Hyperbolic Manifolds and Degenerating Handle Additions

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Abstract

A 2-handle addition on the boundary of a hyperbolic 3-manifold M is called degenerating if the resulting manifold is not hyperbolic. There are examples that some manifolds admit infinitely many degenerating handle additions. But most of them are not "basic". (See section 1 for definitions.) Our first main theorem shows that there are only finitely many basic degenerating handle additions. We also study the case that one of the handle additions produces a reducible manifold, and another produces a ∂-reducible manifold, showing that in this case either the two attaching curves are disjoint, or they can be isotoped into a once punctured torus. A byproduct is a combinatorial proof of a similar known result about degenerating hyperbolic structures by Dehn filling.

1 Main results and examples

In this paper, all 3-manifolds are assumed orientable. We work on smooth or PL category. If α is a subset of a 3-manifold M (generally a properly embedded submanifold), denote by $N(\alpha)$ a closed regular neighborhood of α, and by $\eta(\alpha)$ an open regular neighborhood, i.e, $\eta(\alpha) = N(\alpha) - \overline{M - N(\alpha)}$. If A is a subset of M, we use $|A|$ to denote the number of components in A. All the concepts and notations not defined in the paper are standard, see for example [H, J].

Given an essential simple closed curve α on the boundary of M, we define $M[\alpha]$ to be the manifold obtained from M by attaching a 2-handle to M along α, then capping off a possible 2-sphere component of the resulting manifold by a 3-ball. More explicitly, if α if an essential curve on a torus component of ∂M, $M[\alpha]$ is the Dehn filling along the slope α, while if α is on a nontoral component, $M[\alpha]$ is obtained from M simply by attaching a 2-handle along α.

Denote by DM the double of M along nontoral, nonspherical boundary components. M is called hyperbolic if the interior of DM admits a complete hyperbolic structure of finite volume. By Thurston's Geometrization Theorem, if M is Haken, (in particular if

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M has some nonsphere boundary components,) then M is hyperbolic if and only if M is irreducible, ∂-irreducible, atoroidal and anannular.

Definition. If M is hyperbolic but $M[\alpha]$ is not, then the handle addition or Dehn filling along α is called a *degenerating* handle addition or Dehn filling. The curve α on ∂M is called a *degenerating curve.*

Suppose T is a torus component of ∂M, and suppose M is hyperbolic. By a theorem of Thurston, there are only finitely many Dehn fillings on T which yield nonhyperbolic manifolds. In our language, there are only finitely many degenerating curves on T. If there is more than one boundary component, one can do Dehn fillings step by step. Since at each step there are only finitely many ways to degenerate the hyperbolic structure, one can say that most Dehn fillings along torus components of ∂M yield hyperbolic manifolds.

Now suppose F is a boundary component of M with genus $g > 1$. It is natural to ask whether there are only finitely many handle additions along curves on F that yield nonhyperbolic manifolds. The answer is no in general. Actually, the following is an example that infinitely many handle additions on the boundary of a hyperbolic manifold may yield a solid torus, and the example is easily modified so that infinitely many handle additions yield handlebody of genus $g > 1$. We need the following definition and lemma.

Definition. Two curves α and β on F are called coplanar if some component of $F - \alpha \cup \beta$ is an annulus or once punctured annulus.

Lemma 1.1 *Suppose α is a nonseparating simple closed curve on F. If a separating curve β is coplanar to α, then $M[\alpha] = M[\beta][\alpha]$.*

In other words, to obtain $M[\alpha]$, we can first attach a 2-handle along the curve β, which produces a torus boundary component containing α, then do Dehn filling along α. The lemma is geometrically obvious. We omit the proof.

Example 1.1. Consider the manifold $T \times I$, where T is a torus. By a theorem of Myers [M], there exists an arc γ in it with one endpoint on each of the boundary components, such that the manifold $M = (T \times I) - \eta(\gamma)$ is hyperbolic. Let m be a curve on ∂M which bounds a disk in $N(\gamma)$. Suppose α is a nonseparating curve on ∂M which is disjoint from m. Then by the lemma we have $M[\alpha] = M[m][\alpha]$. But since $M[m] = T \times I$, all Dehn fillings produce solid tori. Therefore, $M[\alpha]$ are solid tori for all such α. One can also use the trivial link instead of the Hopf link. But then many of the handle additions as above produce connected sums of solid torus with lens spaces. The following is a concrete example. The arc γ above corresponds to the arc connecting the two components of the link in the picture. The manifold M is the exterior of the graph, and m is the meridian of the arc. It can be shown that M is hyperbolic.

