational number r_i/s_i. It has one of the three possible parities: odd/odd, odd/even, or even/odd, denoted by o/o, o/e, and e/o, respectively. Note that the three vertices of any simplex in $D(p/q)$ have mutually different parities.

We consider $D(p/q)$ as a graph on a disk D, with all vertices on ∂D, containing ∂D as a subgraph. The boundary of D forms two paths from the vertex $1/0$ to the vertex p/q. The one containing the vertex $0/1$ is called the *top path*, and the one containing the vertex $1/1$ is called the *bottom path.* Edges on the top path are called *top edges*. Similarly for *bottom edges.*

Let Δ_1, Δ_2 be two simplices in $D(p/q)$ with an edge in common. Assume that the two vertices which are not on the common edge are of parity o/o. Then the arcs indicated in Figure 3(a) and (b) are called *channels.* A *path* α in $D(p/q)$ is a union of arcs, each of which is either an edge of $D(p/q)$ or a channel.

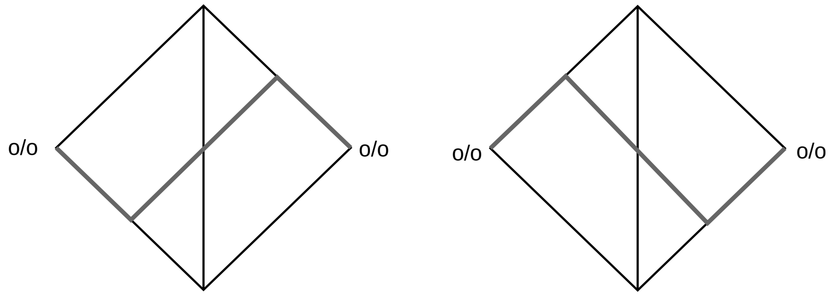

Figure 3

Let v be a vertex on a path α in $D(p/q)$. Let e_1, e_2 be the edges of α incident to v. Then the *corner number* of v in α, denoted by $c(v;\alpha)$ or simply $c(v)$, is defined as the number of simplices in $D(p/q)$ between the edges e_1 and e_2. A path α from $1/0$ to p/q is an *allowable path* if it has at least one channel, and $c(v) \geq 2$ for all v in α.

Now assume that $K = K_{p/q}$ is a 2-bridge knot. Then q is an odd number. Recall that $K_{p/q} = K_{p'/q}$ if $p' \equiv p^{\pm 1} \mod q$, and $K_{-p/q}$ is the mirror image of $K_{p/q}$. We may assume without loss of generality that p is even, and $1 < p < q$. This is because $K_{(q-p)/q}$ is equivalent to the mirror image of $K_{p/q}$, so the result of γ surgery on the first is the same as that of $-\gamma$ surgery on the second. Note that $q - p$ and p have different parity, since q is odd. The following result is due to Delman. See [De1] and [De2, Proposition 3.1].

Theorem 3.1. (Delman) *Given an allowable path α of $D(p/q)$, there is an essential branched surface $\mathcal{F}$ in $S^3 - K$ which remains essential after all nontrivial surgeries on the knot K.* $\square$

Lemma 3.2. *If there is an allowable path α in $D(p/q)$ such that $c(v) > 2$ for some vertex v in α, then $K(\gamma)$ is not a small Seifert fibered space for any γ.*

Proof. It was shown in [Br, Corollary 4] that if $\mathcal{F}$ is an essential branched surface in a small Seifert fibered space M, then each component of $M - \text{Int}N(\mathcal{F})$ is an I-bundle over a compact surface G, such that the vertical surface $\partial_v N(\mathcal{F})$ (also called cusps) is the I-bundle over ∂G. It has been shown in [De1] that for each vertex v of α there is a component W_v of $S^3 - \text{Int}N(\mathcal{F})$ such that W_v is a solid torus whose meridian disk intersects the cusps $c(v)$ times. In particular, if $c(v) > 2$ then W_v is not an I bundle as above. Since $\mathcal{F}$ is an essential branched surface in $K(\gamma)$, it follows that $K(\gamma)$ is not a small Seifert fibered space. $\square$

Lemma 3.3. *Suppose p is even, q is odd, and $1 < p < q-1$. If p/q does not have partial fraction decomposition of type $[r_1, r_2]$, then $D(p/q)$ has an allowable path α such that some vertex v on α has $c(v) > 2$.*

Proof. Let $[a_1, \ldots, a_n]$ be the partial fraction decomposition of p/q such that all a_i are even. Then $a_1 \geq 2$. If $a_i = 2$ for all i, then $p/q = (q-1)/q$, contradicting our assumption. Thus either some $a_i < 0$, or some $a_i > 4$. We separate the two cases.

