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TOROIDAL AND ANNULAR DEHN FILLINGS

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§1. Introduction

Let M be a (compact, connected, orientable) 3-manifold with a torus boundary component T_0. If r is a *slope* (the isotopy class of an essential simple loop) on T_0, then as usual we denote by $M(r)$ the 3-manifold obtained by r-*Dehn filling* on M, that is, attaching a solid torus J to M along T_0 in such a way that r bounds a disk in J.

We shall say that a 3-manifold M is *hyperbolic* if M with its boundary tori removed has a complete hyperbolic structure of finite volume with totally geodesic boundary. If M has non-empty boundary, then Thurston's Geometrization Theorem for Haken manifolds [Th1] [Th2] asserts that M is hyperbolic if and only if it contains no essential surface of non-negative euler characteristic, i.e., sphere, disk, torus or annulus.

There has been a considerable amount of investigation of the question of when a Dehn filling on a hyperbolic 3-manifold fails to be hyperbolic; if $\partial M - T_0$ is non-empty then, as noted above, this is equivalent to the existence of an essential surface of non-negative euler characteristic in the filled manifold. So suppose that M is hyperbolic, and that $M(r_1)$ and $M(r_2)$ contain essential surfaces $\widehat{F}_1$ and $\widehat{F}_2$ respectively, each of which is a sphere, disk, torus or annulus. Then for each of the ten possible pairs of surfaces, one can find upper bounds on $\Delta(r_1, r_2)$, the minimal geometric intersection number of r_1 and r_2. In all cases, the best possible bounds are now known; see [GW] for a fuller discussion. It is also of interest to examine the examples $(M; r_1, r_2)$ that realize values of $\Delta(r_1, r_2)$ at or near the maximal value; it seems to be the case that these are usually quite special.

In the present paper we consider the case where $\widehat{F}_1$ is an annulus and $\widehat{F}_2$ is a torus. Here it is known that $\Delta(r_1, r_2) \le 5$ [Go, Theorem 1.3]. To state our main result, let L_1, L_2, L_3 be the links shown in Figure 7.1 (a), (b) and (c), respectively. These are the Whitehead link, the 2-bridge link associated to the rational number 3/10, and the Whitehead sister link. Let M_i be the exterior of L_i, $i = 1, 2, 3$. Also, let J_α denote the attached solid torus in $M(r_\alpha)$, $\alpha = 1, 2$.

Theorem 1.1. *Suppose M is a hyperbolic 3-manifold, such that $M(r_1)$ contains an essential annulus, $M(r_2)$ contains an essential torus, and $\Delta(r_1, r_2) \ge 4$. Then either*

(1) $\Delta(r_1, r_2) = 4$ and M is homeomorphic to M_1 or M_2; or
(2) $\Delta(r_1, r_2) = 5$ and M is homeomorphic to M_3.

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Moreover, M_i is hyperbolic, $i = 1, 2, 3$, and admits two Dehn fillings $M_i(r_1)$ and $M_i(r_2)$, such that

(i) $\Delta(r_1, r_2) = 4$ for $i = 1, 2$, and $\Delta(r_1, r_2) = 5$ for $i = 3$;

(ii) $M_1(r_1)$ contains an essential annulus and an essential torus, each intersecting J_1 once. All the other $M_i(r_\alpha)$ contain an essential annulus and essential torus, each intersecting J_α twice.

This follows immediately from Theorems 6.10(1) and 7.5.

In particular, the upper bound of 5 on $\Delta(r_1, r_2)$ is realized by the exterior of the Whitehead sister link. The existence of annular and toroidal fillings on this manifold with $\Delta(r_1, r_2) = 5$ has also been proved by Miyazaki and Motegi [MM].

We also show the following.

Theorem 7.8. *There are infinitely many hyperbolic manifolds M_s, which admit two Dehn fillings $M_s(r_1), M_s(r_2)$, each containing an essential annulus and an essential torus, with $\Delta(r_1, r_2) = 3$.*

We now sketch the proof and describe the organization of the paper. Throughout, α, β will denote the numbers 1 or 2, with the convention that if they both appear, then $\{\alpha, \beta\} = \{1, 2\}$.

We may assume that the surface $\widehat{F}_\alpha$ meets the solid torus J_α in n_α meridional disks, and that n_α is minimal over all choices of $\widehat{F}_\alpha$. The arcs of intersection of the corresponding punctured annulus and punctured torus in M define labeled graphs $\Gamma_\alpha \subset \widehat{F}_\alpha$, with n_α vertices, $\alpha = 1, 2$. We assume that $\Delta(r_1, r_2) = 4$ or 5. The proof consists of a detailed analysis of such graphs Γ_α, and the elimination of all but four possible pairs, which are then shown to correspond to the examples listed in Theorem 1.1.

In Section 2 we give some definitions, and establish some properties of the intersection graphs Γ_α, which will be used throughout the paper.

Section 3 deals with the case that one of the graphs has a single vertex. It is shown that if $n_\alpha = 1$ then $\Delta(r_1, r_2) = 4$, $n_\beta = 2$, and M is the exterior of the Whitehead link.

Sections 4 and 5 deal with the case that one of the graphs has two vertices. In Section 4 it is shown that if $n_2 = 2$ then the graph Γ_1 is 2-separable, i.e., there is an essential sub-annulus A' of $\widehat{F}_1$, which contains two vertices of Γ_1 and has boundary disjoint from Γ_1. Note that this is also (trivially) true if $n_1 = 2$. In Section 5 it is shown that if Γ_1 is 2-separable then M must be one of the three manifolds M_i in Theorem 1.1.

Section 6 deals with the generic case. It is shown that if both graphs have at least three vertices, then $\Delta(r_1, r_2) \leq 3$. This completes the proof of the first part of Theorem 1.1. We then prove in Section 7 that the manifolds M_i in Theorem 1.1 do admit two annular and toroidal Dehn fillings with $\Delta(r_1, r_2) = 4$ or 5, and that there are infinitely many manifolds which admit two annular and toroidal Dehn fillings with $\Delta(r_1, r_2) = 3$.

§2. Preliminary Lemmas

Throughout this paper, we will fix a hyperbolic 3-manifold M, with a torus T_0 as a boundary component. Let r_1, r_2 be slopes on T_0. Denote by $\Delta = \Delta(r_1, r_2)$ the minimal geometric intersection number between r_1 and r_2. Recall that α, β

denote the numbers 1 or 2, with the convention that if they both appear, then $\{\alpha, \beta\} = \{1, 2\}$.

Assumption. We will always assume that $\widehat{F}_1$ is an essential annulus in $M(r_1)$, and $\widehat{F}_2$ is an essential torus in $M(r_2)$. Moreover, we assume that $M(r_\alpha)$ contains no essential sphere, i.e. it is irreducible.

The last assumption is necessary in our proof, as some of the properties stated in Lemmas 2.3–2.5 are not true without it. However, it is not necessary in the hypothesis of Theorem 6.10, because if $M(r_1)$ is reducible, then by [Wu1] or [Oh] we have $\Delta \leq 3$. Similarly, if $M(r_2)$ is reducible it follows from [Wu3] that $\Delta \leq 2$.

Denote by F_α the punctured surface $\widehat{F}_\alpha \cap M$. Let n_α, $\alpha = 1, 2$, be the number of boundary components of F_α on T_0. Choose $\widehat{F}_\alpha$ in $M(r_\alpha)$ so that n_α is minimal among all essential surfaces homeomorphic to $\widehat{F}_\alpha$. Minimizing the number of components of $F_1 \cap F_2$ by an isotopy, we may assume that $F_1 \cap F_2$ consists of arcs and circles which are essential on both F_α. Denote by J_α the attached solid torus in $M(r_\alpha)$. Let $u_1, \ldots, u_{n_1}$ be the disks that are the components of $\widehat{F}_1 \cap J_1$, labeled successively when traveling along J_1. Similarly let $v_1, \ldots, v_{n_2}$ be the disks in $\widehat{F}_2 \cap J_2$. Let Γ_α be the graph on $\widehat{F}_\alpha$ with the u_i's or v_j's, respectively, as (fat) vertices, and the arc components of $F_1 \cap F_2$ as edges. The minimality of the number of components in $F_1 \cap F_2$ and the minimality of n_α imply that Γ_α has no trivial loops, and that each disk face of Γ_α in $\widehat{F}_\alpha$ has interior disjoint from F_β.

If e is an edge of Γ_1 with an endpoint x on a fat vertex u_i, then x is labeled j if x is in $u_i \cap v_j$. In this case e is called a *j-edge at u_i*, and in Γ_2 it is an i-edge at v_j. The labels in Γ_1 are considered as integers mod n_2. In particular, $n_2 + 1 = 1$. When going around ∂u_i, the labels of the endpoints of edges appear as $1, 2, \ldots, n_2$ repeated Δ times. Label the endpoints of edges in Γ_2 similarly.

Each vertex of Γ_α is given a sign according to whether J_α passes $\widehat{F}_\alpha$ from the positive side or negative side at this vertex. Two vertices of Γ_α are *parallel* if they have the same sign, otherwise they are *antiparallel.* Note that if $\widehat{F}_\alpha$ is a separating surface, then n_α is even, and v_i, v_j are parallel if and only if i, j have the same parity. We use $val(v, G)$ to denote the valency of a vertex v in a graph G. If G is clear from the context, we denote it by $val(v)$.

Definition 2.1. An edge of Γ_α is a *positive edge* if it connects parallel vertices. Otherwise it is a *negative edge.*

When considering each family of parallel edges of Γ_α as a single edge $\widehat{e}$, we get the *reduced graph* $\widehat{\Gamma}_\alpha$ on $\widehat{F}_\alpha$. It has the same vertices as Γ_α.

We use Γ_α^+ (resp. Γ_α^-) to denote the subgraph of Γ_α whose vertices are the vertices of Γ_α and whose edges are the positive (resp. negative) edges of Γ_α. Similarly for $\widehat{\Gamma}_\alpha^+$ and $\widehat{\Gamma}_\alpha^-$.

A cycle in Γ_α consisting of positive edges is a *Scharlemann cycle* if it bounds a disk with interior disjoint from the graph, and all the edges in the cycle have the same pair of labels $\{i, i+1\}$ at their two endpoints, called the *label pair* of the Scharlemann cycle. A pair of edges $\{e_1, e_2\}$ is an *extended Scharlemann cycle* if there is a Scharlemann cycle $\{e_1', e_2'\}$ such that e_i is parallel and adjacent to e_i'. There is an obvious extension of this definition to extended Scharlemann cycles of arbitrary length, but this is enough for our purposes.

A subgraph G of a graph Γ on a surface F is *essential* if it is not contained in a disk in F. The following lemma contains some common properties of the graphs Γ_α.

Lemma 2.2. (Properties of Γ_α)

(1) (The Parity Rule) *An edge e is a positive edge in Γ_1 if and only if it is a negative edge in Γ_2.*

(2) A pair of edges cannot be parallel on both Γ_1 and Γ_2.

(3) If Γ_α has a set of n_β parallel negative edges, then on Γ_β they form mutually disjoint essential cycles of equal length.

(4) If Γ_α has a Scharlemann cycle, then $\widehat{F}_\beta$ is separating, and n_β is even.

(5) If Γ_α has a Scharlemann cycle $\{e_1, \dots, e_k\}$, then on Γ_β the subgraph consisting of $e_1, \dots, e_k$ and their end vertices, is essential.

(6) If $n_\beta > 2$, then Γ_α contains no extended Scharlemann cycle.

Proof. (1) can be found in [CGLS, Page 279].

(2) is [Go, Lemma 2.1]. It was shown that if a pair of edges is parallel on both graphs, then the manifold M contains a $(1, 2)$ cable space, which is impossible because M is assumed hyperbolic. See also [GLi].

(3) is [Go, Lemma 2.3] and [GLi, Proposition 1.3]. A set of n_β parallel edges with ends v, v' determine a permutation φ on the set of labels $\{1, \dots, n_\beta\}$ such that an edge with label i at v has label $\varphi(i)$ at v'. The edges form a collection of mutually disjoint cycles of length L in Γ_β, where L is the length of the orbits of φ. It was shown that if some cycle is not essential then M is cabled, hence non-hyperbolic.

(4) and (5) follow from the proof of [CGLS, Lemma 2.5.2]. It was shown that using the disk bounded by the Scharlemann cycle one can find another surface $\widehat{F}'_\beta$ in $M(r_\beta)$ which has fewer intersections with J_β, and is cobordant to $\widehat{F}_\beta$, so if $\widehat{F}_\beta$ were nonseparating then $\widehat{F}'_\beta$ would still be essential, which would contradict the minimality of n_β. If the subgraph G consisting of $e_1, \dots, e_k$ and their end vertices is inessential then one can show that $M(r_\beta)$ would be reducible, which contradicts our assumption.

(6) If $\widehat{F}_\beta$ is an annulus, this is [Wu3, Lemma 5.4(3)]. If $\widehat{F}_\beta$ is a torus, see [BZ, Lemma 2.9] or [GLu2, Theorem 3.2]. □

Let $\widehat{e}$ be a collection of at least n_α parallel negative edges on Γ_β, connecting v_1 to v_2. Then $\widehat{e}$ defines a permutation $\varphi : \{1, \dots, n_\alpha\} \to \{1, \dots, n_\alpha\}$, such that an edge e in $\widehat{e}$ has label k at v_1 if and only if it has label $\varphi(k)$ at v_2. Call φ the *permutation associated to* $\widehat{e}$. If we interchange v_1 and v_2, then the permutation is φ^{-1}; hence it is well defined up to inversion.

Lemma 2.3. (Properties of Γ_1) *(1) If a family of parallel edges in Γ_1 contains more than n_2 edges, and if $n_2 > 2$, then all the vertices of Γ_2 are parallel, and the permutation associated to this family is transitive.*

(2) If Γ_1 contains two Scharlemann cycles with disjoint label pairs $\{i, i+1\}$ and $\{j, j+1\}$, then $i \equiv j$ mod 2.

(3) Suppose $n_2 > 2$. Then a family of parallel positive edges in Γ_1 contains at most $n_2/2 + 2$ edges, and if it does contain $n_2/2 + 2$ edges, then $n_2 \equiv 0$ mod 4.

Proof. (1) If the parallel edges are positive, then they contain an extended Scharlemann cycle, which is impossible by Lemma 2.2(6). So assume that the edges are negative. It was shown in [Go, Lemma 4.2] that if the permutation is not transitive, then M contains a cable space, which is impossible in our case because M is hyperbolic. Hence the permutation is transitive. By the parity rule these edges connect parallel vertices in Γ_2, hence all the vertices of Γ_2 are parallel.

(2) and (3) are the same as [Wu1, Lemmas 1.7, 1.4, and 1.8]. □

Lemma 2.4. *Let $\widehat{e}$ be a family of parallel positive edges in Γ_α. If the label i appears twice among the labels of $\widehat{e}$, then there is a pair of edges among this family which form a Scharlemann cycle with i as one of its labels. In particular, if $\widehat{e}$ has more than $n_\beta/2$ edges, then it contains a Scharlemann cycle.*

Proof. Since the edges are positive, by the parity rule i cannot appear at both endpoints of a single edge in this family. Let $e_1, e_2, \ldots, e_k$ be consecutive edges of $\widehat{e}$ such that e_1 and e_k have i as a label. If $k \geq 4$, one can see that these edges contain an extended Scharlemann cycle in the middle, so by Lemma 2.2(6) we have $n_\beta = 2$, in which case e_1, e_2 is a Scharlemann cycle with i as a label, as required. If $k = 2$ then e_1, e_2 is a Scharlemann cycle and we are done. If $k = 3$ the parity rule implies that the endpoints of e_1 and e_3 on the same vertex have the same label, so again $n_\beta = 2$ and e_1, e_2 is a Scharlemann cycle.

If $\widehat{e}$ has more than $n_\beta/2$ edges, then it has more than n_β endpoints, so some label must appear twice. □

Lemma 2.5. (Properties of Γ_2) *(1) Any two Scharlemann cycles on Γ_2 have the same label pair.*

(2) Any family of parallel positive *edges $\widehat{e}$ in Γ_2 contains at most $n_1/2+1$ edges. Moreover, if it does contain $n_1/2+1$ edges, then two of them form a Scharlemann cycle.*

(3) Any family of parallel edges $\widehat{e}$ in Γ_2 contains at most n_1 edges.

(4) No three endpoints of a family of parallel edges in Γ_2 have the same label.

Proof. (1) This is [Wu3, Lemma 5.4(2)].

(2) If $\widehat{e}$ contains $n_1/2+1$ edges then they have n_1+2 endpoints, so some label i appears twice. By Lemma 2.4 i is the label of a Scharlemann cycle in this family. This proves the second sentence of (2).

Now suppose $\widehat{e}$ contains $n_1/2+2$ positive edges. If $n_1 = 2$ then there are only two nonparallel edges on the annulus $\widehat{F}_1$ connecting the two vertices, so a family of $n_1/2+2 = 3$ edges in $\widehat{e}$ would contain two which are parallel on both graphs, contradicting Lemma 2.2(2). If $n_1 > 2$, then 4 labels appear twice among the endpoints of the edges in $\widehat{e}$, so by Lemma 2.4 they are all labels of Scharlemann cycles in $\widehat{e}$, contradicting (1).

(3) If the edges in $\widehat{e}$ are positive, this follows from (2). So assume that $\widehat{e}$ consists of n_1+1 parallel negative edges $e_1, e_2, \ldots, e_{n_1+1}$. By Lemma 2.2(3) the first n_1 edges form several essential cycles in Γ_1. Let $e_1 \cup e_{j_2} \ldots \cup e_{j_k}$ be the one containing e_1. The edges e_1 and e_{n_1+1} have the same label pair at their endpoints in Γ_2, hence have the same end vertices on Γ_1. By Lemma 2.2(2) they cannot be parallel on Γ_1, so after shrinking each vertex to a point, the path e_{n_1+1} is homotopic rel ∂ to the path $e_{j_2} \cup \ldots \cup e_{j_k}$; in other words, the cycle $e_{j_2} \cup \ldots \cup e_{j_k} \cup e_{n_1+1}$ is an inessential cycle in Γ_1, which contradicts Lemma 2.2(3) because it is a cycle corresponding to the n_1 parallel edges $e_2, \ldots, e_{n_1+1}$ in Γ_2.

