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DEHN FILLINGS PRODUCING REDUCIBLE MANIFOLDS AND TOROIDAL MANIFOLDS

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Abstract. If two surgeries on a hyperbolic knot produce a reducible manifold and a toroidal manifold respectively, then the the distance between the surgery slopes is at most three.

0. Introduction

This paper studies one of the problems concerning Dehn fillings producing reducible or toroidal 3-manifolds. Let M be an orientable, irreducible, atoroidal, anannular 3-manifold with T as a torus boundary component. Let γ be an essential simple loop on T. Denote by $M(\gamma)$ the manifold obtained by Dehn filling along the curve γ, i.e. $M(\gamma) = M \cup_\varphi J$, where J is a solid torus, and $\varphi : T \cong \partial J$ is a gluing map sending γ to a meridian curve of J. For two curves γ_1, γ_2 on T, denote by $\Delta = \Delta(\gamma_1, \gamma_2)$ the minimal geometric intersection number between the isotopy classes of γ_1 and γ_2. In [GLu] Gordon and Luecke proved that if both $M(\gamma_i)$ are reducible then $\Delta \leq 1$. Gordon [Go] proved that if both $M(\gamma_i)$ are toroidal then $\Delta \leq 5$ except for a few manifolds M, for which the maximal Δ is 6, 7 or 8. When $M(\gamma_1)$ is reducible and $M(\gamma_2)$ is toroidal, Gordon and Litherland [GLi] showed that $\Delta \leq 5$. This was improved by Boyer and Zhang [BZ] to $\Delta \leq 4$. The main theorem of this paper is:

Theorem 1. *Let M be an orientable, irreducible, atoroidal, anannular 3-manifold. Let γ_1, γ_2 be two slopes on a torus component T of ∂M. If $M(\gamma_1)$ is reducible and $M(\gamma_2)$ is toroidal, then $\Delta(\gamma_1, \gamma_2) \leq 3$.*

As pointed out in [BZ], if we take M to be the manifold obtained by Dehn filling with slope 6 on one boundary component of the exterior of the Whitehead link, then $M(1)$ is reducible and $M(4)$ is toroidal, and the distance between the two Dehn filling slopes is 3. Hence Theorem 1 is the best possible in general. The assumption that M be irreducible and atoroidal is obviously necessary. It is also necessary to assume that M is anannular. For example, if M is the trefoil knot exterior, then $M(0)$ is toroidal, $M(6)$ is reducible, and we have $\Delta(0, 6) = 6$.

The proof of the theorem will use some combinatorial techniques developed in [GLi, CGLS, Wu, Go, BZ]. By the results of [BZ] we may assume that the reducing sphere and the essential torus both intersect the attached solid tori more than twice, and $\Delta = 4$. In Section 1 we give some definitions and conventions, and prove some properties of the intersection graphs. In Section 2 we study the case that some vertex of $\widehat{\Gamma}_2$ has valency equal to 5, where Γ_2 is the graph on the torus, and $\widehat{\Gamma}_2$

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is the reduced graph of Γ_2. Section 3 studies the case that each vertex of $\widehat{\Gamma}_2$ has valency 6. The conclusion of Sections 2 and 3 is that if $\Delta = 4$ then Γ_2 has only 4 vertices, and the torus carrying Γ_2 is separating. Section 4 deals with this case, and shows that it is impossible. The results in these sections are summarized at the end of Section 4 to give the proof of Theorem 1.

Another proof of Theorem 1 has recently been obtained independently by Seungsang Oh. I would like to thank the referee for some helpful comments.

1. Properties of the intersection graphs

We refer the reader to [Ro] for basic concepts about knots and Dehn surgery or Dehn filling, and to [He, Ja] for those of 3-manifold topology. If X is a set in a 3-manifold, we use $N(X)$ to denote its regular neighborhood.

Throughout this paper, we will assume that M, T, γ_1, γ_2 satisfy the assumptions of Theorem 1. So $M(\gamma_1)$ is reducible and $M(\gamma_2)$ is toroidal. Denote by J_i the attached solid torus in $M(\gamma_i)$. Let F_1 be a reducing sphere in $M(\gamma_1)$, and F_2 an essential torus in $M(\gamma_2)$. Let $u_1, \ldots, u_{n_1}$ be the disks of $F_1 \cap J_1$, labeled successively when traveling along J_1. Similarly let $v_1, \ldots, v_{n_2}$ be the disks in $F_2 \cap J_2$. Put $P_i = F_i \cap M$. Thus P_1 is a punctured sphere, and P_2 is a punctured torus. We choose F_i so that n_i is minimal, and the number of components in $P_1 \cap P_2$ is minimal. Let Γ_1 be the graph in F_1 with u_i as (fat) vertices, and the arc components of $P_1 \cap P_2$ as edges. Similarly, Γ_2 is a graph in F_2 with v_j as vertices and the arcs of $P_1 \cap P_2$ as edges. The minimality of the number of components in $P_1 \cap P_2$ and the minimality of n_i imply that Γ_i has no trivial loops, and each disk face of Γ_i in F_i has interior disjoint from P_j, $j \neq i$. It has been shown in [BZ] that $n_1 > 2$, and Theorem 1 is true if $n_2 \leq 2$. Therefore below *we will always assume that* $n_i \geq 3$ *for* $i = 1, 2$. Also since Boyer and Zhang [BZ] have shown that $\Delta \leq 4$, *we will assume that* $\Delta = 4$, and get a contradiction.

If e is an edge of Γ_1 with an end x on a fat vertex u_i, then x is labeled j if x is in $u_i \cap v_j$. In this case e is called a *j-edge at u_i*, and it is an i-edge at v_j. The labels in Γ_1 are considered mod n_2 integers. In particular, $n_2 + 1 = 1$. When going around ∂u_i, the labels of the ends of edges appear as $1, 2, \ldots, n_2$ repeated $\Delta = 4$ times. Label the ends of edges in Γ_2 similarly. Each label in Γ_2 is a mod n_1 integer.

By relabeling the vertices of Γ_i we mean choosing another v_j as v_1, and choosing the direction for going around J_i. Thus the new labels for the vertices $v_1, v_2, \ldots v_{n_1}$ would be $v_k, v_{k+1}, \ldots, v_{k-2}, v_{k-1}$ or $v_k, v_{k-1}, \ldots, v_{k+2}, v_{k+1}$ for some k.

Each vertex of Γ_i is given a sign according to whether J_i passes F_i from the positive side or negative side at this vertex. Two vertices of Γ_i are parallel if they have the same sign, otherwise they are antiparallel. Note that if F_i is a separating surface, then n_i is even, all vertices with even subscripts have the same sign, and all those with odd subscripts have the other sign.