(Figure 1)

Actually, what the example shows is a crucial fact: If β is an essential curve on ∂M bounding a punctured torus P, such that $M[\beta]$ is nonhyperbolic, then except in very special cases, most of the Dehn fillings along a curve α in P are nonhyperbolic, so most curves on P are degenerating curves. This leads us to the following

Definition: A degenerating curve α is called *basic* if either α is separating, or there are no separating degenerating curves coplanar with α.

Clearly, basic degenerating curves are of primary importance in the study of degenerating handle additions, because any degenerating curve must be coplanar to a basic one. Our first main theorem is the following

Theorem 3.4 *Suppose M is a hyperbolic 3-manifold. Let $\mathcal{S}$ be the set of basic degenerating curves on a genus $g > 1$ boundary component F of M. Then $|\mathcal{S}| \leq k_g$, where k_g is a constant depending only on g.*

In particular there are only finitely many separating degenerating curves on ∂M. This theorem has immediate application to the following "handlebody filling" problem. Let F be a genus g boundary component of M. Let H be a handlebody of genus g. We can glue H to M through a homeomorphism $\varphi : \partial H \to \partial M$, and call this process a handlebody filling. Let $D_1, \ldots, D_{g-1}$ be disks cutting H into g solid tori. Then a handlebody filling can be obtained by first attaching regular neighborhood of the D_i, then attaching the solid tori, so it is decomposed into $g - 1$ steps of 2-handle additions along separating curves followed by g Dehn fillings. At each step there are infinitely many possible choices for the attaching curves, but by the above result and Thurston's Theorem [T, Thm 5.8.2], only finitely many of them degenerate the hyperbolic structure. Hence we can say that most of the handlebody fillings along ∂M yield hyperbolic manifolds.

On our way towards the proof of Theorem 3.4, we present a purely combinatorial proof of the following result, which also follows (with better bounds from the Gromov-Thurston 2π theorem [cf BH, Theorem 9]. Suppose $\mathcal{T} = T_1 \cup \ldots \cup T_n$ is a set of tori on the boundary of a hyperbolic 3-manifold M. Let α_i be a simple closed curve on T_i. Write $\alpha = (\alpha_1, \dots, \alpha_n)$. Attaching a solid torus on each T_i along α_i, we get a Dehn filled manifold $M[\alpha] = M[\alpha_1] \dots [\alpha_n]$. It is called a *degenerating Dehn filling* if $M[\alpha]$ is not hyperbolic. In this case we say that α is a *degenerating set*. It is called a *basic degenerating set* if no proper subset of it is degenerating.

Proposition 2.4 *Suppose M is a hyperbolic manifold, and $\mathcal{T}$ is a set of n tori on ∂M. If $\partial M - \mathcal{T}$ is nonempty, then M has at most C_n basic degenerating sets, where C_n is a constant depending only on n.*

Next we make deeper investigation of degenerating handle additions. Given two curves α and β we use $\Delta(\alpha, \beta)$ to denote their geometric intersection number, i.e the minimal intersection number of α with all β' which are isotopic to β. Suppose α and β are degenerating curves, what can we say about α and β? As we have seen in the example, $\Delta(\alpha, \beta)$ can be arbitrarily large. However, in that example either $\Delta(\alpha, \beta) = 0$, or α and β lie in a common once punctured torus. We suspect that this is always true.

In Section 3 we will consider the case that one of the handle additions yields a reducible manifold, and the other one yields a ∂-reducible manifold. Consider the following

Example 1.2. Let K be a cable knot in a ∂-reducible 3-manifold X, such that $X - \eta(K)$ is irreducible and ∂-irreducible. Let α be a meridian of K, and let β be the slope of the cabling annulus. Choose an arc γ from ∂X to the torus $T = \partial N(K)$, complicated enough so that $M = (X - \eta(K)) - \eta(\gamma)$ is hyperbolic. Then we have that $M[\alpha] = X$ is ∂-reducible, and $M[\beta] = (X - \eta(K))[\beta]$ is reducible, but $\Delta(\alpha, \beta) \neq 0$. However, in this case both α, β lie on a common punctured torus $T - \eta(\gamma)$.

