Covering Invariants and Cohopficity of 3-manifold Groups

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Abstract

A 3-manifold M is called to have property C if the degrees of finite coverings over M are determined by the manifolds. Thurston asked the problems of whether there is a real valued invariant of 3-manifolds with a certain covering property, and what 3-manifolds have property C. For geometric manifolds the conjecture is that M has property C if and only if it is not covered by surface$\times S^1$ or torus bundle over S^1. Using the Gromov norm, the conjecture is quickly reduced to the case of graph manifolds.

In this paper we define a new invariant $\omega(M)$ for graph manifolds. It has the covering property that if N is a d-fold covering space of M, then $\omega(N) = d\omega(M)$. Thus in the generic case, when $\omega(M) \neq 0$, the manifold M has property C. We also use the methods to prove two other results. The first classifies all geometric manifolds that admit nontrivial self coverings. The second solves a problem of Gonzales-Acuna and Whitten about the cohopficity of 3-manifold groups, and determines all closed geometric manifolds that have cohopfian fundamental groups.

Definition. A 3-manifold M is called to have property C if, whenever M_1, M_2 are homeomorphic covering spaces of M, the degrees of the coverings are the same. It has property C_r if the above is true for all regular coverings.

We are interested in the following problems of Thurston:

(1) [7, Problem 3.16] Is there a reasonable real valued invariant $C(M)$ such that (a) if $M = M_1 \# M_2$ then $C(M) = C(M_1) + C(M_2)$; (b) the covering property holds, i.e, if M_1 is a k-fold covering of M, then $C(M_1) = kC(M)$? $C(M)$ may have some additional properties.

(2) [7, Problem 3.16(a)] What 3-manifolds have property C?

An invariant of 3-manifolds with the covering property will give a partial solution to the Property C problem: if $C(M) \neq 0$, then the covering degree of $M_1 \to M$ is given by $k = C(M_1)/C(M)$, so M has property C. There are some well known covering invariants. The simplest one is the number of 2-spheres on the boundary of an orientable manifold, and the most interesting one is the Gromov norm $\nu(M)$ defined by Gromov [3] soon after these questions were posed. The following definition is convenient for our purpose.

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Definition. A compact, connected, orientable 3-manifold M is called *geometric*, if M is either a Seifert manifold, or a hyperbolic manifold, or a Haken manifold, or a connected sum of such manifolds.

For a closed geometric 3-manifold M, $\nu(M)$ is essentially the sum of the volumes of all hyperbolic pieces in the geometric decomposition of M; therefore M has property C if it contains some hyperbolic pieces. Note that the Gromov norm is a zero function on the set of all closed 3-manifolds which contain no hyperbolic pieces.

We only need to consider orientable manifolds, for if M is a nonorientable 3-manifold, then it has property C if and only if its orientable double covering does. So we make the following

Convention. Except in section 8, all 3-manifolds in this paper are assumed orientable.

A famous conjecture is that all compact 3-manifolds are geometric. Also, it is known that if M is covered by (surface)$\times S^1$ or torus bundle over S^1, then it does not have property C. Actually such manifolds admit nontrivial self coverings (see Theorem 8.6). Therefore we will concentrate on 3-manifolds in the class $\mathcal{G}$ defined below.

Notation. We use $\mathcal{G}$ to denote the set of all geometric 3-manifolds which are not covered by either (surface)$\times S^1$ or torus bundle over S^1.

Conjecture 1: *Every 3-manifold M in $\mathcal{G}$ has Property C.*

A geometric 3-manifold M is called a *graph manifold* if it has a nonempty canonical splitting surface (see section 1) cutting M into Seifert manifold pieces. It can be shown (Thm 2.5) that Conjecture 1 is true if M is not a graph manifold, so the problem is reduced to whether graph manifolds have property C.

The main result of this paper is to define a new invariant $\omega(M)$ for graph manifolds, which takes values in $\mathbf{R}$. It is defined in terms of the Euler characteristics of orbifolds of the Seifert fibered pieces of M, and the intersection numbers of fibers on splitting surfaces. The central feature of $\omega(M)$ is the covering property. Hence in the generic case, when $\omega(M)$ is nonzero, M has property C.

A closely related problem is the cohopficity of 3-manifold groups. A group is called cohopfian if it is not isomorphic to a proper subgroup of itself. In [1] Gonzales-Acuna and Whitten studied this problem, and determined the cohopficity of the fundamental groups of Haken manifolds whose boundary is a nonempty union of tori. In section 8 we will show that the fundamental group of an irreducible closed geometric 3-manifold is cohopfian if and only if it is not covered by (surface)$\times S^1$ or torus bundle over S^1.

The paper is organized as follows. In section 1 we will review some basic definitions and properties about canonical splitting surfaces and Seifert fibered spaces. In section 2 we reduce Conjecture 1 to the problem of whether graph manifolds have property C. Section 3 is devoted to the study decomposing surfaces, and section 4 to that of the fiber intersection numbers and their basic properties. In section 5 we use these informations to define weighted graphs for all nontrivial graph manifolds, and deduce a relation between the covering degree and the weights of the graph of the manifolds (Proposition 5.1). These are used in section 6 to define a matrix invariant $B(M)$, from which we will define the ω-invariant, and prove its covering property (Theorem 6.6). We give some examples in section 7. The matrix $B(M)$ contains much more information. For example, it will be used to show that a nontrivial graph manifold has no nontrivial covering over itself (Proposition 8.2). We will also show in section 8 that a geometric manifold admits a nontrivial self covering if and only if it is covered by (surface)$\times S^1$ or torus bundle over S^1 (Theorem 8.6). This will be used to solve the cohopficity problem of Gonzales-Acuna and Whitten for closed geometric manifolds (Theorem 8.7). In section 9 we study the symmetry of the matrix $B(M)$, and show that all graph manifolds with at most 3 vertices have property C_r.

1 Preliminaries

Readers are referred to [5, 4] for background material on 3-manifold theory, to [5, 4, 9] for definitions and basic results on Seifert fibered spaces, and to [9] for definitions and properties about orbifolds of Seifert fibered spaces and their Euler characteristics.

For a surface S in a 3-manifold M, we use $N(S)$ to denote a regular neighborhood of S, and use $\eta(S)$ to denote the interior of $N(S)$. A 3-manifold with boundary a (possibly empty) union of tori is called *simple* if every embedded incompressible torus in M is isotopic to a boundary component. The following are some results needed below.

Given a Haken manifold M with ∂M empty or a union of tori, there exists a surface $\mathcal{T}$, unique up to isotopy, satisfying the following conditions:

(a) Each component of $\mathcal{T}$ is a torus;

(b) Each component of $M - \eta(\mathcal{T})$ is either a simple manifold or a Seifert fibered space;

(c) $\mathcal{T}$ is minimal subject to (a) and (b).

This surface $\mathcal{T}$ is called a *canonical splitting surface* [6].

Given a Seifert fibration on M, let OM be the orbifold of M. Each regular fiber corresponds to a regular point in OM, and each singular fiber of type (p, q) corresponds to a singular point of index p. The Euler characteristic of OM is denoted by $\chi(OM)$. It is a rational number.

We say that a Seifert fibered space M has hyperbolic orbifold if $\chi(OM) < 0$. In this case, the Seifert fibration on M is unique up to isotopy [9, Theorem 3.9]. If $\varphi : \tilde{M} \to M$ is a covering map, and $\chi(OM) < 0$, then the number of fibers in $\tilde{M}$ that cover a regular fiber in M is equal to $\chi(O\tilde{M})/\chi(OM)$.

When M has nonempty boundary and is not a solid torus, $\chi(OM) \leq 0$. If $\chi(OM) = 0$, M is either a product I-bundle over a torus or a twisted I-bundle over a Klein bottle.

Lemma 1.1 *If M is a twisted I-bundle over a Klein bottle, it has exactly two Seifert fibrations up to isotopy.*

Proof. It is well known that there are two different Seifert fibrations on M, see for example [5, p87]. So we need only to show that there are at most two fibrations up to isotopy.

Consider a double covering $\psi : N \to M$, where N is a (torus)$\times S^1$. Let $f : N \to N$ be the nontrivial covering transformation. Choose generators $c_1, c_2 \in H_1(N)$ so that $f(c_1) = c_1$, and $f(c_2) = -c_2$. Suppose ξ is a fibration of M, and let $\tilde{\xi}$ be its lifting to N. Consider a fiber α of ξ on ∂M. It lifts to two fibers $\tilde{\alpha}_1, \tilde{\alpha}_2$ of $\tilde{\xi}$ on ∂N. As nonsingular fibers, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ must be isotopic. Suppose $\tilde{\alpha}_1$ represents $ac_1 + bc_2$ in $H_1(N)$. Then $\tilde{\alpha}_2 = f(\tilde{\alpha}_1) = ac_1 - bc_2$. As $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are isotopic, $a = 0$ or $b = 0$. In other words, $\tilde{\alpha}_1 = c_1$ or c_2. Therefore, $\alpha = \psi(c_1)$ or $\psi(c_2)$. The result now follows because Seifert fibrations are determined by fibers on the boundary, see [5, Lemma VI.19]. □

2 Reduction to graph manifolds

Lemma 2.1 *Conjecture 1 is true if every manifold in $\mathcal{G}$ has property C_r.*

Proof. Consider two covering maps $p_1, p_2 : N \to M$. Pick up a subgroup $G \subset p_{1*}\pi_1(N) \cap p_{2*}\pi_1(N)$ which is normal and has finite index in $\pi_1(M)$, and let $\tilde{M}$ be the covering space corresponding to G. Then the inclusion maps of G into $p_{1*}\pi_1(N)$, $p_{2*}\pi_1(N)$ and $\pi_1(M)$ induce finite regular coverings $q_1, q_2 : \tilde{M} \to N$ and $p : \tilde{M} \to M$ such that $p = p_1 q_1 = p_2 q_2$. By assumption N has property C_r, so the degrees of q_1 and q_2 are the same. It follows that p_1 and p_2 also have the same degree. □

By Lemma 2.1, in the text below we need only consider regular coverings.

