# TRIVIAL CYCLES IN GRAPHS EMBEDDED IN 3-MANIFOLDS

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Abstract. A cycle $C$ of a graph $\Gamma$ embedded in a 3-manifold $M$ is said to be trivial in $\Gamma$ if it bounds a disk with interior disjoint from $\Gamma$. Let $e$ be an edge of $\Gamma$ with ends on $C$. We will study the relation between triviality of cycles in $\Gamma$ and that of $\Gamma - e$ and $\Gamma/e$. Let $C_1$ be one of the two cycles in $C \cup e$ containing $e$. The main theorem says that if $C$ is trivial in $\Gamma - e$ and $C_1/e$ is trivial in $\Gamma/e$, then either $C$ or $C_1$ is trivial in $\Gamma$. Some applications to cycle trivial graphs will be given in Section 2.

## 1. The Main Theorem

If $L$ is a link in $S^3$, then a component $K$ of $L$ is said to be trivial if it bounds a disk in $S^3$ which is disjoint from the other components of $L$. This can be generalized to graphs in 3-manifolds. Recall that a cycle $C$ of a graph $\Gamma$ is an embedded circle. Suppose that $\Gamma$ is a graph in a 3-manifold $M$. Then $C$ is called a *trivial cycle in* $\Gamma$ if it bounds a disk $D$ in $M$ with interior disjoint from $\Gamma$. Of course this depends on $\Gamma$ and the way it is embedded in $M$, so we may also say that $C$ is trivial with respect to $(M, \Gamma)$.

Suppose $e$ is a non loop edge in $\Gamma$. Then we have a subgraph $\Gamma - e$ in $M$, and a quotient graph $\Gamma/e$ in $M/e \cong M$. We are interested in the problem of how the triviality of cycles in $\Gamma$ is related to that of $\Gamma - e$ and $\Gamma/e$. If $e$ is disjoint from $C$, then clearly $C$ is trivial in $\Gamma$ if and only if it is trivial in $\Gamma/e$. When $e$ has one end on $C$ the problem was studied in [12], and it was shown that if $C$ is trivial in both $\Gamma/e$ and $\Gamma - e$, then it is trivial in $\Gamma$.

In this paper we study the case that $C \cap e = \partial e$. Let $C_1$ be one of the two cycles in $C \cup e$ that contain $e$. In this case $C/e$ is no longer a cycle, but $C_1/e$ is a cycle in $\Gamma/e$. Consider the following examples.

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1991 *Mathematics Subject Classification.* Primary 57M15, secondary 57N10.

*Key words and phrases.* Trivial cycles, graphs, 3-manifolds.

**Example 1.1.** Let $C$ be the cycle $e_1 \cup e_2$ in Figure 1. $C$ is trivial in $\Gamma - e$, but it is nontrivial in $\Gamma$.

**Example 1.2.** For the graph $\Gamma$ in Figure 2, let $C_1 = e_1 \cup e$. Then $C_1/e$ is trivial in $\Gamma/e$, but $C_1$ is nontrivial in $\Gamma$.

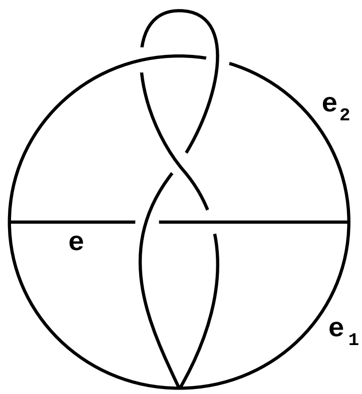


**Figure 1**

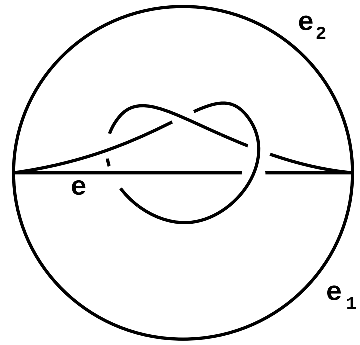


**Figure 2**

The main result of this paper is Theorem 1.4, which says that these two cases can not happen simultaneously: If $C$ is trivial in $\Gamma - e$, and $C_1/e$ is trivial in $\Gamma/e$, then either $C$ or $C_1$ has to be trivial in $\Gamma$. In this section we will prove this theorem. Some applications will be given in the next section.

