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PSEUDO-ANOSOV MAPS AND SIMPLE CLOSED CURVES ON SURFACES

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Abstract. Suppose $\mathcal{C}$ and $\mathcal{C}'$ are two sets of simple closed curves on a hyperbolic surface F. We give necessary and sufficient conditions for the existence of a pseudo-Anosov map g such that $g(\mathcal{C}) \cong \mathcal{C}'$.

1. Introduction

Pseudo-Anosov maps are the most important class among surface homeomorphisms [Th3]. They play important roles in 3-manifold theory, see for example [Th2]. This note is to address the following question: Suppose F is an orientable closed surface with $\chi(F) < 0$, and suppose $\mathcal{C}$ and $\mathcal{C}'$ are sets of mutually disjoint, mutually non-parallel essential circles on F. When does there exists an orientation preserving pseudo-Anosov map $g : F \to F$ such that $g(\mathcal{C}) \cong \mathcal{C}'$?

The problem will be solved in Corollary 1.4, which gives a practical way of determining whether such a map g exists. In the special case that each of $\mathcal{C}$ and $\mathcal{C}'$ contains a single curve, g exists if and only if (i) $(F, \mathcal{C})$ is homeomorphic to $(F, \mathcal{C}')$, and (ii) $\mathcal{C}$ is not isotopic to $\mathcal{C}'$. Note that condition (i) is automatically true if both $\mathcal{C}$ and $\mathcal{C}'$ in are non-separating curves.

The problem is raised by M. Boileau and S. Wang in order to understand degree one maps between hyperbolic 3-manifolds which are surface bundles over the circle [BW]. We need some definitions in order to state and prove our main theorem.

A circle c on a compact surface F is *essential* if c is not contractible or boundary parallel. A set $\mathcal{C} = \{c_1, ..., c_n\}$ of mutually disjoint circles on F is an *independent* set if the curves in $\mathcal{C}$ are essential and mutually non parallel. Write $\mathcal{C} \cong \mathcal{C}'$ if $\mathcal{C}$ is isotopic to $\mathcal{C}'$. Given a homeomorphism f of F, an *f-orbit* in $\mathcal{C}$ is a minimal nonempty subset $\mathcal{C}_1$ of $\mathcal{C}$ such that $f(\mathcal{C}_1) \cong \mathcal{C}_1$. Notice that if no curve of $\mathcal{C}$ is isotopic to a curve in $f(\mathcal{C})$, then $\mathcal{C}$ contains no f-orbit.

Theorem 1.1. *Let $\mathcal{C}$ be an independent set of circles on a surface F with $\chi(F) < 0$, and let $f : F \to F$ be a homeomorphism. Then there is an orientation preserving pseudo Anosov map $g : F \to F$ with $g(c_i) \cong f(c_i)$ for all $c_i \in \mathcal{C}$ if and only if $\mathcal{C}$ contains no f-orbit.*

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Corollary 1.2. *Let c and c' be non-isotopic essential curves on a hyperbolic surface F such that (F, c) is homeomorphic to (F, c'). Then there is an orientation preserving pseudo Anosov map $g : F \to F$ such that $g(c) \cong c'$.*

Proof. This follows immediately from Theorem 1.1, noticing that if $\mathcal{C}$ has only one curve c then it contains an f-orbit if and only if $f(c) \cong c$, where $f : (F, c) \to (F, c')$ is a homeomorphism. □

A set of curves $\mathcal{C}$ on F is *non-separating* if $F - \mathcal{C}$ is connected.

Corollary 1.3. *Suppose $\mathcal{C}$ and $\mathcal{C}'$ are non-separating independent sets on F containing the same number of curves. Then there is an orientation preserving pseudo-Anosov map $g : F \to F$ with $g(\mathcal{C}) \cong \mathcal{C}'$ if and only if $\mathcal{C} \not\cong \mathcal{C}'$.*

Proof. Assume $\mathcal{C} \not\cong \mathcal{C}'$. Then there is a map $f : \mathcal{C} \to \mathcal{C}'$ which has no f-orbit. Since $F - \mathcal{C}$ and $F - \mathcal{C}'$ are connected, there is an orientation preserving map $f' : F \to F$ such that $f'(c_i) \cong f(c_i)$ for all i, hence the result follows from Theorem 1.1. □