CASE 1. *Some $a_i < 0$.*

Let a_i be the first negative number. Then $a_{i-1} > 0$, so there is a sign change. By [De2] there is a channel α_0 in $F_{a_{i-1}} \cup F_{a_i}$ starting at a bottom edge and ending at a top edge, where F_{a_i} is the fan in $D(p/q)$ corresponding to a_i. Let α_1 be the part of the bottom path of $D(p/q)$ from the vertex $1/0$ to the initial point of α_0, and let α_2 be the part of the top path from the end point of α_0 to the vertex p/q. Then $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$ is an allowable path in $D(p/q)$. We need to show that if $c(v) = 2$ for all vertices v on this path, then $p/q = [r_1, r_2]$ for some r_1, r_2.

Consider the vertices on α_1. Since $c(v_i) = 2$ for all v_i, each vertex v_i is incident to exactly one non boundary edge e_i of $D(p/q)$, which must have the other end on a vertex v_i' in the top path. If some of these v_i' are different, then since all faces of $D(p/q)$ are triangles, it is clear that some v_j on α_1 would have at least two non boundary edges, which would be a contradiction. Similarly, each vertex on α_2 has a unique non boundary edge, leading to a common vertex on the bottom path, so the diagram $D(p/q)$ looks exactly as in Figure 4(a). It is the union of two fans F_{r_1} and F_{r_2} with $r_1 > 0$, and $r_2 < 0$. Therefore, $p/q = [r_1, r_2]$.

CASE 2. *Some* $a_i \geq 4$.

In this case there is a channel α_0 with both ends on the bottom path. Construct an allowable path $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$ with α_1, α_2 in the bottom path. Similar to Case 1, it can be shown that each vertex on α_i has a unique non boundary edge leading to a common vertex v_i' on the top path, so $D(p/q)$ looks like that in Figure 4(b). In this case $p/q = [r_1, r_2]$, with both $r_i > 0$. □

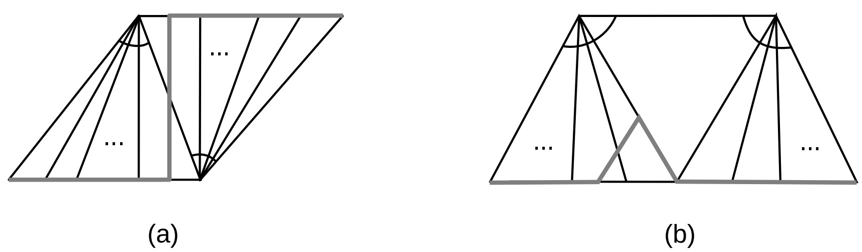

Figure 4

Corollary 3.4. *Let K be a 2-bridge knot. If $K \neq K_{[b_1,b_2]}$ for any b_1, b_2, then $K(\gamma)$ is non-exceptional for all γ.*

Proof. K is not a $(2, q)$ torus knot, otherwise $K = K_{[2q\pm1]} = K_{[2q,\mp1]}$. Hence by Lemmas 2.1 and 2.2, $K(\gamma)$ is irreducible and atoroidal. By Lemmas 3.3 and 3.2, $K(\gamma)$ is not a small Seifert fiber space. Therefore, $K(\gamma)$ is non-exceptional. □

4. Surgery on twisted Whitehead links, proof of Theorem 1.1(2)

A twisted Whitehead link is a two bridge link L associated to a rational number $[2, r, -2]$ for some $r \neq 0$. See Figure 5 for a twisted Whitehead link with $r = -6$. When $r = \pm 2$, L is a Whitehead link. It has been determined exactly which 2-bridge link complements contain persistent laminations [Wu]. The next lemma shows that if $|r| > 2$ then there is a persistent lamination with some desired property.

Recall that a slope γ of a knot K is an integral slope if it intersects the meridian of K exactly once.

Lemma 4.1. *Let $L = k_1 \cup k_2$ be a twisted Whitehead link associated to the rational number $p/q = [2, r, -2]$. Let $L(\gamma_1, \gamma_2)$ be the manifold obtained by γ_i surgery on k_i.*

If $|r| > 2$, and one of the γ_i is not an integral slope, then $L(\gamma_1, \gamma_2)$ is not a small Seifert fiber space.