Consider a graph $\Gamma$ in a 3-manifold $M$. Let $N(\Gamma)$ be a regular neighborhood of $\Gamma$. Define the exterior of $\Gamma$ to be $E(\Gamma) = M - \text{Int}N(\Gamma)$. If $v_1, \ldots, v_r$ are the vertices of $\Gamma$, and $e_1, \ldots e_t$ the edges, then $N(\Gamma)$ is a union of $N(v_i)$ and $N(e_j)$, where $N(e_j)$ is chosen to be small enough so that each $N(e_j)$ intersects $\cup N(v_i)$ in two disks, and all the disks in $\{N(v_i) \cap N(e_j) \,|\, j = 1, \ldots t\}$ are mutually disjoint. Let $\delta(v_i)$ be the punctured sphere $\partial N(v_i) - \cup \text{Int}N(e_j)$, and let $\delta(e_j)$ be the annulus $\partial N(e_j) - \cup \text{Int}N(v_i)$.

If $\Gamma'$ is a subgraph of $\Gamma$, define $\delta(\Gamma')$ to be the union of $\delta(t)$, where $t$ ranges over all vertices and edges of $\Gamma'$. In particular, if $C$ is a cycle, then $\delta(C)$ is a punctured torus. If $C$ is a trivial cycle in $\Gamma$, then it bounds a disk $D$ whose intersection with $E(\Gamma)$ is a disk $D'$. The disk $D$ can be chosen so that $\partial D' \subset \delta(C)$, and for each edge $e_j$ of $C$, $\partial D'$ intersects $\delta(e_j)$ at exactly one essential arc. Conversely, if there is a disk $D'$ in $E(\Gamma)$ satisfying these conditions, then $D'$ can be extended to a disk $D$ in $M$ so that $D \cap \Gamma = \partial D = C$. Therefore we have

**Lemma 1.3.** *A cycle $C$ in $\Gamma$ is trivial if and only if there is a disk $D'$ in $E(\Gamma)$ so that $\partial D' \subset \delta(C)$, and $\partial D' \cap \delta(e_j)$ is an essential arc of $\delta(e_j)$ for all edges $e_j$ of $C$.* $\square$

Now let $e$ be a non loop edge of $\Gamma$. Consider the quotient graph $\Gamma/e$. We use $[e]$ to denote the image of $e$ in $\Gamma/e$. Suppose $e$ is incident to the vertices $v_1$ and $v_r$. We identify the regular neighborhood $N(\Gamma/e)$ with $N(\Gamma)$ by letting $N([e]) = N(e) \cup N(v_1) \cup N(v_r)$. Under this identification we have $E(\Gamma) = E(\Gamma/e)$ and $\delta([e]) = \delta(e) \cup \delta(v_1) \cup \delta(v_r)$.

**Theorem 1.4.** *Let $C$ be a cycle in $\Gamma$. Let $e$ be a non loop edge with $e \cap C = \partial e$. Let $C_1$ be one of the two cycles in $C \cup e$ that contain $e$. If $C$ is trivial in $\Gamma - e$, and $C_1/e$ is trivial in $\Gamma/e$, then either $C$ or $C_1$ is trivial in $\Gamma$.*

*Proof.* Let $e_1, \dots e_n$ and $v_1, \dots, v_n$ be the edges and vertices of $C$, with $\partial e_i = v_i \cup v_{i+1}$, $(v_{n+1} = v_1)$. Without loss of generality we may assume that $\partial e = v_1 \cup v_r$, and $C_1 = e_1 \cup \dots \cup e_{r-1} \cup e$.

By assumption $C$ bounds a disk $D$ in $M$ with $\mathrm{Int}D \cap (\Gamma - e) = \emptyset$. Let $t_1, \dots, t_p$ be the intersection points of $e \cap \mathrm{Int}D$, labeled successively along $e$. Choose $D$ so that $p$ is minimal. If $p = 0$, then $D$ is disjoint from $e$ and we are done. So we assume that $p > 0$.

