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ANNULAR AND BOUNDARY REDUCING DEHN FILLINGS

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§0. Introduction

Surfaces of non-negative Euler characteristic, i.e., spheres, disks, tori and annuli, play a special role in the theory of 3-dimensional manifolds. For example, it is well known that every (compact, orientable) 3-manifold can be decomposed into canonical pieces by cutting it along essential surfaces of this kind [K], [M], [Bo], [JS], [Jo1]. Also, if (as in [Wu3]) we call a 3-manifold that contains no essential sphere, disk, torus or annulus *simple*, then Thurston has shown [T1] that a 3-manifold M with non-empty boundary is simple if and only if M with its boundary tori removed has a hyperbolic structure of finite volume with totally geodesic boundary. For closed 3-manifolds M, the Geometrization Conjecture [T1] asserts that M is simple if and only if M is either hyperbolic or belongs to a certain small class of Seifert fiber spaces.

Because of their importance, a good deal of attention has been directed at the question of when surfaces of non-negative Euler characteristic can be created by Dehn filling. To describe this, let M be a simple 3-manifold, with a torus boundary component $\partial_0 M$. Let α be the isotopy class of an essential simple loop (or *slope*) on $\partial_0 M$. Recall that the manifold obtained from M by α-*Dehn filling* is $M(\alpha) = M \cup V_\alpha$, where V_α is a solid torus, glued to M by a homeomorphism between $\partial_0 M$ and ∂V_α which identifies α with the boundary of a meridian disk of V_α. We are interested in obtaining restrictions on when $M(\alpha)$ fails to be simple. Although clearly little can be said in general about a single Dehn filling, if one considers pairs of non-simple fillings $M(\alpha)$, $M(\beta)$ then it turns out that the *distance* $\Delta(\alpha, \beta)$ between the two slopes α and β (i.e., their minimal geometric intersection number) is quite small, and hence a given M can have only a small number of non-simple fillings. More precisely, if $M(\alpha)$, $M(\beta)$ contain essential surfaces F_α, F_β of non-negative Euler characteristic, then for each of the ten possible pairs of homeomorphism classes of F_α, F_β one can obtain upper bounds on $\Delta(\alpha, \beta)$. In the present paper we deal with the case where F_α is an annulus and F_β is a disk, and prove the following theorem.

Theorem 0.1. *Let M be a simple 3-manifold such that $M(\alpha)$ is annular and $M(\beta)$ is boundary reducible. Then $\Delta(\alpha, \beta) \le 2$.*

The assumption that M is a simple manifold can be replaced by the weaker assumption that it is boundary irreducible and anannular, see Corollary 5.5. The bound is sharp: infinitely many examples of simple 3-manifolds M with $M(\alpha)$ annular, $M(\beta)$ a solid torus, and $\Delta(\alpha, \beta) = 2$ are given in [MM2]. See also [EW].

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M(β) \ M(α)	S	D	A	T
S	1	0	2	3
D		1	2	2
A			5	5
T				8

Table 1: Upper bounds on $\Delta(\alpha, \beta)$

Theorem 0.1 completes the determination of the best possible upper bounds on $\Delta(\alpha, \beta)$ in all ten cases. These are shown in Table 1, where S, D, A and T indicate that the manifold $M(\alpha)$ or $M(\beta)$ contains an essential sphere, disk, annulus or torus, respectively. References for these bounds are: (S, S): [GL3] (see also [BZ]); (S, D): [Sch]; (S, A): [Wu3]; (S, T): [Wu1], [Oh]; (D, D): [Wu2]; (D, T): [GL4]; (A, A), (A, T) and (T, T): [Go]. Examples showing that the bounds are best possible can be found in: (S, S): [GLi]; (D, D): [Be] and [Ga]; (S, A), (D, A) and (D, T): [HM]; (S, T): [BZ2]; (T, A) and (A, A): [GW]; (T, T): [T2] and [Go].

Here is a sketch of the proof of Theorem 0.1. It has been shown by Qiu [Qiu] that $\Delta \leq 3$, so we assume $\Delta = 3$, and try to get a contradiction. Let A and B be an essential annulus and an essential disk in $M(\alpha)$ and $M(\beta)$, and let P and Q be the intersection of A and B with M, respectively. Let p, q be the number of boundary components of P, Q on the torus $\partial_0 M$. Denote by K_β the core of the Dehn filling solid torus V_β in $M(\beta)$.

In Section 2 we consider the special case that $K = K_\beta$ is a *1-arch* knot, which means that it can be isotoped to a union of two arcs C_1 and C_2, such that C_1 lies on $\partial M(\beta)$, and C_2 is disjoint from the compressing disk $B = F_\beta$ of $\partial M(\beta)$. In this case the manifold $M(\beta)$ is homeomorphic to a manifold X_C obtained by adding a 2-handle to a certain manifold X along a curve C. This changes a Dehn surgery problem to a handle addition problem, and we will use a theorem of Eudave-Muñoz to show that in this case the annulus $A = F_\alpha$ can be chosen to intersect the knot K_α at most twice, that is, $p \leq 2$.

As usual, the intersection of $P \cap Q$ defines graphs G_A, G_B on A and B. In Section 3 the "representing all types" technique developed in [GL1–GL4] is modified to suit the case that the intersection graphs have boundary edges. It will be proved that when $\Delta \geq 2$, either G_A represents all types, or G_B contains a great web. The first possibility is impossible because it would lead to a boundary reducing disk of $M(\beta)$ which has less intersection with K_β, hence G_B must contain a great web. This great web is then used in Section 4 to show that if $p \geq 3$ then the knot K_β is a 1-arch knot. Combined with the result of Section 2, this proves Theorem 0.1 in the generic case that $p \geq 3$. Finally in Section 5 the case $p \leq 2$ is ruled out, completing the proof of Theorem 0.1.

The authors would like to thank John Luecke for helpful conversations.

§1. Preliminaries

Recall that a 3-manifold X is *boundary reducible* if its boundary, denoted by ∂X, is compressible in X, in which case a compressing disk of ∂X is also called a *boundary reducing disk* of X. A surface of non-positive Euler characteristic in X is *essential* if it is incompressible, ∂-incompressible, and is not boundary parallel; a sphere (resp. disk) is essential if it is a reducing sphere (resp. boundary reducing disk.)

Let M be a simple 3-manifold, with a torus boundary component $\partial_0 M$. Let α, β be slopes on $\partial_0 M$ such that $M(\alpha)$ is annular and $M(\beta)$ is boundary reducible. Let A be an essential annulus in $M(\alpha)$, and let B be an essential disk in $M(\beta)$. These give rise to a punctured annulus $P = A \cap M$ and a punctured disk $Q = B \cap M$ in M, where $\partial_0 P = P \cap \partial_0 M$ consists of p copies of α, and $\partial_0 Q = Q \cap \partial_0 M$ consists of q copies of β. We assume that A, B, P, Q are chosen so that p and q are minimal. Note that p, q are positive because M is simple. Now isotope P and Q to minimize $|P \cap Q|$, the number of components of $P \cap Q$. Then no arc component of $P \cap Q$ is boundary parallel in P or Q; no circle component of $P \cap Q$ bounds a disk in P or Q; and each component of $\partial_0 P$ meets each component of $\partial_0 Q$ in $\Delta = \Delta(\alpha, \beta)$ points.