Proof. By considering the mirror image of L if necessary we may assume that $r < 0$. If r is even, then $[2, r, -2]$ is a partial fraction decomposition with even coefficient, and $r \leq -4$. There is an allowable path in $D(p/q)$ with two channels, as shown in Figure 6(a), where $r = -4$. If r is odd, then $p/q = [2, r+1, 2]$, in which case $D(p/q)$ also has an allowable path with two channels. See Figure 6(b) for the case $r = -3$.

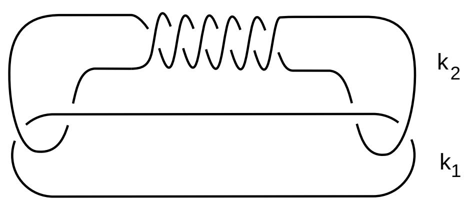

Figure 5

Let $\mathcal{F}$ be the essential branched surface in the link exterior associated to the above allowable path in $D(p/q)$, as constructed in [De2]. There is one solid torus component V_i in $S^3 - \text{Int}N(\mathcal{F})$ for each k_i, containing k_i as a central curve. From the construction of $\mathcal{F}$ one can see that each channel contributes two cusps, one on each ∂V_i. Actually from [De2, Figure 3.5] we see that the two cusps corresponding to a channel are around two points of L on a level sphere with same orientation. Since each k_i intersects the sphere at two points with different orientations, those two cusps must be around different components of L. One is referred to [Wu] for more details about surgery on 2-bridge links.

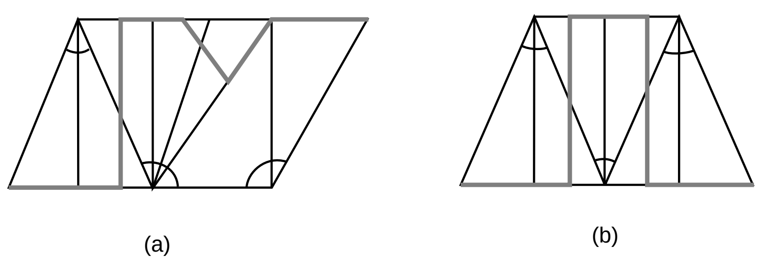

Figure 6

As the allowable path above has two channels, each V_i has two meridional cusps. Thus $\mathcal{F}$ remains an essential branched surface after surgery on L. Moreover, since one of the γ_i is non-integral, after surgery V_i becomes a solid torus whose meridional disk intersects the cusps at least four times. By [Br, Corollary 4], the surgered manifold is not a small Seifert fiber space. □

Corollary 4.2. *Let $K = K_{[b_1,b_2]}$ be a two bridge knot with $|b_i| > 2$ for $i = 1, 2$. Then $K(\gamma)$ is exceptional for only one γ, which yields toroidal manifold. When both b_1 and b_2 are even, $\gamma = 0$. If b_1 is odd and b_2 is even, $\gamma = 2b_2$.*

Proof. By Lemmas 2.1 and 2.2, $K(\gamma)$ is irreducible, and it is toroidal for exactly one γ as described in the corollary. So it remains to show that $K(\gamma)$ is never a small Seifert fiber space.

Since K is a knot, at least one of the b_i is an even number. We may assume without loss of generality that $b_1 = 2n$ for some integer n, because $K_{[b_1,b_2]}$ is equivalent to $K_{[b_2,b_1]}$, by turning the standard diagram for the first knot upside down.

Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p/q = [2, b_2, -2]$. Notice that after $-1/n$ surgery on k_1, the other component k_2 becomes the knot $K = K_{[2n,b_2]}$. Therefore, doing γ surgery on K is the same as doing $-1/n$ surgery on k_1, then doing some γ' surgery on k_2. Since $-1/n$ is non integral, and $|b_2| > 2$, the result follows from Lemma 4.1. □

Corollary 4.3. *Let $K = K_{[b,\pm 2]}$ with $|b| > 2$. If γ is a non integral slope, then $K(\gamma)$ is not a small Seifert fiber space.*

Proof. As above, $K(\gamma) = L(\gamma, \pm 1)$, where $L = L_{[2,b,-2]}$. Since γ is non integral, the result follows from Lemma 4.1. The result also follows from [Br]. □

5. Roberts' construction of essential branched surfaces

In [Ro] Roberts constructed branched surfaces in certain knot complements, which can be extended to essential branched surfaces in $K(\gamma)$ for all γ in an infinite interval. We will describe her results and construction in this section, and apply them to surgery on twist knots in the next section.