(4) If the label i appears three times on the endpoints of a family of parallel edges, then two of those would appear on the same end vertex, so there would be

at least $n_1 + 1$ parallel edges, contradicting (3). □

Let u be a vertex of Γ_α. Let P, Q be two edge-endpoints on ∂u. Let $P = P_0, P_1, \dots, P_{k-1}, P_k = Q$ be the edge-endpoints encountered when traveling along ∂u in the direction induced by the orientation of u. Then the *distance* from P to Q is defined as $\rho_u(P,Q) = k$. To emphasize that the distance is measured on Γ_α, we may also use $\tau_\alpha(P,Q)$ to denote $\rho_u(P,Q)$. Notice that if the valency of u is m, then $\rho_u(Q,P) = m - \rho_u(P,Q)$. The following lemma can be found in [Go].

Lemma 2.6. ([Go, Lemma 2.4]) *Let a, b be components of $\partial F_\alpha \cap T_0$, and x, y components of $\partial F_\beta \cap T_0$.*

(i) Suppose $P, Q \in a \cap x$ and $R, S \in b \cap y$. If $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$ then $\tau_\beta(P,Q) = \tau_\beta(R,S)$.

(ii) Suppose that $P \in a \cap x$, $Q \in a \cap y$, $R \in b \cap x$, and $S \in b \cap y$. If $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$, then $\tau_\beta(P,R) = \tau_\beta(Q,S)$. □

If e_i is an edge with exactly one endpoint on u, $i = 1, 2$, define $\rho_u(e_1, e_2) = \rho_u(P_1, P_2)$, where $P_i = e_i \cap u$.

A pair of edges e_1, e_2 connecting two vertices u, v in Γ_α is an *equidistant pair* if $\rho_u(e_1, e_2) = \rho_v(e_2, e_1)$. If $u = v$, then define e_1, e_2 to be an equidistant pair if $\rho_u(P_1, P_2) = \rho_v(Q_2, Q_1)$, where P_i, Q_i are the endpoints of e_i. One can check that this is independent of the choices of P_i and Q_i.

The following lemma follows immediately from the definition, and is convenient for applications. In particular, it applies to parallel loops based at a vertex.

Lemma 2.7. *If e_1, e_2 is a pair of parallel edges connecting a pair of parallel vertices in Γ_α, then e_1, e_2 is an equidistant pair in Γ_α.* □

Lemma 2.8. *Let e_1, e_2 be a pair of edges with $\partial e_1 = \partial e_2$ in both Γ_1 and Γ_2. Then e_1, e_2 is an equidistant pair in Γ_1 if and only if it is an equidistant pair in Γ_2.*

Proof. Suppose e_i joins vertices a and b in Γ_1, and vertices x and y in Γ_2, $i = 1, 2$. Denote the endpoints of e_1, e_2 in Γ_1 by P, S and Q, R, respectively, as in Figure 2.1(a) or (c). There are two cases, depending on whether or not P and Q have the same label. We give a proof in the case that P and Q have the same label, say x. The proof for the other case is similar, using Lemma 2.6(2) instead of 2.6(1).

In this case we have $P, Q \in a \cap x$ and $R, S \in b \cap y$, so e_1, e_2 being equidistant on Γ_1 implies that

$$\tau_\alpha(P,Q) = \rho_a(e_1, e_2) = \rho_b(e_2, e_1) = \tau_\alpha(R,S).$$

By Lemma 2.6(1), we have $\tau_\beta(P,Q) = \tau_\beta(R,S)$, which implies that $\rho_x(e_1, e_2) = \rho_y(e_2, e_1)$, so e_1, e_2 is also equidistant on Γ_2. Similarly, e_1, e_2 being equidistant on Γ_2 also implies that they are equidistant on Γ_1. □

Definition 2.9. Given two slopes r_1, r_2 on T_0, choose a meridian-longitude pair m, l on the torus T_0 so that $r_1 = m$, and the slope r_2 is represented by $dm + \Delta l$ for some d with $1 \leq d \leq \Delta/2$. Then $d = d(r_1, r_2)$ is called the *jumping number* of r_1, r_2. Notice that if $\Delta = 4$, then $d = 1$, and if $\Delta = 5$, then $d = 1$ or 2.

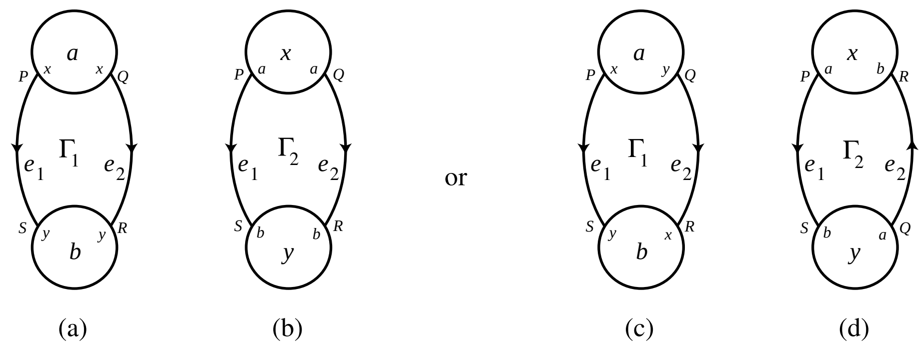


Figure 2.1

Lemma 2.10. *Let $P_1, \dots, P_\Delta$ be the points of $\partial u_i \cap \partial v_j$, labeled so that they appear successively on u_i. Let $d = d(r_1, r_2)$ be the jumping number of r_1, r_2. Then on v_j these points appear in the order of $P_d, P_{2d}, \dots, P_{\Delta d}$, in some direction. In particular, if $\Delta = 4$, then P_1, P_2, P_3, P_4 also appear successively on v_j.*

Proof. This follows by examining the order of the intersection points on the two curves r_1 and r_2. If $\Delta = 4$, we must have $d = 1$, hence the result follows. □

Lemma 2.11. ([HM, Proposition 5.1]) *If C is a cycle in Γ_α consisting of positive i-edges, bounding a disk D which contains no vertices in its interior, then D contains a Scharlemann cycle.* □

Lemma 2.12. *(1) Suppose G is a reduced graph on an annulus A with V vertices and no trivial loops. Then G has at most $3V - 2$ edges, and has a vertex of valency at most 5.*

(2) A reduced graph G on a torus with no trivial loops has at most $3V$ edges.

Proof. (1) By embedding the annulus A in a sphere S, we can consider G as a graph on S. Let E, F be the number of edges and disk faces of G. Then $V - E + F \geq 2$ (the inequality may be strict if there are some non-disk faces.) All but two faces of G have at least three edges, and each of the two exceptional ones has at least one edge. Hence we have

$$2 + 3(F - 2) \leq 2E.$$

Solving those two inequalities, we get $E \leq 3V - 2$.

If each vertex of G has valency at least 6, then $6V \leq 2E$, which is impossible because $E \leq 3V - 2$.

(2) The calculation for G on torus is similar but simpler. We have $V - E + F \geq 0$, and $3F \leq 2E$. The result follows by solving these inequalities. □

§3. Case 1: One of the graphs has a single vertex

From now on we shall denote the essential annulus $\widehat{F}_1$ in $M(r_1)$ by A, and the essential torus $\widehat{F}_2$ in $M(r_2)$ by T. Recall that $M(r_\alpha)$ is assumed to be irreducible. By [Go, Theorem 1.3] we may, and will always, assume that $\Delta \leq 5$. In this section we will consider the case where one of the graphs has only one vertex.

The link in Figure 7.1(a) is called the *Whitehead link*.

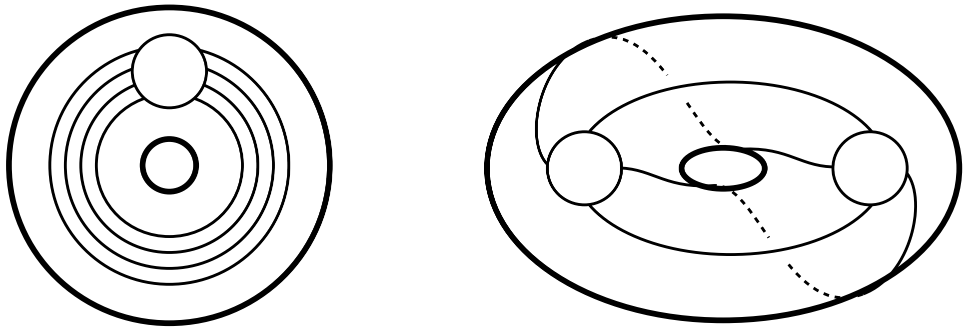


Figure 3.1

Lemma 3.1. *Suppose $n_1 = 1$ and $\Delta \geq 4$. Then $n_2 = 2$, $\Delta = 4$, and the graphs Γ_α are as shown in Figure 3.1. The manifold M is the Whitehead link exterior.*

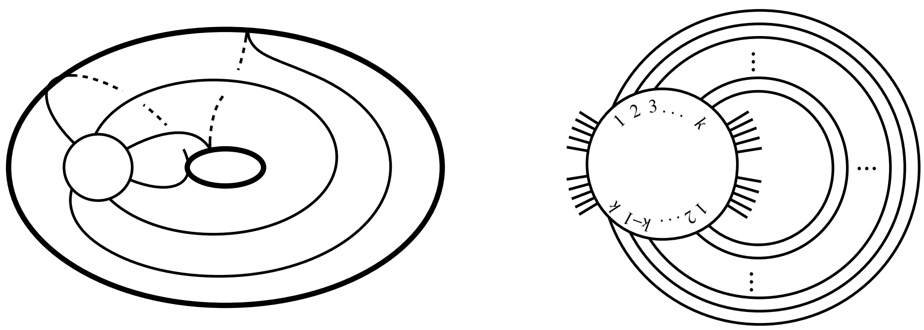
Proof. In this case, there is only one edge in the reduced graph $\widehat{\Gamma}_1$, so all edges of Γ_1 are parallel positive edges. By the parity rule, all edges of Γ_2 are negative. In particular, $n_2 \geq 2$. Moreover, the $\Delta n_2/2$ parallel edges of Γ_1 contain an extended Scharlemann cycle, so by Lemma 2.2(6) we must have $n_2 = 2$. In this case there are exactly Δ edges in each graph. Since they are parallel on Γ_1, by Lemma 2.2(2) no two of them can be parallel on Γ_2. There are at most 4 nonparallel edges on the torus connecting two vertices. Hence $\Delta \leq 4$, and if $\Delta = 4$, then up to homeomorphism of the torus T the graph is that shown in Figure 3.1. We will postpone until Lemma 7.2 the proof that if the intersection graphs are as shown in Figure 3.1 then the manifold M is the exterior of the Whitehead link. This will complete the proof of Lemma 3.1. $\square$

Lemma 3.2. *Suppose $n_2 = 1$ and $\Delta \geq 4$. Then $n_1 = 2$, $\Delta = 4$, and the graphs Γ_α are as shown in Figure 3.3(c) and (d). The manifold M is the Whitehead link exterior.*

Proof. The reduced graph $\widehat{\Gamma}_2$ on the torus is a subgraph of that shown in Figure 3.2(a), which has only three edges. The parity rule implies that $n_1 > 1$. The graph Γ_2 contains $\Delta n_1/2 \geq 2n_1$ edges, and so has a family of more than $n_1/2$ parallel positive edges. By Lemmas 2.5(2) and 2.2(4) n_1 must be even.

We want to show that $n_1 = 2$. If $\Delta = 5$ and $n_1 \geq 4$, then Γ_2 contains a family of $\Delta n_1/6 = \frac{1}{2}n_1 + \frac{1}{3}n_1 > \frac{1}{2}n_1 + 1$ parallel positive edges, which contradicts Lemma 2.5(2). Assume $\Delta = 4$. Then the labels at a family of parallel arcs are as shown in Figure 3.2(b). In particular, the number of parallel edges is even by the parity rule. Thus if $n_1 \geq 4$, then there are at least 4 edges in one of the families. From the labeling in Figure 3.2 we see that this family contains an extended Scharlemann cycle, contradicting Lemma 2.2(6).

Now we assume that $n_1 = 2$. In this case there are only two edges in $\widehat{\Gamma}_1$. Since no two edges can be parallel on both graphs, there are at most two edges in each family of parallel edges in Γ_2. Thus when $\Delta = 5$, there is only one possible configuration, shown in Figure 3.3(a), and when $\Delta = 4$, there are two possibilities, shown in



(a) (b)

Figure 3.2

Figure 3.3(b) – (c). The first two cases can be ruled out because there is an edge with the same label on both endpoints, which contradicts the parity rule. It follows that Figure 3.3(c) is the only possibility, which also uniquely determines the graph Γ_1, shown in Figure 3.3(d).

One can build the manifold M as in Lemma 7.2 to show that M is the exterior of the Whitehead link. However, there is a shortcut. Notice that the union of $A \times I$ with the Dehn filling solid torus V_1 is a genus 3 handlebody. After attaching the 2-handles coming from the faces of Γ_2 and then capping off the spherical boundary components, we get a manifold Y with boundary a torus. Since the original manifold M is assumed to be hyperbolic, this implies that Y must be homeomorphic to $M(\gamma_1)$; in other words, there cannot be anything else attached to Y. Therefore M is uniquely determined by the intersection graphs. Combining this with the above result, we conclude that there is at most one manifold M such that $M(\gamma_1)$ contains a torus intersecting the Dehn filling solid torus only once, $M(\gamma_2)$ is annular, and $\Delta(\gamma_1, \gamma_2) > 3$. On the other hand, the Whitehead link exterior does satisfy those conditions, because the -4 surgery is annular, and the Seifert surface of one component of the link extends, after 0 surgery, to a torus which intersects the Dehn filling solid torus just once. Therefore, this must be the manifold. □

Definition 3.3. The graph Γ_1 is *k-separable* if there is an annulus A' on A, such that (i) A' contains exactly k vertices of Γ_1, (ii) $\partial A'$ consists of two essential curves in the interior of A, and (iii) $\partial A'$ is disjoint from Γ_1. Denote by Γ_1' the graph $\Gamma_1 \cap A'$ on A', and by Γ_2' the subgraph of Γ_2 consisting of edges which, when considered as edges in Γ_1, lie in A'. Notice that by definition Γ_1 is always n_1-separable.

Lemma 3.4. *Suppose $\Delta \geq 4$, and Γ_1 is 1-separable. Then $n_1 = 1$.*

Proof. Let Γ_α' be the graphs on A' and T defined in Definition 3.3. As in the proof of Lemma 3.1, the graphs Γ_α' must be the ones shown in Figure 3.1. Using the construction in the proof of Lemma 7.2, we see that the manifold $W = (A' \times I) \cup J_1 \cup (D \times I)$ is the Whitehead link exterior, where D is a disk face of Γ_2'. Let $T_1 = \partial M - T_0$. Then T_1 contains the boundary of A, so it is nonempty. Let T' be the component of ∂W in the interior of M. Note that T' separates T_0 from T_1,

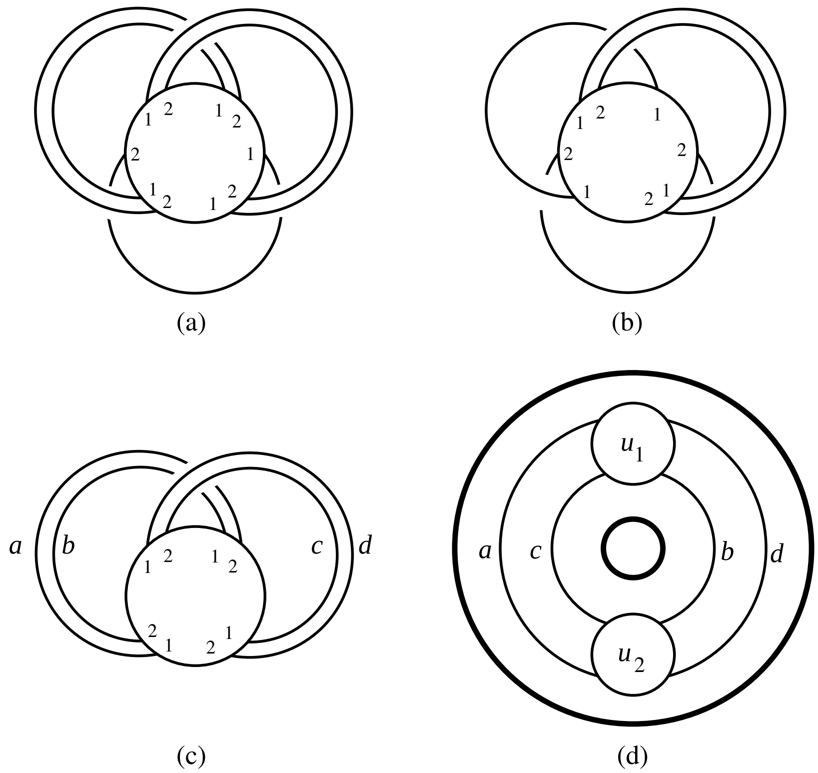


Figure 3.3

so T' is incompressible in M because otherwise M would be reducible. Since M is hyperbolic, T' must be parallel to a boundary component of M. Since W is not a product, we see that T' must be parallel to T_1. In particular, T_1 is a torus. Now the annulus A' can be extended to an essential annulus in $M(r_1)$ intersecting J_1 just once. By the minimality of n_1, we must have $n_1 = 1$. □

Proposition 3.5. *Suppose $\Delta \geq 4$. If $n_\alpha = 1$, then $\Delta = 4$, $n_\beta = 2$, and the manifold M is the exterior of the Whitehead link shown in Figure 7.1.*

Proof. This follows immediately from Lemmas 3.1 and 3.2. □

§4. The case that $n_2 = 2$

So far we have seen that if $\Delta = 4$ and one of the $n_\alpha = 1$ then the manifold M is the exterior of the Whitehead link. The goal of this section is to show that if $n_2 = 2$ then the graph Γ_1 is 2-separable; see Proposition 4.6 below. In the next section it will be shown that if Γ_1 is 2-separable then M must be the exterior of one of the links in Figure 7.1.