The surface $P = D \cap E(\Gamma)$ is a punctured disk with boundary $\partial P = \partial_0 \cup \dots \cup \partial_p$, where $\partial_i$ $(i \geq 1)$ is the boundary of the disk in $N(e) \cap D$ that contains $t_i$, and $\partial_0$ lies on $\delta(C)$, intersecting each of $\delta(e_i)$ and $\delta(v_j)$ at an essential arc. See Figure 3, where $p = 2$.

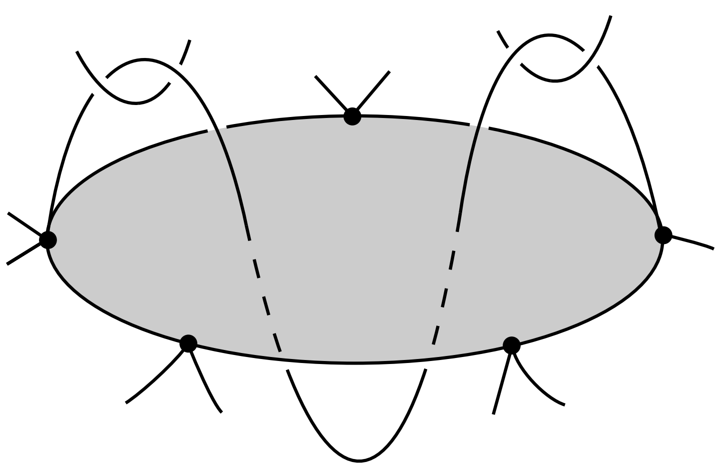

**Figure 3**

Now consider the cycle $C_1/e$ in $\Gamma/e$. By assumption $C_1/e$ is trivial in $\Gamma/e$, so it bounds a disk $D_1$ in $M/e \cong M$ whose interior is disjoint from $\Gamma/e$. The intersection of $D_1$ with $E(\Gamma) = E(\Gamma/e)$ is a disk $Q$. By Lemma 1.3 we may assume that $\partial Q \cap \delta(e_i)$ is an essential curve on $\delta(e_i)$ for $i = 1, \dots, r-1$. If $\partial Q \cap \delta(e)$ is also a single essential curve, then $Q$ extends to a disk in $M$ bounded by $C_1$, and we are done. Therefore we assume that $\partial Q \cap \delta(e)$ is a set of $q > 1$ essential arcs, each intersecting $\partial_i$ at a single point for $i = 1, \dots, p$. The set $G = P \cap Q$ is a properly embedded 1-manifold in both $P$ and $Q$. Moreover, by an isotopy we may push all the ends of $G$ off the annulus $\delta(e_i)$, $i \geq 1$. Thus $\partial G \cap \partial_0 \subset \cup \delta(v_i)$. Choose $P$ and $Q$ so that $|G|$ is minimal subject to the minimality conditions of $p$ and $q$. When considering $G$ as a subset of $P$, we denote it by $G_P$. Similarly for $G_Q$.

CLAIM 1: *$G_P$ is a set of essential arcs in $P$.*

*Proof.* This is essentially an innermost circle outermost arc argument. By surger $P$ along disks on $Q$ bounded by innermost circles of $G_Q$, we can eliminate all circle components of $G_P$. Similarly one can remove all trivial arcs of $G_P$ which has ends on $\partial_i$ with $i \geq 1$, by surger $Q$ along disks on $P$ cut off by such arcs.

We need to be a little careful when an outermost arc $\gamma$ of $G_P$ has ends on $\partial_0$. Let $\Delta$ be the disk on $P$ cut off by $\gamma$. Let $Q_1, Q_2$ be the closure of the two components of $Q - \gamma$. The point here is that, since each of $\partial P$ and $\partial Q$ intersect $\delta(e_i)$ in just one essential arc, one of the $Q_i$, say $Q_1$, has the property that $\partial Q_1$ intersects some $\delta(e_i)$ if and only if $\partial\Delta$ does, so if we replace $Q_1$ by $\Delta$, the new disk $Q' = Q_2 \cup \Delta$ still has the property that it intersects each $\delta(e_i)$ just once for $i = 1, \dots, r-1$. We can thus replace $Q$ by $Q'$ to reduce $|G|$. □