Our second main theorem shows that this is always the case.

Theorem 4.2 *Suppose M is a hyperbolic manifold, α, β are essential simple closed curves on a nontorus boundary component of M, such that $M[\alpha]$ is reducible and $M[\beta]$ is ∂-reducible. Then either $\Delta(\alpha, \beta) = 0$, or both α and β can be isotoped into a once punctured torus P on ∂M.*

As a corollary, we will see that if either α or β is a basic degenerating curve, in particular if one of them is separating on the surface, then $\Delta(\alpha, \beta) = 0$. It will be shown that essentially Example 1.2 has given all the possible manifolds for the second possibility in the theorem.

2 Basic degenerating Dehn fillings

Let P, Q be two properly embedded surfaces in a 3-manifold M. We say that they are in minimal intersection position if

(1) $|\partial P \cap \partial Q| \leq |\partial P \cap \partial Q'|$ for all Q' isotopic to Q, and

(2) $|P \cap Q| \leq |P \cap Q'|$ for all Q' subject to (1).

Let $\hat{P}$ be P with each boundary component identified to a point. The image of $P \cap Q$ in $\hat{P}$ can be considered as a graph, denoted by Γ_P: An edge of the graph is an arc component of $P \cap Q$, and a vertex is the image of a component of ∂P. Similarly we have a graph Γ_Q in $\hat{Q}$. Two edges e_1, e_2 are *parallel* in Γ_P if there is a disk D in $\hat{P}$ such that $\partial D = e_1 \cup e_2$, and Int$D$ contains no vertices of Γ_P. An edge of Γ_P is called a trivial loop if it contains only one vertex, and bounds a disk in $\hat{P}$ with interior disjoint from Γ_P. Recall that a properly embedded surface in M is called *essential* if it is incompressible and ∂-incompressible.

Lemma 2.1 *Let M be an irreducible, ∂-irreducible 3-manifold. Suppose P, Q are essential surfaces properly embedded in M, isotoped so that they are in minimal intersection position. If there are arcs e_1, e_2 in $P \cap Q$ which are parallel in both Γ_P and Γ_Q, then M contains an essential annulus.*

Proof. The arcs e_1, e_2 cut off a disk D_1 from P, and a disk D_2 from Q. By rechoosing the edges e_i if necessary, we may assume that D_1 and D_2 have disjoint interiors. So $A = D_1 \cup D_2$ is a properly embedded annulus or Mobius band in M.

If A is a Mobius band, let $N(A)$ be a regular neighborhood of A. The frontier of $N(A)$ is a properly embedded annulus $B = N(A) \cap E(A)$, where $E(A) = M - \eta(A)$. Notice that $N(A)$ is a solid torus, with B running twice along the longitude direction, so B is incompressible and ∂-incompressible in $N(A)$. If B is compressible in $E(A)$ with D a compressing disk, then the union of $N(A)$ and a regular neighborhood of D will be a punctured lens space, which is impossible because M is irreducible and has nonempty boundary. If B is ∂-compressible, then a ∂-compressing of B would produce a disk, cutting M into two pieces, one of which is a solid torus. Therefore M is ∂-compressible, again a contradiction.

Now suppose A is an annulus. Let c be a boundary component of A. One can see that c is the union of two arcs c_1 and c_2, where $c_1 \subset \partial P$, and $c_2 \subset \partial Q$. If A is compressible, then c bounds a disk in M. Since M is ∂-irreducible, c bounds a disk on ∂M, which can be used to reduce $|\partial P \cap \partial Q|$ by isotoping c_1 off c_2. If A is ∂-reducible, one can choose a ∂-reducing disk D so that the arc $D \cap A$ lies on D_1 and is parallel to the edges e_i, so it is an essential arc on P. The disk D may have other intersections with P, but by an innermost circle – outermost arc argument one can show that P is either reducible or ∂-reducible, which contradicts the hypothesis of P. Therefore A is an essential annulus. $\square$