Lemma 4.1. *Suppose that $\Delta \geq 4$. If all vertices of Γ_2 are parallel, then $n_1 = n_2 = 2$, and hence Γ_1 is 2-separable.*

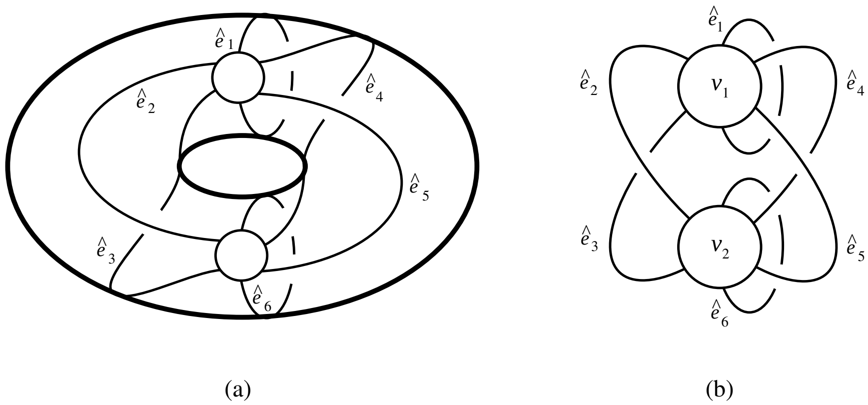


Figure 4.1

Proof. By Lemma 2.12 the reduced graph $\widehat{\Gamma}_2$ has at most $3n_2$ edges. For each i there are Δn_2 i-edges in Γ_2, so two of them are parallel. By Lemma 2.4 this implies that i is the index of some Scharlemann cycle in Γ_2. This is true for any index i from 1 to n_1, but by Lemma 2.5(1) there are at most two labels that can appear as the labels of Scharlemann cycles in Γ_2, hence we have $n_1 = 2$. (We cannot have $n_1 = 1$ because of the parity rule.)

Now on Γ_1 there are only two families of parallel edges. Consider the function

$$\varphi : \{1, 2, \dots n_2\} \to \{1, 2, \dots n_2\}$$

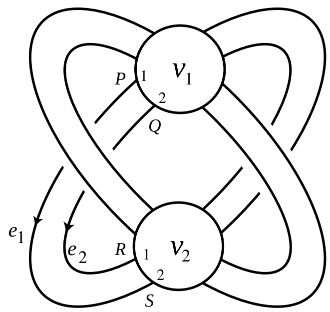
defined so that if e is an edge in Γ_1 with label i on u_1 then it has label $\varphi(i)$ on the vertex u_2. Clearly this is well defined. Note that some family contains at least $2n_2$ parallel edges, so by Lemma 2.3(1) either $n_2 = 2$ and we are done, or the function φ is transitive, i.e. it has only one orbit. In particular, if $n_2 > 2$ and $\varphi(k) = l$, then $\varphi(l) \neq k$.

On Γ_2 there are $\Delta n_2 > 3n_2$ edges, so there is a pair of adjacent parallel edges e_1, e_2 connecting, say, the vertex v_k to v_l. Note that e_1 and e_2 have different labels at their endpoints on v_k, hence in Γ_1 the edge e_1 has label k, say, at the vertex u_1, and label l at the vertex u_2, while e_2 has label l at u_1 and label k at u_2. Therefore $\varphi(k) = l$ and $\varphi(l) = k$, a contradiction. □

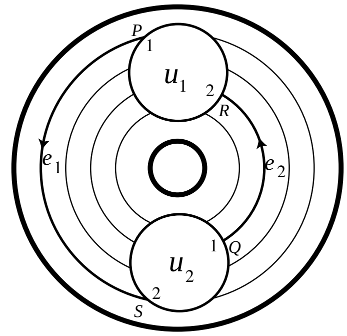
Lemma 4.2. *If $n_2 = 2$ then, up to homeomorphism of T, the reduced graph $\widehat{\Gamma}_2$ is a subgraph of the graph shown in Figure 4.1(a).*

Proof. If $\widehat{\Gamma}_2$ contains two loops based at v_1, then they cut the torus into a disk, hence there would be no loops based at v_2, so any edge with an endpoint on v_2 must have another endpoint on v_1; but since v_1 and v_2 have the same valency in Γ_2, this is impossible. Therefore $\widehat{\Gamma}_2$ has at most one loop based at each v_i. It is now easy to see that $\widehat{\Gamma}_2$ is isomorphic to a subgraph of that shown in Figure 4.1(a). □

We will use Figure 4.1(b) to indicate the reduced graph shown in Figure 4.1(a). The graph Γ_2 is obtained by replacing the edges in Figure 4.1(b) by families of



(a)



(b)

Figure 4.2

parallel edges. Denote by p_i the number of parallel edges in Γ_2 represented by $\widehat{e}_i$ in the reduced graph $\widehat{\Gamma}_2$.

Lemma 4.3. *If $\Delta \geq 4$ and $n_2 = 2$, then the two vertices v_1, v_2 of Γ_2 are antiparallel.*

Proof. Assume that v_1, v_2 are parallel. Then by Lemma 4.1 we have $n_1 = 2$. Since all edges of $\widehat{\Gamma}_2$ are positive, by the parity rule there is no loop on $\widehat{\Gamma}_1$, so $\widehat{\Gamma}_1$ consists of two families of parallel edges connecting u_1 to u_2.

Notice that in this case either all edges of $\widehat{\Gamma}_1$ have the same label on both endpoints, or they all have distinct labels on their two endpoints. In the first case $\widehat{\Gamma}_2$ consists of two families of parallel loops, so some pair of edges would be parallel on both graphs, which is impossible by Lemma 2.2(2). In the second case there are no loops in $\widehat{\Gamma}_2$, so there are only 4 edges in $\widehat{\Gamma}_2$. It follows again from Lemma 2.2(2) that $\Delta = 4$, and $\widehat{\Gamma}_2$ consists of 4 pairs of parallel edges. See Figure 4.2(a).

Consider a pair of parallel edges e_1, e_2 in Γ_2. Since e_1, e_2 are a pair of parallel edges connecting parallel vertices, by Lemma 2.7 they form an equidistant pair in Γ_2, so by Lemma 2.8 they also form an equidistant pair in Γ_1, that is $\rho_{u_1}(e_1, e_2) = \rho_{u_2}(e_2, e_1)$. Also, from Figure 4.2(b) we can see that $\rho_{u_2}(e_2, e_1) = 8 - \rho_{u_1}(e_1, e_2)$, so we must have $\rho_{u_1}(e_1, e_2) = \rho_{u_2}(e_2, e_1) = 4$. On the other hand, observe that the endpoints of e_1, e_2 on u_1 have different labels, so $\rho_{u_1}(e_1, e_2)$ must be an odd number, which is a contradiction. □

Lemma 4.4. *Suppose $\Delta \geq 4$ and $n_2 = 2$. If $p_2 + p_3 \geq p_4 + p_5$, then $p_2 = p_3 = n_1$.*

Proof. By Lemma 2.5(3), $p_i \leq n_1$, $1 \leq i \leq 6$. Since v_1 and v_2 are antiparallel, we shall write $v_1 = v_+$, and $v_2 = v_-$. After relabeling if necessary, we may assume that the labels of the endpoints of the edges in $\widehat{e}_1$ are $1, 2, \ldots, k$ at the bottom of v_+. If $p_2 + p_3 = 2n_1 - r$, then since

$$\Delta n_1 = 2k + (p_2 + p_3) + (p_4 + p_5) \geq 2k + 2(2n - r),$$

we have $r \leq k$, so the labels of the left r endpoints of edges in $\widehat{e}_1$ are $1, 2, \ldots, r$ at the top of v_+; see Figure 4.3(a). If $r > 2$, then the left r edges of $\widehat{e}_1$ would contain an extended Scharlemann cycle, which would contradict Lemma 2.2(6) because $n_1 \geq k \geq r > 2$. Therefore we have $p_2 + p_3 \geq 2n_1 - 2$. Also, the left edge of $\widehat{e}_1$ cannot have the same label at both endpoints because of the parity rule. Hence either $p_2 + p_3 = 2n_1$, which implies $p_2 = p_3 = n_1$, or $p_2 + p_3 = 2n_1 - 2$. We need to rule out the second possibility.

Assume $p_2 + p_3 = 2n_1 - 2$. Since it is assumed that $p_2 + p_3 \geq p_4 + p_5$, we have

$$p_1 = p_6 = \frac{1}{2}(\Delta n_1 - (p_2 + p_3) - (p_4 + p_5)) \geq 2.$$

Examining the labels, we see that the two leftmost edges in $\widehat{e}_1$ form a Scharlemann cycle. Similarly for the two leftmost edges in $\widehat{e}_6$. By Lemma 2.5(1), these two Scharlemann cycles must have the same label pairs, say $\{1, 2\}$; see Figure 4.3(b). The existence of Scharlemann cycles implies that n_1 is even, and all vertices in Γ_1 of the same parity are parallel.

By Lemma 2.5(2) $\widehat{e}_1$ contains at most $n_1/2 + 1$ edges, so $p_4 + p_5 = \Delta n_1 - (p_2 + p_3) - 2p_1 \geq \Delta n_1 - 3n_1$. Since $p_4 + p_5 \leq p_2 + p_3$, we must have $\Delta = 4$.

First assume that $n_1 \geq 4$. In this case, because of the orientation of v_+ and v_-, the labels of the Scharlemann cycles must be as shown in Figure 4.3(b). It is easy to see that we cannot have $p_2 = p_3 = n_1 - 1$, otherwise an edge e in the family $\widehat{e}_2$ would have labels of distinct parities on its two endpoints, which would contradict the parity rule because e would connect antiparallel vertices in both graphs. Hence one of p_2, p_3 is $n_1 - 2$ and the other is n_1. Without loss of generality we may assume that $p_2 = n_1 - 2$, and $p_3 = n_1$. See Figure 4.3(b).

Let $e_1, \ldots, e_{n_1}$ be the parallel edges in the family $\widehat{e}_3$, where e_i has label i at its endpoint on v_+. From Figure 4.3(b) we can see that the other endpoint of e_i is labeled $i + 2$, hence these edges form two cycles $e_1 \cup e_3 \cup e_5 \cup \ldots \cup e_{n_1 - 1}$ and $e_2 \cup e_4 \cup e_6 \cup \ldots \cup e_{n_1}$ on Γ_1. Also, the edges of the Scharlemann cycle in the family $\widehat{e}_1$ form a cycle $e'_1 \cup e'_2$ on Γ_1 with vertices u_1 and u_2. By Lemmas 2.2(3) and 2.2(5) all these three cycles are essential on the annulus A, which implies that the cycle $e'_1 \cup e'_2$ must lie between the other two cycles. See Figure 4.3(c) for a possible picture of $\widehat{\Gamma}_1$, where $n_1 = 6$.

There are $\Delta = 4$ edge endpoints at v_+ labeled 1, among which the two endpoints on e'_1 and e'_2 are non-adjacent. Since $p_1 \leq n_1$, the other two edges labeled 1 at v_+ connect antiparallel vertices in Γ_2, hence connect parallel vertices in Γ_1. Since all the vertices with the same parity as u_1 are on one side of the cycle $e'_1 \cup e'_2$, this implies that all the edges with label + at u_1 are on one side of $e'_1 \cup e'_2$. It follows that the endpoints of e'_1 and e'_2 on u_1 are adjacent among the endpoints of all such edges. Since we have seen that the corresponding endpoints of e'_1 and e'_2 are non-adjacent on v_+ among all endpoints labeled 1, this contradicts Lemma 2.10.

Now assume $n_1 = 2$. Since $p_1 \leq n_1$, and since $p_2 + p_3 = 2n_1 - 2 = 2 \geq p_4 + p_5$, we must have $p_2 + p_3 = p_4 + p_5 = p_1 = 2$. The graph Γ_1 is shown in Figure 4.3(d). Now the same argument as above applies: Among the 4 points of $\partial v_+ \cap \partial u_1$, the two on e'_1 and e'_2 are non-adjacent on ∂v_+ but adjacent on ∂u_1, which contradicts Lemma 2.10, completing the proof of the lemma. □

Lemma 4.5. *Suppose $\Delta \geq 4$, $n_2 = 2$, and $n_1 \geq 2$. Then $p_1 = p_6 > 0$.*

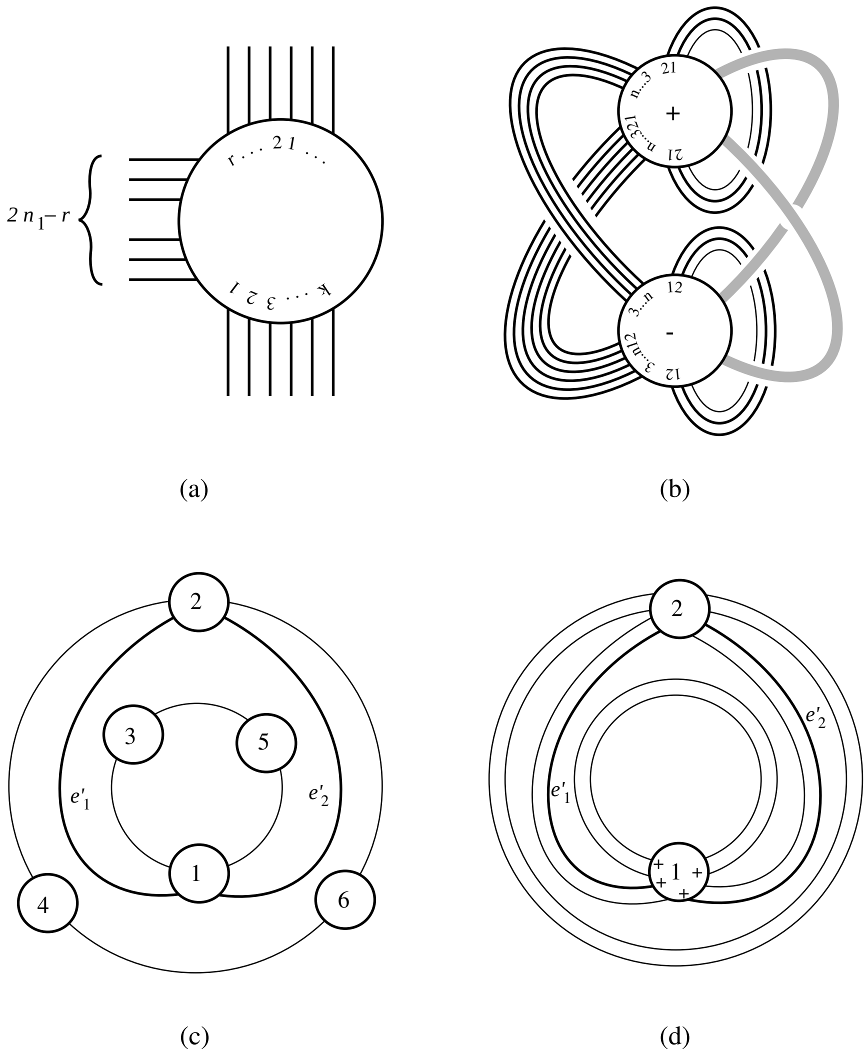


Figure 4.3

Proof. Assume that $p_1 = p_6 = 0$. The reduced graph $\widehat{\Gamma}_2$ has 4 edges $\widehat{e}_2, \widehat{e}_3, \widehat{e}_4, \widehat{e}_5$, each representing at most n_1 edges of Γ_2. The valency of v_+, v_- in Γ_2 is Δn_1, and $\Delta \geq 4$, so we must have $\Delta = 4$, and each $\widehat{e}_i$ represents exactly n_1 edges.

Consider the permutation $\varphi : \{1, \ldots, n_1\} \to \{1, \ldots, n_1\}$ determined by the parallel edges in $\widehat{e}_i$. Since each family has exactly n_1 edges, one can see that φ is independent of the choice of $\widehat{e}_i$. Let L be the length of an orbit of φ. The edges in $\widehat{e}_i$ form cycles in Γ_1, each having length L. Thus the reduced graph $\widehat{\Gamma}_1$ consists of several cycles of length L, and each edge of $\widehat{\Gamma}_1$ represents exactly 4 edges of Γ_1.

CASE 1. $L \geq 3$.