Now consider the arcs $G_Q \subset Q$. For each end $x$ of $G_Q$, define a label $l(x)$ as follows. If $x$ lies on $\partial_i$ with $i \geq 1$, define $l(x) = i$. If $x$ lies on $\delta(v_j)$, define $l(x) = -j$. Thus, when traveling along $\partial(Q_1)$, the labels appear as a string of the form

$$(-1)^{n_1}(-2)^{n_2}\dots(-r)^{n_r}p\dots 21(-1)^{k_1}12\dots p(-r)^{k_2}\dots(-r)^{k_{q-1}}p\dots 21,$$

where $n_i, k_j \geq 0$.

CLAIM 2. *If $\{l_1, l_2\}$ are the labels of an outermost arc $\gamma$ of $G_Q$, then $l_1 = 1$ or $p$, and $l_2$ is between $-2$ and $-r+1$. In particular, $n_1 = n_r = 0$, and all arcs of $G_Q$ are parallel, having at least one negative labels on its ends.*

*Proof.* The labels of an outermost arc must be adjacent in the above label string, so $\{l_1, l_2\}$ has the following possibilities.

(1) $\{l_1, l_2\} = \{i, i+1\}$, $i \geq 1$;

(2) $\{l_1, l_2\} = \{1, -1\}$ or $\{p, -r\}$;

(3) both $l_i$ are negative;

(4) $\{l_1, l_2\} = \{1, 1\}$ or $\{p, p\}$;

(5) $\{l_1, l_2\} = \{1, p\}$;

(6) $l_1 = 1$ or $p$, and $-r + 1 \leq l_2 \leq 2$.

Note that (4) happens only if some $k_i = 0$, and (5) happens only if all $n_i = 0$.

Cases (1) – (3) are easy to rule out. If $\{l_1, l_2\} = \{i, i+1\}$ for some $i \geq 1$, we can isotop the edge $e$ through the outermost disk $\Delta$ on $Q$ cut off by $\gamma$ to reduce $|e \cap D|$, contradicting the minimality of $p$. This also works if $\{l_1, l_2\} = \{1, -1\}$ or $\{p, -r\}$. (But it does not work if $\{l_1, l_2\} = \{1, -2\}$, say, because then part of $\partial\Delta$ lies on the cycle $C$.) If $\{l_1, l_2\}$ are both negative, then one can surger $P$ along $\Delta$ to obtain a new $P'$ which has less intersection with $Q$. Note that $\partial P'$ still intersects each $\delta(e_j)$ just once, so it can be used to replace $P$.

Now assume $\{l_1, l_2\} = \{1, 1\}$. Since each positive label appears an odd number of times, some arc $\alpha$ of $G_Q$ has different labels on its ends. Let $\Delta$ be the component of $Q - \alpha$ that contains the outermost arc $\gamma$. We may choose $\alpha$ to be extremal in the sense that the two labels of each arc of $G_Q$ in $\Delta$ are the same. This means that each arc in $\Delta$ corresponds to a loop in $G_P$. If $p$ is a label of some arc in $\Delta$, then all positive labels appear in $\Delta$, so each vertex of $G_P$ has a loop based on it. But then one of these loops must be an inessential arc of $G_P$, contradicting claim 1. Therefore $p$ does not appear as a label of arcs in $\Delta$. It is now easy to see that $\gamma$ is the only outermost arc in $\Delta$, so all the arcs in $\Delta$ are parallel to each other. But this and the fact that $p$ does not appear on $\Delta$ imply that the labels of $\alpha$ are the same, contradicting the choice of $\alpha$. Therefore (4) can not happen.

If (5) happens, then all $n_i = 0$, so no outermost arc can be of type (6). From the label string one can see that there can be at most one type (5) arc. If $G_Q$ has just one arc, (5) would be the same as (1), which has been ruled out. If $G_Q$ has more than one arcs, then it has at least two outermost arcs, so the ones other than $\gamma$ must be of types (1) – (4), which is impossible by the above. Therefore (5) can not happen either.