Ruifeng Qiu showed in [Qiu] that if M is a simple manifold, $M(\alpha)$ is annular and $M(\beta)$ is boundary reducible, then $\Delta \leq 3$. Thus to prove Theorem 0.1, we need only rule out the possibility that $\Delta = 3$. *In this paper except in Section 3, we will assume that $\Delta = 3$, and proceed to get a contradiction.* Results in Section 3 have been proved in a broader setting, so they can be used in the future.

Regarding the components of $\partial_0 P$, $\partial_0 Q$ as fat vertices, we get graphs G_A, G_B in A, B respectively, where the edges of G_A and G_B are the arc components of $P \cap Q$ that have at least one endpoint on $\partial_0 M$. Let $J = A$ or B. An edge of G_J is an *interior* edge if each of its endpoints lies on a vertex of G_J, and a *boundary* edge if one of its endpoints lies on a vertex of G_J and the other lies on ∂J. The faces of G_J correspond in the usual way to components of $J - \text{Int}N(G_J)$. A face of G_J is an *interior* face if it does not meet ∂J; otherwise it is a *boundary* face. Thus the edges in the boundary of an interior face are interior edges, while the boundary of a boundary disk face contains some boundary edges. Denote by $\widehat{G}_J$ the reduced graph of G_J, in which each parallel family of edges is replaced by a single edge.

Let $u_1, \ldots, u_p$ be the vertices of G_A, labeled successively when traveling along the Dehn filling solid torus V_α. Each u_i is given a sign according to whether V_α passes A from the positive side or negative side at this vertex. Two vertices u_i, u_j are *parallel* if they have the same sign, otherwise they are *antiparallel*. The vertices $v_1, \ldots, v_q$ of G_B are labeled and signed similarly.

If e is an edge of G_A with an endpoint on u_i, then the endpoint is labeled j if it is on $\partial u_i \cap \partial v_j$. Thus when going around ∂u_i, the labels of the edge endpoints appear as $1, 2, ..., q$ repeated Δ times. The edge endpoints of G_B are labeled similarly.

A cycle in G_A or G_B is a *Scharlemann cycle* if it bounds a disk with interior disjoint from the graph, and all the edges in the cycle have the same pair of labels $\{i, i+1\}$ at their two endpoints, called the *label pair* of the Scharlemann cycle. A pair of edges $\{e_1, e_2\}$ is an *extended Scharlemann cycle* if there is a Scharlemann cycle $\{e_1', e_2'\}$ such that e_i is parallel and adjacent to e_i'.

We use $N(X)$ to denote a regular neighborhood of a subset X in a given manifold.

Lemma 1.1. (Properties of G_A.)

(1) (The Parity Rule) *An edge connects parallel vertices on G_A if and only if it connects antiparallel vertices on G_B.*

(2) G_A does not have q parallel interior edges.

(3) G_A contains no Scharlemann cycles.

(4) Each label $x \in \{1, ..., q\}$ appears at most once among the endpoints of a family $\mathcal{E}$ of parallel edges in G_A connecting parallel vertices; in particular, $\mathcal{E}$ contains at most $q/2$ edges.

(5) No pair of edges are parallel on both G_A and G_B.

Proof. (1) This is on [CGLS, Page 279].

(2) If G_A contains q parallel interior edges, then the core of the Dehn filling solid torus in $M(\beta)$ would be a cable knot, in which case M contains an essential annulus, contradicting the assumption. See the proof of [GLi, Proposition 1.3].

(3) This follows from [CGLS, Lemma 2.5.2].

(4) If some label appears twice among the endpoints of a family of parallel edges connecting a pair of parallel vertices, then there is a Scharlemann cycle among this family, contradicting (3). See [CGLS, Lemma 2.6.6].

(5) By [Go, Lemma 2.1], no two interior edges are parallel on both G_A and G_B. If a pair of boundary edges are parallel on both graphs, then they cut off a disk on each of P and Q, whose union is an annulus in M, which is essential because its intersection with $\partial_0 M$ is a curve intersecting α at a single point. This contradicts the assumption that M is simple. □

Lemma 1.2. (Properties of G_B.)

(1) If G_B has a Scharlemann cycle, then A is a separating annulus, and p is even. Moreover, the subgraph of G_A consisting of the edges of the Scharlemann cycle and their vertices is not contained in a subdisk of A.

(2) If $p > 2$, then G_B has no extended Scharlemann cycle. Any two Scharlemann cycles of G_B have the same label pair.

Proof. (1) This follows from the proof of [CGLS, Lemma 2.5.2]. It was shown that using the disk bounded by the Scharlemann cycle one can find another annulus A' in $M(\alpha)$ which has fewer intersections with the Dehn filling solid torus, and is cobordant to A, so if A were nonseparating then A' would still be essential, which would contradict the minimality of p. If the subgraph G consisting of the edges of a Scharlemann cycle and their end vertices is contained in a disk in A then $A' \cup A$ bounds a connected sum of $A \times I$ and a lens space, so A being essential implies that A' is essential, which again contradicts the minimality of p.

(2) This is [Wu3, Lemma 5.4(2) – (3)]. If G_B has an extended Scharlemann cycle or two Scharlemann cycles with distinct label pairs, then one can find another essential annulus in $M(\alpha)$ having fewer intersection with K_α, which would contradict the minimality of p. □

§2. 1-arch knots

Let $K = K_\beta$ be the core of the Dehn filling solid torus in $M(\beta)$. The knot K is a *1-arch* knot (with respect to B) if K is isotopic to a union of two arcs C_1 and C_2, such that C_1 lies on $\partial M(\beta)$, and C_2 is disjoint from a compressing disk B of $\partial M(\beta)$.

Fix an orientation of K so that when traveling along K with this orientation one meets the fat vertices $v_1, \dots, v_q$ successively. Let $K[i]$ be the point $K \cap v_i$, and for

$i \neq j$, let $K[i,j]$ be the oriented arc segment of K starting from $K[i]$ and ending at $K[j]$. Thus $K = K[i,j] \cup K[j,i]$.