Definition 1.1. An edge of Γ_i is a *positive edge* if it connects parallel vertices. Otherwise it is a *negative edge*.

An edge of Γ_1 is an arc component of $P_1 \cap P_2$, so it is also an edge of Γ_2. The parity rule of [CGLS] can be stated as:

Lemma 1.1. *An edge e is a positive edge in Γ_1 if and only if it is a negative edge in Γ_2.* $\square$

When considering each family of parallel edges of Γ_i as a single edge $\widehat{e}$, we get the *reduced graph* $\widehat{\Gamma}_i$ on F_i. It has the same vertices as Γ_i. We will write $\widehat{e} = \{e_1, \dots, e_k\}$ to indicate that $\widehat{e}$ corresponds to the parallel edges $e_1, \dots, e_k$. It is always assumed that e_j is adjacent to e_{j+1}. The *weight of* $\widehat{e}$, denoted by $w(\widehat{e})$, equals the number of edges in $\widehat{e}$. So in the above case we have $w(\widehat{e}) = k$. If v is a vertex of $\widehat{\Gamma}_i$, then the total weight of edges at v, i.e. the sum of weights of edges at v, equals $\Delta n_j = 4n_j$ $(j \neq i)$, which is also the valency of v in Γ_i.

We use Γ_i^+ to denote the subgraph of Γ_i whose vertices are the vertices of Γ_i and whose edges are the positive edges of Γ_i. Similarly for $\widehat{\Gamma}_i^+$.

Let v be a vertex of a graph Γ on a surface F. Let Λ be the component of Γ that contains v. Then v is called an *interior vertex* of Γ if there is no arc α on F connecting v to $\Gamma - \Lambda$ with interior disjoint from Γ. It is clear that if v is an interior vertex of $\widehat{\Gamma}_i^+$, then all edges at v are positive edges.

A pair of adjacent parallel positive edges $\{e_1, e_2\}$ is called a *Scharlemann cycle* if the two labels of the ends of e_1 are the same as those of e_2. This is actually a special case of a Scharlemann cycle as defined in [CGLS]. The two labels of e_i must be $\{j, j+1\}$ for some j. We call $\{j, j+1\}$ the label set of the Scharlemann cycle.

If $\{e_1, e_2\}$ is a Scharlemann cycle on Γ_1, then when shrinking each fat vertex of Γ_i to a single point, $C = e_1 \cup e_2$ will be a loop in F_i. When we say "the loop $e_1 \cup e_2$", we will always refer to the loop obtained this way.

If $\{e_1, e_2, e_3, e_4\}$ are four parallel positive edges with e_i adjacent to e_{i+1} for $i = 1, 2, 3$, and if the two middle edges $\{e_2, e_3\}$ form a Scharlemann cycle, then the set of these four edges is called an *extended Scharlemann cycle*. We need some results from [BZ].

Lemma 1.2. *(1) Neither Γ_1 nor Γ_2 has extended Scharlemann cycles.*

(2) If Γ_2 has a Scharlemann cycle then F_1 is a separating sphere.

(3) If Γ_1 has a Scharlemann cycle $\{e_1, e_2\}$ then F_2 is a separating torus. Moreover, the loop $e_1 \cup e_2$ is an essential simple closed curve on F_2.

(4) There are no n_1 parallel edges on Γ_2.

(5) Any two Scharlemann cycles on Γ_2 have the same label set.

Proof. (1) is Lemma 2.5(2) and Lemma 2.9 of [BZ]. (2) and (3) are Lemma 2.2 and Lemma 2.8 of [BZ]. (4) is Lemma 2.6 of [BZ]. See also [GLi, Proposition 1.3]. (5) is Lemma 2.5(1) of [BZ]. See also [Wu, Lemma 2.2–2.4]. □

Lemma 1.3. *Let $\{e_1, \dots, e_k\}$ be parallel positive edges in Γ_i with e_j adjacent to e_{j+1}. If some label t appears twice, then t is the label of some Scharlemann cycle $\{e_j, e_{j+1}\}$, and either $j = 1$ or $j + 1 = k$. In particular, t must be a label of e_1 or e_k.*

Proof. This is essentially Lemma 2.5(4) of [BZ]. Notice that if $\{e_j, e_{j+1}\}$ is a Scharlemann cycle but $j \neq 1$ and $j+1 \neq k$ then $\{e_{j-1}, e_j, e_{j+1}, e_{j+2}\}$ would be an extended Scharlemann cycle. □

Lemma 1.4. (Properties of $\widehat{\Gamma}_1$) *If $\widehat{e}$ is a positive edge of $\widehat{\Gamma}_1$, then $w(\widehat{e}) \leq n_2/2 + 2$. Moreover, if $w(\widehat{e}) = n_2/2 + 1$ or $w(\widehat{e}) = n_2/2 + 2$, then up to relabeling of vertices of Γ_2, the edges of $\widehat{e}$ in Γ_1 have labels as in Figure 1.1(a) and (b), respectively.*

Proof. Suppose $\widehat{e} = \{e_1, \dots, e_k\}$. If $k > n_2/2 + 2$, the edges have at least $n_2 + 5$ ends, so some label appears twice among the ends of edges other than e_1 and e_k, which is impossible by Lemma 1.3.

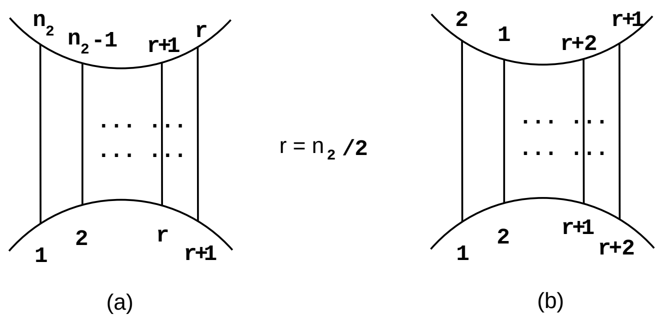


Figure 1.1

If $k = n_2/2 + 1$, there are two labels repeated among the ends of the edges. By Lemma 1.3 there is a Scharlemann cycle which is either $\{e_1, e_2\}$ or $\{e_{k-1}, e_k\}$. Up to relabeling we may assume that at one of the vertices the label of e_j is j, and the Scharlemann cycle is $\{e_{k-1}, e_k\}$, so the labels must be as shown in Figure 1.1(a). The proof for the case that $k = n_2/2 + 2$ is similar. □

Lemma 1.5. (Properties of $\widehat{\Gamma}_2$)

(1) If $\widehat{e}$ is a positive edge of $\widehat{\Gamma}_2$, then $w(\widehat{e}) \leq n_1/2 + 1$. Moreover, if $w(\widehat{e}) = n_1/2 + 1$, then $\widehat{e}$ contains a Scharlemann cycle, and up to relabeling of vertices of Γ_1, the edges of $\widehat{e}$ in Γ_2 have labels as in Figure 1.1(a), with n_2 replaced by n_1.