Lemma 2.2 *A 3-manifold M in $\mathcal{G}$ admits a finite normal covering $\tilde{M}$ which has torsion free fundamental group.*

Proof. Let $P_1, ..., P_k$ be the prime factors of M which have finite fundamental groups. Recall that if a prime factor of M has infinite fundamental group, then it is torsion free [4, Cor 9.9]. Thus any element of finite order in $\pi_1(M)$ is conjugate to an element in $\pi_1(P_i)$ for some i. In particular, $\pi_1(M)$ has only finitely many elements of finite order up to conjugacy.

Since $\pi_1(M)$ is residually finite (see [10, 3.3]), for each torsion element x there is a normal subgroup G_x of finite index which does not contain x (and hence any conjugate of x). The intersection of all these G_x is a finite intersection, and hence contains a normal subgroup G which is of finite index in $\pi_1(M)$, and is torsion free. Let $\tilde{M}$ be the finite covering of M corresponding to the subgroup $G \subset \pi_1(M)$. Then $\tilde{M}$ is as required. □

Lemma 2.3 *Suppose a non-prime geometric 3-manifold M has torsion free fundamental group, and suppose ∂M contains no 2-spheres. Then M has property C.*

Proof. Let $p : \tilde{M} \to M$ be a finite covering with degree d.

Since $\pi_1(M)$ contains no elements of finite order and ∂M contains no 2-sphere components, we may assume that the prime factorization of M consists of c_1 aspherical prime factors and c_2 $S^2 \times S^1$ prime factors; therefore there are $c(M) = c_1 + c_2 - 1$ factorization 2-spheres. Denoted by S the union of these spheres. Suppose the prime factorization of $\tilde{M}$ consists of $c(\tilde{M})$ factorization 2-spheres.

Now $p^{-1}(S)$ is a union of $dc(M)$ two spheres. Since the prime factorization of M contains no prime factors with finite fundamental group, no component of the closure of $\tilde{M} - p^{-1}(S)$ is a punctured 3-sphere. Hence $p^{-1}(S)$ is a family of factorization 2-spheres of $\tilde{M}$. By the uniqueness of prime factorization, we have $dc(M) = c(\tilde{M})$. Since M is non-prime, $c(M)$ is not zero. Therefore,

$$d = \frac{c(\tilde{M})}{c(M)},$$

which is uniquely determined by $\tilde{M}$ and M. □

Remark. The fact that $P^3 \# P^3$ is an n-fold covering over itself for all $n > 0$ shows that the formula above does not apply to the case when M contains prime factors with finite fundamental groups. Here P^3 is the real projective 3-space.

Theorem 2.4 *Any non-prime 3-manifold M in $\mathcal{G}$ has property C.*

Proof. If ∂M contains 2-sphere components, then the covering degree of $N \to M$ is the number of 2-spheres in ∂N divided by the number of 2-spheres in ∂M. So we may assume ∂M has no sphere components. Suppose there are coverings $p_1, p_2 : N \to M$ such that the degrees of p_1 and p_2 are not the same. By the argument in proving Lemma 2.1, there are finite regular coverings $q_1, q_2 : \tilde{N} \to N$ such that the degrees of q_1 and q_2 are not the same. By Lemma 2.2, after passing to a common regular covering of M and $\tilde{N}$ if necessary, we may assume that $\pi_1(\tilde{N})$ has no torsion elements. For the same reason as above, there are coverings $r_1, r_2 : \tilde{M} \to \tilde{N}$ such that the degrees of r_1 and r_2 are not equal. By Lemma 2.3, this could happen only if $\tilde{N}$ is a prime 3-manifold; so either it is an $S^1 \times S^2$, or it is irreducible. But the first case implies that M is not in $\mathcal{G}$ by definition, and the second case contradicts the fact that $\pi_2(\tilde{N}) = \pi_2(M) \neq 0$. □

Theorem 2.5 *If a 3-manifold M in $\mathcal{G}$ is not a graph manifold, then M has property C.*

Proof. Let $\tilde{M} \to M$ be a regular covering map of degree d.

By Theorem 2.4, we may assume M is prime. In particular ∂M contains no 2-sphere components. If $\chi(\partial M) < 0$, then $\chi(\partial \tilde{M}) < 0$ and $d = \chi(\partial \tilde{M})/\chi(\partial M)$. So we may assume that $\chi(\partial M) = 0$. Thus ∂M consists of a union of tori (possibly an empty set). The canonical splitting surface (see §1) cuts M into simple manifolds and Seifert fibered spaces. By Thurston's theorem, a simple manifold admits a hyperbolic structure, which is unique up to isometries. Let $v(M)$ be the sum of the volumes of the hyperbolic pieces. Then it is an invariant of M, and it has the covering property. Therefore, if $v(M) \neq 0$, then M has property C.

Now we consider the case when M is a Seifert manifold.

If the Euler number $e(M)$ of M is zero (see [9, Section 3] for definitions), or if the Euler characteristic $\chi(OM)$ of the orbifold of M is zero, then it is known that M is finitely covered by either (surface)$\times S^1$ or a torus bundle over S^1. In this case, M is not in $\mathcal{G}$.

If $e(M) \neq 0$ and $\chi(OM) > 0$, then M is covered by S^3. So $\pi_1(M)$ is finite, and for any covering p from $\tilde{M}$ to M, the degree of p is equal to $|\pi_1(M)|/|\pi_1(\tilde{M})|$.

If $e(M) \neq 0$ and $\chi(OM) < 0$, then M admits a unique Seifert fibration. So we may assume that p is fiber preserving. Let the orbifold covering degree and the fiber covering degree of p be l and m respectively. Then $d = m\,l$, and $l = \chi(O\tilde{M})/\chi(OM)$. By Theorem 3.6 of [9], we have

$$e(\tilde{M}) = \frac{e(M)\,l}{m}$$

So we have

$$d = \frac{l^2 e(M)}{e(\tilde{M})} = \frac{\chi(O\tilde{M})^2\,e(M)}{\chi(OM)^2\,e(\tilde{M})}$$

Therefore M has property C. Clearly, $C(M) = \chi(OM)^2/e(M)$ is an invariant with the covering property.

The remaining case is when M has a nonempty canonical splitting surface, and all pieces are Seifert fibered spaces. By definition M is a graph manifold. □

3 Decomposing surfaces of graph manifolds

The canonical splitting surface of M cuts M into Seifert fibered pieces. But from our point of view, it has two disadvantages:

(1) It is not preserved by covering maps. If $\varphi : \tilde{M} \to M$ is a covering map, and $\mathcal{T}$ is the canonical splitting surface in M, then $\varphi^{-1}(\mathcal{T})$ is usually not a canonical splitting surface for $\tilde{M}$.

(2) Seifert fibrations on $M - \eta(\mathcal{T})$ may not be unique.

These problems happen when some component T of $\mathcal{T}$ bounds a twisted I-bundle over a Klein bottle. It is possible that T bounds twisted I-bundles on both sides, or that the complement of $\eta(T)$ is a (torus)$\times I$. We call such a manifold a *trivial graph manifold*. Note that a trivial graph manifold is closed, and is covered by a torus bundle over S^1. Since it is not Seifert fibered, by [9, Theorem 5.3], it admits a Sol geometry. On the other hand, if M is a nontrivial graph manifold, then some Seifert fibered piece of M has a hyperbolic orbifold, so it is not covered by torus bundle over S^1. Therefore, a graph manifold is trivial if and only if it is covered by some torus bundle over S^1.

Now consider a nontrivial graph manifold M. We define a *decomposing surface* of M to be a surface $\mathcal{S}$, which is the union of those tori in $\mathcal{T}$ that do not bound twisted I-bundles, and the central Klein bottles in the twisted I-bundle components of $M - \eta(T)$. In other words, we replace each torus T which bounds a twisted I-bundle N by the central Klein bottle in N. The following lemma follows from the above definition and the uniqueness of canonical splitting surfaces.

Lemma 3.1 *If M is not covered by a torus bundle over S^1, then it has a decomposing surface, which is unique up to ambient isotopy.* □

Consider a canonical splitting surface $\mathcal{T}$ of M. Let $\mathcal{S}$ be the corresponding decomposing surface. Notice that a twisted I-bundle over Klein bottle is a regular neighborhood of its central Klein bottle, so it is easy to see that $M - \eta(\mathcal{S})$ can be obtained from $M - \eta(T)$ by deleting the twisted I-bundle components. Therefore each component of $M - \eta(\mathcal{S})$ has hyperbolic orbifold.

On the other hand, if we have a surface $\mathcal{S}$ in M consisting of tori and Klein bottles, such that $M - \eta(\mathcal{S})$ consists of Seifert fibered spaces with hyperbolic orbifolds, then we can construct a union of tori $\mathcal{T}$ by reversing the above process, i.e $\mathcal{T}$ is the union of the tori in $\mathcal{S}$ and the boundary of the regular neighborhood of Klein bottles in $\mathcal{S}$. It is clear that $M - \eta(\mathcal{T})$ is a Seifert fibered space. Moreover, $M - \eta(\mathcal{T}')$ is Seifert fibered for some proper subset $\mathcal{T}'$ of $\mathcal{T}$ if and only if $M - \eta(\mathcal{S}')$ is Seifert fibered for some proper subset $\mathcal{S}'$ of $\mathcal{S}$. Therefore, from the definition of canonical splitting surfaces we have the following characterization of decomposing surfaces.