It follows that $\gamma$ must be of type (6). From the label string it is clear that there are at most two such outermost arcs. Therefore all arcs of $G_Q$ are parallel. □

Now consider the curves in $G_P$. By assumption, for $i \geq 1$, $\partial Q$ intersect each $\partial_i$ exactly $q$ times, $q \geq 3$, so each $\partial_i$ is incident to $q$ arcs of $G_P$. By Claim 2, all of these arcs have the other ends on $\partial_0$. (Cf. Figure 4.) Therefore, there is a pair of parallel arcs $b_1, b_2$ on $G_P$, which is outermost in the sense that they cut off a disk $\Delta$ with interior disjoint from $Q$. Let $\Delta'$ be the disk on $Q$ between $b_1, b_2$. Form a disk $Q' = (Q - \Delta') \cup \Delta$.

One can see that $Q'$ can be isotoped to have less that $q$ intersection arcs with $\delta(e)$, and $\partial Q'$ intersects $\delta(e_j)$ just once for $j = 1, \dots, r-1$. This contradicts the choice of $Q$, completing the proof of Theorem 1.4. □

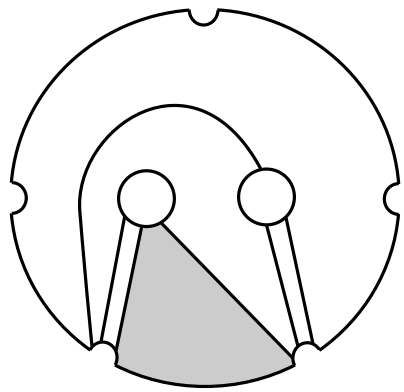

**Figure 4**

## 2. Some Applications

The following theorem was proved in [12].

**Theorem 2.1.** *Suppose $\Gamma$ is a graph embedded in a 3-manifold $M$. Let $C$ be a cycle in $\Gamma$, and let $e$ be an edge of $\Gamma$ with at most one end on $C$. If $C$ is trivial in both $\Gamma - e$ and $\Gamma/e$, then it is trivial in $\Gamma$.*

If $e$ has both ends on $C$, then $C/e$ is a union of two cycles. The following result has a similar nature as that above. The difference is that when considering the quotient graph, we need to consider both cycles in $C/e$.

**Corollary 2.2.** *Suppose $\Gamma$ is a graph embedded in a 3-manifold $M$. Let $C$ be a cycle in $\Gamma$. Let $e$ be a non loop edge of $\Gamma$ such that $\partial e = e \cap C$. If $C$ is trivial in $\Gamma - e$, and if the two cycles in $C/e$ are trivial in $\Gamma/e$, then $C$ is trivial in $\Gamma$.*

*Proof.* Let $C_1, C_2$ be the two cycles in $C \cup e$ that contains $e$. By Theorem 1.4, either $C$ is trivial in $\Gamma$ and we are done, or both $C_1, C_2$ are trivial in $\Gamma$. Since $C_1 \cap C_2 = e$ is connected, by Lemma 1.1 of [12], $C_i$ bounds a disk $D_i$ with interior disjoint from $\Gamma$, such that $D_1 \cap D_2 = e$. Let $D$ be $D_1 \cup D_2$ with interior pushed off $e$, then $D \cap \Gamma = C$, therefore $C$ is trivial in $\Gamma$. □

A subgraph $\Gamma'$ of $\Gamma$ is said to be *cycle trivial in* $\Gamma$ if all cycles of $\Gamma'$ are trivial in $\Gamma$. When $\Gamma' = \Gamma$, we simply say that $\Gamma$ is cycle trivial. The following corollary follows immediately from Theorem 2.1 and Corollary 2.2, by applying the results to each cycle of $\Gamma'$.