Lemma 2.1. *If G_A contains q parallel boundary edges, then $K = K_\beta$ is a 1-arch knot.*

Proof. Without loss of generality we may assume that the interior endpoints of the parallel boundary edges $e_1, \ldots, e_q$ are successively labeled $1, \ldots, q$. Let D be the disk on P cut off by e_1 and e_q. Then D can be extended into the Dehn filling solid torus $N(K)$ to get a disk D' in $M(\beta)$ such that $\partial D' = e'_1 \cup K[1,q] \cup e'_q \cup C_1$, where e'_i is an arc on B containing e_i, connecting $K[i]$ to the endpoint of e_i on ∂A, and $C_1 = D \cap \partial A$ lies on $\partial M(\beta)$. Now K is isotopic to $C_1 \cup (e'_1 \cup K[q,1] \cup e'_q)$ via the disk D'. Let $C_2 = e'_1 \cup K[q,1] \cup e'_q$. After a slight isotopy one can make B disjoint from C_2, as desired. □

Lemma 2.2. *Suppose $\widehat{G}_A$ has a vertex u of valency 4, such that one of the four edges of $\widehat{G}_A$ incident to u is a boundary edge, and the two edges adjacent to it are interior edges. Then either G_B contains a Scharlemann cycle, or $K = K_\beta$ is a 1-arch knot.*

Proof. Let $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4$ be the four edges of $\widehat{G}_A$ incident to u, and assume that $\widehat{e}_2$ is a boundary edge. By Lemmas 2.1 and 1.1(2) we may assume that each $\widehat{e}_i$ represents at most $q-1$ parallel edges of G_A. Now each label appears at most twice among the endpoints at u of edges represented by $\widehat{e}_2$ or $\widehat{e}_4$, hence all labels appear on endpoints at u of edges represented by $\widehat{e}_1$ or $\widehat{e}_3$. Suppose $\widehat{e}_1, \widehat{e}_3$ connect u to u' and u'', respectively. If both u' and u'' are antiparallel to u, then by the parity rule each vertex v on G_B is incident to an edge connecting it to a parallel vertex, with label u at its endpoint at v. By [CGLS, Lemmas 2.6.3 and 2.6.2] this implies that G_B contains a great u-cycle, hence a Scharlemann cycle, and we are done. Also, notice that u' and u'' cannot both be parallel to u, otherwise by Lemma 1.1(4) each of $\widehat{e}_1$ and $\widehat{e}_3$ represents at most $q/2$ edges, and since each of $\widehat{e}_2$ and $\widehat{e}_4$ represents at most $q-1$ edges, this would contradict the fact that the total valency of u in G_A is $\Delta q = 3q$. (This also takes care of the case that $\widehat{e}_1 = \widehat{e}_3$ is a loop at u.) Therefore, we may assume that u' is parallel to u, and u'' is antiparallel to u. Since the total number of edges represented by $\widehat{e}_1 \cup \widehat{e}_2 \cup \widehat{e}_3$ is more than $2q$, we can choose $2q$ successive edges at u, forming a subgraph as shown in Figure 2.1. One can now use [Wu2, Lemma 2.2] and the proof of [Wu2, Lemma 3.4] to show that there is a disk D in $M(\beta)$ with $\partial D = K[1,q] \cup \alpha_1 \cup C_1 \cup \alpha_2$, where α_1 and α_2 are arcs on the compressing disk B connecting $K[1]$ and $K[q]$ to ∂B, and C_1 is an arc on $\partial M(\beta)$. As in the proof of Lemma 2.1, this implies that K is a 1-arch knot. □

We need the following result of Eudave-Muñoz in the proof of Proposition 2.4. If C is a simple loop on the boundary of a 3-manifold X, denote by X_C the manifold obtained by adding a 2-handle to X along the curve C.

Lemma 2.3. *Let X be an irreducible, orientable 3-manifold with ∂X compressible, and C a simple closed curve on ∂X such that $\partial X - C$ is incompressible. Suppose X_C contains an essential annulus A'. Then it contains an essential annulus A which intersects the attached 2-handle in at most two disks. Furthermore, if A' is nonseparating, then A can be chosen to be disjoint from the attached 2-handle.*

Proof. This is essentially [Eu, Theorem 1]. The theorem there says that under the above assumption, either one can find A to be disjoint from the attached 2-handle,

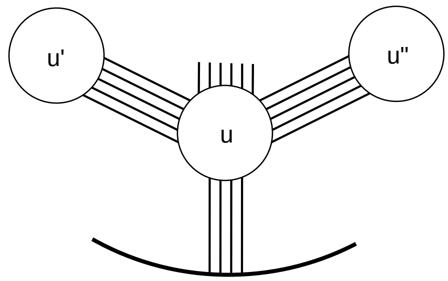


Figure 2.1

or after sliding the cocore σ of the attached 2-handle over itself to get a 1-complex τ, one can find an essential annulus A which intersects τ at a single point. Moreover, if A' is nonseparating, then A is disjoint from τ (see also [Jo2, Sch]). Sliding τ back to σ, we see that A is isotopic to an annulus intersecting σ at most twice. See also the remarks after the statement of Theorem 2 in [Eu]. □

Proposition 2.4. *If $K = K_\beta$ is a 1-arch knot in $M(\beta)$, then $p \leq 2$, and A is a separating annulus in $M(\alpha)$.*

Proof. The first part of the proof here is the same as that in the proof of [Wu2, Proposition 1]. Suppose K is isotopic to $C_1 \cup C_2$ as in the definition of 1-arch knot. Let Y be the manifold obtained by adding a 1-handle H_1 to $M(\beta)$ along two disks centered at ∂C_1, and let C be a simple closed curve on ∂Y obtained by taking the union of $C_1 \cap \partial Y$ and an arc on ∂H_1. Let K' be the union of C_2 and the core of the 1-handle H_1. Then after adding a 2-handle H_2 to Y along C the 1-handle and the 2-handle cancel each other and we get a manifold M' homeomorphic to the original manifold $M(\beta)$, with the knot K identified to K'; hence we have a homeomorphism of pairs $(M(\beta), K) \cong (M', K')$. Let W (denoted by Q in [Wu2]) be the manifold obtained from Y by Dehn surgery on K' along the slope α. Then $M(\alpha)$ is homeomorphic to the manifold W_C obtained by adding the 2-handle H_2 to W along the curve C. It was shown in [Wu2, Lemmas 1.2 and 1.3] that ∂W is compressible, and $\partial W - C$ is incompressible in W when $\Delta \geq 2$.

Since $M(\beta)$ is ∂-reducible, by [Sch] the manifold $M(\alpha) = W_C$ is irreducible. This implies that W is irreducible because a reducing sphere in a manifold always remains a reducing sphere after 2-handle additions. Therefore we can apply Lemma 2.3 and conclude that there is an essential annulus A in $M(\alpha) = W_C$ intersecting the attached 2-handle H_2 in $n \leq 2$ disks; moreover, if A is nonseparating, then it is disjoint from H_2. Our goal is to show that A also intersects the knot K_α in $M(\alpha)$ in two or zero points, respectively.

