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RELATIVE EULER NUMBER AND FINITE COVERS OF GRAPH MANIFOLDS

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1. Introduction

In this paper we show that every non-trivial graph manifold M has a finite cover that contains a foliation which is very close to a fibration over the circle — the foliation restricted to each vertex manifold of M is a fibration over the circle. On the other hand, we show that this foliation can not always be made to be a fibration over the circle, by exhibiting a large class of graph manifolds that have no finite cover that fibers over the circle. These computations are done using a "relative Euler number" for framed Seifert fibered spaces that are assigned to the vertex manifolds of M. This relative Euler number behaves naturally under finite covers. Using the relative Euler number we define a torsion for each vertex manifold of M, and show that if the torsions are all positive then M is not a surface bundle over S^1. Using the naturality of Euler number with respect to covers, we show that under the same conditions in fact no finite cover of M will fiber over the circle. This situation is analogous to that of Seifert fibered spaces. It follows from [EHN] that every Seifert fibered space with base orbifold of negative characteristic is covered by a manifold admitting a horizontal foliation. On the other hand a Seifert fibered space is covered by a surface bundle over S^1 if and only if its Euler number or its orbifold characteristic is zero.

We also use this relative Euler number to define a covering invariant $\sigma(M)$ (in the sense of [WW] and [K]). In particular it follows that if $\sigma(M)$ is non-zero then M will have the property that homeomorphic covers of M will have the same covering degree. It seems rare that $\sigma(M)$ is zero. For example, we show that if M is a knot complement in the 3-sphere then $\sigma(M)$ is non-zero.

We refer the reader to [J,H,Sc] for definitions and basic facts about Seifert fibered spaces, and to [Sc] for the theory of 2-dimensional orbifolds, especially their Euler characteristic. If N is a Seifert fibered space, we use $O(N)$ to denote its orbifold, and use $\chi(O(N))$ to denote the Euler characteristic of its orbifold, called the orbifold characteristic of N.

After Waldhausen [W, page 87], an orientable, connected, compact 3-manifold is a *graph manifold* if and only if there is a collection of tori $\mathcal{T} = T_1 \cup \ldots \cup T_k$ in the interior of M such that the each component of $M - \text{Int}N(\mathcal{T})$ is a circle

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bundle. Here $N(\mathcal{T})$ denotes a regular neighborhood of $\mathcal{T}$. Waldhausen then defines a *simple* graph manifold and shows that any graph manifold is a connected sum of simple graph manifolds, lens spaces, and $S^1 \times S^2$'s. A simple graph manifold is characterized by the fact that there is a collection of tori $\mathcal{T}$ such that each component of $M - \text{Int}N(\mathcal{T})$ is a Seifert fibered space with incompressible boundary. When k is minimal, $\mathcal{T}$ is called a *canonical splitting surface*, and each component of $M - \text{Int}N(\mathcal{T})$ is called a *vertex manifold*. We call a simple graph manifold M *trivial* if either $\mathcal{T}$ is empty or it consists of a single torus cutting M into two twisted I-bundles over a Klein bottle. In the first case M is a Seifert fibered space; in the second case M is covered by a torus bundle over S^1, so it admits either Nil or Sol geometry [Sc].

Suppose M is a nontrivial graph manifold. It is easy to show that M has a double cover $\widetilde{M}$ which has the property that each vertex manifold of $\widetilde{M}$ has negative orbifold characteristic. Since we only discuss properties concerning finite covers, we make the following

Convention. All graph manifolds are assumed simple and nontrivial, and have the property that all of their vertex manifolds have negative orbifold characteristic. We also assume $\mathcal{T}$ is a canonical splitting surface.

Given a graph manifold M, define an associated graph $\Gamma = \Gamma(M)$ as follows. Each vertex manifold M_v of M determines a vertex v of Γ, and each component T_e of $\mathcal{T}$ determines an edge e of Γ, so that if $N(T_e)$ has one boundary component in each of M_v and $M_{v'}$ then e has endpoints on v and v'. Denote by M_V the union of the vertex manifolds. The product structure of $N(\mathcal{T})$ determines an involution ϕ on ∂M_V, and M is homeomorphic to M_V/ϕ.

The paper is organized as follows. In section 2 we define and explore the relative Euler number of a Seifert fibered space. In section 3 we define the torsion of a vertex manifold and use it to give examples of graph manifolds that have no finite cover that fibers over the circle. In section 4, we construct covers of graph manifolds that are needed in sections 3 and 5. In section 5 we show that a graph manifold has a finite cover containing a horizontal foliation. Finally in section 6 we define the covering invariant, $\sigma(M)$, and apply it to the question of when homeomorphic covers of M must have the same covering degree.

2. Relative Euler number of a framed Seifert fibered space

In this section we will define a relative Euler number for Seifert fibered spaces with a specified set of curves, called a framing, on their boundary. Proposition 2.1 gives a relation between relative Euler numbers corresponding to different framings. Proposition 2.3 shows that the relative Euler number has a natural property under finite covers, which is similar to the naturality of the Euler number of a closed Seifert fibered space.

If α, β are oriented curves on an oriented surface T, we use $\alpha \cdot \beta$ to denote the algebraic intersection number of α with β. Note that $\alpha \cdot \beta = -\beta \cdot \alpha$.

We recall the definition of the Euler number of a Seifert fibered space.

Suppose M is a closed Seifert fibered space. Let $V_1, \ldots, V_n$ be a nonempty set of disjoint fibered solid tori in M such that $X = M - \bigcup \text{Int}V_i$ has no singular fibers. Then there is a surface F in X intersecting each fiber exactly once. Let T_i be the torus ∂V_i, let m_i be the curve $F \cap T_i$, and l_i a fiber of M on T_i. Orient m_i, l_i so that

$l_i \cdot m_i = 1$ with respect to the orientation of T_i induced from that of X. Now let α_i be a meridian curve of V_i. Then α_i represents some $p_i m_i + q_i l_i$ in $H_1(T_i)$. The *Euler number* of M is defined by $e(M) = \sum q_i/p_i$. This definition is equivalent to the standard one [NR,Sc]. It turns out that $e(M)$ is independent of the choice of V_i and F, so it is an invariant of the Seifert fibration of M.

A covering $p : \widetilde{M} \to M$ of Seifert fibered spaces is called a (u, v) covering if each regular fiber of M is covered by u regular fibers of $\widetilde{M}$, and the restriction of p on each regular fiber has degree v. Thus the covering degree of $p : \widetilde{M} \to M$ is uv. One of the important properties of $e(M)$ is the naturality: If $p : \widetilde{M} \to M$ is a (u, v) covering, then $e(\widetilde{M}) = e(M)u/v$.

Now consider a Seifert fibered space M with nonempty boundary $\partial M = T_1 \cup \ldots \cup T_n$. A *framing* of M is a set of curves $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$, such that $\alpha_i = \alpha \cap T_i$ is not isotopic to a fiber of M. Denote by $M(\alpha)$ the manifold obtained from M by Dehn filling along α, that is, $M(\alpha)$ is obtained from M by gluing a solid torus V_i to each T_i so that α_i bounds a disk in V_i. The Seifert fibration of M has a unique extension over $M(\alpha)$.

Definition. The *relative Euler number* of the framed Seifert fibered space (M, α) is defined to be $e(M, \alpha) = e(M(\alpha))$.

Suppose a, b, c are curves on an oriented torus T so that neither of a, b is isotopic to c. Write

$$\delta(a, b, c) = \frac{a \cdot b}{(a \cdot c)(b \cdot c)}.$$

Note that we need to choose an orientation for the curves to define $\delta(a, b, c)$, but since each curve appears twice in the formula, $\delta(a, b, c)$ is independent of such choice.

If α and β are framings of a Seifert fibered space M, we define

$$\Delta(\alpha, \beta) = \sum_i \delta(\alpha_i, \beta_i, l),$$

where α_i and β_i are the curves of α and β on the torus T_i, and l is a fiber of M.

Proposition 2.1. *If α and β are framings of a Seifert fibered space M, then $e(M, \alpha) = e(M, \beta) + \Delta(\alpha, \beta)$.*

Proof. Suppose $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$. Let $V_{n+1}, \ldots, V_m$ be fibered solid tori in M such that $X = M - \bigcup \mathrm{Int} V_j$ has no singular fibers. For $j = n+1, \ldots, m$, let α_j be a curve on $T_j = \partial V_j$ which bounds a meridian disk in V_j. Write $\alpha' = \alpha \cup \alpha_{n+1} \cup \ldots \cup \alpha_m$, and $\beta' = \beta \cup \alpha_{n+1} \cup \ldots \cup \alpha_m$. By definition we have $e(M, \alpha) = e(M(\alpha)) = e(X(\alpha'))$. Similarly $e(M, \beta) = e(X(\beta'))$.