Lemma 3.2 *A surface $\mathcal{S}$ in M is a decomposing surface if and only if*

(a) *$\mathcal{S}$ is a union of tori and Klein bottles;*

(b) *$M - \eta(\mathcal{S})$ consists of Seifert fibered spaces with hyperbolic orbifolds;*

(c) *$\mathcal{S}$ is minimal subject to (a) and (b).* □

4 Fiber intersection number

Consider $N = T \times I$, a trivial I-bundle over a torus T. Suppose ∂N has been given an S^1 fibration structure ξ. Choose a fiber α_i on $T \times i$ for $i = 0, 1$, and let α_1' be the curve on

$T \times 0$ which is isotopic to α_1 in N. We define the fiber intersection number on T as

$$\Delta(T) = \Delta_\xi(T) = |\alpha_0 \cdot \alpha_1'|$$

where $\alpha_0 \cdot \alpha_1'$ is the algebraic intersection number of the two curves on $T \times 0$.

Suppose $\varphi : \tilde{N} \to N$ is a k-fold covering map. Then $\tilde{N}$ is also a trivial I-bundle over a torus $\tilde{T}$. Endow $\partial\tilde{N}$ with the S^1 fibration structure induced by φ.

Lemma 4.1 *Let s_i be the number of fibers in $\partial\tilde{N}$ that cover a fiber in $T \times i$. Then*

$$s_0 s_1 \Delta(\tilde{T}) = k\Delta(T).$$

Proof. Choose a fiber α_i on $T \times i$. It lifts to a union $\tilde{\alpha}_i$ of fibers. Let β_i be a component of $\tilde{\alpha}_i$. Clearly, $\tilde{\alpha}_1$ is isotopic to the lifting $\tilde{\alpha}_1'$ of α_1'. Each intersection point of α_0 and α_1' lifts to k points of intersection of $\tilde{\alpha}_0$ and $\tilde{\alpha}_1'$, so we have

$$\tilde{\alpha}_0 \cdot \tilde{\alpha}_1' = k(\alpha_0 \cdot \alpha_1').$$

Since $\tilde{\alpha}_i$ consists of s_i copies of β_i, we have

$$\tilde{\alpha}_0 \cdot \tilde{\alpha}_1' = s_0 s_1 (\beta_0 \cdot \beta_1').$$

By definition, $\Delta(\tilde{T}) = |\beta_0 \cdot \beta_1'|$. Therefore, $s_0 s_1 \Delta(\tilde{T}) = |\tilde{\alpha}_0 \cdot \tilde{\alpha}_1'| = k\Delta(T)$. □

Now suppose N is a twisted I-bundle over a Klein bottle K. Then ∂N is a single torus. Up to isotopy N has exactly two Seifert fibered structures (Lemma 1.1). We denote by c_1 a curve on ∂N that is a fiber of the fibration whose orbifold is a Möbius band, and denote by c_2 a fiber on ∂N of the other fibration. They form a coordinate system on ∂N. Given an arbitrary S^1 fibration ξ on ∂N, a fiber α of ξ represents a unique element $ac_1 + bc_2$ in $H_1(\partial N)$. We define the fiber intersection number on the Klein bottle K to be

$$\Delta(K) = |4ab|.$$

Consider a finite covering map $\varphi : \tilde{N} \to N$. $\tilde{N}$ is an I-bundle over a surface $\tilde{S}$, which is either a torus or a Klein bottle. Endow $\partial\tilde{N}$ with the induced S^1 fibration $\varphi^{-1}(\xi)$. We have

Lemma 4.2 *Suppose $\varphi : \tilde{N} \to N$ is a regular covering. Let s be the number of fibers in $\partial\tilde{N}$ that cover a fiber α of the S^1 fibration ξ in ∂N. Then*

$$s^2 \Delta(\tilde{S}) = k\Delta(K).$$

Proof. We divide the lemma into several cases.

CASE 1: *$\tilde{S}$ is a Klein bottle.*

In this case $\tilde{N}$ is a twisted I bundle over $\tilde{S}$. Consider the lifting $\tilde{c}_i$ of c_i. Suppose $\tilde{c}_i$ has t_i components, and let $\tilde{e}_i$ be one of them. Since the lifting of a Seifert fibration of N is a Seifert fibration of $\tilde{N}$, we know that $\tilde{e}_1, \tilde{e}_2$ are fibers of the Seifert fibrations on $\tilde{N}$, so they form the required coordinate system. Since α represents $ac_1 + bc_2$ in $H_1(\partial N)$, its lifting represents

$$\tilde{\alpha} = a\tilde{c}_1 + b\tilde{c}_2 = (at_1)\tilde{e}_1 + (bt_2)\tilde{e}_2.$$

Since $\tilde{\alpha}$ is a union of simple closed curves, the number of curves in $\tilde{\alpha}$ is $s = gcd(at_1, bt_2)$. Therefore each component β of $\tilde{\alpha}$ represents

$$\beta = (\frac{at_1}{s})\tilde{e}_1 + (\frac{bt_2}{s})\tilde{e}_2$$

in $H_1(\partial\tilde{N})$. By definition,

$$\Delta(\tilde{S}) = |4(\frac{at_1}{s})(\frac{bt_2}{s})| = \frac{t_1t_2}{s^2}\Delta(K).$$

The result now follows from the fact that the covering degree $k = t_1t_2$.

CASE 2: *$\tilde{N}$ is the double covering corresponding to the map $\pi_1(\partial N) \to \pi_1(N)$ induced by the inclusion map.*

In this case, $\tilde{S}$ is a torus, and $\tilde{N} = \tilde{S} \times I$. The restriction of φ to each component of $\partial\tilde{N}$ is a homeomorphism. We use c_i^j to denote the curve $\varphi^{-1}(c_i)$ on $\tilde{S} \times j$, $j = 0, 1$.

It is easy to see that c_1^0 is isotopic to $-c_1^1$ in $\tilde{N}$, while c_2^0 is isotopic to c_2^1. A fiber β_0 of $\varphi^{-1}(\xi)$ in $\tilde{S} \times 0$ represents $ac_1^0 + bc_2^0$, while a fiber β_1 in $\tilde{S} \times 1$ represents $ac_1^1 + bc_2^1$. Since in $\tilde{N}$ the curve β_1 is isotopic to $(-a)c_1^0 + bc_2^0$, we have

$$\Delta(\tilde{S}) = |\det \begin{pmatrix} a & b \\ -a & b \end{pmatrix}| = |2ab| = \frac{1}{2}\Delta(K).$$

Since in this case $k = s = 2$, the result follows.

CASE 3: *$\tilde{S}$ is a torus, $\varphi : \tilde{N} \to N$ is a map of degree $k > 2$.*

Any such map factors through a covering

$$\tilde{N} \to \hat{N} \to N,$$

where $\hat{N}$ is the double covering in Case 2. Since $\tilde{N}$ is orientable, it is a product $\tilde{S} \times I$. By assumption, φ is a regular covering, so there is a covering transformation sending $\tilde{S} \times 0$

to $\tilde{S} \times 1$. Thus the number of fibers in $\tilde{S} \times 0$ covering a fiber α on ∂N is the same as that in $\tilde{S} \times 1$. Since the total number is s, each $\tilde{S} \times i$ has $s/2$ fibers covering α. Write $\hat{N}$ as $\hat{T} \times I$. Let β_i be the curve in $\hat{T} \times i$ that covers α. Then each β_i is covered by $s/2$ fibers in $\tilde{S} \times i$. By Lemma 3.1 and Case 2 above, we have

$$
\begin{aligned}
(s/2)(s/2)\Delta(\tilde{S}) &= (k/2)\Delta(\hat{T}) \\
\Delta(\hat{T}) &= (1/2)\Delta(K)
\end{aligned}
$$

Combining these together yields $s^2\Delta(\tilde{S}) = k\Delta(K)$, as required. □

Lemma 4.3 *Suppose N is an I-bundle over S, where S is either a torus or a Klein bottle. Let ξ be an S^1 fibration on ∂N. Then ξ extends to a Seifert fibration on N if and only if $\Delta(S) = 0$.*

Proof. If S is a torus, then the extension exists if and only if the fibers of ξ on the two components of ∂N are isotopic to each other, which is equivalent to $\Delta(S) = 0$. If S is a Klein bottle, then N has exactly two fibrations up to isotopy, and ξ extends to one of them if and only the a or b in the definition of $\Delta(S)$ is 0. □

Consider a graph manifold M. Let $\mathcal{S}$ be a surface satisfying (a) and (b) of Lemma 3.2. Then $M - \eta(\mathcal{S})$ has a unique Seifert fibration structure, which gives rise to an S^1 fibration structure on $\partial N(\mathcal{S})$, so each component S of $\mathcal{S}$ has a well defined fiber intersection number $\Delta(S)$. The surface $\mathcal{S}$ satisfies (c) of Lemma 3.2 if and only if the fibration on $\partial N(\mathcal{S})$ cannot be extended to Seifert fibrations over any $N(S)$. By Lemma 4.3, this is true if and only $\Delta(S) \neq 0$ for all S in $\mathcal{S}$.