We assume $n = 2$, the cases $n = 0$ or 1 are similar. Let D_1, D_2 be the disks $A \cap H_2$, and let F be the twice punctured annulus $A - \text{Int}(D_1 \cup D_2)$ in W. A meridian disk D of the 1-handle H_1 gives rise to a nonseparating essential annulus $D_0 = D \cap X$ in the manifold $X = Y - \text{Int}N(K') = W - \text{Int}N(K_\alpha)$. Let $F_0 = F \cap X$. Form intersection graphs G_D and G_F in the usual way, i.e, G_F has $F \cap N(K_\alpha)$ as fat vertices, G_D has a single vertex $D \cap N(K_\beta)$, and the edges of G_D and G_F are

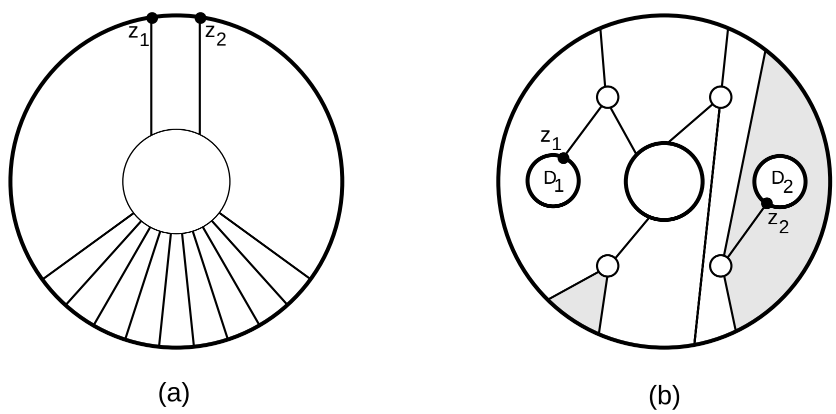


Figure 2.2

the arc components of $D_0 \cap F_0$ which has at least one endpoint on the fat vertices. See Figure 2.2. Choose D and F so that D_0 intersects F_0 minimally. Then each fat vertex of G_F has valency $\Delta = 3$, and the only vertex x of G_D has valency $3t$, where t is the number of vertices of G_F. Note that, since A is essential in $M(\alpha)$, we have $t \geq p$. As usual, there are no trivial loops. Hence each edge of G_D connects x to ∂D.

Each of ∂D_1 and ∂D_2 intersects ∂D at a single point, which we denote by z_1, z_2, as indicated by the dark dots in Figure 2.2(a) and (b). They divide ∂D into two arcs α_1, α_2, one of which, say α_1, lies in $N(C)$, which is the attaching region of the two handle H_2 above. Hence the interior of α_1 is disjoint from ∂F. It follows that all the endpoints of the edges of G_D on ∂D lie on the arc α_2, as shown in Figure 2.2(a).

Now suppose G_F has $t \geq 3$ vertices. Since each of ∂D_1 and ∂D_2 is adjacent to at most one edge, there is a vertex of G_F with two edges connecting it to the same component of $\partial A \subset \partial F$. An outermost such vertex has a pair of edges a_1, a_2 on G_F, cutting off a region B_1 on F_0 which is either a disk, or a once punctured disk containing one of $\partial D_1, \partial D_2$, as shown by the two shaded regions in Figure 2.2(b). (If B_1 contains both ∂D_i, choose another outermost vertex.) They also cut off a disk B_2 on D_0 which, by the property in the last paragraph, has boundary disjoint from α_1. Therefore, $B_1 \cup B_2$ is either an annulus or a once punctured annulus in X, with one boundary component γ a curve on $\partial N(K)$, another a curve on $\partial X - \partial N(K)$ disjoint from C, and a possible third curve parallel to C. After capping off the last component by a disk in the attached 2-handle H_2, the surface becomes an annulus in X_C. However, since $X_C = M$, and since the boundary component γ of the annulus on the torus $\partial N(K) = \partial_0 M$ is an essential curve, (essential because γ is the union of an arc in α and an arc in β, and α intersects β minimally,) this contradicts the fact that M is ∂-irreducible and anannular.

When $n = 0$, since $t \geq p > 0$, there is a pair of edges which are parallel on both graphs G_D and G_F. As shown above, this would give rise to an essential annulus in M, which would contradict the simplicity of M. Hence this case does not happen.

In particular, this and Lemma 2.3 show that $M(\alpha)$ cannot contain a nonseparating annulus. □

§3. Representing types

Denote by $\mathbf{q} = \{1, ..., q\}$ the set of labels of the vertices of G_B. We have the concept of a $\mathbf{q}$-*type* etc. from [GL1]. An interior face of G_A *represents* a $\mathbf{q}$-type τ if it is a disk and represents τ in the sense of [GL1]. We say G_A *represents* τ if some interior face of G_A represents τ.

Theorem 3.1. *G_A does not represent all $\mathbf{q}$-types.*

Proof. See [GL4, Proof of Theorem 2.2]. The proof works for any essential surface F in $M(\alpha)$ (in [GL4] F was a torus). A set of representatives of all $\mathbf{q}$-types contains a set $\mathcal{D}$ of interior faces of G_F which can be used to surger Q tubed along the annuli corresponding to the corners of the faces in $\mathcal{D}$, contradicting the minimality of q. □

A *web* in G_B is a non-empty connected subgraph Λ of G_B such that all the vertices of Λ have the same sign, and such that there are at most p edge endpoints at vertices of Λ which are not endpoints of edges in Λ. Note that a web may have boundary edges.

Let U be a component of $B - N(\Lambda)$ that meets ∂B. Then $D = B - U$ is a *disk bounded by* Λ. Λ is a *great web* if there is a disk bounded by Λ such that Λ contains all the edges of G_B that lie in D.

Remark. If there are no boundary edges, then these definitions coincide with those in [GL2, Section 2]. The following is the analog in our present setting of [GL2, Theorem 2.3].

Theorem 3.2. *Suppose $\Delta \geq 2$. Let L be a subset of $\mathbf{q}$, and τ be a non-trivial L-type such that*

(i) all elements of $C(\tau)$ have the same sign, and

(ii) all elements of $A(\tau)$ have the same sign.

If $G_A(L)$ does not represent τ then G_B contains a web Λ such that the set of vertices of Λ is a subset of either $C(\tau)$ or $A(\tau)$.

Proof. Regard $G_A(L)$ as a graph in S^2, by capping off the boundary components of A with two additional fat vertices v_1, v_2.

Define a directed graph $\Gamma = \Gamma(\tau)$ as follows. The vertices of Γ are the fat vertices of $G_A(L)$ plus v_1, v_2, together with dual vertices of $G_A(L)$ (one in the interior of each face of $G_A(L)$.) The edges of Γ join each dual vertex to the fat vertices in the boundary of the corresponding face. The edges of Γ are oriented as follows: If an edge e has an endpoint on a vertex of $G_A(L)$, then it is oriented according to the type τ, (as in [GL1], where Γ is denoted by $\Gamma(T)^*$); if e has an endpoint on v_1 or v_2, orient e so that no dual vertex in a boundary face of $G_A(L)$ is a sink or source of Γ. See Figure 3.1.