Let $E(K) = S^3 - \text{Int}N(K)$ be the exterior of a knot K in S^3, let R' be a (possibly non orientable) compact surface in S^3 with $\partial R' = K$. Let $R = R' \cap E(K)$.

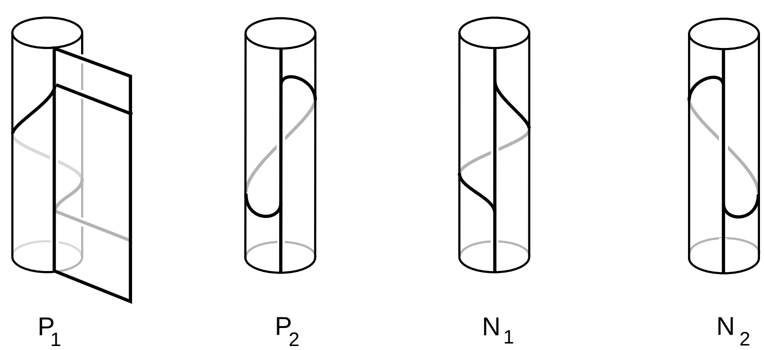

Figure 7

Let S be a surface in $E(K)$ which has interior disjoint from R, and has a single boundary curve $\partial S = a_1 \cup b_1 \cup \ldots \cup a_n \cup b_n$, where b_i are mutually disjoint arcs on R, and a_i are arcs on $T = \partial N(K)$. By specifying a cusp at each b_i, the union of R and S becomes a branched surface $B = \langle R, S \rangle$ in $E(K)$. The cusps will be assigned in such a way that each a_i on T is one of the four types indicated in Figure 7. Note that ∂B is a train track on T, and each component of $T - \partial B$ is a digon, i.e a disk with two cusps.

Remark. The pictures on Figure 7 are mirror images of that in [Ro, Figure 22]. Thus for example, type P_1 here is of type N_1 in [Ro]. Apparently we are using different coordinate systems. This paper adapts the convention that the meridian-longitude pair (m, l) on T is chosen so that when K is endowed with the same orientation as that of l, the linking number $lk(m, K) = 1$, measured using the right hand rule. See [R]. With this convention, types P_1, P_2 in Figure 7 will have positive contributions to any slope γ carried in the train track.

Let $N(B)$ be a regular neighborhood of the branched surface B in $E(K)$ with the natural I-bundle structure. A *surface of contact* is a properly embedded compact surface P in $N(B)$, transverse to the I-fibers, with $\partial P \subset \partial_v N(B) \cup T$, such that the intersection of ∂P with each component of the vertical surfaces $\partial_v N(B)$ is either empty or a single arc.

Let p_1, p_2, n_1, n_2 be the numbers of a_i of type P_1, P_2, N_1, N_2 respectively. Let r be the slope of ∂R on T. Let

$$J = \{r + (p_1 - n_1)\frac{x}{x+1} + (p_2 - n_2)x \mid x > 0\}$$

Then Roberts' theorem [Ro, Theorem 2.3] can be stated as

Theorem 5.1. (Roberts) *If $B = \langle R, S\rangle$ constructed above is an essential branched surface in $E(K)$, and has no planar surface of contact, then B extends to an essential branched surface B_γ in $K(\gamma)$ for all slope $\gamma \in J$.*

The construction of the extended branched surface is as follows. Let $T \times I$ be a small neighborhood of T in $E(K)$ with $T = T \times 0$, such that $B \cap (T \times I) = \partial B \times I$. Add the digons $T \times 1 - B$ to B, and branched so that the cusps on the two edges of each digon lies on different sides. The definition of J guaranteed that the train track ∂B on T can be split to produce a curve C of slope γ on T. Split $\partial B \times I$ accordingly and, after Dehn filling, cap off C by a meridian disk in the Dehn filling solid torus, one obtains a branched surface B_γ in $K(\gamma)$. It was shown in [Ro] that B' carries an essential lamination in $K(\gamma)$.

Denote by $E(B)$ the exterior of B in $E(K)$, i.e $E(B) = E(K) - \text{Int}N(B)$.