(2) Any edge $\widehat{e}$ of $\widehat{\Gamma}_2$ has $w(\widehat{e}) < n_1$.

Proof. The proof of (1) is the same as that of Lemma 1.4, only notice that if $k = n_1/2 + 2$ there would be two Scharlemann cycles with different label sets, which would contradict Lemma 1.2(5).

(2) is a restatement of Lemma 1.2(4). □

Let $\{e_1, e_2\}$ be a Scharlemann cycle in Γ_1. There is a disk D on F_1 with $\partial D = e_1 \cup \alpha \cup e_2 \cup \beta$, where α and β are arcs on the boundary of the fat vertices at the ends of e_i. The disk D is called the *Scharlemann disk* between e_1 and e_2. Consider $M(\gamma_2)$. Let K_i be the part of the attached solid torus J_2 between v_i and v_{i+1}. By Lemma 1.2(3) F_2 is separating, so it cuts $M(\gamma_2)$ into two components. Let X be the one containing K_i. Let V be a regular neighborhood of $D \cup K_i$ in X.

Lemma 1.6. *V is a solid torus in X. The loop $C = e_1 \cup e_2$ is a simple closed curve on ∂V running twice along the longitude, so V has a Seifert fibration with C as a regular fiber, which extends to a Seifert fibration of X with orbifold a disk with two cone points.*

Proof. V can be considered as obtained from $N(C \cup K_i)$ by attaching the 2-handle $N(D)$. Since ∂D runs over e_1 only once, it is a primitive curve on the genus two handlebody $N(C \cup K_i)$. Hence V is a solid torus. By calculating the homology one can see that C runs twice along the longitude.

Let W be the closure of $X - V$. Then ∂W is a torus intersecting J_2 only $n_2 - 2$ times. By the choice of F_2, ∂W must be compressible. Since F_2 is incompressible in $M(\gamma_2)$, and the annulus $A = W \cap V$ is incompressible in V, it is easy to see that ∂W must be compressible in W. Since $M(\gamma_2)$ is irreducible (otherwise by [GLu] we would have $\Delta \leq 1$), W is a solid torus. Thus X is the union of two solid tori along an annulus A. The annulus A can not be meridional or longitudinal on ∂W, otherwise F_2 would be compressible. Hence the fibration of V extends over W, and both V and W have a singular fiber in its interior. □

We call V the *solid torus produced by the Scharlemann cycle* $\{e_1, e_2\}$.

Lemma 1.7. *If* $\{e_1, e_2\}$ *and* $\{e_3, e_4\}$ *are two Scharlemann cycles on* Γ_1 *with disjoint label sets* $\{i, i+1\}$ *and* $\{j, j+1\}$*, then* i *and* j *have the same parity.*

Proof. If i and j have different parity, then the solid tori produced by them lie on different sides of the torus F_2. Let X_1, X_2 be the closures of the components of $M(\gamma_2) - F_2$. As the label sets are disjoint, the loops $C_1 = e_1 \cup e_2$ and $C_2 = e_3 \cup e_4$ are disjoint simple closed curves on F_2. By Lemma 1.6 each X_i is a Seifert fiber space with orbifold a disk with two cone points. Since the regular fibers C_i on X_i are disjoint, hence isotopic to each other on F_2, the fibrations on the two sides of F_2 can be combined to a single Seifert fibration of $M(\gamma_2)$, whose orbifold is a sphere with 4 cone points.

Let V_i be the solid torus produced by the Scharlemann cycle $\{e_1, e_2\}$. Let W_i be the closure of $X_i - V_i$. Then both $V_1 \cup W_2$ and $V_2 \cup W_1$ are Seifert fiber space with orbifold a disk with two cone points, so their common boundary torus F' is an essential torus in $M(\gamma_2)$. Since F' intersects J_2 in only $n_2 - 4$ disks, this contradicts the choice of F_2. □

Corollary 1.8. *If* $\widehat{\Gamma}_1$ *has a positive edge* $\widehat{e}$ *with* $w(\widehat{e}) = n_2/2 + 2$*, then* $n_2 \equiv 0$ *mod 4.*

Proof. By Lemma 1.4 and Figure 1.1, up to relabeling of the vertices of Γ_2, the graph Γ_1 has two Scharlemann cycles with label sets $\{1, 2\}$ and $\{n_2/2+1, n_2/2+2\}$. By Lemma 1.7 the label $n_2/2 + 1$ must be odd. □

Lemma 1.9. *Let* $\{e_1, e_2\}$ *and* $\{e_3, e_4\}$ *be Scharlemann cycles in* Γ_1 *with label sets* $\{1, 2\}$ *and* $\{2, 3\}$*. If the two loops* $C_1 = e_1 \cup e_2$ *and* $C_2 = e_3 \cup e_4$ *on* F_2 *are homotopic to each other on* F_2*, then* $e_1 \cup e_2 \cup e_3 \cup e_4$ *bounds a disk* D *in* F_2 *which contains some vertices of* Γ_2 *in its interior.*

Proof. The proof is similar to that of Lemma 1.7. The two loops C_1 and C_2 intersect at a single vertex v_2 of Γ_2, so if they are homotopic on F_2, they must cobound a disk D on F_2, as shown in Figure 1.2. Let X_i, V_i, W_i, C_i be as in the proof of Lemma 1.7. Since C_1 and C_2 are isotopic on F_2, $M(\gamma_2)$ is a Seifert fiber space with orbifold a sphere with 4 cone points. Now $Y = V_1 \cup V_2 \cup N(D)$ is a fibered subset with orbifold a disk with two cone points, so ∂Y is an essential torus in $M(\gamma_2)$. If the interior of D is disjoint from J_2, then ∂Y intersects J_2 only twice, which would contradict the choice of the torus F_2. □

Lemma 1.10. Γ_1 *can not have three Scharlemann cycles with mutually disjoint label sets.*

Proof. Let V_1, V_2, V_3 be the solid tori produced by these Scharlemann cycles. By Lemma 1.7 they must all be in the same closed up component X of $M(\gamma_2) - F_2$. By Lemma 1.6 X has a Seifert fibration with orbifold a disk with two cone points, and with V_i as a fibered solid torus. Recall that the fibration on such a manifold X is unique [Ja, Theorem VI.17]. So there is a fibration of X so that all the V_i are

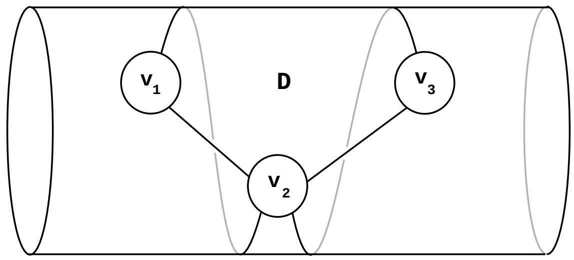


Figure 1.2

fibered solid tori. But since each V_i contains a singular fiber, this contradicts the fact that M has only two singular fibers. □

2. The case that some vertex of $\widehat{\Gamma}_2$ has valency 5

In this section we study the case that some vertex v_t of $\widehat{\Gamma}_2$ has valency 5. Notice that the valency of v_t can not be lower than 5, otherwise some edge $\widehat{e}$ of $\widehat{\Gamma}_2$ would have $w(\widehat{e}) \geq n_1$, contradicting Lemma 1.5.