Now suppose $\varphi : \tilde{M} \to M$ is a finite covering over a graph manifold M which is not covered by torus bundles over S^1. Then M has a nonempty decomposing surface $\mathcal{S}$. Let $\tilde{\mathcal{S}}$ be the surface $\varphi^{-1}(\mathcal{S})$ in $\tilde{M}$. Clearly it satisfies (a) of Lemma 3.2. Since $\tilde{M} - \eta(\tilde{\mathcal{S}})$ covers $M - \eta(\mathcal{S})$, it also satisfies (b) of Lemma 3.2. By Lemmas 4.1 and 4.2, $\Delta(S) \neq 0$ for all S in $\mathcal{S}$ if and only if $\Delta(\tilde{S}) \neq 0$ for all $\tilde{S}$ in $\tilde{\mathcal{S}}$. Therefore, $\tilde{\mathcal{S}}$ satisfies (c) of Lemma 3.2. So we have

Corollary 4.4 *If $\mathcal{S}$ is a decomposing surface for M, then $\tilde{\mathcal{S}} = \varphi^{-1}(\mathcal{S})$ is a decomposing surface for $\tilde{M}$.* □

5 Graph of Manifolds

Suppose M is a nontrivial graph manifold. As we have seen above such a manifold has a decomposing surface $\mathcal{S}$. We define a graph $\Gamma(M)$ as follows: For each component N_i of $M - \eta(\mathcal{S})$ is assigned a vertex v_i, and for each component S_j of $\mathcal{S}$ is assigned an edge e_j, so that

(1) If S_j is a torus, and $\partial N(S_j)$ has one component in each of N_i and N_k (i may be equal to k), then e_j has endpoints on v_i and v_k;

(2) If S_j is a Klein bottle, and $\partial N(S_j)$ is in N_i, then e_j has both endpoints on v_i.

We define a "weight" for each vertex or edge of $\Gamma(M)$ as follows. If v_i is a vertex of $\Gamma(M)$ corresponding to a component M_i of $M - \eta(\mathcal{S})$, let the weight $x_i = x(v_i)$ be the Euler characteristic of the orbifold of N_i. If e is an edge corresponding to a surface S in $\mathcal{S}$, let the weight $\Delta(e)$ be the fiber intersection number $\Delta(S)$. Since each component of $M - \eta(\mathcal{S})$ has hyperbolic orbifold, the weights of vertices are nonzero rational numbers. Also, by the remark after Lemma 4.3, the weights of edges are all nonzero integers. Since the decomposing surface and the Seifert fibrations on the pieces are unique, this weighted graph is well defined, and is an invariant of M.

Consider a covering map $\varphi : \tilde{M} \to M$. As we have seen before, if $\mathcal{S}$ is a decomposing surface for M, then $\tilde{\mathcal{S}} = \varphi^{-1}(\mathcal{S})$ is a decomposing surface for $\tilde{M}$. Therefore φ induces a map on the graphs

$$\varphi : \Gamma(\tilde{M}) \to \Gamma(M).$$

It is defined in the obvious way. A vertex $\tilde{v}_p$ is mapped to v_i if the corresponding component $\tilde{N}_p$ in $\tilde{M} - \eta(\tilde{\mathcal{S}})$ is mapped to N_i, and an edge $\tilde{e}_q$ is mapped to e_j if the corresponding surface $\tilde{S}_q$ covers S_j.

We now assume φ is a regular covering. The main result of this section is Proposition 5.1 below, which expresses the covering degree d in terms of the weights of the graphs. We need the following notations.

For each vertex v_i in $\Gamma(M)$, define

$$I_i = \{p \mid \varphi(\tilde{v}_p) = v_i\}.$$

We use u_i to denote the number of elements in I_i. In other words, u_i is the number of components in $\tilde{M} - \eta(\tilde{\mathcal{S}})$ that covers the component N_i in $M - \eta(\mathcal{S})$.

Suppose $\tilde{v}_p, \tilde{v}_q$ are two vertices covering v_i Since φ is a regular covering, there is a covering transformation $f : \tilde{M} \to \tilde{M}$ which maps the component $\tilde{N}_p$ homeomorphically onto $\tilde{N}_q$. Therefore, $x(\tilde{v}_p) = x(\tilde{v}_q)$. Hence we can use $\tilde{x}_i$ to denote $x(\tilde{v}_p)$ without ambiguity.

Proposition 5.1 *Let e be an edge in $\Gamma(M)$ with endpoints on v_i and v_j. Let t be the number of edges in $\Gamma(\tilde{M})$ covering e, and let $\tilde{e}$ be one of these edges. Then*

$$d = \frac{\Delta(\tilde{e})\, \tilde{x}_i\, \tilde{x}_j\, u_i\, u_j}{t\, \Delta(e)\, x_i\, x_j}. \tag{1}$$

Proof. Let $\tilde{v}_p, \tilde{v}_q$ be the endpoints of $\tilde{e}$. As before, we use $\tilde{N}_p, \tilde{N}_q$ to denote the corresponding components of $\tilde{M} - \eta(\tilde{\mathcal{S}})$. There are several cases.

CASE 1: *The surface S corresponding to e is a torus.*

Let T_i, T_j be the tori on $\partial N(S)$, with α_i and α_j a fiber on T_i and T_j respectively. Denote by $\tilde{S}$ the torus in $\tilde{M}$ corresponding to $\tilde{e}$. By Lemma 4.1 we have

$$s_i s_j \Delta(\tilde{S}) = k\Delta(S), \tag{2}$$

where k is the covering degree of $\tilde{S} \to S$, and s_i is the number of fibers on $\partial N(\tilde{S})$ that cover α_i.

By definition t is the number of edges in $\Gamma(\tilde{M})$ that is mapped to e, so equivalently it is the number of tori in $\tilde{M}$ covering T_i. Since there are u_i vertices in $\Gamma(\tilde{M})$ covering v_i, there are t/u_i tori in $\tilde{N}_p$ that cover T_i. The orbifold covering degree of $\tilde{N}_p$ over N_i is $\tilde{x}_i/x_i$, so on each torus there are $(\tilde{x}_i/x_i)/(t/u_i)$ fibers covering α_i. That is, $s_i = \tilde{x}_i u_i/t x_i$. Substitute this into equation (2), we have

$$\tilde{x}_i \tilde{x}_j u_i u_j \Delta(\tilde{S}) = k\, t^2 x_i x_j \Delta(S) = d\, t x_i x_j \Delta(S). \tag{3}$$

CASE 2: *Both S and $\tilde{S}$ are Klein bottles.*

The computation in this case is quite similar. We omit the details.

CASE 3: *S is a Klein bottle, $\tilde{S}$ is a torus.*

Recall that $s_i = s_j$ is the number of fibers on $\partial N(\tilde{S})$ that cover a fiber α in ∂N. As the covering is regular, there is a covering transformation sending one torus of $\partial N(\tilde{S})$ to the other, so on each torus there are $s_i/2$ fibers covering α. Since each $\tilde{S}$ corresponds to 2 tori covering ∂N, there are $2t$ tori in $\tilde{M}$ that cover ∂. Thus on each $\tilde{N}_p$ there are $2t/u_i$ tori covering ∂N. Therefore, on each torus, there should be $(\tilde{x}_i/x_i)/(2t/u_i)$ fibers covering

α. That is, $s_i/2 = (\tilde{x}_i/x_i)/(2t/u_i)$. Hence $s_i = \tilde{x}_i u_i / x_i t$. Equation (3) now follows from Lemma 4.2. □

6 The invariant

Proposition 5.1 gives a formula about the degree of a covering map. However, it depends on u_i, u_j and t, which usually depend on the covering. In this section we will simplify the formula, then obtain an invariant from a system of linear equations.

For each edge e in the graph $\Gamma(M)$, define

$$a(e) = \frac{1}{\Delta(e) x_i x_j}$$

Then equation (1) can be rewritten as

$$d\, t a(\tilde{e}) = a(e) u_i u_j$$

or

$$d \sum a(\tilde{e}) = a(e) u_i u_j \tag{4}$$

where $\sum$ is taken over all $\tilde{e}$ that is mapped to e. For all i, j, define

$$a_{ij} = a_{ij}(\Gamma(M)) = \begin{cases} \sum\{a(e) \mid e \text{ has endpoints on } v_i \text{ and } v_j\} \\ 0 \qquad \text{if there is no such } e \end{cases}$$

Then the equation (4), when sum over all such e, becomes

$$d \sum a(\tilde{e}) = a_{ij} u_i u_j \tag{5}$$

where the sum is taken over all edges $\tilde{e}$ in $\Gamma(\tilde{M})$ that cover some e connecting v_i to v_j. Note that $\tilde{e}$ is such an edge if and only if it has endpoints on some $\tilde{v}_p$ and $\tilde{v}_q$, where $\tilde{v}_p$ is mapped to v_i, and $\tilde{v}_q$ is mapped to v_j. Use $\tilde{a}_{pq}$ to denote $a_{pq}(\Gamma(\tilde{M}))$. Then the above sum can be written as

$$\begin{aligned} \sum a(\tilde{e}) &= \sum\{\tilde{a}_{pq} \mid \tilde{v}_p \text{ covers } v_i,\ \tilde{v}_q \text{ covers } v_j\} \\ &= \sum\{\tilde{a}_{pq} \mid p \in I_i,\ q \in I_j\}. \end{aligned} \tag{6}$$

When $i \neq j$, the two sets I_i and I_j are disjoint, so we have

$$\sum\{\tilde{a}_{pq} \mid p \in I_i,\ q \in I_j\} = \sum_{p \in I_i} \sum_{q \in I_j} \tilde{a}_{pq}. \tag{7}$$

When $i = j$, the two sets I_i and I_j are equal, so

$$\sum\{\tilde{a}_{pq} \mid p \in I_i,\, q \in I_j\} = \frac{1}{2}\sum_{p\in I_i}\ \sum_{q\in I_j-\{p\}} \tilde{a}_{pq} + \sum_{p=q\in I_i} \tilde{a}_{pq}. \tag{8}$$