Now suppose $\mathcal{C} = \{c_1, \dots, c_n\}$ and $\mathcal{C}' = \{c'_1, \dots, c'_n\}$ are two independent sets on F. For each permutation $\tau \in S_n$, define a map $\phi_\tau : \mathcal{C} \to \mathcal{C}'$ by $\phi_\tau(c_i) = c'_{\tau(i)}$. A map ϕ_τ is *realizable (by homeomorphism)* if there is a homeomorphism $f : F \to F$ such that $f(c_i) \cong \phi_\tau(c_i)$. In this case an f-orbit is also called a ϕ_τ-orbit. Note that whether a subset of $\mathcal{C}$ is a ϕ_τ-orbit is independent of the choice of the realization map f. Cutting F along $\mathcal{C}$ and $\mathcal{C}'$ respectively and looking at the components, it is easy to determine if a map ϕ_τ is realizable by homeomorphism. Applying Theorem 1.1 to each realizable ϕ_τ, we get the the following corollary, which gives a practical way to determine whether there exists a pseudo-Anosov map $g : F \to F$ such that $g(\mathcal{C}) \cong \mathcal{C}'$.

Corollary 1.4. *Let $\mathcal{C}, \mathcal{C}'$ be as above. Then there is an orientation preserving pseudo-Anosov map $g : F \to F$ with $g(\mathcal{C}) \cong g(\mathcal{C}')$, if and only if for some $\tau \in S_n$, ϕ_τ is realizable by homeomorphism, and $\mathcal{C}$ contains no ϕ_τ-orbit.* □

Example 1.5. Consider independent sets $\mathcal{C} = \{c_1, \dots, c_4\}$ and $\mathcal{C}' = \{c'_1, \dots, c'_4\}$ on a closed orientable surface F of genus at least 3, such that each of c_2, c_3, c'_2, c'_3 cuts off a once punctured torus containing c_1, c_4, c'_1, c'_4, respectively. Assume $c_i = c'_i$, for $i = 1, 2$, and $c_j \not\cong c'_j$ for $j = 3, 4$.

Since c_1, c_4, c'_1, c'_4 are nonseparating and c_2, c_3, c'_2, c'_3 are separating, the only realizable ϕ_τ are $\phi_{(1)}$ and $\phi_{(14)(23)}$. Now $\mathcal{C}$ contains some $\phi_{(1)}$-orbit, but it does not contain any $\phi_{(14)(23)}$-orbit. Therefore $\phi_{(14)(23)}$ is realizable by an orientation preserving pseudo-Anosov map g, which maps c_i to $c'_{\tau(i)}$, where $\tau = (14)(23)$.

Problem 1.6. Are similar results to Theorem 1.1 and its corollaries true for non-orientable surfaces or orientation reversing pseudo-Anosov maps?

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2. Proof of the main theorem

It is not difficult to see that the condition in Theorem 1.1 is necessary. Below we will assume that $\mathcal{C}$ contains no f-orbit, and proceed to show that there is a pseudo-Anosov map $g : F \to F$ with $g(c_i) \cong f(c_i)$. We remark that it is possible that $\mathcal{C}$ contains no f-orbit, and yet $f^k(c) \cong c$ for some $k > 1$, in which case we can still find a pseudo-Anosov map g such that $g(c_i) \cong f(c_i)$.

Let $\mathcal{C}$ and F be as in Theorem 1.1. There is an orientation reversing homeomorphism $h : F \to F$ which sends each $c_i \in \mathcal{C}$ to itself. This can been seen by cutting F along $\mathcal{C}$ to get a surface with boundary, which admits an orientation reversing map sending each boundary component to itself, inducing the required map h. Replacing f in Theorem 1.1 by $f \circ h$ if necessary, *we may assume from now on that f is orientation preserving.*

Given a set of curves $\mathcal{C}$ on F, denote by $N(\mathcal{C})$ a regular neighborhood of $\mathcal{C}$ in F. The set $\mathcal{C}$ is a *maximal* independent set if $\mathcal{C}$ is independent and no component of $F - \text{Int}N(\mathcal{C})$ contains essential circles, that is, each component of $F - \text{Int}N(\mathcal{C})$ is a pair of pants. The following lemma allows us to replace $\mathcal{C}$ in the theorem with a maximal independent set.