Let F be a surface in X intersecting each fiber of X once. Suppose $\alpha_i = p_i m_i + q_i l_i$, with l_i a fiber of X and m_i a component of ∂F. Choose orientations for m_i, l_i so that $l_i \cdot m_i = 1$ with respect to the orientation of T_i induced from that of X. Suppose $\beta_i = r_i m_i + s_i l_i$. Since the framings of α' and β' are the same on ∂V_j, we have

$$e(X(\alpha')) - e(X(\beta')) = \sum_{i=1}^{n} \frac{q_i}{p_i} - \sum_{i=1}^{n} \frac{s_i}{r_i} = \sum_{i=1}^{n} \frac{q_i r_i - p_i s_i}{p_i r_i}.$$

Since $l_i \cdot m_i = - m_i \cdot l_i = 1$, we have

$$\alpha_i \cdot l_i = (p_i m_i + q_i l_i) \cdot l_i = -p_i$$
$$\beta_i \cdot l_i = (r_i m_i + s_i l_i) \cdot l_i = -r_i$$
$$\alpha_i \cdot \beta_i = (p_i m_i + q_i l_i)(r_i m_i + s_i l_i) = q_i r_i - p_i s_i$$

Therefore,

$$e(M, \alpha) = e(M, \beta) + \sum_i \frac{\alpha_i \cdot \beta_i}{(\alpha_i \cdot l_i)(\beta_i \cdot l_i)} = e(M, \beta) + \Delta(\alpha, \beta). \quad \square$$

A surface F properly embedded in M is called a *transverse surface* if it is transverse to the fibers of M. A framing α is called a *boundary framing* (with respect to F) if there is a transverse surface F so that on each torus T_i of ∂M the curve α_i is a component of $F \cap T_i$. It is well known that a closed Seifert fibered space N has $e(N) = 0$ if and only if there is a closed transverse surface. It follows that β is a boundary framing if and only if $e(M, \beta) = 0$. This fact and Proposition 2.1 can be used to compute $e(M, \alpha)$ without deleting the singular fibers:

Corollary 2.2. *Suppose β is a boundary framing. Then $e(M, \alpha) = \Delta(\alpha, \beta)$.* $\square$

Remark 1. There is another definition of $e(M)$ for closed Seifert fibered space which makes it clearer that $e(M)$ is the obstruction to a cross section of the Seifert fibration: Let V be a regular neighborhood of a regular fiber. Let (α, l) be the meridian–fiber pair of $T = \partial V$. Choose a proper surface in $X = M - \text{Int}V$ transverse to the fibers and let β be the corresponding boundary framing. If $\beta = p\alpha + ql$, then $e(M) = e(X, \alpha) = e(X, \beta) + \Delta(\alpha, \beta) = -q/p$. In other words, M has Euler number $-q/p$ if the boundary framing of X is a (p, q) curve with respect to the meridian–fiber pair of T. From the point of view of Dehn surgery, M has Euler number $-q/p$ if and only if a p/q surgery (with respect to the meridian-fiber coordinate) on a regular fiber produces a manifold with zero Euler number.

Remark 2. Proposition 2.1 and Corollary 2.2 can be thought of as describing a Dehn surgery formula for Euler number. For example, to compute the Euler number for the (r, s) surgery on a (p, q) torus knot, one notices that the longitude l is a boundary framing, and the fiber α of the Seifert fibration of the knot complement is a $(pq, 1)$ curve with respect to the meridian-longitude pair, so one can use Corollary 2.2 to show that the Euler number of the Seifert fibered space after (r, s) surgery is $r/pq(r - pqs)$ up to convention of orientations.

A framed Seifert fibered space $(\widetilde{M}, \widetilde{\alpha})$ is called a cover of (M, α) if there is a covering $\varphi : \widetilde{M} \to M$ so that each $\widetilde{\alpha}_i$ is a component of $\varphi^{-1}(\alpha)$. It is a (u, v) cover if $\varphi : \widetilde{M} \to M$ is such. If φ can be extended to a cover of $\widetilde{M}(\widetilde{\alpha})$ over $M(\alpha)$, then by the naturality of Euler number we have $e(\widetilde{M}, \widetilde{\alpha}) = e(M, \alpha)u/v$. In general there is no such extension of φ, but the naturality still holds.

Proposition 2.3. (Naturality of relative Euler number.) *Suppose $(\widetilde{M}, \widetilde{\alpha})$ is a (u, v) cover of (M, α). Then $e(\widetilde{M}, \widetilde{\alpha}) = e(M, \alpha)u/v$.*

Proof. Let β be a boundary framing of M with respect to a transverse surface F, let $\widetilde{\beta}$ be the boundary framing in $\widetilde{M}$ corresponding to the lifting $\widetilde{F}$ of F. Suppose

$\widetilde{T}_j$ is a component of $\partial\widetilde{M}$ covering a torus T_i on ∂M. Let u_j be the number of fibers on $\widetilde{T}_j$ that cover a fiber of T_i. Let $\varphi_*[\widetilde{\alpha}] = n_\alpha[\alpha]$, $\varphi_*[\widetilde{\beta}] = n_\beta[\beta]$. Then by counting the total number of intersections of one component of the preimage of β with all preimages of l we have $(\widetilde{l}\cdot\widetilde{\alpha})u_j = (l\cdot\alpha)n_\alpha$. Similarly, $(\widetilde{l}\cdot\widetilde{\beta})u_j = (l\cdot\beta)n_\beta$. Since the preimage of α has $u_j v/n_\alpha$ components, for the same reason we get $(\widetilde{\alpha}\cdot\widetilde{\beta})(u_j v/n_\alpha) = (\alpha\cdot\beta)n_\beta$. Therefore

$$\delta(\widetilde{\alpha_j},\widetilde{\beta}_j,\widetilde{l}_j) = \frac{(\widetilde{\alpha}\cdot\widetilde{\beta})}{(\widetilde{l}\cdot\widetilde{\beta})(\widetilde{l}\cdot\widetilde{\alpha})} = \frac{(\alpha\cdot\beta)(n_\alpha n_\beta/u_j v)}{(l\cdot\beta)(l\cdot\alpha)(n_\beta n_\alpha/u_j^2)}$$
$$= \frac{(\alpha\cdot\beta)}{(l\cdot\beta)(l\cdot\alpha)}\frac{u_j}{v} = \delta(\alpha_i,\beta_i,l_j)\frac{u_j}{v}.$$

Summing over all $\widetilde{T}_j$ that cover T_i, we have $\sum u_j = u$. Hence

$$\sum\{\delta(\widetilde{\alpha}_j,\widetilde{\beta}_j,\widetilde{l}_j) \mid \widetilde{T}_j \text{ covers } T_i\} = \delta(\alpha_i,\beta_i,l_i)u/v.$$

Summing both sides of the above formula over i, we get $\Delta(\widetilde{\alpha},\widetilde{\beta}) = \Delta(\alpha,\beta)u/v$. The result now follows from Corollary 2.2. □

We apply the relative Euler number of framed Seifert fibered spaces to graph manifolds via their vertex manifolds. Let M be a graph manifold and M_i a vertex manifold of M. If T_λ is a torus on ∂M_i, let α^i_λ be a curve on T_λ such that $\phi(\alpha^i_\lambda)$ is a fiber of M_V, where $\phi : \partial M_V \to \partial M_V$ is the involution defined in section 1. In other words, if T_λ is a torus between M_i and M_j, then α^i_λ is a fiber of M_j on T_λ. Since the canonical splitting surface is minimal, none of the α^i_λ is a fiber of M_i. When M_i does not contain boundary components of M, $\alpha^i = \bigcup \alpha^i_\lambda$ is a framing on ∂M_i, called the *natural framing* of M_i. Define the *relative Euler number* of M_i as $E(M_i) = e(M_i,\alpha^i)$. When M_i contains some components of ∂M, $M_i(\alpha^i)$ has zero Euler number, so it is natural to define $E(M_i) = 0$.

3. An obstruction to fibration

The Euler number of Seifert fibered spaces is the obstruction to the space admitting a horizontal surface, hence usually the obstruction to its fibering over the circle. In fact a Seifert fiber space has a finite cover that fibers over the circle if and only if either its Euler number or its orbifold characteristic is zero [G]. The relative Euler number can be used to define a torsion for each vertex manifold of a graph manifold. Proposition 3.1 shows that if all vertex manifolds of M have positive torsion, then M is not a surface bundle over S^1. The main result is Theorem 3.2, which says that under the same assumption, M is not even covered by any surface bundle over S^1. Some examples will be given at the end of the section.