There are 4 edges in Γ_1 from u_1 to $u_{\varphi(1)}$, two of which have label $+$ at u_1. On

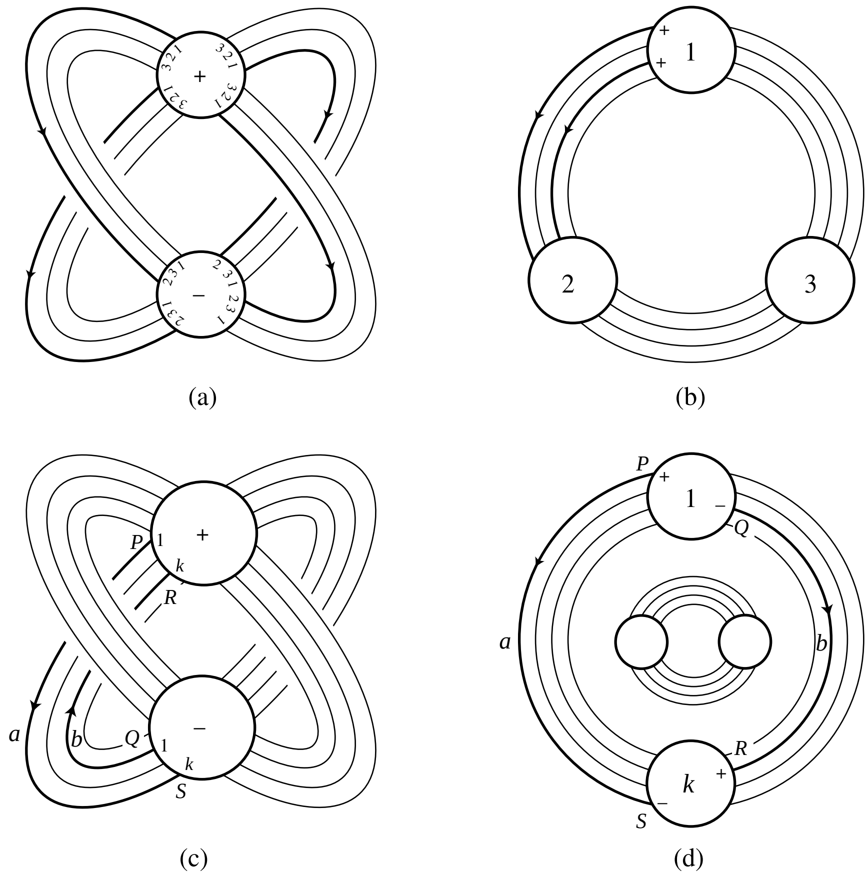


Figure 4.4

the other hand, there are 4 edges in Γ_2 having label 1 at v_+ and label $\varphi(1)$ at v_-, which is a contradiction. See Figure 4.4(a)–(b) for the case that $L = n_1 = 3$ and $\varphi(1) = 2$.

CASE 2. $L = 2$.

Let $k = \varphi(1) = n_1/2 + 1$. There is a pair of edges a, b in the family $\widehat{e}_3$, such that a has label 1 at v_+ and label k at v_-, while b has label k at v_+ and label 1 at v_-. Let P, S, Q, R be the endpoints of a, b, as shown in Figure 4.4(c). Notice that the vertices u_1 and u_k are parallel, so from Figure 4.4(d) we can see that $\rho_1(a, b) = \rho_k(b, a)$. Hence a, b are an equidistant pair in Γ_1. By Lemma 2.8 a, b are also an equidistant pair in Γ_2. However, this is impossible because $\rho_+(a, b) = k - 1$ and $\rho_-(b, a) = 7(k - 1)$.

CASE 3. $L = 1$.

In this case all edges of Γ_1 are loops, hence the reduced graph $\widehat{\Gamma}_1$ consists of n_1 disjoint loops. Since $n_1 > 1$, Γ_1 is 1-separable, which contradicts Lemma 3.4. $\square$

Proposition 4.6. *Suppose $\Delta \geq 4$, $n_2 = 2$, and $n_1 > 2$. Then Γ_1 is 2-separable.*

Proof. By Lemmas 4.3–4.5 we may assume that $v_1 = v_+$ and $v_2 = v_-$ are antiparallel, $p_1 = p_6 > 0$, and $p_2 = p_3 = n_1$. We may assume without loss of generality that the labels of the edge endpoints are as shown in Figure 4.5(a).

The families of parallel edges in $\widehat{e}_2$ and $\widehat{e}_3$ determine the same permutation φ on the set of labels $\{1, \dots , n_1\}$. Let L be the length of an orbit of φ. Let $k = \varphi(1)$. The argument splits into two cases: $L \geq 2$, and $L = 1$. We will show that the first case leads to a contradiction, while the second implies that Γ_1 is 2-separable.

CASE 1. $L \geq 2$.

The parallel edges in $\widehat{e}_2$ form cycles in Γ_1. There is a loop (namely C in Figure 4.5(a)) in Γ_2 at v_+ with endpoints labeled 1 and n_1, so by the parity rule u_1 and u_{n_1} are of opposite signs, hence 1 and n_1 belong to distinct orbits of φ. Let θ be the cycle containing u_1 and u_k, and let θ' be the one containing u_{n_1} and u_{k-1}. By Lemma 2.2(3) all these cycles are essential in A. Since there is an edge C in Γ_2 with endpoints labeled 1 and n_1 (see Figure 4.5(a)), which connects u_1 to u_{n_1} in Γ_2, there are no vertices inside the annulus between θ and θ'.

First assume that $\Delta = 4$. Let A, B, C, D be the edges with an endpoint on v_+ labeled 1, and let A, B, F, G be those with an endpoint on v_- labeled k. See Figure 4.5(a). On Γ_2 these edges form a subgraph as shown in Figure 4.5(b). The edges C and G cut the annulus between θ and θ' into two disks U_1 and U_2, as shown in the figure.

The endpoints of the edges A, B, C, D appear successively on ∂v_+ in Γ_2, so by Lemma 2.10 they also appear successively on ∂u_1 in Γ_1. Similarly, the endpoints of A, B, F, G appear successively on ∂u_k. From Figure 4.5(b) one can see that exactly one of the edges D and F, say D, lies inside of U_1; in particular, we must have $D \neq F$. On the other hand, in U_1 the only vertices which are parallel to u_1 are u_k and u_1 itself, so D must have its other endpoint on u_k. But this is a contradiction, because in Γ_2 the endpoint of D on v_- would then have label k, implying that $D = F$.

Now assume that $\Delta = 5$. Let A, B, C, D, E be the edges with an endpoint on v_+ labeled 1, and let A, B, F, H, G be those with an endpoint on v_- labeled k, as shown in Figure 4.5(a). By Lemma 2.10 there are two possible orders in which the endpoints of these edges appear on ∂u_1 and ∂u_k respectively. (i) The jumping number $d = d(r_1, r_2) = 1$. In this case the endpoints appear as A, B, C, D, E on ∂u_1, and the proof is similar to that given above, noticing that D, E are in U_1 if and only if F and H are outside of U_1. (ii) The jumping number $d = 2$, so the endpoints appear in the order A, D, B, E, C on ∂u_1, and A, H, B, G, F on ∂u_k. See Figure 4.5(c). One can see that the edges E and F lie in different disks U_1, U_2, while D and H lie between A and B; hence the argument above can be applied to the edges E and F to give a similar contradiction.

CASE 2. $L = 1$.

In this case all the edges in $\widehat{e}_2$ and $\widehat{e}_3$ are loops in Γ_1. Orient all the loops from label $+$ to label $-$. Let e_i and e'_i be the edges in $\widehat{e}_2, \widehat{e}_3$ which form loops based at u_i. There are two possibilities, depending on whether they have the same orientation; see Figure 4.6. We claim that *in either case all the non-loop edges with an endpoint at u_i are on one side of the loops.*

First assume $\Delta = 4$. If the two edges have different orientations, then since their $+$ endpoints are adjacent among all endpoints on ∂v_+ which are labeled i, by Lemma 2.10 these two endpoints are also adjacent on ∂u_i among all endpoints labeled $+$. Since the $+$ and $-$ labels alternate around ∂u_i, it follows that all the

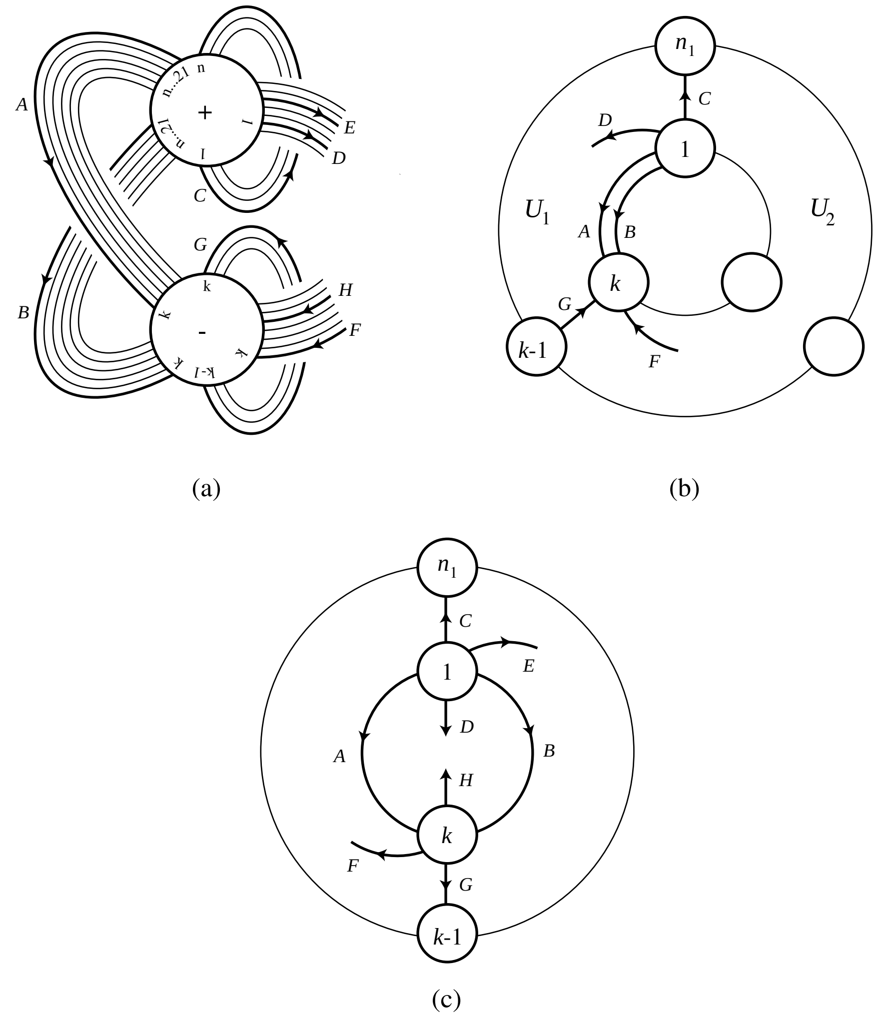


Figure 4.5

endpoints besides those of e_i and e_i' are on one side of the loops. If the two edges have the same orientation, then there is another edge between them. Since the $+$ and $-$ labels alternate, from Figure 4.6(b) we see that the number of labels on each side of the loops must be even. Therefore the remaining two endpoints must be on the same side of the loops.

Now assume $\Delta = 5$. We have $p_4 + p_5 + 2p_1 = 3n_1$. Since $p_4, p_5 \le n_1$, and $p_1 \le n_1/2 + 1$, we have only two possibilities: Either $p_4 = p_5 = n_1$ and $p_1 = n_1/2$, or $p_4 + p_5 = 2n_1 - 2$ and $p_1 = n_1/2 + 1$. In the first case there are four loops based at each vertex, so there are only two adjacent edge endpoints left, which must be on the same side of the loops. In the second case, the argument beginning at the second paragraph of the proof of Lemma 4.4, with p_2, p_3 replaced by p_4, p_5, applies (although here we don't have $p_4 + p_5 \ge p_2 + p_3$), to show that the reduced

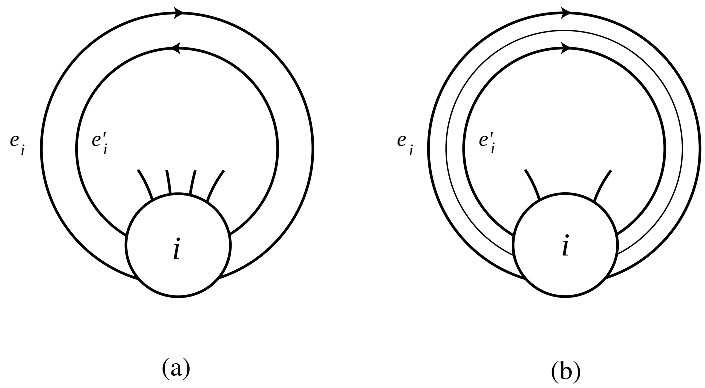


Figure 4.6

graph Γ_1 contains a subgraph as shown in Figure 4.3(c), (which illustrates the case $n_1 = 6$.) However, since $p_1 = n_1/2 + 1 \geq 3$, there is an edge in $\widehat{e}_1$ adjacent to the Scharlemann cycle, with endpoints labeled 3 and n_1. On Γ_1 this edge connects u_3 to u_{n_1}, which is impossible because u_3 and u_{n_1} are on different sides of the loop $e'_1 \cup e'_2$ in Figure 4.3(c). This completes the proof of the claim.

Consider the reduced graph $\widehat{\Gamma}_1$. We have shown that each vertex is the base of a loop. Let u', u'' be two vertices that are connected by an edge. The above claim says that there are no edges connecting u', u'' to any other vertices. Thus there is an annulus A' containing only the vertices u', u'', and with boundary disjoint from Γ_1. By definition Γ_1 is 2-separable. $\square$

§5. The case that Γ_1 is 2-separable

Assume that $n_1, n_2 \geq 2$, and $\Delta \geq 4$. In Section 4 we have shown that if $n_2 = 2$ then Γ_1 is 2-separable. Lemma 5.1 below shows that the converse is also true. The goal of this section is to show that if Γ_1 is 2-separable, (in particular, if one of the $n_\alpha = 2$), then $n_1 = n_2 = 2$, and the manifold M is the exterior of one of the links in Figure 7.1(b) or (c). See Proposition 5.5.

Throughout this section we will assume that Γ_1 is 2-separable. We use A', Γ'_1, Γ'_2 to denote the annulus and graphs given in Definition 3.3. Let u_1, u_2 be the vertices of Γ'_1. The labeling of u_i in Γ'_1 may not be the same as that in Γ_1, however this does not affect the proof below. Notice also that a pair of edges is an equidistant pair in Γ'_α if and only if it is an equidistant pair in Γ_α, hence Lemma 2.8 still applies to Γ'_α. Also, since Γ'_α are subgraphs of Γ_α, most of the properties of Γ_α still hold for Γ'_α. For example, there is no extended Scharlemann cycle in Γ'_1.

Lemma 5.1. *Suppose $\Delta \geq 4$. If Γ_1 is 2-separable, then $n_2 = 2$.*

Proof. Let A', Γ'_α be as in Definition 3.3. The reduced graph $\widehat{\Gamma}'_1$ is a subgraph of the graph shown in Figure 5.1. If some edge represents more than n_2 parallel edges of Γ'_1, then by Lemma 2.3(1) either $n_2 = 2$ and we are done, or all vertices of Γ_2 are parallel, which by Lemma 4.1 again implies that $n_2 = 2$. We can thus assume that $\Delta = 4$, and that each edge in Figure 5.1 represents exactly n_2 parallel edges of Γ'_1.

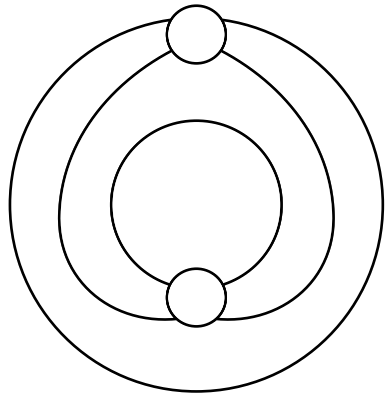


Figure 5.1

The n_2 parallel loops based at u_1 contain a Scharlemann cycle, so by Lemma 2.2(4) n_2 is even. Also, notice that the labels of the $2n_2$ endpoints of these loops appear successively, so if $n_2 \geq 4$ then the 4 edges in the middle would form an extended Scharlemann cycle, which is impossible by Lemma 2.2(6). □

Lemma 5.2. *Suppose $\Delta \geq 4$, and Γ_1 is 2-separable. Then Γ_2' must be one of the graphs shown in Figure 5.2(d) or (e).*

Proof. By Lemma 5.1 we have $n_2 = 2$, by Lemma 4.3 the two vertices of Γ_2', v_+ and v_-, are antiparallel, and by Lemma 4.2 the reduced graph $\widehat{\Gamma}_2'$ is a subgraph of the graph shown in Figure 4.1. Denote by p_i' the number of edges of Γ_2' in the family $\widehat{e}_i$ shown in Figure 4.1(b). Recall that p_i is the number of edges of Γ_2 in $\widehat{e}_i$. Without loss of generality we may assume that $p_2 + p_3 \geq p_4 + p_5$. Then by Lemma 2.5(3) we have $p_i \leq 2$, and by Lemma 4.4 we have $p_2 = p_3 = n_1$. By considering the edges of Γ_2' in each family $\widehat{e}_i$, it follows that $p_i' \leq 2$, and $p_2' = p_3' = 2$. According as $p_1' = 0$, 1 or 2, and $\Delta = 4$ or 5, we have the following possibilities.

(a) $p_1' = 0$, $\Delta = 4$, $p_4' + p_5' = 4$;
(b) $p_1' = 1$, $\Delta = 4$, $p_4' + p_5' = 2$;
(c) $p_1' = 1$, $\Delta = 5$, $p_4' + p_5' = 4$;
(d) $p_1' = 2$, $\Delta = 4$, $p_4' + p_5' = 0$;
(e) $p_1' = 2$, $\Delta = 5$, $p_4' + p_5' = 2$;

Using the parity rule, each of the above uniquely determines Γ_2' up to a homeomorphism of the underlying torus T. See Figure 5.2(a)–(e). We want to show that cases (a), (b) and (c) cannot occur.