Denote by τ_c a right hand Dehn twist along a circle c on F.

Lemma 2.1. *Let $f : F \to F$ be a homeomorphism and $\mathcal{C}$ an independent set of circles on F. If $\mathcal{C}$ contains no f-orbit, then there is a homeomorphism $f' : F \to F$ and a maximal independent set $\mathcal{C}'$ of circles such that*

(1) $\mathcal{C} \subset \mathcal{C}'$,

(2) $f'(c) = f(c)$ for each $c \in \mathcal{C}$, and

(3) $\mathcal{C}'$ contains no f'-orbit.

Proof. Assume that $\mathcal{C}$ is not maximal, so there is a component S of $F - \text{Int}N(\mathcal{C})$ with $\chi(S) < 0$ which is not a disc with two holes. Now one can find a pair of essential circles c' and c'' in S such that their minimum geometric intersection number is positive. Let $\mathcal{C}' = \mathcal{C} \cup c'$. Then $\mathcal{C}'$ is independent. Let $f_k = f \circ \tau_{c''}^k$, where k is an integer. Then for any k, $f_k(c) = f(c)$ for $c \in C$. Since the minimum geometric intersection number of c' and c'' is positive, there are infinitely many isotopy classes of essential circles on F in the family $\{\tau_{c''}^k(c') \mid k \in \mathbb{Z}\}$. Since there are only finitely many curves in the set $f^{-1}(\mathcal{C}')$, we may pick k large enough so that $\tau_{c''}^k(c')$ is not isotopic to any curve in $f^{-1}(\mathcal{C}')$. Let $f' = f_k$ for this k. Then $f'(c') = f(\tau_{c''}^k(c'))$ is not isotopic to any curve in $\mathcal{C}'$.

If $\mathcal{C}'$ would contain an f'-orbit $\mathcal{C}_1$, then $f'(\mathcal{C}_1) \cong \mathcal{C}_1$, so $c' \notin \mathcal{C}_1$ because $f'(c')$ is not isotopic to any curve in $\mathcal{C}'$. But this would imply that $\mathcal{C}_1 \subset \mathcal{C}$, and since $f = f'$ on $\mathcal{C}$, it would contradict our assumption that $\mathcal{C}$ contains no f-orbit. Hence $\mathcal{C}'$ contains no f'-orbit. The proof now follows by induction because the number of curves in an independent set is bound above by $3g + h - 3$, where g is the genus and h the number of boundary components of F. □

Because of Lemma 2.1, we may assume from now on that the independent set $\mathcal{C}$ in Theorem 1.1 is maximal. Let $M = F \times [0,1]/f$ be the surface bundle over S^1 with fiber F and gluing map f, that is, it is the quotient of $F \times [0,1]$ obtained by identifying $(x,0)$ with $(f(x),1)$). Let $q : F \times [0,1] \to M$ be the quotient map. Denote by F^* the surface $q(F \times 0) = q(F \times 1)$, by $\mathcal{C}^*$ the curves $q(\mathcal{C} \times 0) = q(f(\mathcal{C}) \times 1)$ in M, and by c_i^* the curves $q(c_i \times 0) = q(f(c_i) \times 1)$ in $\mathcal{C}^*$, $i = 1, ..., n$. Let $N(\mathcal{C}^*)$ be a tubular neighborhood of $\mathcal{C}^*$ in M; let $M^* = M - \text{Int}N(\mathcal{C}^*)$; let T_i be the torus $\partial N(c_i^*)$ on ∂M^*.

Pick a meridian-longitude pair for each T_i, with longitude a component of $F^* \cap T_i$. Thus the slopes (i.e isotopy classes of simple closed curves) on T_i are in one to one correspondence with the numbers in $\mathbb{Q} \cup \{\infty\}$. (See [R] for more details.) Define $M^*(q_1, ..., q_n)$ to be the manifold obtained by q_i Dehn filling on T_i, $i = 1, \ldots, n$.

Lemma 2.2. *Let* $g(k_1, ..., k_n) = f \circ \tau_{c_1}^{k_1} \circ ... \circ \tau_{c_n}^{k_n}$. *Then*
(1) $g(k_1, ..., k_n)(c_i) = f(c_i)$ *for all* $i = 1, ..., n$ *and* $k_j \in \mathbb{Z}$.
(2) $F \times [0, 1]/g(k_1, ..., k_n) = M^*(1/k_1, ..., 1/k_n)$.