By Glass' index formula (see [Gl]) applied to Γ, we have

$$\sum_{\text{vertices}} I(v) + \sum_{\text{faces}} I(f) = \chi(S^2) = 2.$$

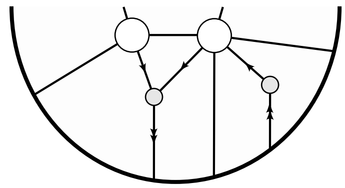


Figure 3.1

Assume $G_A(L)$ does not represent τ. Then no dual vertex in Γ is a sink or source. Hence

$$\sum_{v \text{ dual}} I(v) \leq 0.$$

Let

$c(\tau) = \#$ clockwise switches $(= \#$ anticlockwise switches$)$ of τ

$c(v_i) = \#$ clockwise switches $(= \#$ anticlockwise switches$)$ at v_i, $i = 1, 2$

For v a vertex of $G_A(L)$, we have

$$I(v) = 1 - \Delta c(\tau).$$

Also,

$$I(v_i) = 1 - c(v_i), \qquad i = 1, 2.$$

Therefore, the number of switch edges, including all switch boundary edges, is at least

$$\begin{aligned}\sum I(f) &\geq 2 + p(\Delta c(\tau) - 1) + (c(v_1) - 1) + (c(v_2) - 1)\\ &= p(\Delta c(\tau) - 1) + c(v_1) + c(v_2) \qquad (*)\end{aligned}$$

Since the number of switch edge endpoints is twice the number of switch edges, this is also a lower bound for the number of (say) clockwise switch edge endpoints. The total number of clockwise switches is $p\Delta c(\tau) + c(v_1) + c(v_2)$, so the number of clockwise switches that are not endpoints of clockwise switch edges is at most p.

Since $\Delta \geq 2$, the right hand side of (*) is positive. Let Λ be a component of the subgraph of G_B consisting of the edges corresponding to the clockwise switch edges of $G_A(L)$. Then at most p edge endpoints at vertices of Λ do not belong to edges of Λ. Thus Λ is a web, as described. $\square$

The following is the analog of [GL2, Theorem 2.5].

Theorem 3.3. *Suppose $\Delta \geq 2$. Let Λ be either (i) a web in G_B, or (ii) the empty set. In case (i), let D be a disk bounded by Λ, and in case (ii), let $D = B$. Let L be the set of vertices of $G_B - \Lambda$ that lie in D. Then either G_B contains a great web or $G_A(L)$ represents all L-types.*

Proof. Basically, this follows from the proof of [GL2, Theorem 2.5]. We indicate briefly how this goes.

We prove the result by induction on $|L|$.

Let τ be an L-type. We show that if $G_A(L)$ does not represent τ then G_B contains a great web. There are two cases.

CASE 1. *τ is trivial.* Proceed as in [GL2, Proof of Theorem 2.5]. Let $\widehat{\Lambda}$ be a component of the subgraph of G_B consisting of vertices J, all interior edges with both endpoints on vertices in J, and all boundary edges with one endpoint on a vertex in J.

(a) $\widehat{\Lambda}$ is a web. Argue as in [GL2], with "faces" meaning "interior faces".

(b) $\widehat{\Lambda}$ is not a web. Again the argument in [GL2] remains valid. More precisely, since $\widehat{\Lambda}$ is not a web, there are more than p edges of G_B connecting a vertex of $\widehat{\Lambda}$ to an antiparallel vertex. Let Σ be the subgraph of G_A consisting of the vertices of G_A together with those edges. Note that these are interior edges of G_A, connecting parallel vertices. Applying Euler's formula to Σ, a graph in A, gives

$$V - E + \sum \chi(f) = 0.$$

Therefore

$$\sum \chi(f) = E - V > p - p = 0.$$

Hence Σ has a disk face, which must be an interior face. This face then contains a face of $G_A(L)$ representing the trivial type.

CASE 2. *τ is non-trivial.* Here the argument in [GL2] goes through essentially without change, (using Theorem 3.2), where we always interpret "face" as "interior face". In particular, [GL2, Lemmas 2.4 and 2.6] carry over in this way. □

Theorems 3.1 and 3.3 (in case (ii)) imply:

Corollary 3.4. *If $\Delta \geq 2$, then G_B contains a great web.* □

§4. The generic case

Let Λ be a great web in G_B given by Corollary 3.4, and let D be a disk bounded by Λ with the property in the definition of a great web. Let x be a label of the vertices of G_A, and let Λ_x be the subgraph of Λ consisting of all vertices of Λ and all edges in Λ with an endpoint labeled x. Let V be the number of vertices of Λ. A *ghost endpoint of* Λ is an endpoint, at a vertex of Λ, of an edge of G_B which does not belong to Λ. A *ghost endpoint of* Λ_x is a ghost endpoint of Λ labeled x. (It is called a ghost label in [GL2].) By the definition of a web, Λ has at most p ghost endpoints.

By a *monogon* we mean a disk face with one edge in its boundary, and by a *bigon* we mean a disk face with two edges in its boundary.

Lemma 4.1. (Cf. [GL2, Lemma 4.2]) *If Λ_x has at least $3V-2$ edges then Λ_x contains a bigon in D.*

Proof. Let Ω be the graph in S^2 obtained from Λ_x by regarding ∂B as a vertex. Then Ω has $V+1$ vertices, E edges (= number of edges of Λ_x), and the faces of Ω are the faces of Λ_x in D together with an additional face f_0. Note that f_0 is not a monogon. Suppose Λ_x contains no bigon in D.

First suppose f_0 is not a bigon. Then $2E \geq 3F$, where $F = \sum \chi(f)$ summed over all faces of Ω. Also,

$$(V+1) - E + F = 2.$$

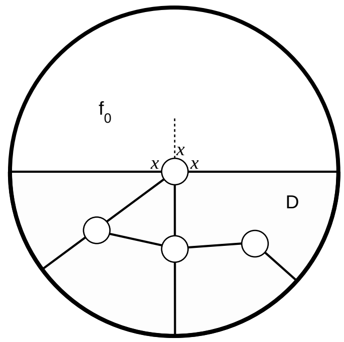


Figure 4.1

Hence

$$1 = V - E + F \leq V - E + \frac{2E}{3} = V - \frac{E}{3},$$

giving $3 \leq 3V - E$, i.e, $E \leq 3V - 3$, contrary to assumption.

Now suppose f_0 is a bigon; see Figure 4.1. Then Λ_x has at most one ghost endpoint. Therefore $E \geq 3V - 1$. Also, Since Λ_x has no bigon in D, we have

$$2E \geq 3F - 1.$$

Hence, as before,

$$1 = V - E + F \leq V - E + \frac{2E+1}{3} = V - \frac{E}{3} + \frac{1}{3}.$$

Therefore $3 \leq 3V - E + 1$, implying $E \leq 3V - 2$, a contradiction. □

Remark. One can show that the conclusion of Lemma 4.1 still holds if we only assume that Λ_x has at least $3V - 3$ edges, but Lemma 4.1 will suffice for our purposes.

Lemma 4.2. *Λ_x contains a bigon in D for at least $2p/3$ labels x.*

Proof. (Cf. [GL2, Theorem 4.3]). By Lemma 4.1, if Λ_x does not contain a bigon in D then Λ_x has at most $3V - 3$ edges. Since the vertices of Λ_x are all parallel, by the parity rule no edge of Λ_x has both endpoints labeled x, so among the endpoints of edges of Λ_x, at most $3V - 3$ are labeled x. Since $\Delta = 3$, this means that Λ_x has at least 3 ghost endpoints. Since the total number of ghost endpoints in Λ is at most p by the definition of a great web, there can be at most $p/3$ such labels x. Hence for at least $2p/3$ labels x, Λ_x does contain a bigon in D. □

Note that a boundary bigon in Λ_x gives $p + 1$ parallel boundary edges in G_B.