Consider the graph in Figure 5.2(a). It is easy to see that either all edges have the same label at both endpoints, or each edge has distinct labels at its two endpoints. In the first case Γ_1' consists of loops, so Γ_1 is 1-separable. By Lemma 3.4 we must then have $n_1 = 1$, which is a contradiction. In the second case, the two vertices of Γ_1' are parallel by the parity rule, the reduced graph $\widehat{\Gamma}_1'$ consists of two edges from u_1 to u_2, so one can see that any pair of edges is an equidistant pair on Γ_1'. On the other hand a pair of parallel edges $\{e_1, e_2\}$ is not an equidistant pair on Γ_2' because

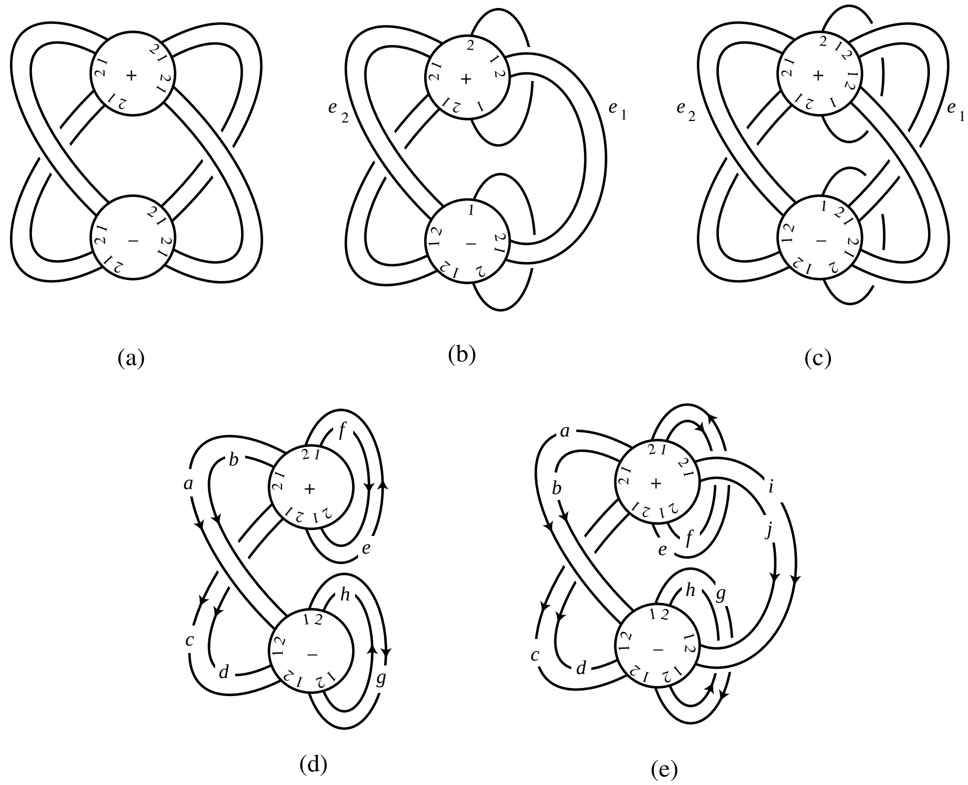


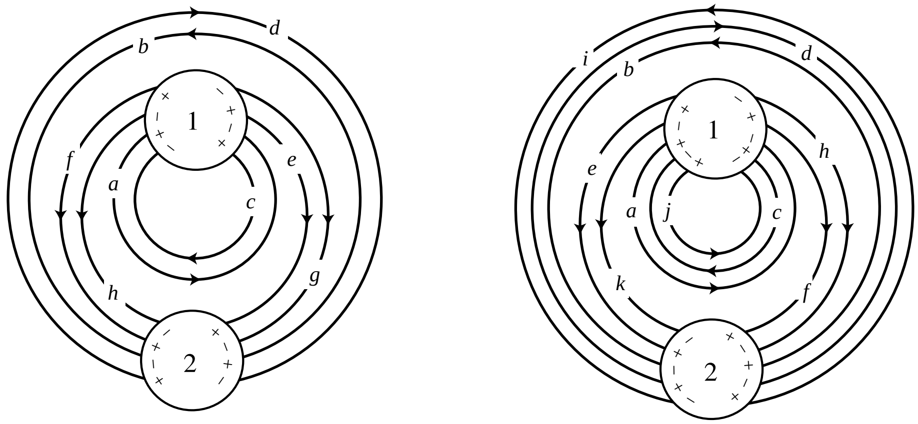
Figure 5.2

$\rho_{v_+}(e_1, e_2) = 1$ and $\rho_{v_-}(e_2, e_1) = 7$. This contradicts Lemma 2.8, showing that Γ_2' cannot be the graph in Figure 5.2(a).

In all the remaining cases, $p_1' \geq 1$, so there is a loop based at v_+. By the parity rule, the two vertices of Γ_1' are antiparallel. Thus any edge in Γ_2' from v_+ to v_- must have the same label at its two endpoints, and is a loop in Γ_1'. The reduced graph of Γ_1' is a subgraph of that in Figure 5.1, so any two loops based at u_1 are parallel. In particular, the edges e_1, e_2 shown in Figure 5.2(b)–(c) are parallel loops in Γ_1', so by Lemma 2.7 they form an equidistant pair in Γ_1', and by Lemma 2.8 they must also be an equidistant pair in Γ_2'. However, in both cases we have $\rho_{v_+}(e_1, e_2) = 2$ and $\rho_{v_-}(e_2, e_1) = 4$. Hence cases (b) and (c) cannot happen. □

Lemma 5.3. *Each of the graphs Γ_2' in Figure 5.2(d) and (e) uniquely determines the graph Γ_1' up to symmetry, as shown in Figure 5.3(a) and (b), respectively. The correspondence between the edges in Γ_1' and Γ_2' is also uniquely determined up to symmetry.*

Proof. The reduced graph $\widehat{\Gamma}_1'$ is a subgraph of the graph shown in Figure 5.1. The edges of Γ_2' come in parallel pairs, which are nonparallel in Γ_1' by Lemma 2.2(2). Using the fact that any loop in Γ_2' is a non-loop in Γ_1', and vice versa, we see that the graphs Γ_1' corresponding to Γ_2' in Figure 5.2(d) and (e) are the ones in Figure



(a) (b)

Figure 5.3

5.2(a) and (b), respectively. It remains to show that the correspondence between the edges of Γ'_1 and Γ'_2 is unique up to symmetry.

Label the edges of Γ'_2 as in Figure 5.2(d). Up to symmetry we may assume that the edge f is the one shown in Figure 5.3(a). The points of $\partial v_+ \cap \partial u_1$ appear in the order e, f, a, c on ∂v_+, so by Lemma 2.10 they must also appear in this order on ∂u_1. This determines these edges in Γ'_1. Similarly, the points of $\partial v_+ \cap \partial u_2$ appear in the order f, e, b, d on ∂v_+, so they also appear in this order on ∂u_2. This determines the edges b, d. By looking at the points of $\partial v_- \cap \partial u_1$, one can further determine the edges h, g. Hence in case (d) the edge correspondence is determined by Γ'_2.

The proof of case (e) is similar, except that in this case there is a jumping number $d = d(r_1, r_2)$ given by Definition 2.9, so if $P_1, P_2, \ldots, P_5$ are the points of $\partial v_i \cap \partial u_j$, numbered succesively on ∂v_i, (here $i = +$ or $-$), then they appear as $P_d, P_{2d}, \ldots, P_{5d}$ on ∂u_j. Label the edges of Γ'_2 as in Figure 5.2(e), and fix a choice of e on Γ'_1. Then there is only one more edge from u_1 to u_2 with endpoint labeled $+$ on u_1, so that must be the edge f. The points of $\partial v_+ \cap \partial u_1$ appear as e, j, f, a, c on ∂v_+, and as $e, *, *, *, f$ on ∂u_1, so we must have $d = 2$, and the edges are in the order e, a, j, c, f. Similarly one can determine the other edges of Γ'_1. See Figure 5.3(b). This completes the proof of Lemma 5.3. $\square$

Let F'_1 be the punctured annulus $A' - \mathrm{Int}(u_1 \cup u_2)$. Let F'_2 be the subsurface of F_2 consisting of disk faces of Γ'_2. Consider the submanifold X of M which is a regular neighborhood of $T_0 \cup F'_1 \cup F'_2$ with spherical boundary components capped off by 3-balls. Since M is irreducible, each sphere on $\partial N(T_0 \cup F'_1 \cup F'_2)$ does bound a 3-ball in M, hence X is indeed a submanifold of M. The manifold X can be constructed as follows. Let $Y = (F'_1 \times I) \cup (T_0 \times I)$. For each disk face D_i of Γ'_2 attach a 2-handle $H_i = D_i \times I$ to Y along the boundary curve of D_i on $F'_1 \cup T_0$, and finally cap off the spherical boundaries.

Lemma 5.4. *The manifold X constructed above is uniquely determined by the* *graph* Γ_2'.

Proof. By Lemma 5.3 the graph Γ_1' and the correspondence between the edges of Γ_1' and Γ_2' are uniquely determined by Γ_2'. Thus for each face D_i of Γ_2', the boundary curve of D_i on $F_1' \cup T_0$ is determined. The lemma now follows from the above construction of X, because X is uniquely determined by the way the 2-handles H_i are attached to Y. □

Proposition 5.5. *Suppose* $\Delta \geq 4$*, and* Γ_1 *is 2-separable. Then* $n_1 = n_2 = 2$*, and there are exactly two such manifolds* M*. If* $\Delta = 4$*,* M *is the exterior of the link in Figure 7.1(b), and if* $\Delta = 5$*,* M *is the exterior of the link in Figure 7.1(c).*

Proof. By Lemma 5.1 we have $n_2 = 2$. Let F_1', X, Y be as above. The surface F_1' separates Y into two sides, the black side Y_1 and the white side Y_2. Let P_i be the punctured torus $(\partial Y - T_0) \cap Y_i$. Each face of Γ_2' is colored according to the side it is on. Since F_2 is transverse to F_1', the colors on the faces of Γ_2' alternate. From Figure 5.2(d) and (e) we see that all the white disk faces have three edges, so their boundaries are nonseparating curves on the punctured torus P_2. Thus after attaching the 2-handles and capping off the spheres, the surface P_2 becomes an annulus. On the black side, there is a Scharlemann cycle on Γ_2', so after attaching the H_i and capping off the spheres, P_1 become either an annulus or a pair of disks. The second case does not happen because otherwise the boundary of X contains a sphere separating the boundary components of M, which contradicts the fact that M is irreducible. It follows that X has exactly two boundary components, each being a torus. The punctured annulus F_1' is an essential surface in X, with one boundary component on the outside torus T', and two other boundary components on T_0, hence X is not a product $T_0 \times I$.

The torus T' is incompressible in the global manifold M, because otherwise after compression it would become a sphere separating the boundary components of M, contradicting the irreducibility of M. Since M is atoroidal, and X is not a product, it follows that T' must be parallel to another boundary component T_1 of M. That is, $M - \text{Int}X$ is a product $T_1 \times I$. In particular, by Lemma 5.4 $M \cong X$ is determined by Γ_2'. Since there is only one possible Γ_2' for each of $\Delta = 4, 5$, given in Figure 5.2(d) and (e), we see that there is at most one such manifold M for each of $\Delta = 4, 5$. The existence of such manifolds will be established by Theorem 7.5.

Since $M - \text{Int}X \cong T_1 \times I$, the annulus A' in $X(r_1)$ can be extended to an essential annulus A'' in $M(r_1)$ intersecting J_1 just twice. By the minimality of n_1, we must have $n_1 = 2$. This completes the proof of the proposition, modulo Theorem 7.5. □

§6. The generic case

After the results in the previous sections, we may now assume that $n_\alpha \geq 3$ for $\alpha = 1, 2$. The purpose of this section is to show that in this case we must have $\Delta \leq 3$.

Lemma 6.1. *If* $n_1 \geq 3$*, and* $n_2 = 3$*, then* $\Delta \leq 3$*.*

Proof. By Lemma 4.2 this is true if all vertices of Γ_2 are parallel, so we may assume that v_1, v_2 are positive vertices, and v_3 is a negative vertex of Γ_2. Assume $\Delta \geq 4$.

First assume that there is only one family of parallel loops based at v_3, containing k edges. By Lemma 2.5(3), $k \leq n_1$. These loops contribute $2k$ endpoints at v_3, so there are $\Delta n_1 - 2k$ negative edges at v_3, hence the number of 3-edges in Γ_1 is

$$E = (\Delta n_1 - 2k) + k = \Delta n_1 - k \geq 3n_1.$$

By Lemma 2.2(2) no two negative 3-edges are parallel in Γ_1 because they are parallel positive edges in Γ_2. Since $n_2 = 3$, T is nonseparating, so by Lemmas 2.4 and 2.2(4) no two positive 3-edges are parallel in Γ_1. Thus all the 3-edges are mutually nonparallel, so the reduced graph $\widehat{\Gamma}_1$ has at least $3n_1$ edges, which contradicts Lemma 2.12(1).

Now assume that there are at least two families of parallel loops based at v_3. These (together with v_3) cut T into pieces, one of which is a disk containing a vertex v_1, say, such that there are no loops based at v_1, and there is at most one family of parallel edges from v_1 to v_2. The argument in this case is now similar to that given above, considering the 1-edges of Γ_1 instead of the 3-edges. This completes the proof of Lemma 6.1. □

Let G be a graph on a disk D. A vertex v of G is an *interior vertex* if there is no arc from v to ∂D with interior disjoint from G. It is a *cut vertex* if there is an arc on D, cutting D into D_1, D_2, such that the graphs $G_i = D_i \cap G$ have the properties that $G_1 \cap G_2 = v$, and neither G_i is a single vertex. In this case write $G = G_1 *_v G_2$. A vertex v is a *boundary vertex* if it is neither an interior vertex nor a cut vertex. Define

$a_i = a_i(G)$: the number of boundary vertices v with $val(v, G) = i$;
$l = l(G)$: the number of trivial loops in G;
$p = p(G)$: the number of pairs of adjacent parallel edges;
$\sigma(G) = 3a_1 + 2a_2 + a_3 + 4l + 2p$;
$\tau(G) = a_1 + a_2 + l + p$.

Lemma 6.2. *Let G be a connected graph on a disk D, which is not a single vertex. Suppose each interior vertex v of G has $val(v) \geq 6$. Then (i) $\sigma(G) \geq 6$, and (ii) if G has no interior vertices then $\tau(G) \geq 2$.*

Proof. The result is obvious if G has only one edge. Also, if $G = G_1 *_v G_2$ then the vertex v contributes at most 3 to $\sigma(G_i)$ and at most 1 to $\tau(G_i)$, so

$$\sigma(G) \geq (\sigma(G_1) - 3) + (\sigma(G_2) - 3) \geq \sigma(G_1) + \sigma(G_2) - 6;$$
$$\tau(G) \geq (\tau(G_1) - 1) + (\tau(G_2) - 1) \geq \tau(G_1) + \tau(G_2) - 2;$$

hence the result follows by induction. We can thus assume that G contains more than one edge, and has no cut vertices. By considering a sub-disk if necessary, we may assume that $\partial D \subset G$. Notice that in this case $a_1(G) = 0$.

(i) Taking the double of D, we get a graph G' on a sphere S, with $a_0(G') = a_1(G') = a_3(G') = a_5(G') = 0$, $a_2(G') = a_2(G)$, and $a_4(G') = a_3(G)$. There are $2l$ faces with 1 edge, $2p$ faces with 2 edges, and all the others have at least 3 edges. Denote by V, E, F the number of vertices, edges, and faces of G'. We have the following inequalities:

$$2a_2 + 4a_3 + 6(V - a_2 - a_3) \leq 2E$$
$$2l + 2(2p) + 3(F - 2l - 2p) \leq 2E$$
$$V - E + F = 2$$

Solving these inequalities gives $\sigma(G) = 2a_2 + a_3 + 4l + 2p \geq 6$.

(ii) This is obvious if all edges of G are either loops or on the boundary of D. So assume that e is a nonloop edge which is not on the boundary of D. Since G has no interior vertices, e cuts D into two disks D_1, D_2, and G into G_1, G_2. If $l(G_i) + p(G_i) \geq 1$ for both i, then $\tau(G) \geq 2$ and we are done. Suppose $l(G_i) = p(G_i) = 0$ for some i. Then by induction we have $\tau(G_i) = a_2(G_i) \geq 2$, i.e. it has at least two boundary vertices of valency 2. If both endpoints of e have valency 2 in G_i, then shrinking e to a point gives a graph G_i', which by induction must have another boundary vertex of valency 2. Hence G_i must have a boundary vertex of valency 2 which is not on ∂e. Similarly for the other graph G_j. Hence we have $\tau(G) \geq 2$. □

Lemma 6.3. *Suppose $n_1 \geq 3$, $n_2 = 4$, and $\Delta \geq 4$. Then Γ_1 cannot have 4 length 2 Scharlemann cycles on mutually distinct label pairs.*

Proof. There are only four possible label pairs $\{i, i+1\}$, $i = 1, \ldots, 4$, so if the lemma were false then they would all appear as label pairs of Scharlemann cycles in Γ_1. Denote by V_i the part of the Dehn filling solid torus between v_i and v_{i+1}. The existence of Scharlemann cycles in Γ_1 implies that the torus T in $M(r_2)$ is separating (Lemma 2.2(4)). Without loss of generality we may assume that V_1, V_3 lie on the side containing some boundary components of $M(r_2)$.

Let $\mathcal{E}^i = \{e_1^i, e_2^i\}$ be the Scharlemann cycle on Γ_1 with label pair $\{i, i+1\}$, and let D_i be the disk bounded by $\mathcal{E}^i$ on the punctured annulus F_1. Then $e_1^1 \cup e_2^1$ is a cycle on Γ_2 with endpoints on v_1, v_2. Consider the manifold $X = (T \times I) \cup V_1 \cup (D_1 \times I)$. It is easy to see that X is a cable space, whose boundary consists of two tori $T \cup T'$, with $T' \cap J_2$ consisting of two disks. If T' is compressible, then after compression it becomes a sphere containing T on one side and some boundary component of M on the other, which is therefore a reducing sphere in $M(r_2)$; by [Wu3] in this case we would have $\Delta \leq 2$. So assume that T' is incompressible. Then $X' = (T' \times I) \cup V_3 \cup (D_3 \times I)$ is also a cable space, hence T' is not parallel to a torus in $\partial M(r_2)$. It follows that T' is an essential torus in $M(r_2)$ intersecting J_2 in fewer than n_2 disks, contradicting the choice of the essential torus T. □

A vertex u in Γ_1 is a *full vertex* if its valency in Γ_1^+ is Δn_2. In other words, u is full if all the edges incident to it are positive.