Lemma 2.1. *If v_t is a vertex of $\widehat{\Gamma}_2$ with valency 5, then it is adjacent to at most one positive edge in $\widehat{\Gamma}_2$.*

Proof. By Lemma 1.5 each positive edge of $\widehat{\Gamma}_2$ has weight at most $n_1/2+1$ and each negative edge has weight at most $n_1 - 1$. The total weight of edges at v_t is $4n_1$. So if v_t has k positive edges in $\widehat{\Gamma}_2$, then we have

$$4n_1 \leq k(n_1/2+1) + (5-k)(n_1-1) = 5n_1 - 5 + k(2 - n_1/2)$$

that is, $n_1 - 5 \geq k(n_1/2 - 2)$. Since $n_1 > 2$, this implies that either $k \leq 1$ or $n_1 = 3$. But if $n_1 = 3$, then F_1 is a nonseparating sphere, so by Lemma 1.2(2) Γ_2 has no Scharlemann cycles. Hence each positive edge of $\widehat{\Gamma}_2$ has weight at most $n_1/2$ by Lemma 1.5, and the above inequality becomes $n_1 - 5 \geq k(n_1/2 - 1)$, which again leads to the conclusion that $k \leq 1$. □

Recall that if a vertex is an interior vertex of $\widehat{\Gamma}_1^+$ then it has no negative edges in Γ_1.

Lemma 2.2. *If $\widehat{\Gamma}_2$ has a vertex v_t of valency 5, and if some interior vertex u_i of $\widehat{\Gamma}_1^+$ has valency at most 5, then $n_2 = 4$, and F_2 is separating.*

Proof. Consider the reduced graph $\widehat{\Gamma}_1$. Since the total weight of edges at u_i is $4n_2$, in this case at least one edge $\widehat{e}$ at u_i has weight more than $n_2/2$. Since all edges of $\widehat{\Gamma}_1$ at u_i are positive, by Lemma 1.3 Γ_1 has a Scharlemann cycle, so F_2 is separating. In particular, n_2 must be an even number.

If $n_2 \geq 8$, then one of the edges at u_i would have weight at least $4n_2/5 = n_2/2 + 3n_2/10$, which is greater than $n_2/2+2$, contradicting Lemma 1.4.

If $n_2 = 6$, then the total weight at u_i is $4 \times 6 = 24$, so one of the edges e at u_i has weight at least $5 = n_2/2 + 2$. But since $n_2 \equiv 2 \bmod 4$, this contradicts Corollary 1.8.

Since we have assumed $n_2 > 2$, we must have $n_2 = 4$. $\square$

A component Λ of a graph Γ in S^2 is an *extremal component* if it is contained in a disk D which is disjoint from the other components of Γ. It is easy to see that each graph in S^2 has at least one extremal component, and it has exactly one extremal component if and only if the graph is connected. A vertex v of Γ is a *cut vertex* if $\Gamma - v$ has more components than Γ. An edge e of Γ is a *free edge* if one of its ends has valency 1.

Lemma 2.3. *Let Λ be an extremal component of $\widehat{\Gamma}_1^+$, and assume that Λ is not a single point. If each interior vertex of Λ has valency at least 6, then Λ has at least two vertices of valency at most 3 which are not cut vertices.*

Proof. First assume that Λ has no cut vertices. By assumption Λ is not a single vertex. If Λ is a single edge the result is obvious. If Λ has more than one edge, then Λ having no cut vertices implies that there is a disk D in the sphere F_1 such that D contains Λ, and ∂D is a subgraph of Λ. Taking two copies of (D, Λ) and gluing along their boundary, we get a graph Λ' on a 2-sphere S^2. Our assumption says that any vertex which is not on ∂D has valency at least 6. If a vertex v on ∂D has valency more than 3 in Λ, then it also has valency at least 6 on Λ'. Hence if the lemma is not true, then all but one vertex v of Γ' have valency at least 6, and v has valency 2 or 4. Let V, E, F be the number of vertices, edges and faces of Λ'. What we have shown is $6(V-1)+2 \leq 2E$, so $V < E/3+1$. Since Λ' is a reduced graph, each face has at least three edges, so we also have $3F \leq 2E$. Therefore,

$$V - E + F < (\frac{E}{3} + 1) - E + (\frac{2E}{3}) = 1$$

which contradicts the fact that the Euler number of a sphere is 2.

In general, let v be a cut vertex of Λ. Let $\Lambda_1, \ldots, \Lambda_k$ be the components after cutting along v. By induction each Λ_i has at least two vertices v_i^1, v_i^2 of valency at most 3 which are not cut points of Λ_i. At most one of them could be v, so for each i, at least one of the v_i^j has valency at most 3 in Λ, and is not a cut vertex of Λ. Since $k \geq 2$, the result follows. $\square$

Lemma 2.4. *If some vertex of $\widehat{\Gamma}_2$ has valency 5, then $n_2 = 4$, and F_2 is separating.*

Proof. Assume that the vertex v_t of $\widehat{\Gamma}_2$ has valency 5. By Lemma 2.1, at most one edge of $\widehat{\Gamma}_2$ at v_t is positive, which by Lemma 1.5 has weight at most $n_1/2+1 < n_1$. Thus at the vertex v_t in the graph Γ_2, at least three i-edges are negative for each i. By the parity rule this implies that each vertex u_i of Γ_1 has at least three positive t-edges.