In the remainder of this section, *we assume that* $p \geq 3$.

Lemma 4.3. *For some label x, Λ_x contains a boundary bigon.*

Proof. A bigon face of Λ_x in D is either a boundary bigon or an interior bigon. The latter is either an order 2 Scharlemann cycle in G_B, or contains an extended Scharlemann cycle. The second is impossible by Lemma 1.2(2). When p is odd, the first is also impossible (Lemma 1.2(1)), and when p is even, any two Scharlemann cycles have the same label pair (Lemma 1.2(2)). Hence, by Lemma 4.2, the number of labels x such that Λ_x contains a boundary bigon is at least

$$\begin{cases} 2p/3 \geq 2 \times 3/3 = 2, & p \text{ odd;} \\ 2p/3 - 2 \geq 2 \times 4/3 - 2 = 2/3, & p \text{ even.} \end{cases}$$

Hence there is at least one label x with the stated property. □

Corollary 4.4. *(a) Every vertex of G_A has a boundary edge incident to it.*
(b) G_A has a vertex with two non-parallel boundary edges.

Proof. (a) By Lemma 4.3, Λ_x contains a boundary bigon for some x, which gives rise to $p+1$ parallel boundary edges in G_B. Hence each label of $\mathbf{p} = \{1, ..., p\}$ appears at the endpoint of some boundary edge of G_B, and the result follows.

(b) By Lemma 1.1(5), the two boundary edges in a boundary bigon of Λ_x are nonparallel on G_A. □

Lemma 4.5. *$\widehat{G}_A$ has no vertex of valency at most 3.*

Proof. G_A has at most $q-1$ parallel interior edges by Lemma 1.1(2), and at most $q-1$ parallel boundary edges by Lemma 2.1, Proposition 2.4, and the assumption that $p \geq 3$. Since the total valency of each vertex of G_A is $3q$, the result follows. □

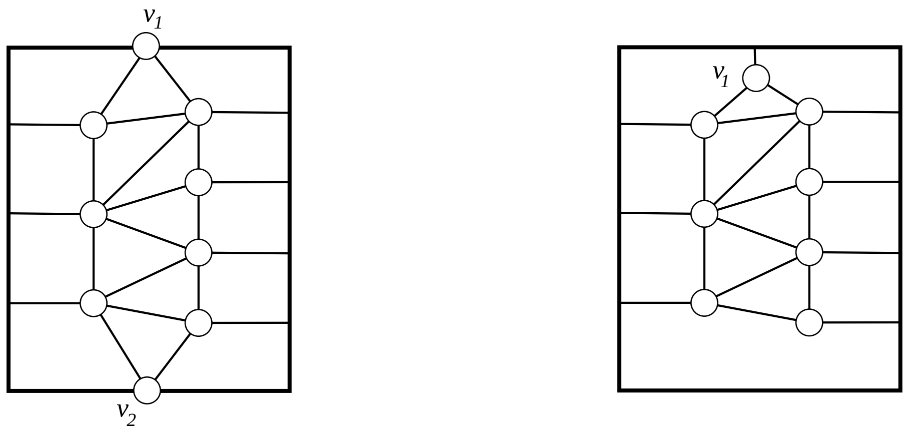
Corollary 4.6. *$\widehat{G}_A$ has no vertex with two boundary edges going to the same component of ∂A.*

Proof. Consider an outermost such vertex, with E the corresponding subdisk of A. Doubling E along the two boundary edges in question and applying [CGLS, Lemma 2.6.5] gives a vertex in the interior of E of valency at most 3, contradicting Lemma 4.5. □

Lemma 4.7. *$\widehat{G}_A$ has a vertex v of valency 4, such that no two boundary edges of $\widehat{G}_A$ at v have endpoints adjacent on ∂v.*

Proof. By Corollaries 4.4(b) and 4.6, $\widehat{G}_A$ has at least one vertex with two boundary edges going to different components of ∂A. Cut A along all such pairs of edges; we get a certain number (≥ 1) of disk regions. If there are no vertices in the interior of any of these regions, then every vertex of $\widehat{G}_A$ satisfies the conclusion of the lemma; (recall that there is no vertex of valency ≤ 3 by Lemma 4.5). So consider a region with a non-zero number of vertices in its interior; see Figure 4.2(a). Note that each vertex v in the interior of the region is incident to exactly one boundary edge, hence we need only show that some v has valency 4.

If the number of vertices in the interior of the region is 1 or 2, the result is obvious. So suppose there are at least 3 such vertices. Delete v_2 and all edges incident to it, and push v_1 inwards and attach a boundary edge to it, as shown in Figure 4.2(b). Applying assertion (*) in the proof of Lemma 2.6.5 in [CGLS] to the resulting graph, we conclude that there is a vertex $v \neq v_1$ of valency at most



(a) (b)

Figure 4.2

3. Since there is at most one edge joining v to v_2, v has valency at most 4 (hence exactly 4) in the original graph $\widehat{G}_A$. $\square$

Proposition 4.8. *Theorem 0.1 is true if* $p \geq 3$.

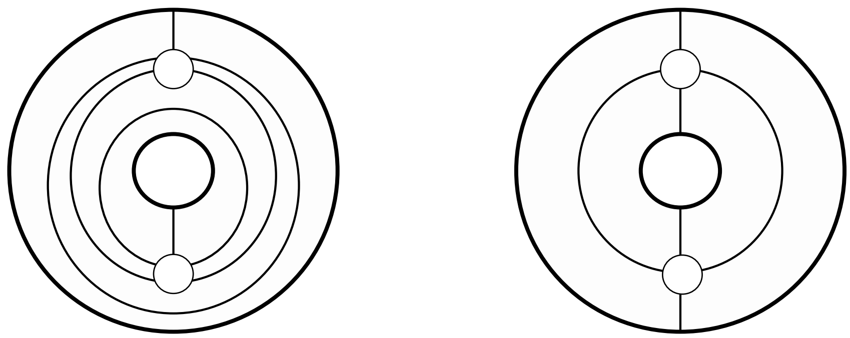
Proof. Let v be the vertex of valency 4 given by Lemma 4.7. Since we have assumed $p \geq 3$, by Lemma 2.2 and Proposition 2.4, G_B contains a Scharlemann cycle. Suppose the Scharlemann cycle has label pair $\{1, 2\}$. Then by Lemma 1.2(1), p is even, hence $p \geq 4$, and the edges of the Scharlemann cycle are not contained in a disk on A. Thus in $\widehat{G}_A$ there are two edges connecting v_1 to v_2, as shown in Figure 5.1(b). They separate the two boundary components of the annulus A, so no other vertex is incident to two edges going to different boundary components of A. It follows from Corollary 4.6 that the only possible vertices of $\widehat{G}_A$ with two boundary edges are v_1 and v_2. Since there are no Scharlemann cycles on any other label pair, and no extended Scharlemann cycles (Lemma 1.2(2)), the only labels x for which Λ_x has a bigon in D are 1 and 2. Hence by Lemma 4.2, we have $2p/3 \leq 2$, i.e. $p \leq 3$. But we have just shown that $p \geq 4$, which is a contradiction. $\square$

§5. The case that $p \leq 2$

After Proposition 4.8, it remains to consider the case that the graph G_A on the annulus A has at most two vertices. In this section we will consider this remaining case, and complete the proof of Theorem 0.1. As before, we assume that $\Delta = 3$.