To simplify the formulas, we define

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j \\ \\ 2a_{ij} & \text{if } i = j \end{cases}$$

Define $\tilde{b}_{pq}$ in the same way. Then the above equations (5), (6) (7) and (8) can be combined into

Theorem 6.1 *Suppose $\varphi : \tilde{M} \to M$ is a regular covering over a nontrivial graph manifold. Then the degree of φ satisfies the equation*

$$d\sum_{p\in I_i}\sum_{q\in I_j} \tilde{b}_{pq} = b_{ij}u_iu_j. \tag{9}$$

□

Definition Given a nontrivial graph manifold M, let $B = B(M) = (b_{ij})$ be the matrix with b_{ij} defined as above. Let $J = (1, 1, \ldots, 1)^T$. The ω-invariant of M is defined by

$$\omega(M) = \begin{cases} 0 & \text{if } By = J \text{ has no solution} \\ \\ \sum y_i & \text{if } By = J \text{ has a solution } y = (y_1, y_2, \ldots)^T \end{cases}$$

Proposition 6.2 *$\omega(M)$ is an invariant of M.*

Proof: Recall that the weighted graph $\Gamma(M)$ is an invariant of M. The definition of $B = B(M)$ depends only on $\Gamma(M)$ and the indexing of its vertices, so up to simultaneous permutations of rows and columns it is well defined. A row permutation of B will not change the solutions to $By = J$, while a column permutation of B will result in a permutation of the components of the solutions, which has no affect on $\sum y_i$. Therefore the Proposition is true if either $By = J$ has no solution or if the solution if unique. Now suppose y, z are two solutions. Then

$$B(y - z) = 0, \text{ and } (y^T - z^T)B = 0;$$

Therefore

$$\begin{aligned}\sum y_i &= y^T J = y^T By = (z^T + (y^T - z^T))B(z + (y - z)) \\ &= z^T Bz + z^T B(y - z) + (y^T - z^T)By = z^T Bz \\ &= \sum z_i\end{aligned}$$

□

Suppose $\tilde{M}$ is a covering space over M. As before, we use $\tilde{B} = B(\tilde{M}) = (\tilde{b}_{pq})$ to denote the corresponding matrix of $\tilde{M}$. Given i, j, and $q \in I_j$, define

$$c_{iq} = \sum_{p \in I_i} \tilde{b}_{pq}. \tag{10}$$

We need the following

Lemma 6.3 *If $\varphi : \tilde{M} \to M$ is a regular covering, then c_{iq} is independent of the choice of $q \in I_j$.*

Proof: Suppose $q, q' \in I_j$. Since φ is a regular covering, there is a covering transformation f sending a component $\tilde{M}_q$ of $\tilde{M} - \eta(\tilde{S})$ to the component $\tilde{M}_{q'}$. Suppose f sends $\tilde{M}_p$ to $\tilde{M}_{p'}$. Then an edge e having endpoints on $\tilde{v}_p$ and $\tilde{v}_q$ is sent to the edge $f(e)$ with endpoints on $\tilde{v}_{p'}$ and $\tilde{v}_{q'}$. Since f is a homeomorphism, it induces a weight preserving isomorphism of the graph $\Gamma(\tilde{M})$; so $a(e) = a(f(e))$. Summing over all edges connecting $\tilde{v}_p$ and $\tilde{v}_q$, we have $\tilde{a}_{pq} = \tilde{a}_{p'q'}$; therefore $\tilde{b}_{pq} = \tilde{b}_{p'q'}$. While v_p ranges over all vertices with $p \in I_i$, $v_{p'} = f(v_p)$ also ranges over all such vertices. Hence,

$$\sum_{p \in I_i} \tilde{b}_{pq} = \sum_{p' \in I_i} \tilde{b}_{p'q'} = \sum_{p \in I_i} \tilde{b}_{pq'}.$$

□

Proposition 6.4 *Suppose $\varphi : \tilde{M} \to M$ is a d-fold regular covering. If $\tilde{B}z = J$ has a solution, then so does $By = J$, and $\omega(\tilde{M}) = d\,\omega(M)$.*

Proof: Since c_{iq} is independent of $q \in I_j$, we may write it as c_{ij}. By Theorem 6.1, we have

$$\begin{aligned}b_{ij} u_i u_j &= d \sum_{p \in I_i} \sum_{q \in I_j} \tilde{b}_{pq} = d \sum_{q \in I_j} c_{ij} \\ &= d u_j c_{ij}\end{aligned} \tag{11}$$

Suppose z is a solution to $\tilde{B}z = J$. Let

$$y_j = \frac{1}{d}\sum_{q\in I_j} z_q$$

Then

$$\begin{aligned}\sum_j b_{ij}y_j &= \sum_j \frac{dc_{ij}}{u_i}\frac{1}{d}\sum_{q\in I_j} z_q \\ &= \sum_j\sum_{q\in I_j}\frac{c_{iq}z_q}{u_i} = \sum_q \frac{c_{iq}z_q}{u_i} \\ &= \sum_{p\in I_i}\sum_q \frac{\tilde{b}_{pq}z_q}{u_i} = \sum_{p\in I_i}\frac{1}{u_i} = 1.\end{aligned}$$

Therefore y is a solution to $By = J$, and

$$d\omega(M) = d\sum_j y_j = \sum_q z_q = \omega(\tilde{M}).$$

□

Proposition 6.5 *Suppose $\varphi : \tilde{M} \to M$ is a d-fold regular covering. If $By = J$ has a solution, then $\tilde{B}z = J$ also has a solution.*

Proof. Since B is symmetric, by Lemma 6.3, $\sum_{q\in I_j}\tilde{b}_{pq}$ is independent of the choice of $p \in I_i$. Therefore

$$\sum_{q\in I_j}\tilde{b}_{pq} = \sum_{p\in I_i}\sum_{q\in I_j}\tilde{b}_{pq}/u_i$$

For $q \in I_j$, let $z_q = dy_j/u_j$. Suppose $p \in I_i$. Then

$$\begin{aligned}\sum_q \tilde{b}_{pq}z_q &= d\sum_q \tilde{b}_{pq}y_j/u_j = d\sum_j(\sum_{q\in I_j}\tilde{b}_{pq})y_j/u_j \\ &= d\sum_j(\sum_{p\in I_i}\sum_{q\in I_j}\tilde{b}_{pq}/u_i)y_j/u_j \\ &= \sum b_{ij}y_j = 1.\end{aligned}$$

□

Theorem 6.6 *Suppose $\tilde{M} \to M$ is a d-fold covering over a nontrivial graph manifold. Then*

$$\omega(\tilde{M}) = d\,\omega(M).$$

Proof. By the above two propositions, the theorem is true if it is a regular covering. (In particular $\omega(M) = 0$ if and only if $\omega(\tilde{M}) = 0$.) The general case follows by passing to a common regular covering of M and $\tilde{M}$. □

Corollary 6.7 *If $\omega(M) \neq 0$, then the degree of a covering over M is determined by the manifolds, i.e. M has Property C.* □

7 Some examples

Example 1. The simplest graph manifolds are those whose graphs have only one vertex. Since M is not Seifert fibered, there must be some edges. Therefore the matrix $B(M)$ has only one entry $b_{11} \neq 0$, and

$$\begin{aligned}\omega(M) &= 1/b_{11} = 1/(2\sum \frac{1}{a(S)x(v)^2}) \\ &= x(v)^2/(2\sum \frac{1}{a(S)})\end{aligned}$$

where the sum is taken over all the components in a decomposing surface.

Example 2. The graph $\Gamma(M)$ of M has two vertices. In this case there are three possibilities for the reduced graph Γ' of $\Gamma(M)$, according to the number its of edges.

If Γ' has only one edge, then $b_{12} \neq 0$, and $b_{11} = b_{22} = 0$. We have

$$\omega(M) = 2/b_{12} = 2x(v_1)x(v_2)/\sum \frac{1}{\Delta(S)}$$

where the sum is taken over all surfaces in a decomposing surface of M.

Suppose Γ' has two edges. Without loss of generality we may assume that $b_{22} = 0$. Thus the matrix is

$$B(M) = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & 0 \end{pmatrix}$$

In this case the matrix is also nonsingular, and a solution to $Bx = J$ is given by $x_1 = 1/b_{12}$ and $x_2 = (b_{12} - b_{11})/b_{12}^2$. Hence

$$\omega(M) = \frac{1}{b_{12}} + \frac{b_{12} - b_{11}}{b_{12}^2} = \frac{2b_{12} - b_{11}}{b_{12}^2}$$

which is nonzero unless $b_{11} = 2b_{12}$.

When Γ' has three edges, all b_{ij} are nonzero. If $\det B = b_{11}b_{22} - b_{12}^2 \neq 0$, we have

$$\omega(M) = \frac{b_{11} + b_{22} - 2b_{12}}{b_{11}b_{22} - b_{12}^2}$$

If $\det B = 0$, then $\omega(M) = 0$ unless $b_{11} = b_{22} = b_{12}$, in which case $\omega(M) = 1/b_{11}$.

Example 3. The reduced graph of $\Gamma(M)$ is a chain, i.e $b_{ij} = 0$ unless $i = j \pm 1$.