Consider the graph $\widehat{\Gamma}_1^+$ consisting of all vertices and all positive edges of $\widehat{\Gamma}_1$. Let Λ be an extremal component of $\widehat{\Gamma}_1^+$. We have just seen that each vertices of $\widehat{\Gamma}_1^+$ has at least 3 positive t-edges, so Λ is not an isolated vertex. By Lemma 2.2 we may assumed that each interior vertex of $\widehat{\Gamma}_1^+$ has valency at least 6. By Lemma 2.3, some vertex u of Λ has valency at most 3, and is not a cut vertex. Now u not being a cut vertex means that the boundary of the fat vertex v can be divided into two arcs, one containing only the ends of positive edges, the other containing only the ends of negative edges. As at least three positive edges are labeled t at u, Γ_1 contains at least $2n_2+1$ positive edges adjacent to u. Since u has valency at most

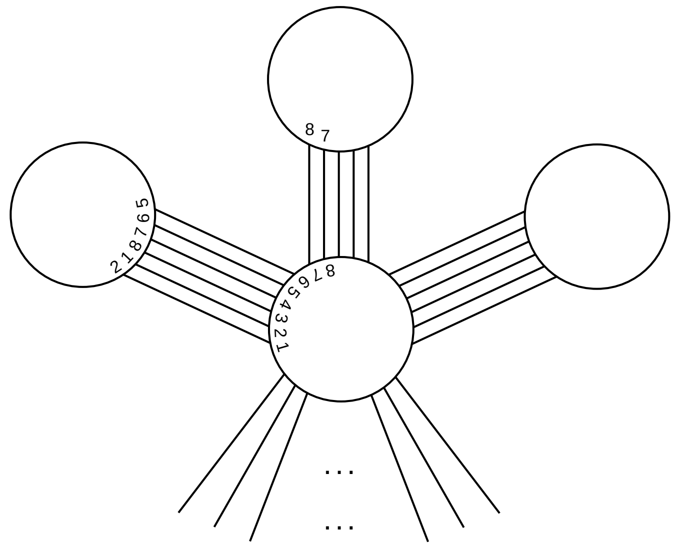


Figure 2.1

3 in $\widehat{\Gamma}_1^+$, one of the edges of $\widehat{\Gamma}_1^+$ at u has weight at least $k = (2n_2 + 1)/3$. Because $k > n_2/2$, by Lemma 1.2 the torus F_2 in $M(\gamma_2)$ is separating, and n_2 is even. Also, by Lemma 1.4 we have $(2n_2+1)/3 \leq n_2/2+2$, so $n_2 \leq 10$. It remains to show that $n_2 \neq 6, 8, 10$.

If $n_2 = 6$ or 10, by Corollary 1.8 the weight of each positive edge in $\widehat{\Gamma}_1^+$ is at most $n_2/2 + 1$, so in Γ_1 the total number of positive edges at u is no more than $3(n_2/2 + 1)$. Thus $3(n_2/2 + 1) \geq 2n_2 + 1$. But this is impossible for $n_2 = 6$ or 10.

Now assume $n_2 = 8$. Then there are 17 edges divided into 3 adjacent collections $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ of parallel positive edges at u. By Lemma 1.4 each $\widehat{e}_i$ has at most $n_2/2+2 = 6$ edges. Thus two of the collections have 6 edges, and the other has 5 edges. If $\widehat{e}_2$ has 5 edges, it has a Scharlemann cycle on one side of it. If it has 6 edges, then it has Scharlemann cycles on both sides. In either case one of the $\widehat{e}_1$ or $\widehat{e}_3$ has the property that it has 6 edges, and is adjacent to a Scharlemann cycle of the middle collection. Without loss of generality we may assume that $\widehat{e}_1$ has this property, and the labels of the ends are as in Figure 2.1. Then we see that Γ_1 has three Scharlemann cycles with label sets $\{1, 2\}$, $\{5, 6\}$, and $\{7, 8\}$. By Lemma 1.10 this is impossible. $\square$

3. The case that each vertex of $\widehat{\Gamma}_2$ has valency 6

In this section we assume that all vertices of $\widehat{\Gamma}_2$ have valency 6. Recall that $\widehat{\Gamma}_1^+$ denotes the subgraph of $\widehat{\Gamma}_1$ consisting of all vertices and positive edges. We first consider the case that $\widehat{\Gamma}_1^+$ has an interior vertex.

Lemma 3.1. *If each vertex of $\widehat{\Gamma}_2$ has valency 6, and $\widehat{\Gamma}_1^+$ has an interior vertex u_i, then for each vertex v_t of Γ_2, there is a Scharlemann cycle in Γ_1 with t as one of its labels.*

Proof. Recall that if a vertex u_i is an interior vertex of $\widehat{\Gamma}_1^+$ then all edges of $\widehat{\Gamma}_1$ adjacent to it are positive edges. By the parity rule, all i-edges on Γ_2 are negative edges. Consider the four i-edges at a vertex v_t of Γ_2. By Lemma 1.5, no two of them can be parallel (otherwise there would be n_1+1 parallel edges in Γ_2), so they correspond to distinct edges in $\widehat{\Gamma}_2$. Thus at each vertex of $\widehat{\Gamma}_2$ there are at least 4 negative edges, and at most 2 positive edges. Recall from Lemma 1.5 that each positive edge of $\widehat{\Gamma}_2$ has weight at most $n_1/2+1$. In Γ_2 this means that at each vertex v_t there are at most n_1+2 positive edges, and at least $3n_1-2$ negative edges. Dually, in Γ_1 we have at least $3n_1-2$ positive t-edges.

Consider $\widehat{\Gamma}_1$. Let $\widetilde{\Gamma}$ be $\widehat{\Gamma}_1$ with a minimal number of edges added to make it connected. Let V, E, F be the number of vertices, edges and faces of $\widetilde{\Gamma}$. Then $F \leq 2E/3$. By calculating the Euler number of the sphere F_1 we have

$$2 = V - E + F \leq V - E + 2E/3 = V - E/3$$

So $E \leq 3V-6$. Since $\widehat{\Gamma}_1$ has no more edges than $\widetilde{\Gamma}$, it has at most $3n_1-6$ edges. We have just shown that there are at least $3n_1-2$ positive t-edges in Γ_1, so there exists a pair of parallel positive t-edges in Γ_1. By Lemma 1.3 they form a Scharlemann cycle with t as a label. □

Lemma 3.2. *Let Λ be an extremal component of $\widehat{\Gamma}_1^+$. If Λ is not a single vertex, and has no interior vertices, then it has at least two vertices of valency at most 2, which are not cut vertices of $\widehat{\Gamma}_1^+$.*

Proof. We need only prove the lemma for the case that Λ has no cut vertices and is not a single edge. The proof of the general case follows as in the proof of Lemma 2.3

As in the proof of Lemma 2.3, we may choose a disk D containing Λ, with ∂D a subgraph of Λ. If $\Lambda = \partial D$, then since Γ_1 has no trivial loops, there are at least two vertices on ∂D, which are the required vertices. So assume that Λ has some edges not on ∂D. Using an outmost arc argument, we see that there are at least two arcs α_1, α_2 on ∂D which are parallel to some edge of Λ in the interior of D. Since Λ is part of the reduced graph $\widehat{\Gamma}_1^+$, it has no parallel edges, so α_i can not be an edge of Λ. It follows that each α_i contains at least one vertex of Λ in its interior, which is then a vertex of valency 2. □