Lemma 6.4. *Suppose $n_1 \geq 3$, $n_2 \geq 4$, and $\Delta \geq 4$. Then each full vertex u_j of Γ_1 has valency at least 6 in $\widehat{\Gamma}_1$.*

Proof. Assume that $val(u_j, \widehat{\Gamma}_1) \leq 5$. There are a total of Δn_2 edge endpoints at u_j, and each family of parallel edges contains at most $n_2/2 + 2$ edges (Lemma 2.3(3)), so we have

$$\frac{\Delta n_2}{5} \leq \frac{n_2}{2} + 2.$$

This immediately gives $n_2 \leq 6$. If $n_2 = 6$, there is a family of 5 parallel edges, with endpoints labeled $1, 2, 3, 4, 5$ at u_j, say. Then one can see that this family contains two Scharlemann cycles on label pairs $\{1, 2\}$ and $\{4, 5\}$, respectively, which contradicts Lemma 2.3(2). If $n_2 = 5$ then T is non-separating, so by Lemmas 2.4 and 2.2(4) each family of parallel edges contains at most $n_2/2$ edges. Replacing the right hand side of the above inequality with $n_2/2$, we see that $\Delta < 3$, contradicting the assumption. Hence we must have $n_2 = 4$.

Consider the reduced negative graph $\widehat{\Gamma}_2^-$. Let E, F be the number of its edges and disk faces. There are four edges in Γ_2^- labeled j at each v_i, no two of which are parallel because otherwise there would be n_1+1 parallel edges in Γ_2, which would contradict Lemma 2.5(3). Thus the valency of v_i in $\widehat{\Gamma}_2^-$ is at least 4 for any i, so

$$4n_2 \le 2E.$$

Each face of $\widehat{\Gamma}_2^-$ must have at least 4 edges because two adjacent vertices on the boundary of the face have opposite signs; hence

$$4F \le 2E.$$

Calculating the Euler number we have

$$n_2 - E + F \ge 0.$$

These three inequalities hold if and only if they are all equalities. In particular, each vertex has valency 4 in $\widehat{\Gamma}_2^-$, and there are exactly $2n_2 = 8$ edges in $\widehat{\Gamma}_2^-$.

The four edges of $\widehat{\Gamma}_2^-$ incident to v_2 have the other endpoint on v_1 or v_3. If there were three edges of $\widehat{\Gamma}_2^-$ connecting v_2 to v_1, then they would cut the torus T into a disk. Since $val(v_3, \widehat{\Gamma}_2^-) = 4$, some pair of edges of $\widehat{\Gamma}_2^-$ incident to v_3 would be parallel, which is absurd because $\widehat{\Gamma}_2^-$ is a reduced graph. Therefore there are exactly two edges of $\widehat{\Gamma}_2^-$ with endpoints on v_1 and v_2. Similarly, for each $i = 1, \ldots, 4$, there are exactly two edges in $\widehat{\Gamma}_2^-$ connecting v_i to v_{i+1}.

In the remaining of this proof, we count an edge of $\widehat{\Gamma}_2$ with label j at both its endpoints as two j-edges; similarly, on $\widehat{\Gamma}_1$ a loop at u_j is counted twice. With this convention there are exactly 16 edges adjacent to u_j, so there are exactly 16 j-edges in $\widehat{\Gamma}_2$.

The 8 edges of $\widehat{\Gamma}_2^-$ contain the 16 j-edges, and by Lemma 2.5(4) each edge of $\widehat{\Gamma}_2^-$ contains at most two j-edges, and hence exactly two j-edges. Combining this with the conclusion of the last paragraph, we see that there are 4 j-edges connecting v_i to v_{i+1} for any i. On Γ_1 this means that among the 16 edges incident to u_j, exactly 4 of them have label pairs $\{i, i+1\}$, for each i.

By Lemma 6.3, some pair $\{i, i+1\}$, say $\{1, 2\}$, is not the label pair of a length 2 Scharlemann cycle on Γ_1. Thus by Lemma 2.4 the 4 edges with label pair $\{1, 2\}$ must be mutually nonparallel. Notice that if an edge e is parallel to an edge with label pair $\{1, 2\}$, then it has label pair either $\{1, 2\}$ or $\{3, 4\}$. Since there are only 5 families of parallel edges at u_j, (again, a loop is counted twice), 4 of which contain edges labeled $\{1, 2\}$, it follows that all the 8 edges labeled $\{2, 3\}$ or $\{4, 1\}$ are in the remaining (nonloop) family. However, this is impossible because by Lemma 2.3(3) each family contains at most $n_2/2 + 2 = 4$ edges. □

Lemma 6.5. *Suppose $n_1 \ge 3$, $n_2 \ge 4$, $\Delta \ge 4$, and suppose Γ_1^+ has a full vertex u_k. Then*

(i) $\widehat{\Gamma}_1^+$ has no vertices of valency at most 1;

(ii) if u_i is a boundary vertex of $\widehat{\Gamma}_1^+$ with valency at most 3 in $\widehat{\Gamma}_1^+$, then i is a label of a Scharlemann cycle in Γ_2; and

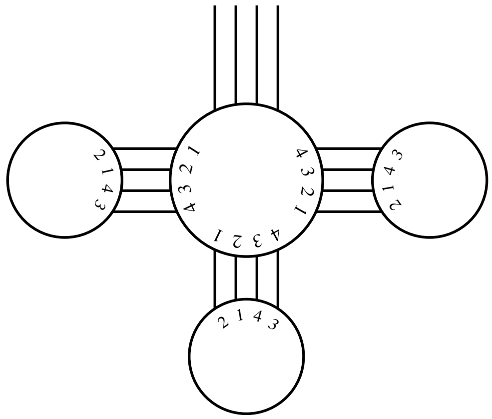


Figure 6.1

(iii) each component of $\widehat{\Gamma}_1^+$ contains at most one boundary vertex of valency at most 3.

Proof. Since u_k is a full vertex, all the k-edges in Γ_2 are negative edges. By Lemma 2.5(4) each edge of $\widehat{\Gamma}_2$ contains at most two endpoints labeled k. Since there are Δn_2 such endpoints, we see that the reduced graph $\widehat{\Gamma}_2$ contains at least $\Delta n_2/2 \geq 2n_2$ negative edges.

(i) By Lemma 2.12(2) there are at most $3n_2$ edges in the reduced graph $\widehat{\Gamma}_2$. We have shown that at least $2n_2$ of the edges are negative, so Γ_2 has at most n_2 positive edges. By Lemma 2.3(3) each edge of $\widehat{\Gamma}_1^+$ represents at most $n_2/2+2 \leq n_2$ edges of Γ_1, so if some u_i has valency at most 1 in $\widehat{\Gamma}_1^+$, then it has more than $2n_2$ negative edges, so there are more than $2n_2$ positive i-edges in Γ_2, which then implies that 3 of those edges are parallel in Γ_2. This contradicts Lemma 2.5(4).

(ii) If u_i has more than n_2 negative edges, then two of them are parallel positive edges on Γ_2, hence by Lemma 2.4, i is a label of some Scharlemann cycle in Γ_2, and we are done. Now assume that u_i has at most n_2 negative edges, and hence at least $(\Delta - 1)n_2$ positive edges. By assumption it has at most three families of parallel positive edges, and by Lemma 2.3(3) each family contains at most $n_2/2+2$ edges, so we have $(\Delta - 1)n_2 \leq 3(n_2/2+2)$, which implies that $\Delta = n_2 = 4$, and each family contains exactly $n_2/2+2 = 4$ edges. Since u_i is a boundary vertex in $\widehat{\Gamma}_1^+$, these three families are adjacent, as shown in Figure 6.1.

By Lemma 2.2(6) Γ_1 contains no extended Scharlemann cycle, so the labels at the other endpoints of the positive edges are as shown in Figure 6.1. From the figure we can see that there are three Scharlemann cycles with label pair $\{1,2\}$ and another three with label pair $\{3,4\}$. On Γ_2, this implies that there are 6 i-edges connecting v_1 to v_2, and another 6 connecting v_3 to v_4. Since $n_2 = 4$, there are no other vertices, so it is easy to see that there exist three parallel i-edges, contradicting Lemma 2.5(4).

(iii) If G is a component of $\widehat{\Gamma}_1^+$, and u_i, u_j are boundary vertices of valency at most 3 in G, then by (ii) both i, j are labels of Scharlemann cycles in Γ_2. Since

u_i, u_j are parallel, no label pair of a Scharlemann cycle could contain both i and j; hence there must be two Scharlemann cycles in Γ_2 with distinct label pairs, contradicting Lemma 2.5(1). This completes the proof of Lemma 6.5. □

Consider a graph Γ on a sphere S. A component G of Γ is called an *extremal component* if there is a disk D in S, such that $D \cap \Gamma = G$.

Lemma 6.6. *Suppose $n_1 \geq 3$, $n_2 \geq 4$, and $\Delta \geq 4$. Then Γ_1 has no full vertices.*

Proof. Suppose Γ_1 has a full vertex u_k. Embed the annulus A in a sphere S, and consider $\widehat{\Gamma}_1$ as a graph on S. If $\widehat{\Gamma}_1^+$ is connected, then all vertices of Γ_1 are full. By Lemma 2.12(1) there is a vertex of valency at most 5 in $\widehat{\Gamma}_1^+$, which would contradict Lemma 6.4. Therefore $\widehat{\Gamma}_1^+$ is not connected, so it has at least two extremal components G_1, G_2. Let D_i be a disk on S such that $D_i \cap \widehat{\Gamma}_1^+ = G_i$. By Lemma 6.5(1) all vertices of G_i have valency at least 2. That is, G_i is not an isolated vertex, and $a_1(G_i) = 0$. (Recall the notations before Lemma 6.2.)

If some D_i contains no component of $S - A$, then G_i has no loops or parallel edges, so by Lemma 6.2(i), it has at least two vertices of valency at most 3, which contradicts Lemma 6.5(iii). It follows that each D_i contains exactly one component of $S - A$, hence $l(G_i) + p(G_i) \leq 1$. This also shows that G_1, G_2 are the only extremal components, otherwise choosing another extremal component as G_1 would give a contradiction. Applying Lemma 6.2(i) to G_i, we have $\sigma(G_i) = 3a_1 + 2a_2 + a_3 + 4l + 2p \geq 6$. Since $a_1 = 0$, and $l + p \leq 1$, we have either $a_3 \geq 2$ or $a_2 \geq 1$. The first case is impossible by Lemma 6.5(iii). Hence $a_2 \geq 1$, that is, each G_i contains a boundary vertex u^i of valency 2. By Lemma 6.5(ii), the index of u^i is a label of all Scharlemann cycles in Γ_2. Without loss of generality, we may assume that the vertex u^i has label i, that is, $u^i = u_i$, $i = 1, 2$. Thus all the Scharlemann cycles in Γ_2 have label pair $\{1, 2\}$.

Sublemma. *There are exactly 4 Scharlemann cycles of length 2 in Γ_2, and there are no other positive edges in Γ_2.*

Let k be the number of positive i-edges in Γ_2, $i = 1, 2$. These are the negative edges in Γ_1 incident to u_i. Each edge of $\widehat{\Gamma}_1^+$ incident to u_i represents at most $n_2/2+2$ parallel edges. Since u_i has valency 2 in $\widehat{\Gamma}_1^+$, we have $k \geq \Delta n_2 - 2(n_2/2+2) \geq 3n_2 - 4$. On the other hand, as shown in the proof of Lemma 6.5(i), the existence of a full vertex of $\widehat{\Gamma}_1^+$ implies that there are at most n_2 positive edges in the reduced graph $\widehat{\Gamma}_2$, and by Lemma 2.5(4) each such edge represents at most two i-edges. Therefore, we must have $k \leq 2n_2$. These inequalities give $n_2 = \Delta = 4$, and $k = 2n_2 = 8$. Moreover, each of the 4 positive edges in $\widehat{\Gamma}_2$ must contain exactly two i-edges, hence the 8 i-edges form 4 pairs of Scharlemann cycles. As shown before the sublemma, they all have the label pair $\{1, 2\}$.

By Lemma 2.2(5) the edges of a Scharlemann cycle form an essential loop in Γ_1, which separates the two extremal components G_1, G_2 because u_1, u_2 are boundary vertices and neither G_i is contained in a disk in the annulus A. Since all the 8 negative edges from u_1 end up on u_2, there are no other negative edges in Γ_1. This completes the proof of the sublemma. In particular, all vertices of Γ_1 other than u_1, u_2 are full vertices.

Note that since there is an arc connecting the only two extremal components G_1, G_2 of $\widehat{\Gamma}_1^+$, we must have $\widehat{\Gamma}_1^+ = G_1 \cup G_2$. We have assumed that $n_1 \geq 3$, hence one of the G_i, say G_1, contains more than one vertex.

Embed A in a sphere S and consider G_1 as a graph on S. Let V, E, F be the number of vertices, edges and faces of G_1. By Lemma 6.4 all the vertices but u_1 have valency at least 6, so

$$2 + 6(V-1) \le 2E.$$

There are two faces D_1, D_2 of G_1 which may contain some components of $S - A$, all others are incident to at least three edges. Since u_1 has valency 2, and G_1 is connected and contains at least two vertices, there is no loop based at u_1, so the disk face D_i containing u_1 on its boundary has at least two edges. Hence we have

$$2 + 1 + 3(F-2) \le 2E.$$

Solving these two inequalities, we have $3V - 3E + 3F \le 5$, which contradicts the Euler number formula $V - E + F \le 2$. □

Lemma 6.7. *Suppose $n_1 \ge 3$, $n_2 \ge 4$, and $\Delta \ge 4$. If $\widehat{\Gamma}_1^+$ has no interior vertex, then $n_2 = \Delta = 4$. Moreover, each vertex of Γ_1 has at most 8 negative edges incident to it, and there are 8 negative edges connecting a pair of vertices u_1, u_2, which form 4 Scharlemann cycles in Γ_2.*

Proof. As in the proof of Lemma 6.6, we may embed A into a sphere S, and use Lemma 6.2(ii) to show that some extremal component of $\widehat{\Gamma}_1^+$ contains a vertex u_i of valency at most 2. By Lemma 2.3(3) at u_i there are at most $2(n_2/2 + 2) = n_2 + 4$ endpoints of positive edges, hence at least $(\Delta - 1)n_2 - 4$ negative edges. Choose a vertex u_j such that the number m of negative edges incident to it is maximal among all the vertices in Γ_1. The above shows that $m \ge (\Delta - 1)n_2 - 4$. The parity rule says that there are m positive j-edges in Γ_2.

Let G be the subgraph of Γ_2 consisting of the positive j-edges and all vertices of Γ_2. Then G has n_2 vertices and m edges. Denote by F the number of disk faces of G. Calculating the Euler number, we have $n_2 - m + F \ge 0$, hence

$$F \ge m - n_2.$$

By Lemma 2.11, each disk face of G contains a Scharlemann cycle, so there are at least $t = 2F \ge 2(m - n_2)$ edges of Scharlemann cycles. By Lemma 2.5(1), all these Scharlemann cycles have the same label pair, say $\{1, 2\}$. Let $p \in \{1, 2\}$. Then in Γ_1 there are at least t negative edges incident to the vertex u_p. By the definition of m, u_p also has at least $\Delta n_2 - m$ endpoints of positive edges. Thus the valency of u_p is at least

$$\begin{aligned} t + (\Delta n_2 - m) &\ge 2(m - n_2) + \Delta n_2 - m = m - 2n_2 + \Delta n_2 \\ &\ge (\Delta - 1)n_2 - 4 - 2n_2 + \Delta n_2 \\ &= (\Delta - 3)n_2 - 4 + \Delta n_2 \\ &\ge \Delta n_2 \end{aligned}$$

Since the valency of u_p is Δn_2, all the above inequalities are equalities, so we must have (i) $\Delta = n_2 = 4$, (ii) $m = (\Delta - 1)n_2 - 4 = 8$, hence there are at most 8 negative edges at each vertex in Γ_1, and (iii) there are $t = 2(m - n_2) = 8$ edges of Scharlemann cycles in Γ_2 with label pair $\{1, 2\}$, hence on Γ_1 there are exactly 8 negative edges at u_i, $i = 1, 2$, all connecting u_1 to u_2. □

Lemma 6.8. *Suppose $n_1 \geq 3$, $n_2 \geq 4$, $\Delta \geq 4$, and suppose $\widehat{\Gamma}_1^+$ has no interior vertex. If Γ_1 has at least three vertices incident to 8 negative edges, then Γ_1 is 2-separable.*

Proof. We have seen from Lemma 6.7 that $\Delta = n_2 = 4$, and u_1, u_2 have 8 negative edges, which form 4 Scharlemann cycles in Γ_2 with label pair $\{1, 2\}$. Assume u_i is another vertex incident to 8 negative edges. By Lemma 2.5(1) i is not a label of a Scharlemann cycle, so by Lemma 2.4 no two of the 8 positive i-edges in Γ_2 are parallel. Thus the reduced graph $\widehat{\Gamma}_2$ has at least 8 positive edges. By Lemma 2.12 $\widehat{\Gamma}_2$ has at most $3n_2 = 12$ edges, so it has at most $12 - 8 = 4$ negative edges.