Lemma 5.1. *If* $p \leq 2$*, then* $p = 2$*, and the two vertices of* G_A *are antiparallel.*

Proof. First assume $p = 1$. Then A is a nonseparating annulus in $M(\alpha)$. The reduced graph $\widehat{G}_A$ consists of one vertex, at most one loop, and at most two boundary edges. By Lemma 1.1(4) the number of endpoints of loops is at most q. Since the total valency of the vertex is $3q$, there exist q parallel boundary edges. By Lemma



(a) (b)

Figure 5.1

2.1 K_β is a 1-arch knot. However, by Proposition 2.4 in this case $M(\alpha)$ contains no nonseparating annulus, a contradiction.

Now assume $p = 2$ and the two vertices of G_A are parallel. Then again A is nonseparating in $M(\alpha)$. The reduced graph $\widehat{G}_A$ is a subgraph of one of the two graphs shown in Figure 5.1, depending on whether or not $\widehat{G}_A$ has a loop. Since the two vertices are parallel, by Lemma 1.1(4) each family of parallel interior edges contains at most $q/2$ edges, hence in both cases there is a family of at least q parallel boundary edges. As above, this implies that K_β is a 1-arch knot, hence contradicts Proposition 2.4 and the fact that A is nonseparating. □

We may now assume that $p = 2$ and the two vertices of G_A are antiparallel. Suppose W is a submanifold of $M(\alpha)$ containing K_α. We use $\partial_i W$ to denote the closure of $\partial W - \partial M(\alpha)$, which is the frontier of W in $M(\alpha)$, and call it the *interior boundary.*

If D is a disk embedded (improperly) in $M(\alpha)$ such that $D \cap W = D \cap \partial_i W$ is a single arc C on the boundary of D, and $\partial D - C$ lies on $\partial M(\alpha)$, then the pair $(W \cup N(D), K_\alpha)$ is homeomorphic to (W, K_α), with $\partial_i(W \cup N(D))$ identified to $\partial_i W$ cut along the arc C. This observation will be useful in the proof of Lemma 5.4.

For the purpose of this section, we define an *extremal component* of a subgraph Λ of G_B to be a component Λ_0 such that there is an arc γ cutting B into B_1 and B_2, with $B_1 \cap \Lambda = \Lambda_0$.

Lemma 5.2. *If $p = 2$ and the two vertices of G_A are antiparallel, then each vertex of G_B is incident to a boundary edge. In particular, each face of G_B is a disk.*

Proof. The reduced graph $\widehat{G}_A$ is a subgraph of that shown in Figure 5.1(a) or (b). In case (b), each of the interior edges of $\widehat{G}_A$ represents at most q edges of G_A, hence each label appears at most four times at endpoints of interior edges. It follows that each vertex of G_B is incident to at least two boundary edges.

In case (a), consider the edge endpoints at a vertex v of G_A. Let s be the number of boundary edges at v, and let t be the number of loops based at v. Observe that if $s < q$ but $s + 2t > q$ then some label would appear twice among the endpoints of the parallel loops, which would contradict Lemma 1.1(4). If $s + 2t \le q$, then the two nonloop edges of $\widehat{G}_A$ would represent $3q - (s + 2t) \ge 2q$ edges, which would contradict Lemma 1.1(2). Therefore we must have $s \ge q$, which implies that each vertex of G_B is incident to a boundary edge.

If some face f of G_B is not a disk, then the vertices inside of a nontrivial loop in f would have no boundary edges, which would contradict the above conclusion. □

Lemma 5.3. *Suppose $p = 2$ and the two vertices of G_A are antiparallel. Then there is a vertex v_0 of $\widehat{G}_B$ with the following properties.*

(1) v_0 has valency 2 or 3 in $\widehat{G}_B$, and belongs to a single boundary edge e of $\widehat{G}_B$.

(2) If the valency of v_0 is 3, then the face opposite to the boundary edge is an interior face.

(3) One of the two faces of $\widehat{G}_B$ containing e intersects ∂B in a single arc.

Proof. By Lemma 5.2 each vertex of G_B belongs to a boundary edge. Consider an extremal component C of G_B, and let $\widehat{C}$ be its reduced graph. Let $\widehat{C}'$ be its corresponding component in $\widehat{G}_B$. Note that $\widehat{C}'$ and $\widehat{C}$ are almost identical, except that one of the vertices v' of $\widehat{C}'$ may have two parallel boundary edges, in which case $\widehat{C}$ can be obtained from $\widehat{C}'$ by amalgamating these two edges together.

Note that $\widehat{C}$ must have at least two vertices, for otherwise, since C is extremal, the vertex would have 6 parallel boundary edges in G_B, two of which would also be parallel on G_A because $\widehat{G}_A$ has at most four boundary edges, which would then contradict Lemma 1.1(5). Hence each vertex of C is incident to at least one interior edge and one boundary edge, so the valency of each vertex of $\widehat{C}$ is at least 2. Modify $\widehat{C}$ as follows. If some vertex v of $\widehat{C}$ satisfies condition (1) but not (2), add a boundary edge to v in the face opposite to the boundary edge at v. Having done this for all v, we get a graph $\widehat{C}''$, which is still a reduced graph, with at least one boundary edge incident to each vertex. Now using (*) in the proof of [CGLS, Lemma 2.6.5] and arguing directly when $\widehat{C}''$ has only two or three vertices, we see that $\widehat{C}''$ contains at least two vertices, each of which has valency 2 or 3 in $\widehat{C}''$ and belongs to a single boundary edge of $\widehat{C}''$. At least one of these two vertices, say v_0, is not the vertex v' above, hence it has property (1) when considered as a vertex in $\widehat{G}_B$. By the definition of $\widehat{C}''$, v_0 automatically has property (2). To prove (3), notice that if both faces containing e intersect ∂B in more than one arc, then C would not be an extremal component. □

Lemma 5.4. *If $p = 2$, and the two vertices of G_A are antiparallel, then ∂M is a union of tori.*

Proof. Let W_0 be a regular neighborhood of $A \cup K_\alpha$. Since K_α intersects A in two points of different signs, $\partial_i W_0$ has two components F_b, F_w, each being a twice punctured torus. The annulus A cuts W_0 into two components W_0^b and W_0^w (with $W_0^b \supset F_b$), which will be called the black region and the white region, respectively. If E is a disk face of G_B or more generally a disk in $M(\alpha)$, then E is said to be *black* (resp. *white*) if $E \cap W_0$ lies in the black (resp. white) region.