Let n be the number of vertices. If n is even, then $\det B(M) = b_{12}^2 + b_{34}^2 + \ldots + b_{n-1,n}^2$. The ω invariant can be computed explicitly. For example, when $n = 6$,

$$\omega(M) = 2(\frac{1}{b_{12}} + \frac{1}{b_{34}} + \frac{1}{b_{56}} - \frac{b_{23}}{b_{12}b_{34}} - \frac{b_{45}}{b_{34}b_{56}} + \frac{b_{23}b_{45}}{b_{12}b_{34}b_{56}})$$

If n is odd, $\det B(M) = 0$, so $\omega(M) = 0$ unless

$$\det \begin{pmatrix} b_{12} & & & \ldots & 1 \\ b_{23} & b_{34} & & \ldots & 1 \\ & b_{45} & b_{56} & \ldots & 1 \\ & & b_{67} & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} = 0$$

Example 4. Let $f : S \to S$ be an orientation preserving homeomorphism of a compact orientable surface S. f is called reducible if there is a nontrivial simple closed curve C so that $f(C)$ is isotopic to C. Otherwise f is irreducible. A set $\mathcal{C} = C_1 \cup \ldots \cup C_k$ of mutually disjoint, mutually nonparallel, nontrivial simple closed curves on S is called a canonical reducing set of f if

(a) $f(\mathcal{C})$ is isotopic to $\mathcal{C}$. We can thus choose a map g isotopic to f with $g(\mathcal{C}) = \mathcal{C}$. Then for some n, g^n maps each component S_i of $S - \mathcal{C}$ to itself.

(b) For each i, $g^n|S_i$ is either irreducible or is isotopic to a map of finite order.

(c) the number of curves in $\mathcal{C}$ is minimal subject to (a) and (b).

It is known that $\mathcal{C}$ is unique up to isotopy [12]. The mapping torus $M = M_f = S \times I/\{(x,0) = (f(x),1)\}$ is a graph manifold if and only if $\mathcal{C}$ is nonempty, and $g^n|S_i$ is isotopic to a map of finite order for each i. Since M_f is determined by the isotopy class of f, we may assume that $f(\mathcal{C}) = \mathcal{C}$, and f is periodic on $S - N(\mathcal{C})$. The image of $\mathcal{C} \times I$ in M is then the canonical splitting surface of M. Since M contains no twisted I-bundles, it is also the decomposing surface.

To compute $\omega(M)$, notice that M_{f^n} is an n-fold covering of M. By the covering property of ω, we need only to compute $\omega(M_{f^n})$. Therefore without loss of generality we may assume that f is an identity map on $S - N(\mathcal{C})$. Hence it is a composition of Dehn twists on the C_i's. Let τ_i be a Dehn twist on the curve D_i. Then $f = \tau_1^{n_1} \dots \tau_k^{n_k}$, and all the n_i's are nonzero because $\mathcal{C}$ is minimal by (c). Now each component S_i of $S - \mathcal{C}$ corresponds to a vertex v_i of $\Gamma(M)$, with weight $x(v_i) = \chi(S_i)$, and each curve C_r incident to S_i and S_j corresponds to an edge e_r connecting v_i to v_j, with weight $\Delta(e_r) = |n_r|$. The ω-invariant of M is now easily computed from $\Gamma(M)$. For example, if S is a genus 2 closed surface and f is a single Dehn twist along a nontrivial separating curve, then $\Gamma(M)$ has two vertices and one edge, $b_{12} = 1$, and $\omega(M) = 2$.

8 Self coverings and cohopficity

The ω-invariant determines the covering degrees as long as it is nonzero. But the matrix $B(M)$ contains more information. We have the following

Conjecture: *The degree of a covering map $\varphi : \tilde{M} \to M$ is determined by $B(\tilde{M})$ and $B(M)$.*

Here is an example to show that some other invariants induced from $B(M)$ may help to determine the degrees of covering maps.

Let $v(M)$ be the number of vertices in $\Gamma(M)$. Define $b(M) = \sum b_{ij}$, where b_{ij} are the entries of $B(M)$, and the sum is taken over all the entries. Suppose $\varphi : \tilde{M} \to M$ is a regular covering map, and suppose $v(\tilde{M}) = v(M)$. Then each vertex of $\Gamma(M)$ is covered by exactly one vertex of $\Gamma(\tilde{M})$. In other words, $u_i = |I_i| = 1$ for all i. Theorem 6.1 now reduces to $d\tilde{b}_{ij} = b_{ij}$, where $\tilde{b}_{ij}$ are the entries of $B(\tilde{M})$. Therefore, we have

Proposition 8.1 *Suppose $\varphi : \tilde{M} \to M$ is a regular covering map over a nontrivial graph manifold. If $v(\tilde{M}) = v(M)$, then the covering degree is given by $d = b(M)/b(\tilde{M})$.* □

Consider a (possibly non-regular) covering $\varphi : \tilde{M} \to M$. Let $\tilde{\mathcal{T}}$ and $\mathcal{T}$ be the decomposing surfaces of $\tilde{M}$ and M respectively. In the proof of Proposition 5.1 we needed to use covering transformations to show that some of the weights of the graph $\Gamma(\tilde{M})$ are repeated. However, if $\tilde{\mathcal{T}}$ and $\mathcal{T}$ have the same number of tori and the same number of Klein bottles, then every torus (resp. Klein bottle) in $\mathcal{T}$ is covered by a single torus (resp. Klein bottle) in $\tilde{\mathcal{T}}$, so we do not need to use covering transformations. Hence the conclusion of

Proposition 5.1 is still true in this setting, and is reduced to

$$d = \frac{\Delta(\tilde{e})\tilde{x}_i\tilde{x}_j}{\Delta(e)x_ix_j} = \frac{a(e)}{a(\tilde{e})}$$

Summing over all edges gives $d = b(M)/b(\tilde{M})$. In particular we have

Proposition 8.2 *If M is a 3-manifold in $\mathcal{G}$, then it admits no covering maps of degree > 1 over itself.* □

This result has also been obtained in [11]. The following lemmas will be used to prove the converse for geometric manifolds. Lemma 8.3 is essentially due to Gonzales-Acuna, Litherland and Whitten [2]. For Nil manifolds, Lemma 8.5 has also been proved in [2] with different method.

In the rest of this section (only), we will consider nonorientable manifolds as well.

Lemma 8.3 *Suppose M is either geometric, or is nonorientable and its orientable double covering is geometric. If M is covered by (surface)$\times S^1$, it admits nontrivial self coverings.*

Proof. In this case, the Euler number e of the fibration is 0. First suppose M is closed. If M is orientable, by [9, §4], M admits one of the geometric structures of $S^2 \times R$, E^3 and $H^2 \times R$. If M is nonorientable, by [8] and the references there, M also admits such a geometric structure. By the discussion in [9, §4], M has the structure of a surface bundle over a 1-dimensional orbifold, so there is a two sided surface F cutting M into one or two (possibly twisted) I-bundles over surfaces. We notice that such a surface F also exists in any compact Seifert manifold with boundary. For if a manifold M is covered by $Q \times S^1$, where Q is a compact surface, then the double DM of M is covered by $DQ \times S^1$, so by the above argument DM has a two sided embedded surface F' transverse to all fibers. Let F be a component of $F' \cap M$. Then F cuts M into one or two I-bundles over some surfaces.

Fix the surface F above. Notice that if an I-bundle M_1 over surface has F as a boundary component, then either M_1 or its double cover is homeomorphic to $F \times I$. Therefore M is covered by some F-bundle $\tilde{M} = F \times I/f$. Since the I-bundle structure arises from a Seifert fibration, the gluing map f is periodic. After passing to another covering if necessary, we may assume f is the identity map, i.e it sents $(x, 0)$ to $(x, 1)$, and the covering map $\varphi : \tilde{M} \to M$ sends (x, i) to $x \in F$, $i = 0, 1$. Let $p : F \times I \to \tilde{M}$ be the quotient map. Now it is easy to construct a self covering of M: Cut along F to get a manifold N. Gluing $\partial(F \times I)$ to the two copies of F on ∂N by the identity map, we get a manifold $M' = (N \cup F \times I / \cong)$ homeomorphic to M. The covering map $M' \to M$ is given by mapping N to M by the identity map, and mapping $F \times I$ to M by the composition $\varphi \circ p$. □

We refer the readers to [9] for definitions and properties of Sol or Nil manifolds. The following result supplements [9, Theorem 5.3]

Lemma 8.4 *If M is a nonorientable Sol manifold, then it is either a union of two twisted I-bundles over a torus along their boundaries, or is a torus bundle over S^1.*

Proof. By [9, Theorem 5.3], a Sol manifold is either a bundle over S^1 with fiber the torus or Klein bottle, or is the union of two twisted I-bundles over the torus or Klein bottle.

Notice that up to isotopy a Klein bottle K has only one orientation preserving nonseparating simple closed curve, say α. (It represents the torsion element in $H_1(K)$.) Thus, if f is a homeomorphism of K, then up to isotopy we may assume that $f(\alpha) = \alpha$. If M is a Klein bottle bundle over S^1, then $M = K \times I/f$, where $f : K \times 0 \to K \times 1$ is a homeomorphism. We may assume that $f(\alpha \times 0) = \alpha \times 1$. Thus the S^1-fibration of $K \times I$ by circles parallel to α gives rise to a Seifert fibration of M. This is impossible because a Sol manifold does not admit Seifert fibrations.

Similarly, if $M = M_1 \cup_f M_2$, with M_i twisted I-bundles over a Klein bottle such that the ∂M_i are Klein bottles, then the orientation preserving nonseparating simple closed curves α_i extend to Seifert fibrations on M_i, and f maps α_1 to α_2. Thus M is Seifert fibered, again a contradiction.