Corollary 3.3. *Suppose $\widehat{\Gamma}_1^+$ has no interior vertices. Then there is a vertex u_i which is not a cut vertex of $\widehat{\Gamma}_1^+$, and has valency at most 2 in $\widehat{\Gamma}_1^+$, such that i is not the label of any Scharlemann cycle in Γ_2.*

Proof. $\widehat{\Gamma}_1^+$ can not be connected, otherwise all of its vertices are interior vertices, contradicting the assumption. Hence it has at least two extremal components. By Lemma 3.2, each extremal component either is an isolated vertex (which has valency 0), or has at least two vertices of valency at most 2 in $\widehat{\Gamma}_1^+$, which are not cut vertices. By Lemma 1.2, at most 2 numbers can occur as the labels of Scharlemann cycles in Γ_2. If the corollary were false, then $\widehat{\Gamma}_1^+$ must have exactly two extremal components u_r, u_s, each being an isolated vertex, and $\{r, s\}$ would be the label pair of some Scharlemann cycle in Γ_2, whose edges connect u_r to u_s. On the other hand, since $n_1 > 2$, $\widehat{\Gamma}_1^+$ contains some other components, each of which must separate u_r from u_s as otherwise there would be another extremal component. Hence there is no arc in Γ_1 connecting u_r to u_s, a contradiction. □

Lemma 3.4. *If each vertex of $\widehat{\Gamma}_2$ has valency 6, and $\widehat{\Gamma}_1^+$ has no interior vertices, then $n_2 = 4$ and F_2 is separating.*

Proof. By Corollary 3.3, there is a vertex u_i which is not a cut vertex of $\widehat{\Gamma}_1^+$, and has valency at most 2 in $\widehat{\Gamma}_1^+$, such that i is not the label of any Scharlemann cycle in Γ_2. By Lemma 1.4, an edge of $\widehat{\Gamma}_1^+$ has weight at most $n_2/2+2$, so u_i is adjacent to at most n_2+4 positive edges in Γ_1. Let k be the number of negative edges at u_i. Then $k \geq 3n_2 - 4$. Note that k is the number of positive i-edges in Γ_2.

Consider the subgraph $\Gamma_2(i)$ of Γ_2 consisting of all i-edges. Since i is not the label of any Scharlemann cycle of Γ_2, by Lemma 1.3 no two positive edges of $\Gamma_2(i)$ are parallel, and by Lemma 1.5 at most two negative i-edges could be parallel (otherwise there would be more than n_1 parallel edges). Because there are k positive i-edges, and $4n_2 - k$ negative i-edges in $\Gamma_2(i)$, the reduced graph $\widehat{\Gamma}_2$ has at least $k + (4n_2 - k)/2 = 2n_2 + k/2 \geq (7/2)n_2 - 2$ edges. As each vertex of $\widehat{\Gamma}_2$ has valency 6, there are actually $3n_2$ edges in $\widehat{\Gamma}_2$. So we have $(7/2)n_2 - 2 \leq 3n_2$, i.e $n_2 \leq 4$.

We want to show that F_2 is a separating surface. If this were not true, by Lemma 1.2 Γ_1 contains no Scharlemann cycles, hence each positive edge of $\widehat{\Gamma}_1$ represent no more than $n_2/2$ edges, so $k \geq 3n_2$. Using this inequality in the above calculation, we have $3n_2 \geq 2n_2 + k/2 = (7/2)n_2$, which is absurd. Therefore F_2 is separating. This also implies that $n_2 \neq 3$, hence $n_2 = 4$. □

4. The case that $n_2 = 4$ and F_2 is separating

Notice that since F_2 is separating, all v_t with t odd have the same sign, and all v_t with t even have the other sign. So we may assume that v_1, v_3 are positive vertices, and v_2, v_4 are negative vertices.

Lemma 4.1. *$\widehat{\Gamma}_1^+$ must have an interior vertex.*

Proof. Suppose $\widehat{\Gamma}_1^+$ has no interior vertex. By Corollary 3.3, there is a vertex u_i in $\widehat{\Gamma}_1^+$ which is not a cut vertex, and has valency at most 2 in $\widehat{\Gamma}_1^+$, such that i is not the label of any Scharlemann cycle in Γ_2. Since each positive edge of $\widehat{\Gamma}_1^+$ contains at most $n_2/2+2 = 4$ edges, there are at least 8 negative edges incident to u_i, which give at least 8 positive i-edges in Γ_2. By Lemma 1.3, no two positive i-edges of Γ_2 are parallel. Also, since u_i is not a cut vertex, at least 4 of the these positive i-edges in Γ_2 have ends on $v_1 \cup v_3$, and 4 others have ends on $v_2 \cup v_4$. Denote by G the subgraph of Γ_2 consisting of all vertices and positive i-edges of Γ_2.

There are at most two edges of G connecting a vertex to another, because 3 mutually nonparallel edges connecting v_1 to v_3 would cut F_2 into a disk containing the other two vertices v_2 and v_4, so there could not be 4 mutually nonparallel edges with ends on $v_2 \cup v_4$. Similarly, there is at most one loop based at each vertex. It is now easy to see that there must be one loop based at each vertex, two edges connecting v_1 to v_3, and two others connecting v_2 to v_4, so the graph G is isomorphic to the one drawn on Figure 4.1, where the torus F_2 is obtained by identify the two ends of the cylinder together. Notice that any positive edge in Γ_2 must be parallel to one of the edges in G. In other words, we have $G = \widehat{\Gamma}_2^+$.

Consider the graph $\widehat{\Gamma}_2$. From Figure 4.1 one can see that each vertex has at most 2 negative edges. We claim that each negative edge has weight at most $n_1 - 2$. This follows from the proof of [GLi, Proposition 1.3]. It was shown that Γ_2 can not have more than $n_2 - 1$ parallel negative edges. Moreover, if it has $n_2 - 1$ such edges

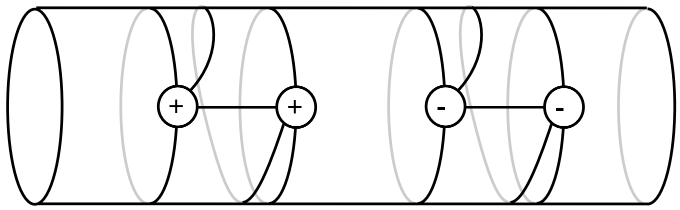


Figure 4.1

then on Γ_1 these edges form some cycles and at most one arc. If there are some cycles then there is one bounding a disk whose interior contains no vertices of Γ_1, in which case it was shown that M contains an essential annulus, contradicting our assumption on M. If there is no cycle, then all the vertices of Γ_1 would be parallel, which would contradict the fact that there are some negative edges in Γ_1.