Suppose D is a compressing disk of F_b in $M(\alpha) - \text{Int} W_0$. If ∂D is a nonseparating curve on F_b, then after adding the 2-handle $N(D)$ to W_0, the surface F_b becomes an annulus. If ∂D is separating on F_b, then it is not parallel to a boundary curve of F_b because ∂F_b is parallel to ∂A and A is incompressible in $M(\alpha)$; thus ∂D must cut

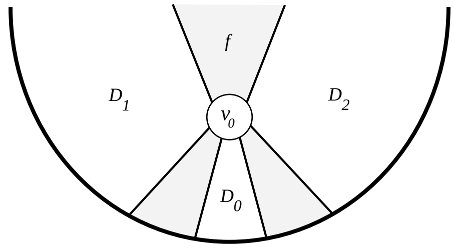


Figure 5.2

F_b into a once punctured torus and a thrice punctured sphere, and after adding the 2-handle $N(D)$ the surface F_b becomes the union of a torus S_1 and an annulus S_2. Since M is simple, S_1 either is boundary parallel or bounds a solid torus, and S_2 must be boundary parallel, because ∂S_2 is parallel to ∂A and A is incompressible, which implies that S_2 is incompressible. In any case, we have shown that if F_b is compressible in $M(\alpha) - \mathrm{Int}W_0$ then there is a component C_b of $M(\alpha) - \mathrm{Int}W_0$ such that $C_b \cap W_0 = F_b$ and $C_b \cap \partial M(\alpha)$ is either an annulus or the union of an annulus and a torus. Similarly for F_w. In particular, if both F_b and F_w are compressible in $M(\alpha) - \mathrm{Int}W_0$, then $\partial M(\alpha)$ is a union of tori, and we are done. From now on, we will assume that F_w is incompressible in $M(\alpha) - \mathrm{Int}W_0$ and show that this will lead to a contradiction. Note that the assumption implies that G_B has no interior white face: For, by Lemma 5.2 all faces of G_B are disks, and since G_A has no trivial loops, the boundary of an interior face is always an essential curve on $\partial_i W_0$; hence an interior white face would give rise to a compressing disk of F_w in $M(\alpha) - \mathrm{Int}W_0$.

Let v_0 be a vertex of G_B given by Lemma 5.3. By Lemma 5.3(1), v_0 has valency 2 or 3 in $\widehat{G}_B$. First assume that v_0 has valency 2 in $\widehat{G}_B$. Then the interior edge e of $\widehat{G}_B$ incident to v_0 must represent exactly two edges of G_B: It cannot represent more than two edges, otherwise there would be two interior faces of different colors, contradicting the fact that G_B has no interior white face. It cannot represent only one edge of G_B, otherwise v_0 would have five parallel boundary edges, which would contradict Lemma 1.1(5) because $\widehat{G}_A$ has at most four boundary edges. Thus the part of G_B near v_0 is as shown in Figure 5.2, where f is the interior (black) face bounded by the two edges represented by e.

Now assume that v_0 has valency 3 in $\widehat{G}_B$. Then by Lemma 5.3(2) the face f of $\widehat{G}_B$ opposite to the boundary edge at v_0 is an interior face. Thus f is a black face, and each of the interior edges of $\widehat{G}_B$ incident to v_0 represents only one edge of G_B as otherwise there would be a white interior face. Hence again the part of G_B near v_0 is as shown in Figure 5.2.

Consider the white boundary faces D_0, D_1, D_2 as shown in Figure 5.2. By Lemma 5.3(3), we may assume that D_1 intersects ∂B in a single arc. Let C_i, $i = 0, 1$, be the arc $D_i \cap F_w$. Then C_0, C_1 are essential arcs on F_w. Moreover, since C_0 intersects a meridian of K_α exactly once, while C_1 intersects it at least twice, they are nonparallel. Recall that $\partial_i(W_0 \cup N(D_0 \cup D_1))$ is obtained from $\partial_i W_0$ by cutting along $C_0 \cup C_1$. Since C_0 and C_1 are nonparallel, they cut F_w into one or two annuli, which must be boundary parallel because we have assumed that F_w is incompressible and M is simple. It follows that the whole surface F_w is boundary parallel. Now a meridian disk of $N(K_\alpha)$ in the white region corresponds to a disk in $M(\alpha)$ intersecting the curve K_α in a single point, which gives rise to an essential annulus in M, contradicting the fact that M is anannular. □

Proof of Theorem 0.1. By Proposition 4.8, we may assume that $p \leq 2$. By Lemmas 5.1 and 5.4, ∂M is a union of tori. Since $M(\beta)$ is ∂-reducible, either it is reducible or it is a solid torus. In the first case the result follows from [Wu3, Theorem 5.1]. So we assume that $M(\beta)$ is a solid torus. In particular, $\partial M(\alpha)$ is a single torus T. The boundary of the annulus A cuts T into two annuli A_1, A_2. If some $A \cup A_i$ is an essential torus in $M(\alpha)$ then $M(\alpha)$ is toroidal, so the result follows from [GL4]. If each $A \cup A_i$ is inessential, then it bounds a solid torus (note that it cannot be boundary parallel, otherwise A would be boundary parallel). It follows that $M(\alpha)$ is a Seifert fiber space with orbifold a disk with two singular points. It was shown in [MM1, Theorem 1.2] that if $M(\alpha)$ is a Seifert fiber space and $M(\beta)$ is a solid torus then $\Delta \leq 1$. This completes the proof of Theorem 0.1. □

In the proof of Theorem 0.1, we assumed that the manifold M is simple. However, the conditions that M is irreducible and atoroidal can be removed from the assumptions.

Corollary 5.5. *Suppose M is anannular and boundary irreducible. If $M(\alpha)$ is annular and $M(\beta)$ is boundary reducible, then $\Delta(\alpha, \beta) \leq 2$.*

Proof. First assume that M is irreducible but toroidal. Since M is anannular, by the canonical splitting theorem of Jaco-Shalen-Johannson (see [JS, p. 157]) there is a set of essential tori $\mathcal{T}$ cutting M into a manifold M' such that each component of M' is either a Seifert fiber space or a simple manifold. If the component X containing the boundary torus $\partial_0 M$ is Seifert fibered, then it contains an essential annulus consisting of Seifert fibers, with both boundary components on $\partial_0 M$, so M would be annular, contradicting our assumption. So assume X is simple. Since $M(\beta)$ is boundary reducible, by looking at a boundary reducing disk B which has minimal intersection with $\mathcal{T}$, one can see that $X(\beta)$ must be boundary reducible. Similarly one can show that $X(\alpha)$ is either boundary reducible or annular. Applying Theorem 0.1 and [Wu2, Theorem 1] to X, we have $\Delta \leq 2$.

If M is reducible, split along a maximal set of reducing spheres to get an irreducible manifold M'. By an innermost circle argument one can show that $M'(\alpha)$ is annular and $M'(\beta)$ is boundary reducible, so the result follows from that for irreducible manifolds. □

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