Definition. Given a framed Seifert fibered space (M,α) with $\partial M = T_1 \cup \ldots \cup T_m$ and l_j a fiber of M on T_j, define the *torsion* of (M,α) to be

$$t(M,\alpha) = |e(M,\alpha)| - \sum_{j=1}^{m} \frac{1}{|\alpha_j\cdot l_j|}.$$

If M is a graph manifold, and M_i is a vertex manifold of M, let α^i be the natural framing on ∂M_i defined in section 2. Define the torsion of M_i to be the torsion with respect to the natural framing, i.e. $t(M_i) = t(M_i,\alpha^i)$.

Proposition 3.1. *If $t(M_i) > 0$ for each vertex manifold M_i of M, then M is not a surface bundle over S^1.*

Proof. Assuming the contrary, write $M = F \times I/h$, where $h : F \times 0 \to F \times 1$ is the monodromy map. By Thurston's classification theorem of surface homeomorphisms [T2] we may choose h and a set of curves $\mathcal{C} = C_1 \cup \ldots \cup C_k$ with $h(\mathcal{C}) = \mathcal{C}$, so that h^n restricted to each component of $F_V = F - \mathrm{Int}N(\mathcal{C})$ is either periodic or pseudo-Anosov, and maps each component of F_V to itself. Since M is a graph manifold, no component of $F_V \times I/h$ is hyperbolic, so $h^n|_{F_V}$ has no pseudo-Anosov component [T1]. Hence $h^n|_{F_V}$ is periodic, and the I bundle structure of $F_V \times I$ gives rise to a Seifert fibration of $F_V \times I/h$. When $\mathcal{C}$ is minimal, $\mathcal{C} \times I/h$ is the canonical splitting surface of M, so $M_V = F_V \times I/h$. In particular, F is transverse to all fibers of M_V.

Let M_i be a component of M. Let k_i be the number of times a regular fiber of M_i intersects F. Choose i so that $k_i \geq k_j$ for all j. For each T_λ on ∂M_i, let β_λ be a component of $T_\lambda \cap F$. The union of these β_λ forms a framing β of M_i. Since β_λ are boundaries of $F \cap M_i$, β is a boundary framing, so $e(M_i, \beta) = 0$. By Proposition 2.1,

$$e(M_i, \alpha^i) = e(M_i, \beta) + \Delta(\alpha^i, \beta) = \sum_{T_\lambda \subset \partial M_i} \delta(\alpha^i_\lambda, \beta_\lambda, l_\lambda) = \sum_{T_\lambda \subset \partial M_i} \frac{\alpha^i_\lambda \cdot \beta_\lambda}{(\alpha^i_\lambda \cdot l_\lambda)(\beta_\lambda \cdot l_\lambda)},$$

where l_λ is a fiber of M_i on T_λ, and α^i is the natural framing on ∂M_i. The sum is taken over all λ such that T_λ is on ∂M_i. Let $r_\lambda = |F \cap T_\lambda|$. Then $k_i = |F \cap l_\lambda| = |F \cap T_\lambda|\, |\beta_\lambda \cap l_\lambda| = r_\lambda\, |\beta_\lambda \cdot l_\lambda|$. Suppose the involution $\phi : \partial M_V \to \partial M_V$ maps T_λ to a torus on ∂M_j. Then similarly we have $k_j = r_\lambda\, |\alpha^i_\lambda \cdot \beta_\lambda|$. Hence,

$$|e(M_i, \alpha^i)| \leq \sum_\lambda \left| \frac{\alpha^i_\lambda \cdot \beta_\lambda}{(\alpha^i_\lambda \cdot l_\lambda)(\beta_\lambda \cdot l_\lambda)} \right| = \sum_\lambda \frac{k_j}{|\alpha^i_\lambda \cdot l_\lambda| k_i} \leq \sum_\lambda \frac{1}{|\alpha^i_\lambda \cdot l_\lambda|}.$$

This contradicts the assumption that $t(M_i, \alpha^i) > 0$. □

Definition. An s^2-fold covering $f : \widetilde{T} \to T$ of tori is called a *characteristic* covering if there is a basis $(\widetilde{a}, \widetilde{b})$ for $\pi_1(\widetilde{T})$, and a basis (a, b) for $\pi_1(T)$, such that $f_*(\widetilde{a}) = sa$ and $f_*(\widetilde{b}) = sb$.

A covering $p : \widetilde{M} \to M$ of Seifert fibered spaces is called *characteristic* or *s-characteristic* if the restriction of p to each boundary component is an s^2-fold characteristic covering. If $\widetilde{M}$ and M are graph manifolds, then p is a *characteristic* covering if its restriction to each piece of $\widetilde{M}$ is characteristic.

In the next section we will prove the following result.

Proposition 4.4. *Let $p : \widetilde{M} \to M$ be a covering of graph manifolds. Then there is a covering $\widetilde{p} : \widehat{M} \to \widetilde{M}$ so that $p \circ \widetilde{p} : \widehat{M} \to M$ is a characteristic covering.*

Theorem 3.2. *If $t(M_i) > 0$ for each vertex manifold M_i of M, then M is not covered by a surface bundle over S^1.*

Proof. If $p : \widetilde{M} \to M$ is a cover such that $\widetilde{M}$ is a surface bundle over S^1, by the above proposition we can find a further cover $\widetilde{p} : \widehat{M} \to \widetilde{M}$ so that $\widehat{p} = p \circ \widetilde{p} : \widehat{M} \to M$ is an s-characteristic cover for some s. Suppose $\widehat{M}_j$ is a vertex manifold that covers a vertex manifold M_i of M, and suppose a component $T_\lambda \subset \partial M_i$ is covered by k components of $\partial \widehat{M}_j$. Then $\widehat{p}$ is a (ks, s) cover, so by Proposition 2.3, $E(\widehat{M}_j) = E(M_i)k$. Also, since $\widehat{p}$ is a characteristic cover, if $\widehat{T}_\mu \subset \partial \widehat{M}_j$ covers T_λ then $\widehat{\alpha}_\mu \cdot \widehat{l}_\mu = \alpha_\lambda \cdot l_\lambda$, where $\widehat{\alpha}_\mu$ is the natural framing of $\widehat{M}_j$, and $\widehat{l}_\mu$ a fiber. Since each torus in ∂M_i is covered by k tori in $\partial \widehat{M}_j$, we have

$$\sum_{\widehat{T}_\mu \subset \partial \widehat{M}_j} \frac{1}{|\widehat{\alpha}_\mu \cdot \widehat{l}_\mu|} = k \sum_{T_\lambda \subset \partial M_i} \frac{1}{|\alpha_\lambda \cdot l_\lambda|}.$$

Therefore $t(\widehat{M}_j) = t(M_i)k$. It follows that if all vertex manifolds of M have positive torsions, then the vertex manifolds of $\widehat{M}$ have positive torsions also. Since $\widehat{M}$ is a cover of the surface bundle $\widetilde{M}$, it is also a surface bundle, which contradicts Proposition 3.1. □

Examples. (1). Let M_1, M_2 be the product $F \times S^1$, where F is an orientable surface with a single boundary component. On each component of ∂M_i, define $m_i = \partial F \times 1$, and $l_i = \{\text{point}\} \times S^1$. Define a map $\phi : \partial M_1 \to \partial M_2$ so that

$$\phi(l_1, m_1) = (l_2, m_2) \begin{pmatrix} a & ad+1 \\ 1 & d \end{pmatrix}$$

where a, d are arbitrary integers. Since the determinant of the matrix is -1, such ϕ do exist. Let $M = M_1 \cup_\phi M_2$. Using Corollary 2.2 one computes that $E(M_1) = -d$, $E(M_2) = a$, and $|l_1 \cdot l_2| = 1$. Therefore, if both $|a|$ and $|d|$ are greater than 1, then $t(M_i) > 0$, so M is not covered by surface bundles.

(2). Suppose M is a closed graph manifold. From the remarks after Corollary 2.2, we see that if we do p/q surgery (with respect to the meridian-fiber coordinate) on a regular fiber of M_i to get M_i', then $E(M_i') = E(M_i) - q/p$. This operation certainly does not change the second term in the definition of the torsion $t(M_i)$. Therefore, if q is sufficiently large, then $t(M_i') > 0$. It follows that for any closed graph manifold M, we can always do surgery on fibers of vertex manifolds to obtain a new graph manifold M' which is not covered by surface bundles.

(3). Let $\mathcal{D}$ be the plumbing of $\{\mathcal{D}_i\}$, where each $\mathcal{D}_i$ is a 2-disk bundle over an orientable closed surface other than 2-sphere [NR]. Let Γ be the plumbing diagram. Then the boundary of $\mathcal{D}$ is a graph manifold M. Γ is also the underlying graph of M. For each vertex v_i of Γ, the relative Euler number of M_i is the Euler number of the corresponding disk bundle $\mathcal{D}_i$. Furthermore, the absolute value of the intersection number of circle fibers at each edge of Γ is 1. Therefore, the torsion of M_i is the absolute value of the Euler number of $\mathcal{D}_i$ minus the valence of v_i.