The total weight of v_j in $\widehat{\Gamma}_2$ is $4n_1$, each of the two negative edge has weight $\leq n_2 - 2$, and each of the four positive edge has weight $\leq n_2/2 + 1$, so all the inequalities must be equalities. By Lemma 1.5, each positive edge of $\widehat{\Gamma}_2^+$ contains a Scharlemann cycle. The two adjacent negative edges of $\widehat{\Gamma}_2^+$ at v_j represents $2n_1 - 4 > n_1$ edges in Γ_2, so each label appears as the label of at least one negative edge at v_j. It follows that the four Scharlemann cycles at v_j can not all have the same label pairs, which contradicts Lemma 1.2(5). □

Lemma 4.2. *(1) If some vertex u_i of Γ_1 has 5 negative edges, then Γ_2 has a Scharlemann cycle with i as a label.*

(2) u_i can not have 9 negative edges.

Proof. By Lemma 4.1 $\widehat{\Gamma}_1^+$ has an interior vertex u_j. Thus all the 16 j-edges of Γ_2 are negative. By Lemma 1.2(4) no three of these j-edges are parallel. Hence there are at least 8 negative edges in $\widehat{\Gamma}_2$. By calculating the Euler number, one can see that $\widehat{\Gamma}_2$ has at most 12 edges, hence it has at most 4 positive edges.

If u_i is incident to 5 negative edges, then two of the positive i-edges on Γ_2 would be parallel, so by Lemma 1.3 Γ_2 has a Scharlemann cycle with i as a label. If u_i is incident to 9 negative edges, then three of the positive i-edges on Γ_2 would be parallel. But this implies that there are at least $n_1 + 1$ parallel edges in Γ_2, contradicting Lemma 1.5. □

Lemma 4.3. *Let $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ be successive positive edges in $\widehat{\Gamma}_1$ at a vertex u_i. Then they can not all have weight 4.*

Proof. Otherwise the graph Γ_1 would look like that in Figure 4.2. By Lemma 1.4, up to relabeling of the vertices in Γ_2, the labels must look like those in the figure. There are 6 edges with labels $\{1, 2\}$ and 6 with labels $\{3, 4\}$. Thus on Γ_2 there are 6 edges connecting v_1 to v_2, and 6 connecting v_3 to v_4. It is easy to see that at least three of these i-edges must be parallel, so there would be at least $n_1 + 1$ parallel edges in Γ_2 between them, which contradicts Lemma 1.5. □

Corollary 4.4. *$\widehat{\Gamma}_1^+$ has no interior vertex of valency ≤ 4.*

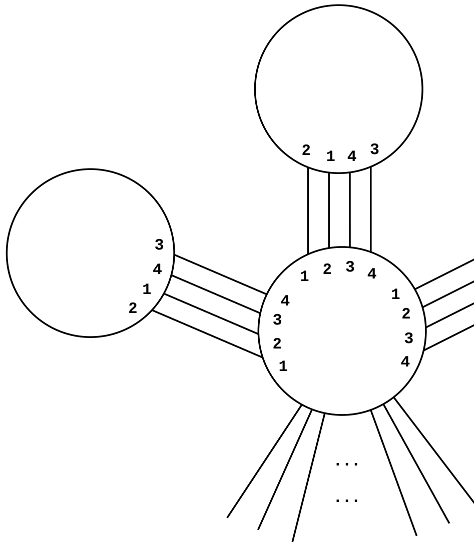


Figure 4.2

Proof. The total weight of $\widehat{\Gamma}_1^+$ at an interior vertex u_i is $\Delta n_2 = 16$. By Lemma 1.4 each edge at u_i has weight at most 4. So u_i has 4 edges in $\widehat{\Gamma}_1^+$, each having weight 4. But this contradicts Lemma 4.3. □

Lemma 4.5. *$\widehat{\Gamma}_1^+$ must have an interior vertex of valency 5.*

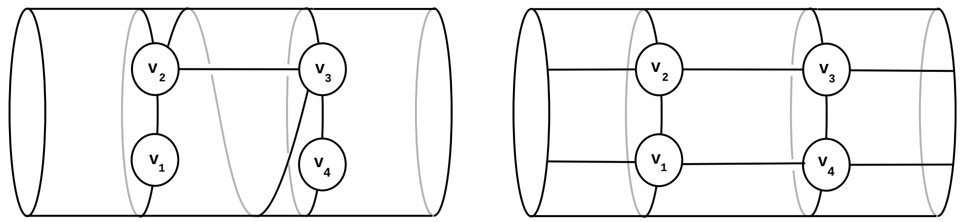
Proof. Otherwise by Corollary 4.4 each interior vertex of $\widehat{\Gamma}_1^+$ has valency at least 6. By counting the Euler number it is easy to show that a connected reduced graph on S^2 must have a vertex of valency at most 5. If $\widehat{\Gamma}_1^+$ were connected, each vertex would be an interior vertex, hence our assumption implies that $\widehat{\Gamma}_1^+$ is not connected. Thus there are at least two extremal components.

Consider an extremal component C of $\widehat{\Gamma}_1^+$. It can not be a single vertex, otherwise all 16 edges at this vertex would be negative, contradicting Lemma 4.2(2). Hence by Lemma 2.3 each extremal component of $\widehat{\Gamma}_1^+$ has at least 2 vertices which have valency at most 3. Let u_i be such a vertex. Let $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ be the edges of $\widehat{\Gamma}_1^+$ at u_i. By Lemma 1.4 the weight of each $\widehat{e}_i$ is at most 4. If all $\widehat{e}_i$ have weight 4, we get a contradiction to Lemma 4.3. So one of the $\widehat{e}_i$ has weight less than 4. In this case u_i has at least 5 negative edges, therefore by Lemma 4.2(1) there is a Scharlemann cycle in Γ_2 with i as a label.

There are at least two extremal components, each having at least two vertices u_i such that i is the label of a Scharlemann cycle in Γ_2, so at least 4 numbers appear as labels of Scharlemann cycles in Γ_2. In particular there would be two Scharlemann cycles with different label sets, which contradicts Lemma 1.2(5). □

Lemma 4.6. *$\widehat{\Gamma}_1^+$ can not have an interior vertex of valency 5.*

Proof. Assume $u_i \in \widehat{\Gamma}_1^+$ is an interior vertex with valency 5. Since the total weight at u_i is 16, one of the edges has weight 4. In Γ_1 this means there are 4 parallel positive edges adjacent to u_i. Since $4 = n_2/2 + 2$, by Lemma 1.4 up to relabeling



(a) (b)

Figure 4.3

the edges in Γ_1 look like that in Figure 1.1(b). These edges form two Scharlemann cycles $\{e_1, e_2\}$ and $\{e_3, e_4\}$ with label sets $\{1, 2\}$ and $\{3, 4\}$, respectively. On the torus F_2 they form two parallel loops $C_1 = e_1 \cup e_2$ and $C_2 = e_3 \cup e_4$ (consider vertices as points). They cut F_2 into two annuli.