**Corollary 2.3.** *Let $\Gamma'$ be a subgraph of $\Gamma$, and let $e$ be a non loop edge of $\Gamma - \Gamma'$. If $\Gamma'$ is cycle trivial in $\Gamma - e$, and $\Gamma'/e$ is cycle trivial in $\Gamma/e$, then $\Gamma'$ is cycle trivial in $\Gamma$.* $\square$

Suppose $F$ is a surface in the boundary of a 3-manifold $X$, and $\alpha$ a simple closed curve in $F$. We use $X_\alpha$ to denote the manifold obtained from $X$ by attaching a 2-handle to $X$ along $\alpha$, and use $F_\alpha$ to denote the corresponding surface in $X_\alpha$. More explicitly, $X_\alpha = X \cup_\varphi (D^2 \times I)$, where $\varphi$ identifies $\partial D^2 \times I$ to a regular neighborhood $A$ of $\alpha$ in $F$, and $F_\alpha = (F - A) \cup (D^2 \times \partial I)$. The following result was proved in [11]. When $K = \emptyset$, it reduces to Jaco's Handle Addition Lemma [2, Lemma 1]. In [11] the conclusion was stated as $|\partial D' \cap K| \leq |\partial D \cap K|$, but the proof there has actually given the following stronger version.

**Proposition 2.4.** *Let $F$ be a surface on the boundary of a 3-manifold $X$, and $K$ a 1-manifold in $F$ with $F - K$ compressible in $X$. Let $\alpha$ be a simple loop in $F - K$. If $F_\alpha$ has a compressing disk $D$ in $X_\alpha$, then $F - \alpha$ has a compressing disk $D'$ in $X$ such that $\partial D' \cap K \subset \partial D \cap K$.*

**Lemma 2.5.** *Let $\Gamma$ be a graph in a 3-manifold $M$ which has no lens space or $S^1 \times S^2$ summand. Suppose $\Gamma$ has only one vertex $v$ and $n$ edges $e_1, \ldots e_n$. Then $\Gamma$ is cycle trivial if and only if $\partial N(\Gamma')$ is compressible in $E(\Gamma')$ for all subgraphs $G' \neq v$ of $G$.*

*Proof.* If $n = 1$, then $\Gamma$ is a knot in $M$. Let $D$ be a compressing disk of $\partial N(\Gamma)$ in $E(\Gamma)$. If $\partial D$ is not a longitude of $\Gamma$, then $N(\Gamma) \cup N(D)$ is a punctured lens space or punctured $S^1 \times S^2$, contradicting our assumption about $M$. Hence it can be extended to a disk in $M$ bounded by $\Gamma$, so $\Gamma$ is a trivial cycle.

Now assume that $\delta(v)$ is compressible in $E(\Gamma)$. Then a compressing disk can be extended to a sphere $S$ in $M$ intersecting $\Gamma$ only at $v$, with both sides of $S$ containing parts of $\Gamma$. Note that $S$ must be separating because $M$ has no $S^1 \times S^2$ summands. Let $\Gamma', \Gamma''$ be the subgraphs of $\Gamma$ in the two sides of $S$. By induction they are cycle trivial, so each cycle $C$ of $\Gamma'$ bounds a disk $D$ with interior disjoint from $\Gamma'$. Isotop $D$ so that $D \cap S$ is a union of circles which are mutually disjoint except possibly at the point $v$. By doing 2-surgeries along disks in $S$ bounded by innermost circles of $C \cap S$, we can change $D$ to a disk with interior disjoint from $\Gamma' \cup S$, and hence disjoint from $\Gamma$. Similarly, every cycle in $\Gamma''$ is trivial in $\Gamma$.

It remains to show that if $n > 1$ then $\delta(v)$ is compressible. Let $m_i$ be the central curve of the annulus $\delta(e_i)$. Let $F = \partial N(\Gamma)$. Let $K$ be a maximal subset of the $m_i$'s such that $F - K$ is compressible. Notice that if $K$ contains all the $m_i$'s, then we are done because $F - K$ retracts to $\delta(v)$. So assume that $m_1$ is not in $K$. Then $F - K$ is compressible in $E(\Gamma)$, but $F - (K \cup m_1)$ is incompressible. After attaching a 2-handle along $m_1$, the surface $F_{m_1} = \partial N(\Gamma - e_1)$, and the manifold $E(\Gamma)_{m_1} = E(\Gamma - e_1)$. By induction we may assume that $\Gamma - e_1$ is cycle trivial, so the loop $e_2$ is trivial in $\Gamma - e_1$. This means that there is a compressing disk $D$ of $F_{m_1}$ intersecting $m_2$ just once, and is disjoint from the other $m_i$'s, so it intersects $K$ at most once, at a point in $m_2$. Applying Proposition 3, we know that $E(\Gamma)$ contains a compressing disk $D'$ of $F$ which is disjoint from $m_1$, and intersects $K$ at most at one point in $m_2$. If $D'$ is disjoint from $m_2$, then $F - (K \cup m_1)$ is compressible. If $D'$ intersects $m_2$ at one point, then the boundary of a regular neighborhood of $\partial D' \cup m_1$ in $F$ is a compressing disk of $F - (K \cup m_1)$. Either case contradicts the assumption that $F - (K \cup m_1)$ is incompressible. □