The set of torsions of the vertex manifolds of M gives an obstruction for M to be covered by surface bundles. But the obstruction disappears when M has boundary. A vertex manifold M_i of M that contains components of ∂M has $E(M_i) = 0$, so $t(M_i)$ is negative.

Question. *Suppose a graph manifold M has nonempty boundary. Is M always covered by some surface bundle over S^1?*

4. Characteristic covering

In this section we will prove Proposition 4.4, which says that whenever we have a cover of graph manifold, there is a further cover which is characteristic. The result is needed in sections 3 and 5.

Lemma 4.1. *Let S be surface with negative Euler characteristic. To each component d of ∂S assume there is associated an integer $n_d > 1$. Then there is a covering $p : \widetilde{S} \to S$ so that if d is a component of ∂S and $\widetilde{d}$ is a component of $p^{-1}(d)$ then $p : \widetilde{d} \to d$ is the n_d-fold cover.*

Proof. One can prove this by direct construction, but we provide the following proof using known results about 2-dimensional orbifolds. One is referred to [Sc] for definitions and theorems regarding orbifolds. Recall that an orbifold is called a good orbifold if it is neither a 2-sphere with a single cone point nor a 2-sphere with 2 cone points of different cone angle.

Gluing to each boundary component d of ∂S a cone C_d of angle $2\pi/n_d$, we get an orbifold $\widehat{S}$. Since S has negative Euler characteristic, it either has nonzero genus, or is planar with at least 3 boundary components, so $\widehat{S}$ is a good orbifold. Thus there is an orbifold covering $p : F \to \widehat{S}$, where F has no cone point, i.e. it is a surface [Sc, Theorem 2.5]. The preimage of each cone C_d is a set of disjoint disks, so the restriction of p to each component of $p^{-1}(d)$ is an n_d fold cover. Define $\widetilde{S} = p^{-1}(S)$. Then the restriction of p to $\widetilde{S}$ is the required cover. $\square$

Lemma 4.2. *Suppose N is a Seifert fibered space with non-empty boundary and negative orbifold characteristic. Then there is a characteristic covering $N' \to N$ so that N' is a product.*

Proof. For each component T_i of ∂N, choose a curve α_i which intersects each fiber of ∂N once. Then $\alpha = \cup \alpha_i$ is a framing on N. As before, let $e(N, \alpha)$ be the relative Euler number of (N, α). It is a rational number.

Let S be the orbifold of N. Glue a cone of angle $2\pi/k$ to each component of ∂S, giving an orbifold $\overline{S}$. Since $\chi(S) < 0$, we can choose k big enough so that $\chi(\overline{S}) \leq 0$. (Actually $k = 6$ is enough.) Let $q : \widehat{S} \to \overline{S}$ be a covering such that $\widehat{S}$ is an orientable surface without cone points. Since $\chi(\widehat{S}) \leq 0$, $\widehat{S}$ has positive genus. Thus by going to a further cover if necessary, we may assume that $e(N, \alpha) \deg(q)$ is an integral multiple of k.

Let $\widetilde{S} = q^{-1}(S) \subset \widehat{S}$. Then $q : \widetilde{S} \to S$ is a covering of orbifolds. Note that q restricted to any boundary component of $\widetilde{S}$ has degree k. By pulling back the fibration $N \to S$, we get a fibration $\widetilde{N} \to \widetilde{S}$, and a map $p : \widetilde{N} \to N$ that induces $q : \widetilde{S} \to S$, (see [Sc, Page 431].) The orbifold covering degree of p is $\deg(q)$, and the fiber covering degree is 1. By Proposition 2.3, $e(\widetilde{N}, \widetilde{\alpha}) = e(N, \alpha) \deg(q)$, so it is a multiple of k, where $\widetilde{\alpha} \subset p^{-1}(\alpha)$ is the pull-back framing. Since α intersects each fiber of ∂N once and since the fiber covering degree is 1, $\widetilde{\alpha}$ also intersects each fiber of $\partial \widetilde{N}$ once. The orbifold map q is a degree k map on each boundary of $\widetilde{S}$, so $\deg(p|_{\widetilde{\alpha}_j}) = k$ for all $\widetilde{\alpha}_j \subset \widetilde{\alpha}$.

Since $\widetilde{S}$ is an orientable surface with boundary, $\widetilde{N}$ can be written as a product $\widetilde{N} = \widetilde{S} \times S^1$. For each torus $\widetilde{T}_j \subset \partial \widetilde{N}$, let $\widetilde{m}_j$ be the component of $\widetilde{S} \times 1$ on $\widetilde{T}_j$, let $\widetilde{l}_j$ be a fiber of $\widetilde{N}$, and let $\widetilde{\alpha}_j$ be the framing curve $\widetilde{\alpha} \cap \widetilde{T}_j$. Since $\widetilde{\alpha}_j$ intersects

$\widetilde{l}_j$ once, $\widetilde{\alpha}_j$ can be written as $\widetilde{m}_j + k_j\widetilde{l}_j$ for some integer k_j. Notice that $\cup\widetilde{m}_j$ is a boundary framing, so by Corollary 2.2 we have

$$e(\widetilde{N}, \widetilde{\alpha}) = \Delta(\widetilde{\alpha}, \cup\widetilde{m}_j) = \sum \frac{\widetilde{\alpha}_j \cdot \widetilde{m}_j}{(\widetilde{\alpha}_j \cdot \widetilde{l}_j)(\widetilde{m}_j \cdot \widetilde{l}_j)} = \sum k_j.$$

Suppose T_i is the torus in ∂N covered by $\widetilde{T}_j$. Then $p : \widetilde{T}_j \to T_i$ is given by $p(\widetilde{\alpha}_j) = k\alpha_i$, and $p(\widetilde{l}_j) = l_i$. We need to find a covering $p' : N' \to \widetilde{N}$, so that if T'_r covers $\widetilde{T}_j$ and if α'_r, l'_r are components of the preimage of $\widetilde{\alpha}_j$ and $\widetilde{l}_j$ respectively, then $p'(\alpha'_r) = \widetilde{\alpha}_j$, and $p'(l'_r) = k\widetilde{l}_j$. It will then follow that $p \circ p' : N' \to N$ is a k-characteristic covering. Since $\widetilde{N}$ is a product and N' is a cover of $\widetilde{N}$, N' will also be a product.

To find such a covering p', it suffices to find a map $f_* : \pi_1(\widetilde{N}) \to \mathbb{Z}_k$ so that $f_*(\widetilde{\alpha}_j) = 0$, and $f_*(\widetilde{l}_j) = 1$. N' will then be the covering corresponding to the kernel of f_*.

Consider $\widetilde{N} = \widetilde{S} \times S^1$. We have

$$\pi_1(\widetilde{N}) = \langle a_1, b_1, \ldots, a_g, b_g, \widetilde{m}_1, \ldots, \widetilde{m}_r \mid \Pi[a_i, b_i]\Pi\widetilde{m}_i = 1\rangle \times \langle\widetilde{l}\rangle,$$

where a_i, b_i are the standard generators for the corresponding closed surface, $\widetilde{l}$ is represented by a fiber, and $\widetilde{m}_j$ by the boundaries of $\widetilde{S}$. Define $f_* : \pi_1(\widetilde{N}) \to \mathbb{Z}_k$ by

$$\begin{aligned} f_*(a_i) &= f_*(b_i) = 0 \\ f_*(\widetilde{l}) &= 1 \\ f_*(\widetilde{m}_j) &= -k_j. \end{aligned}$$

Since $\sum k_j = e(\widetilde{N}, \widetilde{\alpha})$ is a multiple of k, the map is well defined. Also, $f_*(\widetilde{\alpha}_j) = f_*(\widetilde{m}_j + k_j\widetilde{l}_j) = 0$. Hence this is the required map. □

Lemma 4.3. *Suppose $p : \widetilde{N} \to N$ is a covering of Seifert fibered spaces with nonempty boundary and negative orbifold characteristic. Then there is an integer r such that for any multiple s of r there is a cover $\widetilde{p} : \widehat{N} \to \widetilde{N}$ so that the composition $p \circ \widetilde{p}$ is an s-characteristic cover. Furthermore, $\widehat{N}$ is a product.*

Proof. First assume $N = F \times S^1$ for some compact orientable surface F. Then by passing to a further cover we may assume that $\widetilde{N}$ is also a product $\widetilde{N} = \widetilde{F} \times S^1$, and p can be written as $p = p_1 \times p_2 : \widetilde{F} \times S^1 \to F \times S^1$. For each component d of $\partial\widetilde{F}$, let $m_d = \deg(p_1|_d)$. Let $m_t = \deg(p_2)$ be the circle fiber covering degree. Let $r' = \text{l.c.m.}(\{m_d| \; d \text{ is a component of } \partial\widetilde{F}\ \} \cup \{m_t\})$, and let $r = 2r'$.