For each label j, by Lemma 2.5(4) each of the 4 negative edges in $\widehat{\Gamma}_2$ contains at most two edge endpoints labeled j, so Γ_2 has at most 8 negative edge endpoints labeled j. On Γ_1 this implies that there are at most 8 positive edge endpoints incident to u_j. On the other hand, by Lemma 2.7 there are also at most 8 negative edges at u_j. Since there are a total of 16 edge endpoints at each vertex, this implies that there are exactly 8 positive and 8 negative edge endpoints at each vertex u_j. In particular, both Γ_α have the same number of positive edges and negative edges. There are a total of $\Delta n_1 n_2 / 2 = 8n_1$ edges, so Γ_2 has $4n_1$ negative edges. By Lemma 2.5(3) each of the 4 negative edges in $\widehat{\Gamma}_2$ represents exactly n_1 negative edges.

Let $\mathcal{E}$ be a set of n_1 parallel negative edges of Γ_2. By Lemma 2.2(3) they form several essential cycles of length L on Γ_1, which we will call the $\mathcal{E}$-cycles. Choose $\mathcal{E}$ so that L is maximal. Let e_1, e_2 be a Scharlemann cycle in Γ_2, with label pair $\{1, 2\}$. By Lemma 2.2(5) these edges form an essential cycle in Γ_1, hence cut the annulus A into two annuli A_1, A_2, each containing several $\mathcal{E}$-cycles, and having u_1, u_2 on its boundary. Let p (resp. n) be the number of $\mathcal{E}$-cycles in A_1 whose vertices are positive (resp. negative). Assume that u_1 is a positive vertex. Then on A_1 there are pL positive vertices, and $nL + 1$ negative vertices (including u_2). Recall that each vertex has 8 negative edges, which connect it to vertices of opposite sign. Moreover, all the negative edges at u_1 go to u_2. Counting the number of negative edges, we have

$$8(pL) = 8(nL + 1),$$

which implies that $L = 1$.

When $L = 1$, all the positive edges in Γ_1 are loops. Thus the loops at u_1, u_2 and the edges connecting them form a component of Γ_1, contained in an annulus A' with boundary disjoint from Γ_1. Therefore Γ_1 is 2-separable. □

Lemma 6.9. *Suppose $n_1 \geq 3$, $n_2 \geq 4$, and $\Delta \geq 4$. If $\widehat{\Gamma}_1^+$ has no full vertices, then it is 2-separable.*

Proof. By Lemma 6.7, $n_2 = \Delta = 4$, and there are four Scharlemann cycles on Γ_2, all having the same label pair $\{1, 2\}$, say. Let G_i, $i = 1, 2$, be the component of $\widehat{\Gamma}_1^+$ containing u_i. If there is an extremal component G of $\widehat{\Gamma}_1^+$, $G \neq G_1, G_2$, or if G_i is an extremal component but u_i is a cut vertex, then applying Lemma 6.2(ii) to G_1, G_2 and G we see that there is a vertex $u_j \neq u_1, u_2$, which has valency at most 2 in $\widehat{\Gamma}_1^+$. Such a vertex has at most $2(n_2/2 + 2) = 8$ positive edges, hence at least $\Delta n_2 - 2(n_2/2 + 2) = 8$ negative edges, so the result will follow from Lemma 6.8. Therefore we assume that G_1, G_2 are the only extremal components of $\widehat{\Gamma}_1^+$, and u_1, u_2 are boundary vertices. Since there are edges connecting u_1 to u_2, there are no other components, that is, $\widehat{\Gamma}_1^+ = G_1 \cup G_2$.

The edges e_1, e_2 of a Scharlemann cycle in Γ_2 form an essential cycle on the annulus A, which cuts A into two annuli A_1, A_2. Since u_i is not a cut vertex, each G_i lies in one of the A_j. If the two G_i are on different A_j, then one of the G_i would lie on a disk in A, so by Lemma 6.2(ii) it would have at least two vertices of valency at most 2 in $\widehat{\Gamma}_1^+$. Together with the vertices u_1 or u_2 on the other component G_k, this would give three vertices incident to at least 8 negative edges, and the result follows from Lemma 6.8. Hence we assume that G_i are on different A_j.

Now the only edges connecting G_1 to G_2 must have at least one endpoint on u_1 or u_2; but since all the 8 negative edges from u_1 must go to u_2 (Lemma 6.7), there are no other negative edges. It follows that all vertices other than u_1, u_2 are full vertices. Since $n_1 \geq 3$, this contradicts the assumption. □

Theorem 6.10. *Suppose M is a hyperbolic manifold, such that $M(r_1)$ contains an essential annulus $\widehat{F}_1$, $M(r_2)$ contains an essential torus $\widehat{F}_2$, and $\Delta(r_1, r_2) \geq 4$.*

(1) If $\Delta = 4$, M is the exterior of the link in Figure 7.1(a) or (b); otherwise, $\Delta = 5$, and M is the exterior of the link in Figure 7.1(c).

(2) $\widehat{F}_\alpha$ can be chosen to intersect the Dehn filling solid torus at most twice.

(3) If some $\widehat{F}_\alpha$ intersects the Dehn filling solid torus only once, then M is the exterior of the Whitehead link in Figure 7.1(a).

Proof. If some $n_\alpha = 1$, then by Proposition 3.5 the manifold M is the exterior of the Whitehead link in Figure 7.1(a), $n_\beta = 2$, and $\Delta = 4$.

Assume $n_1, n_2 \geq 2$, and some $n_\alpha = 2$. If $n_1 = 2$, then by definition Γ_1 is 2-separable. If $n_2 = 2$ and $n_1 > 2$, then by Proposition 4.6 Γ_1 is also 2-separable. By Proposition 5.5, in this case we must have $n_1 = n_2 = 2$, and the manifold M is the exterior of the link in Figure 7.1(b) or (c) respectively, according as $\Delta = 4$ or 5.

The remaining case is that $n_\alpha \geq 3$ for $\alpha = 1, 2$. By Lemma 6.1, we cannot have $n_2 = 3$. Assume $n_2 \geq 4$. By Lemma 6.6 the graph $\widehat{\Gamma}_1^+$ has no full vertex, and by Lemma 6.9 this implies that Γ_1 is 2-separable, which by Proposition 5.5 implies that $n_1 = 2$, contradicting the assumption. Hence this case does not happen. □

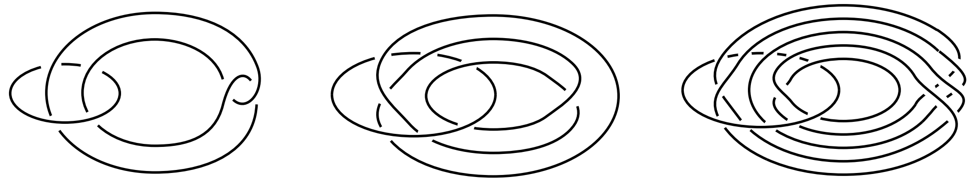
§7. Some Examples

In this section we give some examples of manifolds admitting toroidal and annular surgeries. Consider the three links shown in Figure 7.1. The first one is the Whitehead link. It is the 2-bridge link associated to the rational number 3/8. The second is the 2-bridge link associated to 3/10. The third one is called the Whitehead sister link. It is the $(-2, 3, 8)$ pretzel link. The Whitehead link is well known to be hyperbolic; see for example [Th2]. The other two links are also known to be hyperbolic. In particular, this is proved for the Whitehead sister link in [BFLW]. We will include a simple proof of the hyperbolicity of these two links in Theorem 7.5

First consider the Whitehead link in Figure 7.1(a).

Lemma 7.1. *Let M be the exterior of the Whitehead link L shown in Figure 7.1(a). Then there are two slopes r_1, r_2 such that (i) $M(r_1)$ contains an essential torus and an essential annulus, each intersecting the Dehn filling solid torus J_1 once; (ii) $M(r_2)$ contains an essential annulus and an essential torus, each intersecting J_2 twice; and (iii) $\Delta(r_1, r_2) = 4$.*

Proof. The manifold M is shown in Figure 7.2(a), with $T_0 \subset \partial M$ the inside torus. Let φ be the π-rotation of (M, T_0) about the horizontal axis shown in Figure 7.2(a).



(a) (b) (c)

Figure 7.1

Let α be the axis of φ in M. Then one can see that $(M, \alpha)/\varphi = (X, \eta)$, where X is a product $S^2 \times I$, and η consists of four arcs. See Figure 7.2(b). Thus M is the double cover of X branched along η. By an isotopy (X, η) is equivalent to the pair shown in Figure 7.2(c).

It is well known that, with respect to a certain framing on T_0, the manifold $M(r)$ can be obtained by filling the inner component of X with a rational tangle to get a pair $(B, \eta(r))$, and then taking the double cover of B branched along $\eta(r)$; see [M]. The tangle $(B, \eta(-1/2))$ is shown in Figure 7.2(d). Consider the shaded disk D and Möbius band U shown in Figure 7.2(d). Cutting along D and U, we get a manifold W homeomorphic to $T \times I$, on which D becomes an annulus A, and U becomes a torus T.

The double branched cover $M(-1/2)$ can be obtained by taking two copies of W, gluing T to T, and A to A. Since T and A are incompressible in both sides, they become essential surfaces in $M(-1/2)$. The core of the Dehn filling solid torus J branch covers the core of the rational tangle, shown as K in Figure 7.2(d). Since each of D and U intersects K in a single boundary point, the annulus A and the torus T also intersect J in a single disk. This proves (i).

Now consider the tangle $(B, \eta(1/2))$ shown in Figure 7.2(e). It is equivalent to that in Figure 7.2(f). There is a disk D cutting it into a $(-1/2, -1/2)$-Montesinos tangle and a $(1/2)$-rational tangle. The lift of D is then an essential annulus A in $M(1/2)$, cutting $M(1/2)$ into a Seifert fiber space Y whose orbifold is a disk with two cone points of order 2, and a solid torus on which A runs twice along the longitude. Thus A is an essential annulus, and the boundary of Y pushed into the interior of $M(1/2)$ is an essential torus, which covers a Conway sphere S on the right hand side of the disk D in B. Notice that both D and S intersect the core of the rational tangle just once, hence A and T intersect the core of the Dehn filling solid torus exactly twice. Since $\Delta(-1/2, 1/2) = 4$, this completes the proof of (ii) and (iii).

Notice that the framing we have chosen is not the standard meridian-longitude framing of the Whitehead link. By a more careful examination, one can show that $M(-1/2)$ is the 0-surgery, and $M(1/2)$ is the 4-surgery with respect to the standard framing. $\square$

Lemma 7.2. *Suppose M is a hyperbolic manifold such that $M(r_1)$ is annular, $M(r_2)$ is toroidal, and the intersection graphs Γ_α are as shown in Figure 3.1. Then M is the exterior of the Whitehead link.*

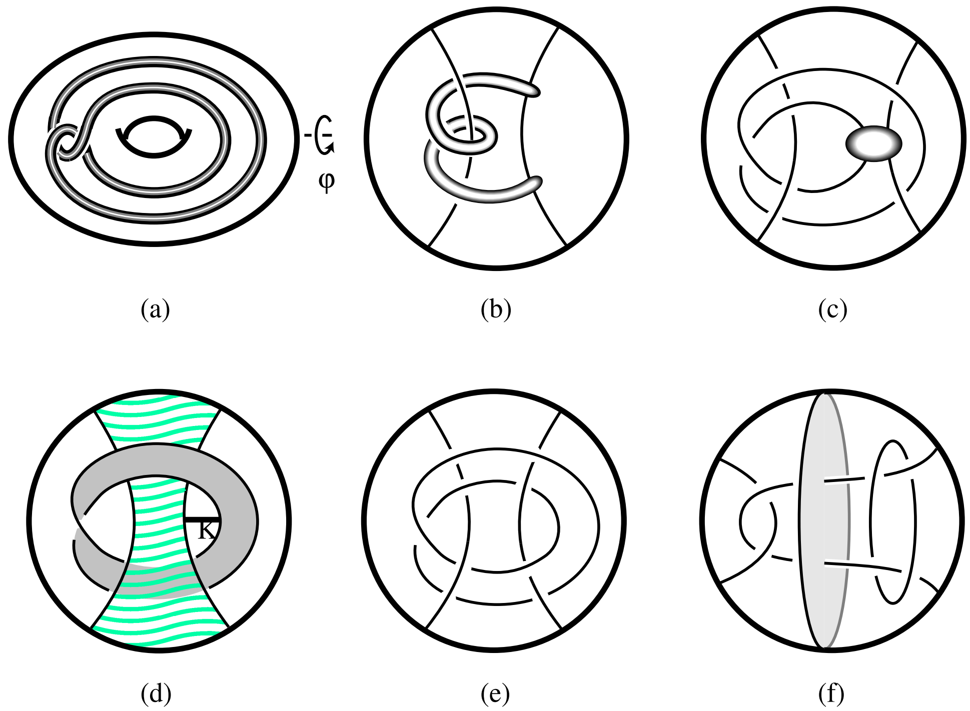
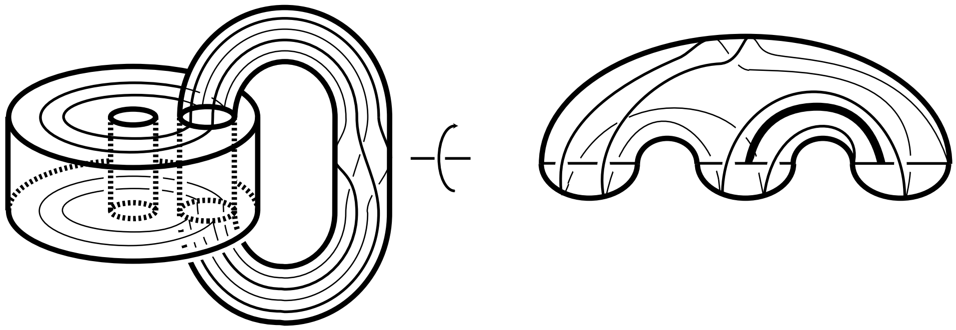


Figure 7.2

Proof. One can prove this by using the argument in the proof of Lemma 3.2. Here we give an explicit construction of M. The method can be used to construct other manifolds from intersection graphs.

Let J_1 be the Dehn filling solid torus in $M(r_1)$. Taking the union of $A \times I$ with J_1, so that the fat vertex of Γ_1 in A is identified with a meridian disk of J_1, we get a genus 2 handlebody X, as shown in Figure 7.3(a). The graph Γ_2 in Figure 3.1(b) cuts the torus T into two disk faces, whose boundary curves are parallel on ∂X, each being isotopic to the curve C shown in Figure 7.3(a). Therefore the regular neighborhood of the union of the handlebody X and the punctured torus F_2, with the spherical boundary component capped off by a 3-ball, is the same as the manifold Y obtained by attaching a 2-handle H to X along the curve C. Let K_1 be the central curve of the solid torus J_1. The manifold M is obtained by removing a regular neighborhood V of K_1 from Y.

There is an involution φ of X, as shown in the figure, which carries C to itself, so φ extends to an involution of $Y = X \cup H$. Let L be the axis of φ in Y. After the involution, the quotient of X is a 3-ball $B = X/\varphi$, and the image of $L \cap X$ consists of three arcs $\alpha_1, \alpha_2, \alpha_3$ in B, as shown in Figure 7.3(b). The quotient of C is an arc α on ∂B connecting the two endpoints of α_3. The image of the attached 2-handle H is a 3-ball B', with $L \cap H$ projecting to an arc which is isotopic rel ∂ to α. It follows that there is a homeomorphism from Y/φ to B, under which the image of L is the union of α and the three arcs α_i, with α pushed into the interior



(a) (b)

Figure 7.3

of B. Denote by β the 1-manifold L/φ. The central curve K_1 of J_1 becomes an arc with endpoints on β, denoted by K. In Figure 7.3(b) K is the thick grey arc inside B. Consider β as a tangle in the 3-ball B. After an isotopy, the triple (B, β, K) is the same as that shown in Figure 7.2(b), where B is the 3-ball, and β is the tangle consisting of two arcs and a circle.

Let $X = B - \text{Int}N(K)$, and $\eta = \beta \cap X$. Then M is the double cover of X branched along η, which has been shown in the proof of Lemma 7.1 to be the exterior of the Whitehead link. □

We need a lemma to test the hyperbolicity of the exteriors of braids in solid tori. A braid is *trivial* if it has only one strand.

Lemma 7.3. *Let K be a knot in a solid torus V, which is a nontrivial braid, and let M be the exterior of K in V. Then M is hyperbolic unless (i) there is a solid torus V' in V whose core is a nontrivial braid in V, and K is a nontrivial braid in V'; or (ii) K is a torus knot in V, i.e. it is isotopic to a curve on ∂V.*

Proof. Clearly M is irreducible and ∂-irreducible. Thus by the Geometrization Theorem of Thurston for Haken manifolds [Th1], M is hyperbolic unless it is toroidal or Seifert fibered.

Suppose M is toroidal, containing an essential torus T. Let D be a meridian disk of V intersecting K in n disks, where n is the braid index of K. Let P be the punctured disk $D - \text{Int}N(K)$. Isotope T so that it has minimal intersection with P. After cutting along P, M becomes a product $P \times I$, and T becomes a set of essential annuli $\mathcal{A}$ with boundary on $P \times \partial I$. Hence $\mathcal{A}$ is a product $\mathcal{C} \times I$, where $\mathcal{C}$ is a set of essential curves on P. It is now clear that after gluing $P \times 0$ to $P \times 1$, the annuli $\mathcal{A}$ are glued to give a torus bounding a solid torus V' whose core is a braid in V, which is nontrivial because T is not boundary parallel. K is now a braid in V', which must be nontrivial because T is not parallel to $\partial N(K)$. Hence (i) holds in this case.