Proof. (1) Since the curves in $\mathcal{C}$ are mutually disjoint, $\tau_{c_j}(c_i) = c_i$ for all i, j; hence $g(k_1, ..., k_n)(c_i) = f \circ \tau_{c_1}^{k_1} \circ ... \circ \tau_{c_n}^{k_n}(c_i) = f(c_i)$.

(2) This is a simple geometric observation: Twisting along the curves will not change the link exterior M^* or the longitude of c_i^*, while a meridian of c_i^* in $F \times [0, 1]/f$ is changed to a $1/k_i$ curve in $F \times [0, 1]/g(k_1, ..., k_n)$. □

We would like to show that $g(k_1, ..., k_n)$ is isotopic to pseudo-Anosov maps for some k_i. By a deep theorem of Thurston (see Otal [Ot] for proof), it suffices to show that $M^*(1/k_1, ..., 1/k_n)$ is hyperbolic. It is attempting to apply the hyperbolic surgery theorem of Thurston [Th1], but we have to be careful because the manifold M^* may not be hyperbolic.

By an isotopy we may assume that either $f(c_i) = c_j$ or $f(c_i)$ is not isotopic to c_j. Define an oriented graph Γ associated to $(\mathcal{C}, f)$ as follows. Each element c_i in $\mathcal{C}$ is a vertex of Γ, still denoted by c_i, and there is an oriented edge $c_i c_j$ in Γ if and only if $c_j = f(c_i)$. A subgraph of Γ is a *chain* if it is homeomorphic to an interval. Since there is no f-orbit and the curves $f(c_i)$ are mutually non-isotopic, we have

Lemma 2.3. Γ *has only finitely many components, each of which is either a single vertex or a chain.* □

For each component γ of Γ, define C_γ in M as follows: If γ is an isolated vertex c_i, let C_γ be the knot c_i^*. If γ has an oriented edge $c_i c_j$, let $C_{c_i c_j}$ be the annulus $q(c_j \times [0, 1])$ in M. Notice that since $f(c_i) = c_j$, we have

$$q(c_j \times 1) = q(f(c_i) \times 1) = q(c_i \times 0) = c_i^*.$$

Thus $C_{c_i c_j}$ is an annulus with "top" boundary curve c_i^* and "bottom" boundary curve c_j^*. Now define C_γ to be the union of $C_{c_i c_j}$ over all edges $c_i c_j$ of γ. Then C_γ is an embedded annulus in M containing all $\{c_i^* \, | c_i \in \gamma\}$. Different components of Γ correspond to disjoint C_γ in M. Let C_Γ be the union of C_γ for all $\gamma \subset \Gamma$.

Lemma 2.4. *Every incompressible torus in* M^* *is isotopic to a torus in* $N(C_\Gamma)$.

Proof. Since F^* is incompressible in the surface bundle M, the surface $F' = F^* \cap M^*$ is also incompressible in M^*. Let T be an incompressible torus in M^*. We assume that T has been isotoped in M^* to meet F' minimally, so $\beta = T \cap F'$ consists of essential curves on both T and F'; in particular, it cuts T into π_1 injective annuli in M^*. Note that $\beta \neq \emptyset$, otherwise T would be an incompressible torus in $F \times [0, 1]$; which is impossible as $\chi(F) < 0$.

Since $\mathcal{C}$ is maximal, each component of F' is a disk with two holes, hence each component of β is parallel, on F^*, to a unique curve c_i^* in $\mathcal{C}$. We may isotope T so that β lies in $N(C_\Gamma)$.

An annulus A in $F \times [0, 1]$ is called a *vertical* annulus if it is isotopic to a product $s \times [0, 1]$ for a simple closed curve s in F; it is a *horizontal* annulus if it is boundary parallel. Each π_1-injective annulus in $F \times [0, 1]$ is either horizontal or vertical.