Suppose s is a multiple of r. For every component d of $\partial\widetilde{F}$ define $n_d = s/m_d$, and $n_t = s/m_t$. By the choice of r, all n_d are even integers, so in particular $n_d > 1$. By Lemma 4.1, there is a cover $f : \widehat{F} \to \widetilde{F}$ so that if d is a component of $\partial\widetilde{F}$ and $\widehat{d}$ a component of $\partial\widehat{F}$ covering d then $f| : \widehat{d} \to d$ is the n_d-fold cover. Let $g : S^1 \to S^1$ be an n_t-fold cover. Then it is easy to see that $\widetilde{p} = f \times g : \widehat{F} \times S^1 \to \widetilde{F} \times S^1$ has the property that $p \circ \widetilde{p} = (p_1 \circ f) \times (p_2 \circ g)$ is an s-characteristic cover.

Now suppose N is not a product. By Lemma 4.2 we can find an s'-characteristic cover $p' : N' \to N$ so that N' is a product. Let N'' be a common finite cover of N'

and $\widetilde{N}$. Consider the cover $p'': N'' \to N'$. By the above, there is an integer r so that for any multiple s of r we have a cover $\widehat{p}: \widehat{N} \to N''$ so that the composition $p'' \circ \widehat{p}: \widehat{N} \to N'$ is an s-characteristic cover. Choose $\overline{r} = s'r$. Then for any multiple $\overline{s} = s's$ of $\overline{r}$, the composition $p' \circ p'' \circ \widehat{p}: \widehat{N} \to N$ is a $\overline{s}$-characteristic cover (because both p' and $p'' \circ \widehat{p}$ are characteristic), and it factors through $\widetilde{N}$, as required. Since N' is a product, N'' is also a product. □

Proposition 4.4. *Let $p: \widetilde{M} \to M$ be a covering of graph manifolds. Then there is a covering $\widetilde{p}: \widehat{M} \to \widetilde{M}$ so that $p \circ \widetilde{p}: \widehat{M} \to M$ is a characteristic covering. Furthermore each vertex manifold in $\widehat{M}$ is a product.*

Proof. For each vertex manifold $\widetilde{M}_{\bar{v}}$ of $\widetilde{M}$, there is a vertex manifold M_v of M so that the restriction of p is a covering of Seifert fibered spaces $p_{\bar{v}}: \widetilde{M}_{\bar{v}} \to M_v$. By Lemma 4.3 there is an integer $r_{\bar{v}}$ and a cover $\widetilde{p}_{\bar{v}}: \widehat{M}_{\bar{v}} \to \widetilde{M}_{\bar{v}}$ so that $p_{\bar{v}} \circ \widetilde{p}_{\bar{v}}: \widehat{M}_{\bar{v}} \to M_v$ is an s-characteristic cover, where s can be any multiple of $r_{\bar{v}}$. Fix an s which is a common multiple of all $r_{\bar{v}}$. Choose an integer $k_{\bar{v}}$ for each vertex $\bar{v}$ in the graph $\widetilde{\Gamma} = \Gamma(\widetilde{M})$ so that $D = k_{\bar{v}} \deg(\widetilde{p}_{\bar{v}})$ are the same for all $\bar{v}$. Let $N_{\bar{v}}$ be $k_{\bar{v}}$ copies of $\widehat{M}_{\bar{v}}$, and let $q_{\bar{v}} = \cup \widetilde{p}_{\bar{v}}: N_{\bar{v}} \to \widetilde{M}_{\bar{v}}$ be the corresponding covering map. Note that $\deg(q_{\bar{v}}) = D$, and the map $p \circ q_{\bar{v}}: N_{\bar{v}} \to M_v$ is an s-characteristic cover for all $\bar{v}$. We need to show that we can glue these $N_{\bar{v}}$ together to form a cover $\widetilde{p}: \widehat{M} \to \widetilde{M}$.

Suppose $\widetilde{M}_v$ and $\widetilde{M}_{v'}$ are vertex manifolds corresponding to the endpoints of an edge e in $\widetilde{\Gamma}$. Let $\widetilde{T}_e$ and $\widetilde{T}'_e$ be the tori in $\widetilde{M}_v$ and $\widetilde{M}_{v'}$ corresponding to e, and let $\widehat{T}_e, \widehat{T}'_e$ be components of $q_v^{-1}(\widetilde{T}_e)$ and $q_{v'}^{-1}(\widetilde{T}'_e)$, respectively. Let $\widetilde{f}: \widetilde{T}_e \to \widetilde{T}'_e$ be the gluing map. We want to find a lifting $\widehat{f}$ of $\widetilde{f}$ so that the following diagram commutes.

$$\begin{array}{ccc} \widehat{T}_e & \xrightarrow{\widehat{f}} & \widehat{T}'_e \\ {\scriptstyle q_v}\downarrow & & \downarrow{\scriptstyle q_{v'}} \\ \widetilde{T}_e & \xrightarrow{\widetilde{f}} & \widetilde{T}'_e \end{array}$$

To simplify the notation, still use e, v, v' to denote the image of e, v, v' in Γ under the map $\widetilde{\Gamma} \to \Gamma$ induced by p. We can extend the above diagram to

$$\begin{array}{ccc} \widehat{T}_e & & \widehat{T}'_e \\ {\scriptstyle q_v}\downarrow & & \downarrow{\scriptstyle q_{v'}} \\ \widetilde{T}_e & \xrightarrow{\widetilde{f}} & \widetilde{T}'_e \\ {\scriptstyle p_v}\downarrow & & \downarrow{\scriptstyle p_{v'}} \\ T_e & \xrightarrow{f} & T'_e \end{array}$$

Pick consistent base points. Then $p_{v'} \circ \widetilde{f} \circ q_v: \widehat{T}_e \to T'_e$ and $p_{v'} \circ q_{v'}: \widehat{T}'_e \to T'_e$ are both the s^2-fold characteristic covers. Thus $\operatorname{im}(p_{v'} \circ \widetilde{f} \circ q_v)_* = \operatorname{im}(p_{v'} \circ q_{v'})_* =$ the characteristic subgroup of index s^2 in $\pi_1(T'_e)$. Since $(p_{v'})_*$ is injective we must have

$$\operatorname{im}(\widetilde{f} \circ q_v)_* = \operatorname{im}(q_{v'})_*.$$

Thus $\widetilde{f} \circ q_v$ and $q_{v'}$ must be equivalent covers, so there is a homeomorphism $\widehat{f}$: $\widehat{T}_e \to \widehat{T}'_e$ making the above diagram commute.

The map $p_v \circ q_v : \widehat{T}_e \to \widetilde{T}_e \to T_e$ is an s^2-fold characteristic map, so $\deg(q_v|_{\widehat{T}_e}) = s^2/\deg(p_v|_{\widetilde{T}_e})$. Thus the number of tori in N_v that cover $\widetilde{T}_e$ is

$$\deg(N_v \to \widetilde{M}_v)/\deg(q_v|_{\widetilde{T}_e}) = D \deg(p_v|_{\widetilde{T}_e})/s^2$$

Similarly for $N_{v'}$. Since $\deg(p_v|_{\widetilde{T}_e}) = \deg(p_{v'}|_{\widetilde{T}'_e})$, it follows that the number of tori in $q_v^{-1}(\widetilde{T}_e)$ is the same as that in $q_{v'}^{-1}(\widetilde{T}'_e)$. Hence we can use the above lifting map $\widehat{f}$ to glue each component of $q_v^{-1}(\widetilde{T}_e)$ to a component of $q_{v'}^{-1}(\widetilde{T}'_e)$. After doing this for all e in $\widetilde{\Gamma}$ we get a manifold $\widehat{M} = (\bigcup N_v)/\{\widehat{f}\}$ and a map $\widetilde{p} : \widehat{M} \to \widetilde{M}$ so that $p \circ \widetilde{p} : \widehat{M} \to M$ is a characteristic map. If $\widehat{M}$ is not connected, just pick a component. □

5. Foliations of finite covers

In section 3 it was shown that many graph manifolds have no finite cover that fibers over the circle. In this section we will show that all graph manifolds (recall our conventions on graph manifolds), on the other hand, have finite covers that almost fiber over the circle. Specifically, we show that there are finite covers which admit "horizontal foliations"− that is, foliations transverse to the Seifert-fibers of each of the vertex manifolds. These horizontal foliations come from fibering each vertex manifold over the circle so that the fibers match up along the canonical splitting surface. Consequently, although these foliations are not often fibrations, they admit a very strong transverse structure.

This situation is analogous to the case of Seifert-fibered spaces. Here again, many Seifert-fibered spaces do not have finite covers that fiber over the circle but it follows from [EHN] that Seifert-fibered spaces with non-positive base orbifold will have finite covers that admit horizontal foliations.