Let $\widehat{e}_1, \dots, \widehat{e}_5$ be the edges of $\widehat{\Gamma}_1$ at u_i. By Lemma 1.4 each $\widehat{e}_i$ has weight at most 4. Moreover, if it has weight 4 then the corresponding edges in Γ_1 contains two Scharlemann cycles, and if it has weight 3 then it gives rise to one Scharlemann cycle. So there are at least $16 - 2 \times 5 = 6$ Scharlemann cycles among the edges at u_i. At least one of them has label sets $\{2, 3\}$ or $\{1, 4\}$, otherwise we would get a contradiction as in the proof of Lemma 4.3. Assume that $\{2, 3\}$ is the label set of a Scharlemann cycle consisting of edges e_5, e_6. The proof of the other case is similar. Let $C_3 = e_5 \cup e_6$. We separate two cases.

CASE 1. *Both e_5 and e_6 are in the same annulus of $F_2 - C_1 \cup C_2$.*

Then the picture looks like that in Figure 4.3(a), where the underlying cylinder represents the torus F_2 cut open along a simple loop. Notice that the two loops C_1 and C_3 are isotopic. But by Lemma 1.9 in this case there would be some vertices in the disk on F_2 bounded by $C_1 \cup C_3$, which is impossible.

CASE 2. *The edges e_5, e_6 are in different annuli of $F_2 - C_1 \cup C_2$.*

Then C_1, C_2, C_3 cut the torus F_2 into two rectangles. Consider the four i-edges at v_1. No two of them can be parallel, otherwise Γ_2 would have more than n_1 parallel edges. Hence there is only one way to arrange these four edges: Two of them belong to the cycle C_1 connecting v_1 to v_2, and the other two connect v_1 to v_4, one in each of the rectangle components of $F_2 - C_1 \cup C_2 \cup C_3$. The graph $\widehat{\Gamma}_2^-$ consisting of negative edges of $\widehat{\Gamma}_2$ now looks like that in Figure 4.3(b). Since an i-edge in Γ_2 is negative, it belongs to one of the edges in $\widehat{\Gamma}_2^-$.

On the boundary of each fat vertex v_t there are four points labeled i. We denote them by $P = \{N, E, S, W\}$, according to whether the point is the end of an edge going north, east, south or west. Let K_t be the part of the attached solid torus J_2 between v_t and v_{t+1}. Then $\partial u_i \cap K_t$ consists of four edges connecting the set P of v_t to that of v_{t+1}, so it determines a map $\varphi_t : P \to P$.

Consider the Scharlemann cycle $\{e_1, e_2\}$. Let D be the Scharlemann disk between then. Then $\partial D = e_1 \cup \alpha \cup e_2 \cup \beta$, where α and β are arcs on ∂K_1. On F_2 the two edges e_1, e_2 connect v_1 to v_2, so in Figure 2(b) they are vertical lines, hence each of their ends is either N or S. The arcs α, β connect $\{N, S\}$ to $\{N, S\}$. By the definition we see that φ_1 preserves the set $\{N, S\}$. Similarly, φ_3 maps each of N

and S to either N or S.

Consider the disk on F_1 between e_2 and e_3. Its boundary consists of 4 arcs: e_2, e_3, and the two arcs α and β on the boundary of the fat vertices at the ends of e_2, e_3. Again, the edges e_2 and e_3 are vertical edges in Figure 4.3(b). Therefore, each end of α is either N or S, so the map φ_2 preserves the set $\{N, S\}$.

Now consider the part of the graph Γ_1 near u_i. There are two subcases.

(1) *None of the 4 arcs on ∂u_i with ends 1 and 4 are between two parallel arcs.*

One can check that in this case the reduced graph $\widehat{\Gamma}_1^+$ has three successive edges of weight 4, which contradicts Lemma 4.3.

(2) *There are two adjacent parallel edges e_7, e_8 whose ends on ∂u_i are 4 and 1, respectively.*

Since e_7 is a positive edge in Γ_1, the other end of e_7 has label either 1 or 3. If it has label 1, then e_7, e_8 is a Scharlemann cycle with label set $\{1, 4\}$. As above, one can see that in this case φ_4 maps each of E and W to either E or W. If the other end of e_7 has label 3, then e_7, e_8 are vertical edges in Figure 4.3(b). Like φ_2, in this case the map φ_4 sends N to either N or S.

In any case, we see that each of the maps φ_i preserves the set $\{N, S\}$, so the map $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ also sends N to either N or S, hence has order 1 or 2. But this is impossible because $\Delta = 4$ implies that the gluing map φ must have order 4. □

Proof of Theorem 1. When $n_2 = 1$ or 2, the theorem has been proved by Boyer and Zhang [BZ]. They have also shown that $\Delta \leq 4$.

When $n_4 = 4$ and F_2 is a separating surface, the theorem follows from the contradiction in the conclusions of Lemma 4.5 and lemma 4.6.

If $n_4 \neq 4$ or F_2 is non separating, by Lemma 2.4 each vertex of $\widehat{\Gamma}_2$ has valency at least 6. Since $\widehat{\Gamma}_2$ is a reduced graph, each of its faces has at least 3 edges, so $F \leq 2E/3$. By calculating the Euler number, we see that each vertex of $\widehat{\Gamma}_2$ has valency exactly 6.

If $\widehat{\Gamma}_1^+$ has no interior vertices, the result follows from Lemma 3.4. If $\widehat{\Gamma}_1^+$ has some interior vertex u_i, then by Lemma 3.1 each number t between 1 and n_2 is the label of some Scharlemann cycle in Γ_1. Thus F_2 is separating, and n_2 is even, so $n_2 \geq 6$. Without loss of generality we may assume that $\{1, 2\}$ is the label set of a Scharlemann cycle in Γ_1. Since 4 is the label of a Scharlemann cycle, either $\{3, 4\}$ or $\{4, 5\}$ is the label set of a Scharlemann cycle. By Lemma 1.7 it can not be $\{4, 5\}$. So it must be $\{3, 4\}$. Similarly we can show that $\{5, 6\}$ is the label set of some Scharlemann cycle. But then we would have 3 Scharlemann cycles with disjoint label sets, contradicting Lemma 1.10. □

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