Corollary 5.2. *For any $\gamma \in J$, the manifold $E(B)$ is homeomorphic to a component W of the exterior of B_γ in $K(\gamma)$, with horizontal surface of B identified to the horizontal surface of B_γ on ∂W.*

Proof. Examine the above construction. After adding the digons of $T \times 1 - B$ to B, the branched surface is topologically homeomorphic to $B \cup (T \times 1)$, which cuts off a region isotopic to $E(B)$. Clearly this region is not affected by the later changes, and its horizontal surface is the restriction of that on $E(B)$. □

6. Surgery on twist knots, and the final proof

As noticed earlier, any twist knot can be written as $K_{[2n,\pm 2]}$ for some n, because if b is odd then $K_{[b,2]} = K_{[b-1,-2]}$. Since $K_{[2n,-2]}$ is the mirror image of $K_{[-2n,2]}$, we need only consider knots of type $K_{[2n,2]}$. We can also assume that $n \neq 0, 1$, otherwise the knot is a trivial knot or a trefoil knot.

Lemma 6.1. *Let $K = K_{[2n,2]}$ be a twist knot with $n \neq 0, 1$. Then $K(\gamma)$ is not a small Seifert fiber space for all $\gamma < -4$.*

Proof. A knot $K = K_{[2n,2]}$ has two spanning surfaces as indicated in Figure 8, where $2n = 4$. The first surface is a punctured Klein bottle, and the second one is a punctured torus.

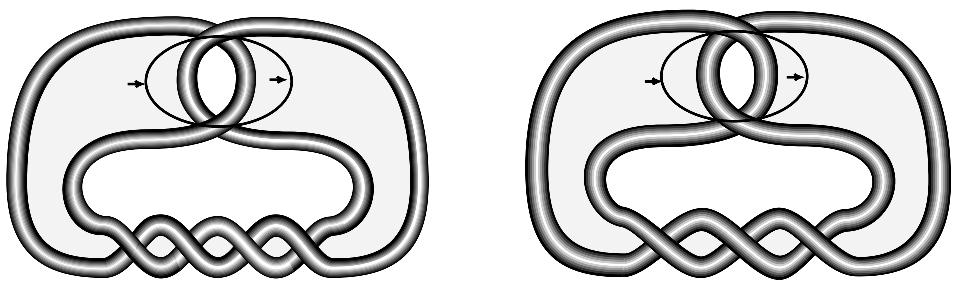

Figure 8

Let R be the punctured Klein bottle of Figure 8(1) in the knot exterior. Add a disk S to R such that the boundary of S is the circle indicated in Figure 8(1). The boundary of S consists of four arcs $a_1 \cup b_1 \cup a_2 \cup b_2$, where b_i lies on R, and a_i on the torus $T = \partial N(K)$. The arrows at the arcs b_i indicate the side of the cusp. This determines the branched surface $B = \langle R, S \rangle$, and hence the type of a_i on T. One of the a_i (the one on top) is of type P_1, and the other one of type N_2. Using the notation in Section 5, we have $p_1 = n_2 = 1$, and $p_2 = n_1 = 0$. By calculating the linking number between K and ∂R, one can see that ∂R has slope $r = -4$ on T. Therefore,

$$J = \{r + (p_1 - n_1)\frac{x}{x+1} + (p_2 - n_2)x \mid x > 0\} = \{-4 + \frac{x}{x+1} - x \mid x > 0\}$$
$$= \{-4 - \frac{x^2}{x+1} \mid x > 0\} = (-\infty, -4)$$

We need to show that B is an essential branched surface in $E(K)$. It is clear that $N(B)$ is topologically a solid torus. By examining the cusp on $\partial N(B)$, one can see that the exterior of B is the same as that of the surface in Figure 9, with cusp the boundary of the surface F. Note that ∂F runs along the meridian of the solid torus $N(F)$ $2n - 1$ times. Hence $E(B)$, the exterior of B, is a solid torus with a single cusp running along the longitude $2n-1$ times. Since $n \neq 0, 1$, it is easy to see that $E(B)$ satisfies all conditions for B to be essential, i.e, $E(B)$ is indecomposable, and the horizontal surface on $\partial E(B)$ is essential. One also needs to check that the branched surface B satisfies all the intrinsic essentiality properties, i.e, it has no disk or half disk of contact, it contains no Reeb branched surfaces, and it fully carries a lamination. This is straight forward.