The main theorem of this section is Theorem 5.5. Propositions 5.1 and 5.4 will be used in the proof of Theorem 5.5.

Proposition 5.1. *Let M be a graph manifold in which every vertex manifold is a product. Let s be a positive integer. Then there is a characteristic cover $p : \widetilde{M} \to M$ such that*

(1) *the preimage under p of any vertex manifold is connected,*
(2) *if e is an edge of $\Gamma = \Gamma(M)$ and T the corresponding component of the canonical splitting surface, then $|p^{-1}(T)|$ is an integral multiple of s.*

Proof. Let M_v be a vertex manifold of M. By hypothesis we may write $M_v = F_v \times S^1$ for some surface F_v with nonempty boundary and nonpositive Euler characteristic. Attach cones of angle $2\pi/3$ to the components of ∂F_v, giving a good orbifold, $\overline{F}_v$, of negative orbifold characteristic. Let $q_v : \widehat{F}_v \to \overline{F}_v$ be a covering such that $\widehat{F}_v$ is an orientable (connected) surface without cone points. $\overline{F}_v$ will have positive genus, so we may assume, after passing to a further cover, that the degree of q_v is a multiple of $3s$. Let $\widetilde{F}_v = q_v^{-1}(F_v) \subset \widehat{F}_v$. Then $q_v : \widetilde{F}_v \to F_v$ is a connected covering which restricts to a 3-fold cover on each component of $\partial \widetilde{F}_v$. Furthermore, there is a positive integer r_v such that if C is a component of ∂F_v, then $|q_v^{-1}(C)| = r_v s$.

Let $p_v : \widetilde{M}_v \to M_v$ be the cover obtained by taking the cover q_v in the F_v direction and taking the 3-fold cover in the circle direction. Then p_v is 3-characteristic, $\widetilde{M}_v$ is connected, and the preimage of any component of ∂M_v consists of $r_v s$ tori.

We apply the above construction to M_v for every vertex $v \in \Gamma$. We may assume, by first passing to further covers q_v, that $r_{v_1} = r_{v_2} = r$ for any vertices v_1, v_2 in Γ. Fix an orientation on the edges of Γ and let e be one such edge. Let M_{v_1} and M_{v_2} be the vertex manifolds at the head and tail of e. Let T_e^1 be the component of ∂M_{v_1} corresponding to e and T_e^2 the component of ∂M_{v_2} corresponding to e. Let f_e be the gluing homeomorphism identifying T_e^1 and T_e^2. If $\widetilde{T}_e^1, \widetilde{T}_e^2$ are components of $p_{v_1}^{-1}(T_e^1), p_{v_2}^{-1}(T_e^2)$ (resp.) then there is a lift, $\widetilde{f}_e : \widetilde{T}_e^1 \to \widetilde{T}_e^2$, of f_e, since both covers are 3-characteristic. Since $|p_{v_1}^{-1}(T_e^1)| = |p_{v_2}^{-1}(T_e^2)| = rs$, we may use such lifts to identify $p_{v_1}^{-1}(T_e^1)$ and $p_{v_2}^{-1}(T_e^2)$. Doing this for all edges e of Γ gives the desired cover $p : \widetilde{M} \to M$. □

Lemma 5.2. *Let* $\begin{pmatrix} a & c \\ b & 0 \end{pmatrix}$ *be an integral matrix with determinant* -1*. Then there is a positive integer,* s*, and a collection of coprime pairs of integers,* $\{(p_i, q_i) \mid i = 1, \dots, s\}$*, such that* $q_i \neq 0$*,* $bp_i + dq_i \neq 0$ *for every* i*, and*

$$\frac{p_1}{q_1} + \cdots + \frac{p_s}{q_s} = 0;$$

$$\frac{ap_1 + cq_1}{bp_1 + dq_1} + \cdots + \frac{ap_s + cq_s}{bp_s + dq_s} = 0.$$

Proof. Since multiplying the matrix by ± 1 will not change the solutions we may assume that $c = b = 1$. Set $s = |a| + 4$.

If $a \geq 0$ set

$$p_1 = p_1 = 1; \qquad p_3 = p_4 = \cdots = p_{|a|+4} = -1;$$
$$q_1 = q_2 = 2; \qquad q_3 = q_4 = \cdots = q_{|a|+4} = a + 2.$$

If $a < 0$ set

$$p_1 = p_1 = -1; \qquad p_3 = p_4 = \cdots = p_{|a|+4} = 1;$$
$$q_1 = q_2 = 2; \qquad q_3 = q_4 = \cdots = q_{|a|+4} = |a| + 2.$$

These values solve the required problem. □

Lemma 5.3. *Let* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *be an integral matrix with determinant* -1 *and assume* $b \neq 0$*,* $d \neq 0$*. Then there are coprime pairs of integers* (p, q) *and* (p', q') *such that* $q, q' \neq 0$ *and*

(a) $$\frac{ap + cq}{bp + dq} + \frac{-ap + cq}{-bp + dq} > 0;$$

(b) $$\frac{ap' + cq'}{bp' + dq'} + \frac{-ap' + cq'}{-bp' + dq'} < 0.$$

Proof. Let $f(x) = 2(cd - abx^2)/(d^2 - b^2x^2)$. If $q \neq 0$, then

$$f(\frac{p}{q}) = \frac{ap + cq}{bp + dq} + \frac{-ap + cq}{-bp + dq}$$

So we must find coprime pairs of integers (p, q) and (p', q') so that $f(p/q) > 0$ and $f(p'/q') < 0$.

Case I: $a \neq 0, c \neq 0$

First notice that $cd/ab > 0$ ($ad = bc - 1$ implies that ad and bc have the same sign, hence so will a/d and b/c). Let $\lambda = \sqrt{cd/ab}$. Then λ is a root of f (note that $f(\lambda)$ is well defined since $\lambda^2 \neq d^2/b^2$). Furthermore,

$$f'(\lambda) = \frac{4\lambda bd}{(d^2 - b^2\lambda^2)^2} \neq 0$$

Thus by taking p/q, p'/q' very close to λ and on opposite sides of λ on the real line we will have $f(p/q) > 0$ and $f(p'/q') < 0$.

Case II: $a = 0$

In this case, $b = c = \pm 1$. Thus

$$f(x) = 2\frac{\pm d}{d^2 - x^2}$$

By choosing $|p/q| < |d|$ and $p'/q' > |d|$, or vice versa, we prove the lemma.

Case III: $c = 0$

In this case, $\{a, d\} = \{+1, -1\}$. Thus

$$f(x) = 2\frac{\pm bx^2}{1 - b^2x^2}$$

If $0 < x \ll 1$ and $y \gg 0$, then $f(x)$ and $f(y)$ will be of opposite sign.

Cases I,II,III exhaust the possibilities of Lemma 5.3. □

Proposition 5.4. *Let* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *be an integral matrix with determinant* -1 *and assume* $b \neq 0$*. Then there is a positive integer,* s*, and a collection of coprime pairs of integers,* $\{(p_i, q_i) \mid i = 1, \ldots, s\}$*, such that* $q_i \neq 0$*,* $bp_i + dq_i \neq 0$ *for every* i*, and*

$$\frac{p_1}{q_1} + \cdots + \frac{p_s}{q_s} = 0;$$

$$\frac{ap_1 + cq_1}{bp_1 + dq_1} + \cdots + \frac{ap_s + cq_s}{bp_s + dq_s} = 0.$$

Proof. The special case that $d = 0$ is treated in Lemma 5.2 above. So we assume that $d \neq 0$. In this case we may apply Lemma 5.3 to find coprime integers (p, q) and (p', q') satisfying conditions (a) and (b) of that lemma.