In the remaining cases, M is either orientable, or is a torus bundle over a 1-orbifold. This completes the proof. □

Lemma 8.5 *If a 3-manifold M is covered by some torus bundle over S^1, then it admits nontrivial self coverings.*

Proof. If the gluing map of the torus bundle is periodic, M is also covered by (surface)$\times S^1$, so the result follows from Lemma 8.3. Otherwise M admits either a Nil or Sol structure. Any Nil manifold is orientable because the Euler number of its Seifert bundle is nonzero. By [9, Theorem 5.3] and Lemma 8.4, we have only three possibilities: (1) M is a torus bundle over S^1; (2) $M = M_1 \cup_f M_2$, where the M_i are orientable, and are twisted I-bundles over a Klein bottle; (3) $M = M_1 \cup_f M_2$, where M_i are twisted I-bundles over a torus.

In Case 1, we have $M = T \times I/f$, where $f : T \times 0 \to T \times 1$ is a homeomorphism. Let $\varphi : T \times I \to T \times I$ be a self covering such that $\varphi_*(x) = x^p$ for all $x \in \pi_1(T \times I)$, where $p > 1$ is a positive integer. Then

$$f_*\varphi_*(x) = f_*(x^p) = (f_*(x))^p \in \varphi_*(\pi_1(T \times 1))$$

i.e $f_*\varphi_*\pi_1(T \times 0)$ is a subgroup of $\varphi_*\pi_1(T \times I)$. Since $f_* : \pi_1(T \times 0) \to \pi_1(T \times 1)$ is an isomorphism, by checking the indices of subgroups, one can see that $f_* : \varphi_*\pi_1(T \times 0) \to \varphi_*\pi_1(T \times 1)$ is actually an isomorphism. Thus f can be lifted to a homeomorphism $\tilde{f} : T \times 0 \to T \times 1$ so that $\varphi \circ \tilde{f} = f \circ \varphi$ on $T \times 0$. Hence φ induces a covering map $\bar{\varphi} : T \times I/\tilde{f} \to T \times I/f$. Also, for all $x \in \pi_1(T \times 0)$,

$$\varphi_*\tilde{f}_*(x) = f_*\varphi_*(x) = f_*(x^p) = (f_*(x))^p = \varphi_*(f_*(x))$$

As φ_* is injective, $\tilde{f}_* = f_*$, so $\tilde{f}$ is isotopic to f. It follows that $T \times I/\tilde{f}$ is homeomorphic to M.

In Case 2, let $\pi_1(M_i) =< a_i, b_i \mid a_i b_i a_i^{-1} = b_i^{-1} >$, where a_i, b_i are represented by simple closed curves on the central Klein bottle of M_i. For odd integers $p > 1$, one can construct p^2 fold coverings $\varphi_i : M_i \to M_i$ so that $\varphi_*(a_i) = a_i^p$ and $\varphi_*(b_i) = b_i^p$. Let α_i, β_i be generators of $\pi_1(\partial M_i)$ such that $j_*(\alpha_i) = a_i^2$, and $j_*(\beta_i) = b_i$, where $j : \partial M \to M$ is the inclusion map. Denote by ψ_i the restriction of φ_i on ∂M_i. Then $\psi_{i*}(\alpha_i) = \alpha_i^p$, and $\psi_{i*}(\beta_i) = \beta_i^p$. Now we can apply the argument in Case 1 to show that f lifts to a map $\tilde{f} : \partial M_1 \to \partial M_2$ such that $\psi_2 \tilde{f} = f \psi_1$, and $\tilde{f}$ is isotopic to f; so the φ_i induce a covering map $\bar{\varphi} : M_1 \cup_{\tilde{f}} M_2 \to M$, and $M_1 \cup_{\tilde{f}} M_2$ is homeomorphic to M.

The proof of Case 3 is similar to that of Case 2. In this case M admits self coverings of degree p^2 for all $p \geq 1$. □

Theorem 8.6 *Suppose M is either geometric, or is nonorientable and is double covered by some geometric manifold. Then M admits a nontrivial self covering if and only if it is covered by (surface)×S^1 or a torus bundle over S^1.*

Proof. The orientable case follows directly from Proposition 8.2, Lemma 8.4 and Lemma 8.5. So suppose M is nonorientable. If M admits a nontrivial self covering, its orientable double cover also does, so the necessity follows from (1). The sufficiency follows from Lemmas 8.3 and 8.5. □

The above results can be used to answer the cohopficity problem of closed geometric manifold groups. A group G is called *cohopfian* if it is not a proper subgroup of itself. In [1] Gonzales-Acuna and Whitten studied the cohopficity of 3-manifold groups. They solved this problem for a Haken manifold M with boundary a nonempty union of tori, and asked what closed irreducible 3-manifolds have infinite cohopfian groups. The reason for restricting to irreducible manifolds is as follows. If $M = S^1 \times S^2$, then $\pi_1(M) = \mathbf{Z}$ is clearly not cohopfian. If $M = M_1 \# M_2$ and $\pi_1(M_i) \neq 1$, then $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$. Pick nontrivial elements $\alpha \in \pi_1(M_1)$ and $\beta \in \pi_1(M_2)$. Then $\pi_1(M)$ has a proper subgroup $(\pi_1(M_1))^{\beta} * (\pi_1(M_2))^{\alpha}$ isomorphic to $\pi_1(M)$ itself, where G^x represents the subgroup $\{xgx^{-1} \mid g \in G\}$. Therefore $\pi_1(M)$ is not cohopfian. Note also that if $\pi_1(M_2) = 1$, then $\pi_1(M) = \pi_1(M_1)$, so $\pi_1(M)$ is cohopfian if and only if $\pi_1(M_1)$ is.

In [2] Gonzales-Acuna, Litherland and Whitten answered the above question for Seifert fibered manifolds. The group of a Seifert bundle is cohopfian if and only if it is not covered by (surface)$\times S^1$ or torus bundle over S^1. Using Theorem 8.6, we can show that this is also true for all irreducible geometric manifolds.

Theorem 8.7 *If M is a closed geometric 3-manifold, then $\pi_1(M)$ is cohopfian if and only if M is irreducible and is not covered by (surface)$\times S^1$ or torus bundle over S^1.*

Proof. As remarked before, we may assume M is irreducible. By [2], the result is true for Seifert fibered spaces. So we may assume that M is either hyperbolic or is a Haken manifold. If M is covered by (surface)$\times S^1$ or torus bundle over S^1, then by Theorem 8.6, M admits nontrivial self coverings, so it is not cohopfian.

Suppose G is a proper subgroup of $\pi_1 M$ which is isomorphic to $\pi_1 M$. Let $\varphi: \tilde{M} \to M$ be the covering map corresponding to G. Then $M, \tilde{M}$ have the same fundamental group. Since $H_3(G) = H_3(M)$, $\tilde{M}$ is closed. It is well known that closed hyperbolic or Haken manifolds are determined by their fundamental groups. Therefore φ is a nontrivial self covering of M. It follows from Theorem 8.6 that M is covered by (surface)$\times S^1$ or torus bundle over S^1. □

Proposition 8.8 *Let M be an orientable irreducible 3-manifold that double covers a closed nonorientable manifold N. If $\pi_1 M$ is cohopfian, so is $\pi_1 N$.*

Proof. Suppose $\pi_1 N$ is not cohopfian. Let G be a proper subgroup of $\pi_1 N$, and let $\varphi: G \to \pi_1 N$ be an isomorphism. We may assume $\pi_1 N$ is infinite, otherwise it is cohopfian. Since M is irreducible, N is aspherical, so we have $H_3(G;\mathbf{Z}_2) = H_3(N;\mathbf{Z}_2) \neq 0$, (see [4] for homology of groups.) Thus the covering space of N corresponding to G is closed, and hence G is of finite index in $\pi_1 N$.

Consider $\pi_1 M$ as a subgroup of index 2 in $\pi_1 N$. Let $G_+ = G \cap \pi_1 M$. Since G_+ is a subgroup of finite index in $\pi_1 M$, it is the group of some closed orientable aspherical 3-manifold, so $H_3(G_+;\mathbf{Z}) = \mathbf{Z}$. Since $H_3(G;\mathbf{Z}) = H_3(N;\mathbf{Z}) = 0$, we have $G_+ \neq G$, i.e G_+ is a proper subgroup of G. As $\pi_1 M$ has index 2 in $\pi_1 N$, $G_+ = G \cap \pi_1 M$ has index 2 in G.

Notice that $\varphi(G_+)$ has index 2 in $\pi_1 N = \varphi(G)$. If it is not contained in $\pi_1 M$, then it contains some elements represented by orientation reversing curves of N, so the covering space $\tilde{M}$ corresponding to $\varphi(G_+)$ is nonorientable. This is impossible because $H_3(\tilde{M};\mathbf{Z}) = H_3(G_+;\mathbf{Z}) = \mathbf{Z}$, as shown above. Therefore $\varphi(G_+) \subset \pi_1 M$. Since they have the same index in $\pi_1 N$, $\varphi(G_+) = \pi_1 M$. Thus φ induces an isomorphism $G_+ \cong \pi_1 M$. Use $[\pi_1 N : G_+]$ to denote the index of G_+ in $\pi_1 N$. Then

$$[\pi_1 N : G_+] = [\pi_1 N : \pi_1 M][\pi_1 M : G_+] = [\pi_1 N : G][G : G_+]$$

As $[\pi_1 N : \pi_1 M] = [G : G_+] = 2$, we have $[\pi_1 M : G_+] = [\pi_1 N : G] \neq 1$, so G_+ is a proper subgroup of $\pi_1 M$, contradicting the assumption that $\pi_1 M$ is cohopfian. □

Theorem 8.9 *If a closed nonorientable 3-manifold N is double covered by an irreducible geometric 3-manifold M, then $\pi_1 N$ is cohopfian if and only if M is not covered by (surface)$\times S^1$ or torus bundle over S^1.*

Proof. if N is not covered by (surface)$\times I$ or torus bundle over S^1, then M is not either, so by Theorem 8.7 and Proposition 8.8, $\pi_1(N)$ is not cohopfian. If N is covered by (surface)$\times I$ or torus bundle over S^1, it admits nontrivial self covering by Theorem 8.6, so $\pi_1(N)$ is not cohopfian. □

We do not know whether $\pi_1 N$ is cohopfian if N is double covered by a reducible manifold.