A subgraph $\Gamma'$ of $\Gamma$ is said to be *essential* if it contains some cycles.

**Theorem 2.6.** *Let $\Gamma$ be a graph in a 3-manifold $M$ which has no lens space or $S^1 \times S^2$ summand. Then $\Gamma$ is cycle trivial if and only if $\partial N(\Gamma')$ is compressible in $E(\Gamma')$ for all essential subgraphs $\Gamma'$ of $\Gamma$.*

*Proof.* By induction on the number of edges we may assume that all subgraphs and quotient graphs of $\Gamma$ are cycle trivial. By the same argument as in the proof of Lemma 2.5, we see that the theorem is true if each component of $\Gamma$ has just one vertex. Therefore, we may assume that $\Gamma$ has some non loop edges.

Let $C$ be a cycle of $\Gamma$. If there is a non loop edge $e$ of $\Gamma$ which is not on $C$, then by applying Corollary 2.3 to $\Gamma' = \Gamma - e$, we see that $C$ is trivial in $\Gamma$. So assume that $C$ contains all the non loop edges of $\Gamma$. If $C$ is a component of $\Gamma$, we may delete the extra vertices on $C$ to reduce to the case that each component of $\Gamma$ has just one vertex. Hence we may assume that there is a loop $e_1$ with $\partial e_1$ on $C$. By the above, $e_1$ is trivial in $\Gamma$, so it bounds a disk $D$ with interior disjoint from $\Gamma$. By induction, $\Gamma - e_1$ is trivial, so $C$ bounds a disk $D'$ with interior disjoint from $\Gamma - e_1$. Now we can isotop $e_1$ through disks on $D$ bounded by outermost arcs of $D \cap D'$ to eliminate all intersections of $e_1$ with $D'$. The reverse isotopy moves $D'$ to a disk with interior disjoint from $\Gamma$. □

*Remark.* A graph in $M^3$ is called planar if it lies on an embedded 2-sphere in $M$. It is abstractly planar if it can be embedded into an abstract 2-sphere. When $M = S^3$, Lemma 2.5 was proved by Gordon [1]. It was also proved that if $\Gamma$ has two vertices, then it is planar if the exterior of all subgraphs are handlebodies. In [8] J. Simon conjectured that if a graph $\Gamma$ is abstractly planar, then Gordon's theorem is still true, that is, $\Gamma$ is planar if and only if the exteriors of all subgraphs of $\Gamma$ has free fundamental group. Simon and Wolcott [9] proved that this is true if $\Gamma$ is the handcuff or double $\theta$ graph. The conjecture was fully proved by Scharlemann and Thompson [4, 5, 7]. In [12] it was noticed that if $\Gamma$ is abstractly planar, then it is planar if and only if it is cycle trivial. Using this fact and Theorem 2.1, a new proof of the Scharlemann-Thompson Theorem was given in [12]. Because of this fact, Theorem 2.6 may be considered as a generalization of the Scharlemann-Thompson Theorem. When $M = S^3$, Theorem 2.6 was first proved by Robertson, Seymour and Thomas [3] under the stronger assumption that the exteriors of any subgraph of $\Gamma$ are connected sums of handlebodies. Scharlemann and Thompson [6] have proved a theorem about sliding arcs in handlebodies, which simultaneously generalize the $S^3$ version of Theorem 2.6 and Waldhausen's Theorem [10].

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