Define integers r, t, u, v as follows
$$\frac{u}{v} = \frac{ap+cq}{bp+dq} + \frac{-ap+cq}{-bp+dq} > 0;$$
$$\frac{r}{t} = \frac{ap'+cq'}{bp'+dq'} + \frac{-ap'+cq'}{-bp'+dq'} < 0.$$
Take $|vr|$ copies of the pair (p, q); $|vr|$ copies of the pair $(-p, q)$; $|tu|$ copies of the pair (p', q'); and $|tu|$ copies of $(-p', q')$ to form a collection of $s = 2|vr| + 2|tu|$ coprime pairs of integers. This collection will satisfy the requirements of the Proposition, for we have
$$\sum_{|vr|}(\frac{p}{q} + \frac{-p}{q}) + \sum_{|tu|}(\frac{p'}{q'} + \frac{-p'}{q'}) = 0;$$
$$\sum_{|vr|}(\frac{ap+cq}{bp+dq} + \frac{-ap+cq}{-bp+dq}) + \sum_{|tu|}(\frac{ap'+cq'}{bp'+dq'} + \frac{-ap'+cq'}{-bp'+dq'}) =$$
$$= |vr|\frac{u}{v} + |tu|\frac{r}{t} = |ru| - |ur| = 0. \quad \square$$

Theorem 5.5. *Let M be a graph manifold. Then there is a finite cover $p : \widetilde{M} \to M$ such that $\widetilde{M}$ admits a foliation that restricts on each vertex manifold of $\widetilde{M}$ to a surface bundle over S^1.*

Proof. By applying Proposition 4.4 to the trivial cover, we may assume that each vertex manifold of M is a product. To each edge of Γ assign an orientation and for each vertex manifold, M_v, pick a product structure $F_v \times S^1$. Let e be an edge of Γ. Let M_v and M'_v be the vertex manifolds corresponding to the tail and head of e, and let T_e and T'_e be the components of ∂M_v and $\partial M'_v$ corresponding to e. Then T_e and T'_e are identified in M by a *gluing homeomorphism* $f_e : T_e \to T'_e$. Now T_e, T'_e are parametrized by the curves $\{l_e, m_e\}, \{l'_e, m'_e\}$ representing $\{\text{circle fiber}, \text{boundary of the surface}\}$ of M_v, M'_v (resp.). In these coordinates f_e is represented by a *gluing matrix* $\begin{pmatrix} a_e & c_e \\ b_e & d_e \end{pmatrix}$ of determinant -1. Note that in these coordinates $b_e \neq 0$ since otherwise this would mean that the Seifert-fibers of M_v and M'_v would be identified under f_e (and this would contradict the minimality of the canonical splitting surface). So we can apply Proposition 5.4 above to this matrix, giving a positive integer s_e and a collection of ordered pairs $\{(p^e_i, q^e_i) \mid i = 1, \dots, s_e\}$.

Let s be the least common multiple of $\{s_e \mid e \in \Gamma\}$. Let $p : \widetilde{M} \to M$ be the cover obtained by applying Proposition 5.1 using this s.

Again let e, M_v, M'_v be as above. Let $\widetilde{M_v} = p^{-1}(M_v)$ and $\widetilde{M'_v} = p^{-1}(M'_v)$. Let $\widetilde{F}_v, \widetilde{F}'_v$ be the preimage of F_v, F'_v in $\widetilde{M}_v, \widetilde{M}'_v$. Take boundary framings on $M_v, M'_v, \widetilde{M}_v, \widetilde{M}'_v$ corresponding to the surfaces $F_v, F'_v, \widetilde{F}_v, \widetilde{F}'_v$ respectively. Let $\widetilde{T}_e, \widetilde{T}'_e$ be components of $p^{-1}(T_e), p^{-1}(T'_e)$ corresponding to the same component of the splitting surface of $\widetilde{M}$. So we have the commuting diagram
$$\begin{array}{ccc} \widetilde{T}_e & \xrightarrow{\widetilde{f}_e} & \widetilde{T}'_e \\ p\big\downarrow & & \big\downarrow p \\ T_e & \xrightarrow{f_e} & T'_e \end{array}$$

The circle fiber and the boundary framing will be denoted l_e, m_e on T_e; $\widetilde{l}_e, \widetilde{m}_e$ on $\widetilde{T}_e$; l'_e, m'_e on T'_e; and $\widetilde{l}'_e, \widetilde{m}'_e$ on $\widetilde{T}'_e$. Note that since p is characteristic, $\widetilde{l}_e$ (resp. $\widetilde{l}'_e$) intersects $\widetilde{m}_e$ (resp. $\widetilde{m}'_e$) once, and the gluing matrix associated to $\widetilde{f}_e$ (in terms of these bases) will be the same as the gluing matrix for f_e. Recall that $s_e, \{(p^e_i, q^e_i) \mid i = 1, \ldots, s_e\}$ are associated to f_e through Proposition 5.4. To $\widetilde{T}_e$ we assign the framing $p^e_i\widetilde{l}_e + q^e_i\widetilde{m}_e$ and to $\widetilde{T}'_e$ we assign the framing $(a_e p^e_i + c_e q^e_i)\widetilde{l}'_e + (b_e p^e_i + d_e q^e_i)\widetilde{m}'_e$ for some i. Note that these curves are identified under $\widetilde{f}_e$. By construction, $|p^{-1}(T_e)| = |p^{-1}(T'_e)| = r_e s_e$ for some integer r_e. We make the assignment of framings so that each ordered pair (p^e_i, q^e_i), $i = 1, \ldots, s_e$, is assigned to exactly r_e components of $p^{-1}(T_e)$.

Doing this for all edges of Γ assigns a framing α_v to each vertex manifold $\widetilde{M}_v$ in $\widetilde{M}$ (where v denotes the vertex of Γ). Since $\partial\widetilde{F}_v$ gives a boundary framing for $\widetilde{M}_v$, Corollary 2.2 says that

$$\begin{aligned} e(\widetilde{M}_v, \alpha_v) &= \Delta(\alpha, \partial\widetilde{F}_v) \\ &= \sum_{v=\text{tail of e}} r_e \sum_{i=1}^{s_e} \frac{p^e_i}{q^e_i} + \sum_{v=\text{head of e}} r_e \sum_{i=1}^{s_e} \frac{a_e p^e_i + c_e q^e_i}{b_e p^e_i + d_e q^e_i} \\ &= 0 \end{aligned}$$

That is, α_v also describes a boundary framing for $\widetilde{M}_v$. A surface corresponding to this boundary framing will be a fiber in a fibration of $\widetilde{M}_v$ over the circle (to see this fibration cut up $\widetilde{M}_v$ along this surface giving an I-bundle over the surface which will be a product bundle). By construction, the fibers in the fibration over the circle that one obtains on $\widetilde{M}_v$ and on an adjacent vertex manifold $\widetilde{M}'_v$ will be isotopic on a splitting torus between them (corresponding to a shared edge $\widetilde{e}$ of $\widetilde{\Gamma}$). Therefore the fibrations on the vertex manifolds piece together to give the desired foliation on $\widetilde{M}$. $\square$

Remark. The resulting foliation of M does not always give a fibration over the circle (it could not by the previous section) for the same reason that the fibers of each fibration do not piece together to give a surface. Let $\widetilde{M}_v$ and $\widetilde{M}'_v$ be adjacent along an edge $\widetilde{e}$ in $\widetilde{\Gamma}$, let $\widetilde{T}$ be the splitting surface corresponding to $\widetilde{e}$, and let $\widetilde{S}_v, \widetilde{S}'_v$ be fibers in the fibrations of $\widetilde{M}_v, \widetilde{M}'_v$. Then $|\widetilde{S}_v \cap \widetilde{T}|$ may differ from $|\widetilde{S}'_v \cap \widetilde{T}|$ (even though each component of $\widetilde{S}_v \cap \widetilde{T}$ is isotopic on $\widetilde{T}$ to any component of $\widetilde{S}'_v \cap \widetilde{T}$). Thus in trying to construct a surface of $\widetilde{M}$ using the fibers of the fibrations of the vertex manifolds one may pick up "holonomy" as one follows a circuit of $\widetilde{\Gamma}$. On the other hand the structure on the foliation one obtains in this construction is very specific and one could capture this phenomenon by saying that the foliation has a transverse affine structure with structure group consisting of rational dilations and contractions.

6. A covering invariant

In this section we will use the relative Euler number to define an invariant $\sigma(M)$ for a graph manifold M. Theorem 6.1 shows that it has a certain covering property. If $\sigma(M)$ is nonzero, then homeomorphic covers of M have the same covering degree.

As an example, we show in Proposition 6.3 that for knot complements this invariant does not vanish.

Let M be a graph manifold. We use χ_i to denote the orbifold characteristic of a vertex manifold M_i.

Definition. The σ invariant of M is defined by

$$\sigma(M) = \sum_{E(M_i)\neq 0} \frac{\chi_i^2}{|E(M_i)|}.$$

When all $E(M_i) = 0$, define $\sigma(M) = 0$.

If M were a Seifert fibered space then this would be a well-known covering invariant. See for example [G,WW].

A question of Thurston [K, Problem 3.16] asks whether there is a real valued 3-manifold invariant C such that if M is a k-fold cover of N then $C(M) = kC(N)$. We call C a *covering invariant* if it satisfies this property. The Gromov norm is a covering invariant of compact 3-manifolds. When M is a graph manifold, however, the Gromov norm vanishes. In [WW] Wang and Wu defined a new invariant $\omega(M)$ for graph manifolds, which was shown to be a covering invariant. The following theorem shows that the σ invariant defined above is a covering invariant.