9 Symmetries of the B-matrix

In this section we study the symmetry of the matrix $B(M)$. This will provide some useful information in determining the covering degrees. Notice that when $\Gamma(M)$ has only one vertex, the ω invariant of M is a positive number, so M has property C. This method does not work for some graph manifolds with more than one vertex, but we will use symmetry to determine the degrees of regular covering maps over graph manifolds with at most 3 vertices.

To clarify the idea, we define a weighted graph $L(M)$ as follows. $L(M)$ has the same vertices as $\Gamma(M)$, and it has one edge e_{ij} for each pair of vertices v_i, v_j (i may equal to j). Assign b_{ij} as the weight of e_{ij}. An *isometry* g of $L(M)$ is an automorphism of the graph which preserves the weights, i.e the weight of e_{ij} equals that of $g(e_{ij})$ for all i, j. Denote the isometry group of $L(M)$ by $G(M)$. It induces an action on the set $V(M)$ of vertices of $L(M)$. We denote by $[v]_G$ the orbit of v under this action, by $n(v)$ the number of elements in the orbit $[v]_G$, and by $O_G(M) = |V(M)/G(M)|$ the number of orbits in $V(M)$.

Suppose $\varphi : \tilde{M} \to M$ is a regular covering map. A covering transformation $f : \tilde{M} \to \tilde{M}$ induces a map $f_\# : L(\tilde{M}) \to L(\tilde{M})$ which is clearly an isometry. Let $F(\tilde{M})$ be the set of isometries induced by covering transformations. Then $F(\tilde{M})$ is a subgroup of $G(\tilde{M})$. Since φ is a regular covering, it induces a one to one correspondence $V(\tilde{M})/F(\tilde{M}) \to V(M)$. Since $F(\tilde{M})$ is a subgroup of $G(\tilde{M})$, for each vertex $v \in V(M)$, $\varphi^{-1}(v)$ is contained in a G-orbit of $V(\tilde{M})$. In other words, if $\tilde{v}$ covers v, then $\varphi^{-1}(v) \subset [\tilde{v}]_G$.

Define

$$c(M) = \sum_{i,j} \frac{b_{ij}}{n(v_i)\, n(v_j)}.$$

Clearly, this is an invariant of M.

Theorem 9.1 *Suppose $\varphi : \tilde{M} \to M$ is a regular covering over a nontrivial graph manifold. If $O_G(\tilde{M}) = v(M)$, then $dc(\tilde{M}) = b(M)$.*

Proof. Recall that $b(M) = \sum b_{ij}$. By Theorem 6.1 we have

$$d \sum_{p \in I_i} \sum_{q \in I_j} \tilde{b}_{pq} = b_{ij} u_i u_j,$$

where u_i is the number of vertices in $L(\tilde{M})$ that cover v_i. Since $O_G(\tilde{M}) = v(M)$, each G-orbit covers only one vertex of $L(M)$. Therefore, $u_i = n(\tilde{v}_p)$ if $p \in I_i$. Now the above formula can be written as

$$d \sum_{p \in I_i} \sum_{q \in I_j} \frac{\tilde{b}_{pq}}{n(\tilde{v}_p)\, n(\tilde{v}_q)} = b_{ij}.$$

Summing over all i, j, we get the required formula $dc(\tilde{M}) = b(M)$. □

In general, we have $1 \leq O_G(\tilde{M}) \leq v(M)$. Since $b(M) \neq 0$, the above theorem determines the degree d in one extreme case. The following result shows that the ω invariant determines d in the other extreme case.

Theorem 9.2 *If $O_G(M) = 1$, then $\omega(M) = v(M)^2/b(M)$.*

Proof. By definition, for any i, j, there is an isometry $g \in G(M)$ sending v_i to v_j. Suppose g maps v_k to $v_{g(k)}$. Since g is an isometry, we have $b_{ik} = b_{jg(k)}$. When v_k ranges over all vertices of $L(M)$, $v_{g(k)}$ also ranges over all vertices. Therefore,

$$\sum_k b_{ik} = \sum_k b_{jg(k)} = \sum_k b_{jk};$$

that is, $c = \sum_k b_{ik}$ is a constant. Since $B(M)$ is a nonzero matrix, c is nonzero. It follows that $x^T = (1/c, \ldots, 1/c)^T$ is a solution to $B(M)x = J$. Therefore, $\omega(\tilde{M}) = \sum x_i = v(M)/c = v(M)^2/\sum b_{ij} = v(M)^2/b(M)$. □

Corollary 9.3 *Suppose $\varphi : \tilde{M} \to M$ is a regular covering over a nontrivial graph manifold. If $O_G(\tilde{M}) = 1$, then $\omega(\tilde{M}) \neq 0$; hence $d = \omega(\tilde{M})/\omega(M)$.* □

Theorem 9.4 *Suppose $\varphi : \tilde{M} \to M$ is a regular covering over a nontrivial graph manifold. If $v(M) \leq 3$, then the degree of φ is determined by $\tilde{M}$ and M.*

Proof. By the above discussion, if $O_G(\tilde{M}) = 1$ or $v(M)$, then the result is true; so we may assume $v(M) = 3$ and $O_G(\tilde{M}) = 2$.

Let $\tilde{V}_i$ be the set of vertices in $L(\tilde{M})$ that cover v_i. Since $O_G(\tilde{M}) = 2$, without loss of generality we may assume that $\tilde{V}_1$ is a G-orbit, and $\tilde{V}_2 \cup \tilde{V}_3$ is the other one. Let $\tilde{v}_p, \tilde{v}_q \in \tilde{V}_2 \cup \tilde{V}_3$ be any two vertices. By assumption there is an isometry g sending $\tilde{v}_p$ to $\tilde{v}_q$. Since $\tilde{V}_1$ is an orbit, g induces a permutation on $\tilde{V}_1$. Therefore,

$$\sum_{r \in I_1} \tilde{b}_{pr} = \sum_{r \in I_1} \tilde{b}_{qr} = c$$

is a constant. Thus we have

$$b_{12}u_1u_2 = d\sum_{p\in I_2}\sum_{r\in I_1}\tilde{b}_{pr} = d\sum_{p\in I_2} c = d\,u_2c,$$

so $b_{12} = d\,c/u_1$. Similarly, we have $b_{13} = d\,c/u_1 = b_{12}$.

To compute the degree, we consider the subgraph $L'(\tilde{M})$ consisting of all edges with one endpoint in $\tilde{V}_1$ and the other in $\tilde{V}_2\cup\tilde{V}_3$, and in $L(M)$ consider $L'(M) = e_{12}\cup e_{13}$. Since $G(\tilde{M})$ is determined by $\tilde{M}$, we see that $L'(\tilde{M})$ is determined by $\tilde{M}$. If $b_{23} \neq b_{12} = b_{13}$, then $L'(M)$ is also well defined. In other words, one can "see" the subgraph directly from $L(M)$ without appealing to φ. If $b_{23} = b_{12} = b_{13}$, we can pick up any two edges, and the weighted graph is always the same. Therefore, as a weighted graph $L'(M)$ is also well defined. That is, it is independent of φ, as long as $O_G(\tilde{M}) = 2$.

Define $\omega(L'(M))$ in the same manner as in the definition of $\omega(M)$. Actually, one can see that if we use M' to denote the manifold obtained from M by cutting along those surfaces in the decomposing surface of M that does not correspond to the edges e_{12} or e_{13}, then $\omega(L'(M)) = \omega(M')$. Thus the covering property is still true, and we have

$$d\,\omega(L'(\tilde{M})) = \omega(L'(M)).$$

Since $\omega(L'(M)) = 2/b_{12} \neq 0$, the degree d is determined by M and $\tilde{M}$. □

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References

[1] F. Gonzales-Acuna and W. Whitten: Imbeddings of 3-manifold groups, *Mem. Amer. Math. Soc.* (to appear).

[2] F. Gonzales-Acuna, R. Litherland and W. Whitten: Cohopficity of Seifert bundle groups, *preprint.*

[3] M. Gromov: Volume and bounded cohomology, *Publ. Math. IHES*, **56** (1983) 5–99.

[4] J. Hempel: *3-manifolds*, Ann. of Math. Study **86**, Princeton University Press 1976.

[5] W. Jaco: *Lectures on three-manifold topology*, Amer. Math. Soc. regional conference series in mathematics **43** (1980).

[6] W. Jaco and P. Shalen: Seifert fibered spaces in 3-manifolds, *Mem. Amer. Math. Soc.* **220** (1979).

[7] R. Kirby: Problems in low dimensional manifold theory, *Algebraic and Geometric topology,* (ed. by R.J. Milgram), Proc. Symposia in Pure Math. **32** (1978) 273–312, Amer. Math. Soc.

[8] W. Meeks and P. Scott: Finite group actions on 3-manifolds, *Invent. Math.* **86** (1986) 287–346.

[9] P. Scott: The geometries of 3-manifolds, *Bull. London Math. Soc.* **15** (1983) 401–487.

[10] W. Thurston: Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* **6** (1982) 357–382.

[11] S. Wang: The π_1-injectivity of selfmaps of nonzero degree on 3-manifolds, *Math. Ann.* (to appear).

[12] Y-Q. Wu: Canonical reducing curves of surface homeomorphisms, *Acta Math. Sinica*, (N.S.) **3** (1987) 305–313.

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