Cutting M^* along F', we get a manifold M' homeomorphic to $F \times [0, 1]$. If a component A_k of $T \cap M'$ is horizontal then its two boundary components are isotopic to the same c_i^* on F^*. Since $\partial A_k \subset \beta$ is already in $N(C_\Gamma)$, by an isotopy rel ∂A_k we may push A_k into $N(c_i^*) \subset N(C_\Gamma)$. If it is vertical, let $c_i^* = c_i \times 0$ and $c_j^* = f(c_j) \times 1$ be the two curves on $F \times \partial[0,1]$ isotopic to ∂A_k. Then the annulus A_k gives rise to an isotopy between c_i and $f(c_j)$ on F, so by our assumption above, we have $c_i = f(c_j)$, so $C_{c_i c_j} \subset C_\Gamma$. It follows that A_j is also rel ∂A_j isotopic into $N(C_\Gamma)$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *$X = M - \text{Int}N(C_\Gamma)$ is a hyperbolic manifold.*

Proof. X is irreducible: A reducing sphere S would bound a ball B in M because M, as an F bundle over S^1, is irreducible. Hence B contains some component of C_Γ. But since each component of C_Γ contains some essential curve c_i^* of F, and since F is π_1 injective in M, this is impossible.

X is not a Seifert fiber space: Each component of $N(C_\Gamma)$ can be shrunk to some $N(c_i^*)$, so X contains a nonseparating, hyperbolic, closed incompressible surface, i.e., the image of $F \times \frac{1}{2}$ under the reverse isotopy of the above shrinking process. No such surface exists in a Seifert fiber space with boundary because an essential surface in such a manifold is either horizontal (hence bounded) or vertical (hence a torus or an annulus).

X is also atoroidal: If T is an essential torus in X, then by Lemma 2.4 there is a torus T' in some $N(C_\gamma) \cap M^*$ such that $T \cup T'$ bounds a product region $W = T \times [0,1]$ in M^*. Since T is essential, T' is π_1 injective in $N(C_\gamma) \cap M^* = P \times S^1$, where P is a planar surface. Note that a π_1 injective torus in $P \times S^1$ is isotopic to $c \times S^1$ for some circle c in P. Since $W \subset M^*$, it contains no component of ∂M^*, so c must be parallel to the boundary curve of P on $\partial N(C_\gamma)$, which means that T', hence T, is parallel to $\partial N(C_\gamma)$.

Since ∂M is a union of tori, the lemma now follows from the Geometrization Theorem of Thurston for Haken manifolds [Th2]. □

Proof of Theorem 1.1. If $\mathcal{C}$ contains an f-orbit $\mathcal{C}_1$, then there is a circle $c \in \mathcal{C}$ such that $f^i(c)$ is isotopic to a curve in $\mathcal{C}_1$ for all i, and $f^k(c) \cong c$ for some $k \neq 0$. Suppose $g : F \to F$ is a homeomorphism such that $g(c') \cong f(c')$ for all $c' \in \mathcal{C}$, then by induction we have

$$g^i(c) = g(g^{i-1}(c)) \cong g(f^{i-1}(c)) \cong f^i(c)$$

for all i. In particular, $g^k(c) \cong c$, so g cannot be a pseudo-Anosov map.

Now suppose $\mathcal{C}$ contains no f-orbit. Let $\gamma_1, ..., \gamma_m$ be the distinct components of Γ. After relabeling, we may assume that c_i is a vertex of γ_i for $i \leq m$. Since $N(C_{\gamma_i})$ is isotopic to $N(c_i^*)$, by performing trivial surgery on c_j^* for all $j > m$ we get a 3-manifold

$$X = M^*(\emptyset, ..., \emptyset, \infty, ..., \infty) = M - \text{Int}N(C_\Gamma).$$

By Lemma 2.5, X is a hyperbolic manifold, therefore, by the Hyperbolic Surgery Theorem of Thurston [Th1], $X(1/k_1, ..., 1/k_m)$ is hyperbolic for sufficiently large k_i. By Lemma 2.2(2) we have

$$X(\frac{1}{k_1}, ..., \frac{1}{k_m}) = M^*(\frac{1}{k_1}, ..., \frac{1}{k_m}, \infty, ..., \infty) = F \times [0,1]/g(k_1, ...k_m, 0, ..., 0).$$

By Lemma 2.2(1) we have $g(k_1, ..., k_m, 0, ..., 0)(\mathcal{C}) = f(\mathcal{C})$. The theorem now follows from Thurston's theorem that $F \times [0,1]/g$ is hyperbolic if and only if g is isotopic to a pseudo Anosov map [Th2, Ot]. □

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