Theorem 6.1. *If $p : \widetilde{M} \to M$ is a k-fold covering of graph manifolds, then $\sigma(\widetilde{M}) = k\sigma(M)$.*

Proof. Suppose $\widetilde{M}_j$ is a component of $p^{-1}(M_i)$, and suppose the restriction of p to $\widetilde{M}_j$ is a (u_j, v_j) cover. The natural framing α^i of M_i lifts to the natural framing $\widetilde{\alpha}^j$ of $\widetilde{M}_j$, so by Proposition 2.3 we have $E(\widetilde{M}_j) = E(M_i)u_j/v_j$. In particular, $E(M_i) \neq 0$ if and only if $E(\widetilde{M}_j) \neq 0$. By definition, u_j is the number of regular fibers in $\widetilde{M}_j$ that cover a regular fiber in M_i, so $u_j = \widetilde{\chi}_j/\chi_i$, where $\widetilde{\chi}_j$ is the orbifold characteristic of $\widetilde{M}_j$. Let $k_j = u_j v_j$ be the covering degree of $\widetilde{M}_j$ over M_i. Then

$$E(\widetilde{M}_j) = \frac{E(M_i)u_j}{v_j} = \frac{E(M_i)u_j^2}{k_j} = \frac{E(M_i)\widetilde{\chi}_j^2}{\chi_i^2 k_j}.$$

Therefore, when $E(M_i) \neq 0$ we have

$$\frac{\widetilde{\chi}_j^2}{E(\widetilde{M}_j)} = \frac{\chi_i^2}{E(M_i)} k_j.$$

Taking absolute value on both sides and summing over all preimages of M_i, we get

$$\sum \left\{ \frac{\widetilde{\chi}_j^2}{|E(\widetilde{M}_j)|} \mid \widetilde{M}_j \text{ covers } M_i \right\} = \frac{\chi_i^2}{|E(M_i)|} \sum k_j = \frac{\chi_i^2}{|E(M_i)|} k.$$

The result now follows by summing both sides over those i with $E(M_i) \neq 0$. □

As is pointed out in [K, Problem 3.16], covering invariants are related to the following problem. What 3-manifolds satisfy

(*): If M_1 and M_2 are homeomorphic finite covers of M, then the covering degrees are equal.

It was conjectured ([WW] and [K, Problem 3.16(A)]) that M satisfies (*) if and only if it is not covered by either a (surface)$\times S^1$ or a torus bundle over S^1. Among the category of geometric 3-manifolds, the conjecture is known to be true except for graph manifolds [WW]. If some covering invariant, for example $\sigma(M)$, is non-zero, then the covering degree is determined by $\deg(p_i) = \sigma(\widetilde{M})/\sigma(M)$, so M satisfies (*). Hence we have the following

Corollary 6.2. *If some vertex manifold M_i of M has non-zero relative Euler number $E(M_i)$, then M satisfies (*).* □

Remark 1. If (N, α) is a framed Seifert manifold then $e(N, \alpha) = 0$ means that α is a boundary framing. In particular, N fibers over a 1-orbifold so that the fiber intersects ∂N in curves isotopic to components of α. Now let M be a closed graph manifold with canonical splitting surface $\mathcal{T} = T_1 \cup \ldots \cup T_k$. Then $\sigma(M) = 0$ implies that each vertex manifold M_v of M fibers over a 1-orbifold so that if T_i is on ∂M_v then the fiber of this fibration of M_v intersects T_i in curves isotopic to the Seifert fiber of the vertex manifold adjacent to M_v across T_i.

Remark 2. The σ invariant is different from the ω invariant defined in [WW]. The ω invariant depends only on the fiber intersection numbers and the orbifold characteristic of vertex manifolds of M, and is insensitive to the relative Euler numbers. So for example if we do some $1/n$ surgeries on regular fibers of vertex manifolds of M, then $\sigma(M)$ will change, but $\omega(M)$ remains the same. Also, if each vertex manifold of M contains some boundary of M, then by definition $\sigma(M)$ vanishes, but $\omega(M)$ may be non-zero.

Example. Suppose a graph manifold M is a closed surface bundle over S^1, so M can be written as $M = F \times I/\varphi$, where $\varphi : F \times 0 \to F \times 1$ is the monodromy map. By passing to a finite cover we may assume that $\varphi = \tau_1^{k_1} \ldots \tau_n^{k_n}$, where the k_i's are non-zero integers, τ_i is a Dehn twist along an essential curve C_i, and $C_1, \ldots, C_n$ are mutually disjoint and nonparallel. A vertex manifold M_i corresponds to a component F_i of $F - \bigcup \mathrm{Int} N(C_i)$. The relative Euler number of M_i can be computed as follows. If $\partial F_i = C_{j_1} \cup \ldots \cup C_{j_r}$, (some C_i may appear twice), then

$$E(M_i) = \frac{1}{k_{j_1}} + \ldots + \frac{1}{k_{j_r}}.$$

Thus, if some of these sums are non-zero, then $\sigma(M) \neq 0$, and M satisfies (*). In particular, if some F_i has only one boundary curve, then $\sigma(M) \neq 0$.

As an application of the σ invariant, we show that the conjecture above is true for knot complements. Write $X(K) = S^3 - \mathrm{Int} N(K)$.

Proposition 6.3. *Let K be a knot in S^3. Then $X(K)$ satisfies (*) if and only if K is not a torus knot.*

Proof. By Theorem 6.1 and [WW, Theorem 2.5], it suffices to show that if $X(K)$ is a graph manifold, then $\sigma(X(K)) \neq 0$. It is well known that $X(K)$ is a graph manifold if and only if K is a connected sum of iterated torus knots but is not a torus knot.

First suppose K is a (p,q) cable of an (r,s) torus knot $K_{r,s}$. In order that K is not just the torus knot $K_{r,s}$, we must have $q > 1$. Let T be a torus decomposing $X(K)$ into $X(K_{r,s})$ and a (p,q) cabling space $C(p,q)$. Let (m,l) be the meridian-longitude pair of $K_{r,s}$ on T. Then l bounds a Seifert surface in $X(K_{r,s})$, which is a transverse surface, so $e(X(K_{r,s}),l) = 0$. Let α be a fiber of $C(p,q)$ on T, let β be a fiber of $X(K_{r,s})$ on T. Then with respect to the (m,l) coordinate α is a (p,q) curve, and β is an $(rs,1)$ curve. By Proposition 2.1, we have

$$\begin{aligned} E(X(K_{r,s})) &= e(X(K_{r,s}),\alpha) = e(X(K_{r,s}),l) + \Delta(\alpha,l) \\ &= \delta(\alpha,l,\beta) = \frac{\alpha\cdot l}{(\alpha\cdot\beta)(l\cdot\beta)} = \frac{p}{(p-qrs)(rs)} \neq 0. \end{aligned}$$

Note that since p,q are coprime, and $q > 1$, we have $p - qrs \neq 0$, so the above computation make sense. In particular, by definition of the σ invariant we have $\sigma(X(K)) \neq 0$.

With the same computation one can show that if K is a nontrivial iteration of torus knot, i.e if K is a (p_1,q_1) cable of a (p_2,q_2) cable of $\ldots$ of a (p_r,q_r) torus knot, and $r > 1$, then the extremal vertex manifold of $X(K)$ has relative Euler number $E = p_r/(p_r - q_r p_{r-1} q_{r-1})(p_{r-1} q_{r-1})$. Hence $\sigma(X(K)) \neq 0$. More generally, if $K = K_1 \# \ldots \# K_r$, and if some K_i is a nontrivial iteration of torus knot, then $X(K_i)$ embeds into $X(K)$, so some vertex manifold of $X(K)$ has non-zero relative Euler number, and $\sigma(X(K)) \neq 0$.

The remaining case is that $K = K_1 \# \ldots \# K_r$, $r > 1$, and all K_i are torus knots. Then $X(K) = X(K_1) \cup \ldots \cup X(K_r) \cup C(r)$, where $C(r)$ is a composition space, i.e, $C(r) = D_r \times S^1$, where D_r is a disk with r holes. Let $T = X(K_1) \cap C(r)$; let (m,l) be a meridian-longitude pair of K_1 on $T = \partial X(K_1)$. Then $X(K_1)$ is glued to $C(r)$ so that m is mapped to some (point)$\times S^1$ on $\partial D_r \times S^1$, and l is mapped to some $C_i \times$(point), where C_i is a boundary component of D_r. Let α be a fiber of $C(r)$, and let β be a fiber of $X(K_1)$. Then with respect to the (m,l) coordinates, α is a $(1,0)$ curve, β is a $(pq,1)$ curve if K_1 is a (p,q) torus knot, and the $(0,1)$ curve l is the boundary of a Seifert surface of K_1. Hence,

$$\begin{aligned} E(X(K_1)) &= e(X(K_1),\alpha) = e(X(K_1),l) + \Delta(\alpha,l) \\ &= \delta(\alpha,l,\beta) = \frac{\alpha\cdot l}{(\alpha\cdot\beta)(l\cdot\beta)} = \frac{1}{1\cdot(pq)} = \frac{1}{pq}. \end{aligned}$$

Therefore $\sigma(X(K)) \neq 0$, and $X(K)$ satisfies (*). □

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