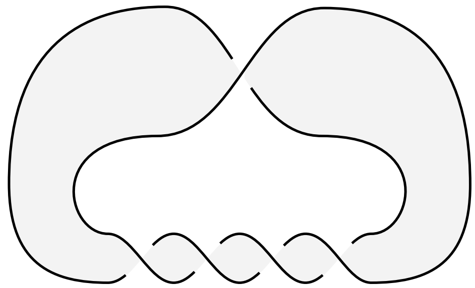

Figure 9

To apply Roberts' theorem, we also need to show that $B = \langle R, S\rangle$ has no planar surface of contact. Assume that P is a compact surface embedded in $N(B)$, transverse to the I-fibers, with $\partial P \subset \partial_v N(B) \cup T$. Cutting along the two arcs b_1, b_2 along which S is glued to R, the branched surface B becomes two surface S and $R' = R - b_1 \cup b_2$. Let u, v be the number of times that P intersects an I-fiber of S and R' respectively. Then the gluing at b_i gives the equation $v = u + v + w$, where w is the times ∂P passes the cusp at b_i. So $u = w = 0$, and P lies in $N(R)$. Since R is nonplanar, P can not be planar.

It now follows from Theorem 5.1 that B extends to an essential branched surface B_γ for all slopes $\gamma < -4$. By Corollary 5.2, $E(B)$ can be considered as a component of the exterior of B_γ. Since $E(B)$ is a solid torus with a cusp running along the longitude $|2n-1| \geq 3$ times, it is not an I-bundle. Therefore, by the result of Brittenham [Br], $K(\gamma)$ is not a small Seifert fiber space for all $\gamma < -4$. □

Lemma 6.2. *Let $K = K_{[2n,2]}$ be a twist knot with $|n| > 2$. Then $K(\gamma)$ is not a small Seifert fiber space for all $\gamma > 0$.*

Proof. The proof is very similar to that of Lemma 6.1, only that instead of using the non orientable surface in Figure 8(1), we use the orientable surface R in Figure 8(2). As a Seifert surface of K, the boundary of R has slope 0. Also, the arcs a_1, a_2 are now of type P_2 and N_1, so $p_1 = n_2 = 0$, $p_2 = n_1 = 1$, and

$$J = \{r + (p_1 - n_1)\frac{x}{x+1} + (p_2 - n_2)x \mid x > 0\} = \{0 - \frac{x}{x+1} + x \mid x > 0\}$$
$$= \{\frac{x^2}{x+1} \mid x > 0\} = (0, \infty)$$

The exterior of the branched surface $B = \langle R, S\rangle$ is the same as that of a band with $2n$ twists. Since $|n| > 2$, it is not an I-bundle. Therefore, one can use the argument in the proof of Lemma 6.1 to obtain a conclusion. □

Proof of Theorem 1.1. Parts (1) and (2) are exactly Corollaries 3.4 and 4.2. For part (3), consider the knot $K_{[2n,2]}$. By Lemma 2.2 and Corollary 2.4, surgeries with $\gamma = 0, 4$ are the only toroidal ones, and $\gamma = -1, -2, -3$ are Seifert fibered. By Corollary 4.3, Lemmas 6.1 and 6.2, there are no other small Seifert fibered surgeries. Since no surgeries on $K_{[2n,2]}$ are reducible (Lemma 2.1), all surgeries with $\gamma \neq 0, -1, -2, -3, -4$ are non-exceptional. Since $K_{[2n,-2]}$ is the mirror image of $K_{[-2n,2]}$, the result follows.

Part (4) follows from the well known result of Thurston about surgery on the Figure 8 knot [Th1], which is stronger as the non-exceptional surgeries are shown to be hyperbolic. The result as stated can also be proved using the techniques in this paper: As toroidal and reducible surgeries are known, we only need to deal with small Seifert fibered surgeries. By [Br] all non-integral surgeries on $K = K_{[-2,2]}$ are not small Seifert fibered. By Lemma 6.1 $K(\gamma)$ are not small Seifert fibered for $\gamma < -4$. Since K is amphicheiral, this is also true for $\gamma > 4$. □

Remark 6.3. Boileau and Porti [BP] proved a version of Thurston's Orbifold Theorem, showing that the geometrization conjecture is true for manifolds which admit a finite group action with nonempty fixed point set. Using this result one can show that non-exceptional surgeries on 2-bridge knots are also hyperbolic. The proof is as follows. A p/q 2-bridge knot can be obtained by taking two arcs of slope p/q on the "pillowcase", then joining the ends with two trivial arcs. From this picture it is easy to see that K is a strongly invertible knot, i.e. there is an involution φ of S^3 such that $\varphi(K) = K$, and the fixed point set of φ is a circle S intersecting K at two points. The map φ restricts to an involution of $E(K) = S^3 - \text{Int}N(K)$, which can be extended to an involution $\widehat{\varphi}$ of the surgered manifold. Since $\widehat{\varphi}$ has nonempty fixed point set, the result follows from [BP].

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