Now suppose M is Seifert fibered. After filling in $N(K)$, the Seifert fibration extends to one on the solid torus V, whose orbifold is a disk with at most one cone point. Since K is nontrivial, it is not a singular fiber, hence it is isotopic to a regular fiber on ∂V, and (ii) follows. □

A braid b can be written as $\sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\dots\sigma_{i_k}^{\epsilon_k}$, where each σ_{i_j} is one of the standard generators of the braid group, and $\epsilon_j = \pm 1$. The *total exponent* of b is defined as $e(b) = \sum \epsilon_j$.

Corollary 7.4. *Suppose a knot K is an n-braid in a solid torus V, with total exponent e. If n is prime, and if e is not a nonzero multiple of $n-1$, then the exterior of K is hyperbolic.*

Proof. If K is an n_1 braid in V' whose core is an n_2 braid in V, then K is an $n_1 n_2$ braid in V. If K is a (n, q) torus knot in V, then the total exponent is $(n-1)q$. Hence the corollary follows from Lemma 7.3. □

Now let M_i, $i = 1, 2, 3$ be the exterior of the three links L_i shown in Figure 7.1. Denote by T_0 the boundary torus corresponding to the right component of L_i in Figure 7.1.

Theorem 7.5. *Each M_i is a hyperbolic manifold, and admits two Dehn fillings $M_i(r_1)$ and $M(r_2)$, such that*

(i) $\Delta = 4$ for $i = 1, 2$, and $\Delta = 5$ for $i = 3$;

(ii) $M_1(r_1)$ contains an essential annulus A and an essential torus T, each intersecting J_1 once. All the other $M_i(r_\alpha)$ contain an essential annulus A and an essential torus T, each intersecting J_α twice.

Proof. If $i = 1$, this is Lemma 7.1. So assume $i = 2$ or 3.

Removing the left component of L_i, we get a solid torus V containing a knot K_i. For K_2, the braid index is 3, and the total exponent is 0. For K_3, the braid index is 5, and the total exponent is 6. It follows from Corollary 7.4 that the exteriors of both K_i are hyperbolic.

The proof of the remainder of the theorem is similar to that of Lemma 7.1, with Figure 7.2 replaced by Figures 7.4 and 7.5. The essential torus and essential annulus in $M_2(r_2)$ are the double covers of the disk D_1 in Figure 7.4(d) and the boundary sphere S_1 on the right hand side of D_1, pushed slightly into the interior. Similarly for the others. In all cases, the tangle $(B, \eta(r_\alpha))$ is the sum of a $(\pm 1/2, \mp 1/3)$ Montesinos tangle and a $(1/2)$-rational tangle, so $M_i(r_\alpha)$ is a graph manifold obtained by gluing a solid torus to a Seifert fiber space along an annulus which runs twice along the longitude of the solid torus. The disks D_i and the Conway spheres S_i intersect the core of the rational tangle at one interior point, so after double covering the annuli and the tori intersect the core of the Dehn filling solid torus at two points. □

Consider the manifold M shown in Figure 7.6(a). It is the exterior of a three component link in S^3. Let $L = l_0 \cup l_1$ be the link in the solid torus V, where l_0 is the component with winding number 1 in V. Let $T_0 = \partial N(l_0)$, with the standard meridian-longitude framing.

Lemma 7.6. *M is hyperbolic, and both $M(0)$ and $M(-3/2)$ are toroidal and annular.*

Proof. L is a braid in V, so M is irreducible and ∂-irreducible. As in the proof of Lemma 7.3, it can be shown that if T is an essential torus in M, then T bounds a solid torus V' in V such that the core of V' is a braid in V, and $L \cap V'$ is a braid in V'. If V' contains the component l_0 which has braid index 1 in V, then the core of V' is a trivial braid in V. So if V' also contains l_1 then $T = \partial V'$ is

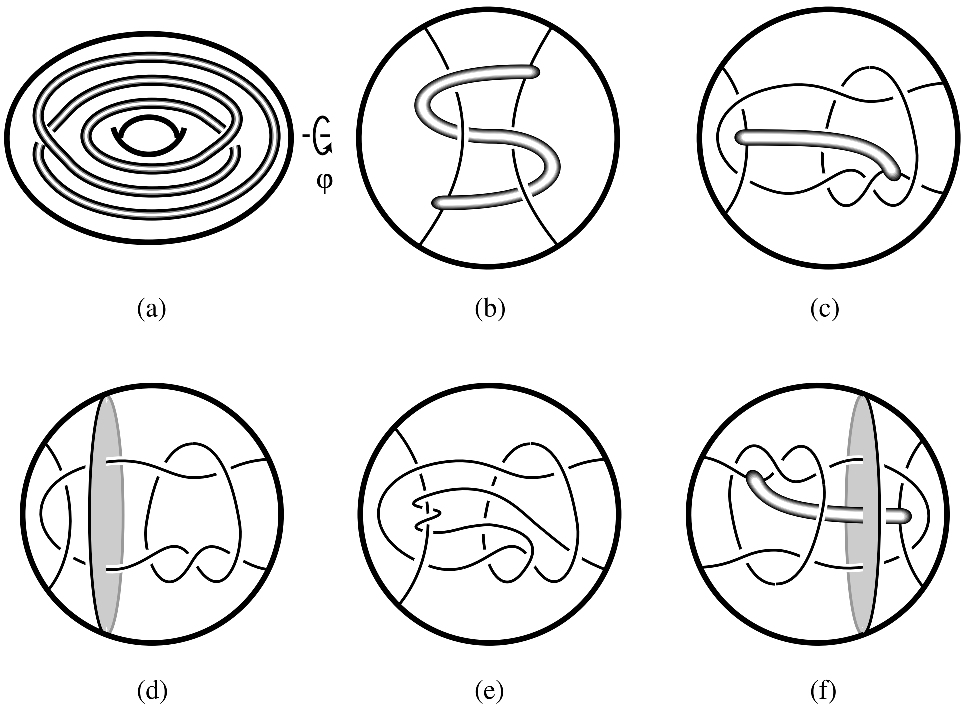


Figure 7.4

parallel to ∂V, and if V' does not contain l_1 then T is parallel to $\partial N(l_0)$, in either case contradicting the assumption that T is essential in M. Hence V' must contain l_1 but not l_0. The braid index of l_1 in V' is 2, otherwise T would be parallel to $\partial N(l_1)$. Let K' be the core of V'. Then the linking number $lk(l_0, l_1) = 2\, lk(l_0, K')$ is even, which is absurd because from the picture we see that $lk(l_0, l_1) = 1$.

To prove that M is hyperbolic, it remains to show that M is not a Seifert fiber space. If M were Seifert fibered, then after the trivial filling on L, the Seifert fibration would extend to one on the solid torus V, so that each l_i is a fiber. Since a Seifert fibration of V has at most one singular fiber, and l_0, l_1 are not isotopic, one can see that l_0 must be a singular fiber while l_1 is a regular fiber, so there would be an isotopy sending l_0 to the core l_0' of V and l_1 to a $(2,1)$ curve l_1' on ∂V. However, the total exponent of $l_0' \cup l_1'$ is 3 but the total exponent of $l_0 \cup l_1$ is 1, so they cannot be isotopic. This completes the proof that M is hyperbolic.

As before, let φ be the π-rotation of M about the horizontal axis shown in Figure 7.6(a), and let α be the axis of φ in M. Then $(M, \alpha)/\varphi = (X, \eta)$, where X is a twice punctured 3-ball, and η consists of 6 arcs, as shown in Figure 7.6(b). M is the double cover of X branched along η. After an isotopy, the picture is shown in Figure 7.6(c), in which the image of T_0 is the left sphere S_0. We have been careful to ensure that the standard meridian and longitude framings induce the ∞ and 0 slopes on ∂S_0. However, notice that the conventions about slopes on ∂S_0 and T_0 are opposite. Denote by $(Y, \eta(r))$ the pair obtained from (X, η) by filling S_0 with

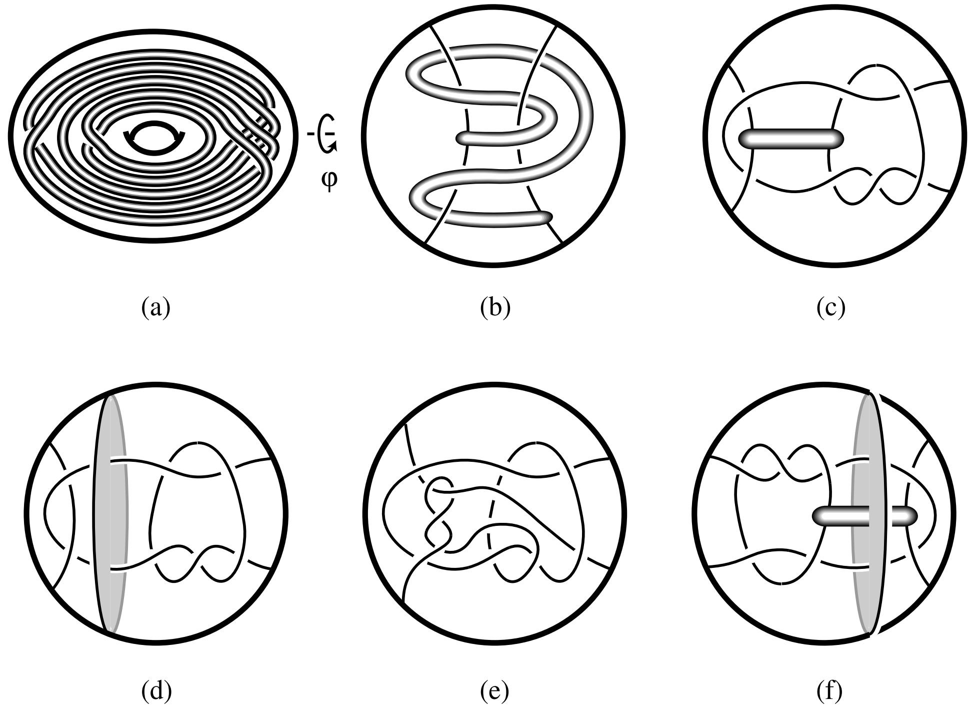


Figure 7.5

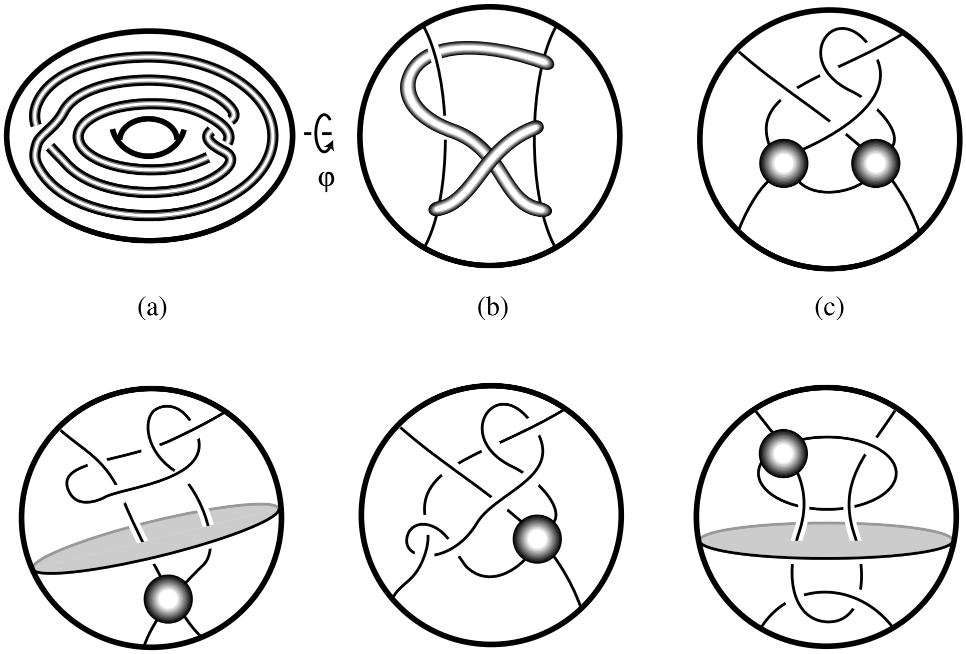
a rational tangle of slope r. Then $M(r)$ is the double cover of $(Y, \eta(-r))$.

It is easy to see that $(Y, \eta(0))$ is the tangle shown in Figure 7.6(d). A disk intersecting $\eta(0)$ twice is shown in the figure. The double branched cover $M(0)$ of $(Y, \eta(0))$ is the union of a trefoil knot exterior and a copy of $T^2 \times I$, glued along an annulus which is meridional on the trefoil knot exterior. Hence $M(0)$ is actually the exterior of the connected sum of a trefoil and a Hopf link. This manifold is toroidal and annular.

Figure 7.6(e) is the picture of $(Y, \eta(3/2))$. After an isotopy it is shown in Figure 7.6(f). In this case $M(-3/2)$ is the union of a solid torus and a $(2,1)$ cable space, glued along an annulus which runs twice along the longitude of the solid torus, hence it is toroidal and annular. One can show that $M(-3/2)$ is actually the exterior of a knot K in a solid torus V, where K is some $(2,q)$ cable of a $(2,1)$ torus knot in V. $\square$

We call the manifold M above the *magic manifold.* Table 7.7 lists some manifolds obtained by Dehn filling on T_0. We use $X(p/q)$ to denote the exterior of the 2-bridge link associated to the rational number p/q, and $X(p,q,r)$ the exterior of the (p,q,r) pretzel link. We have seen from Lemma 7.6 that $M(0)$ and $M(-3/2)$ are toroidal and annular. The first is the exterior of a link which is the connected sum of a trefoil and a Hopf link, and the second one is the exterior of some $(2,q)$ cable of the $(2,1)$ torus knot in the solid torus, so it is also the exterior of a link in S^3.

The first five fillings are non-hyperbolic, and the last three are the exteriors of the



(a) (b) (c)

(d) (e) (f)

Figure 7.6

r	1/0	0	−1	−2	−3/2	−1/2	1	-3
$M(r)$	$X(1/4)$	*	$X(1/2)$	$X(1/6)$	*	$X(3/8)$	$X(3/10)$	$X(-2,3,8)$

Table 7.7

three links in Figure 7.1, which by Theorem 6.10 are the only manifolds admitting annular and toroidal fillings of distance at least 4 apart. It is surprising that all the three exceptional manifolds in Theorem 6.10 arise from Dehn fillings on this manifold.

There is an automorphism of M interchanging any pair of boundary components of M. Let T_1 be the boundary of the neighborhood of the component of L which wraps along V twice. Let (m_i, l_i) be the standard meridian-longitude pair of T_i. The homeomorphism sending T_0 to T_1 will send m_0 to l_1, and l_0 to $-m_1+l_1$. Hence all the above Dehn fillings can also be realized by Dehn filling on T_1, and the slopes can be calculated accordingly. For example, the two toroidal and annular fillings on T_1 have slopes -1 and 2, respectively.

We can apply Theorem 6.10 to prove the following result.

Proposition 7.7. *If $r \neq 0, -1, -3/2, -2, \infty$, then $M(r)$ is hyperbolic.*

Proof. Note that M has three boundary components, so it is not one of the three exceptional manifolds in Theorem 6.10. Let $r_1 = 0$, and $r_2 = -3/2$. By Lemma 7.6 each $M(r_i)$ is toroidal and annular, so if r is a slope such that $\Delta(r, r_i) \geq 4$ for some i, then by Theorem 6.10 the manifold $M(r)$ is atoroidal and anannular, and by [Wu1,Oh] and [GLu] it is also irreducible and ∂-irreducible, and hence $M(r)$ is hyperbolic. Thus if $M(r)$ is non-hyperbolic, then $\Delta(r, r_i) \leq 3$ for $i = 1, 2$.

Suppose $r = a/b$. Without loss of generality we may assume that $a \geq 0$. Then we have

$$\Delta(0, a/b) = a \leq 3;$$

$$\Delta(-3/2, a/b) = |2a + 3b| \leq 3.$$

Solving these inequalities, we have $a/b = 0, \infty, -1, -3/2, -2$, or -3. From Table 7.7 we see that $M(-3)$ is the exterior of the Whitehead sister link shown in Figure 7.1(c), which is hyperbolic by Theorem 7.5. □

Theorem 7.8. *There are infinitely many hyperbolic manifolds M_s, which admit two Dehn fillings $M_s(r_1), M_s(r_2)$, each containing an essential annulus and an essential torus, with $\Delta(r_1, r_2) = 3$.*

Proof. Let M be the manifold in Lemma 7.6. We are now considering Dehn fillings on both T_0 and T_1, so denote by $M(r, *) = M(r)$ the manifold obtained by r-filling on T_0, by $M(*, s) = M_s$ the manifold obtained by s-filling on T_1. Let $r_1 = 0$, and $r_2 = -3/2$.

With respect to certain framings on T_1, the manifold $M(*, s)$ can be obtained by filling the sphere S_1 in Figure 7.6(c) with a rational tangle of slope s, then taking the double branched cover. From Figure 7.6(d) and 7.6(f) we can see that if $s \notin \{0\} \cup \{1/n \,|\, n \in \mathbb{Z}\}$, then for $i = 1, 2$, the essential annulus and essential torus in $M(r_i) = M(r_i, *)$ remain essential in $M(r_i)(s) = M_s(r_i)$. By Thurston [Th2], there is a finite set $\mathcal{S}$ such that if $s \notin \mathcal{S}$ then $M(*, s)$ is hyperbolic. Actually using Proposition 7.7 and the symmetry of M we see that $\mathcal{S}$ contains exactly 5 slopes. Thus if $s \notin \mathcal{S} \cup \{0\} \cup \{1/n \,|\, n \in \mathbb{Z}\}$, then M_s is hyperbolic, and $M_s(r_i)$ is toroidal and annular, $i = 1, 2$. Clearly there are infinitely many such s. Since $\Delta(r_1, r_2) = \Delta(0, -3/2) = 3$, this proves the theorem.

Note that the framing on T_1 above is not the standard meridian-longitude framing. □

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