Lemma 2.2 *Suppose P and Q are essential punctured spheres or punctured tori in an irreducible, ∂-irreducible manifold M. Suppose ∂P has c_1 components parallel to a curve γ_1, and ∂Q has c_2 components parallel to a curve γ_2. If $\Delta(\gamma_1, \gamma_2) > 18|\partial P||\partial Q|/c_1c_2$, then M contains an essential annulus.*

Proof. By an isotopy we may assume that P and Q are in minimal intersection position. As before, use Γ_P to denote the corresponding graph of $P \cap Q$ in $\hat{P}$. If Γ_P has a trivial loop, then some arc of $P \cap Q$ would be boundary parallel in P, so it cuts off a disk from P which can be used to ∂-compress Q. But since Q is ∂-incompressible and M is ∂-irreducible, one can find an isotopy of Q reducing $|\partial P \cap \partial Q|$, which is impossible because P and Q are assumed in minimal intersection position. Hence Γ_P and Γ_Q have no trivial loops. Similarly one can show that $P \cap Q$ has no trivial circle (i.e a circle which bounds a disk in P or Q).

Consider the intersection of ∂P and ∂Q on T_1. Let $\Delta = \Delta(\gamma_1, \gamma_2)$. Since ∂P has c_1 components parallel to γ_1, and ∂Q has c_2 components parallel to γ_2, these components intersect at $c_1c_2\Delta$ points, so $|\partial P \cap \partial Q| \geq c_1c_2\Delta$. It follows that Γ_P contains at least $c_1c_2\Delta/2$ edges.

Denote by $\tilde{\Gamma}_P$ the reduced graph of Γ_P, which by definition is obtained from Γ_P by replacing a set of parallel edges by a single edge. Then $\tilde{\Gamma}_P$ is a graph in $\hat{P}$ with no trivial loops or parallel edges. Denote by v, e, f the number of vertices, edges, and faces of $\tilde{\Gamma}_P$, respectively. (A face of $\tilde{\Gamma}_P$ is a component of $\hat{P} - \tilde{\Gamma}_P$.) The above shows that each face is incident to at least three edges, so we have $3f \leq 2e$. Since $\hat{P}$ is either a sphere or a torus, by counting the Euler characteristic we have

$$v - e + f \geq 0$$

Using the fact that $v = v(P) = |\partial P|$ and $3f \leq 2e$, we get

$$|\partial P| - e + (2/3)e \geq 0,$$

or equivalently, $e \leq 3|\partial P|$. Since Γ_P has at least $c_1c - 2\Delta/2$ edges, we see that some edge of $\tilde{\Gamma}_P$ corresponds to at least $c_1c_2\Delta/6|\partial P|$ parallel edges.

Let $e_1, \ldots e_k$ be a set of parallel edges in Γ_P. Consider the subgraph Γ'_Q of Γ_Q with these e_i's as edges. By the same argument as above one can show that if $k > 3|\partial Q|$ then

Γ'_Q has some parallel edges. If M contains no essential annulus, then by Lemma 2.1 no pair of edges could be parallel in both Γ_P and Γ_Q. Hence $k \leq 3|\partial Q|$. Since Γ_P has a set of at least $c_1 c_2 \Delta/6|\partial P|$ parallel edges, this gives $c_1 c_2 \Delta/6|\partial P| \leq 3|\partial Q|$, or equivalently, $\Delta \leq 18|\partial P||\partial Q|/c_1 c_2$. □

Lemma 2.3 *Let $\mathcal{S}$ be a set of mutually nonisotopic simple closed curves on a torus T. If $\Delta(\alpha, \beta) \leq k$ for all α, β in $\mathcal{S}$, then $\mathcal{S} \leq 2(k+1)^2$.*

Proof. Let m, l be a meridian-longitude pair of T. They form a bases for $H_1(T)$. We write $\alpha = (a, b)$ if $\alpha = am + bl$ in $H_1(T)$, and by reversing the orientation we may always assume $a \geq 0$. If $\alpha = (a, b)$, $\beta = (c, d)$, their intersection number is given by

$$\Delta(\alpha, \beta) = |\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}| = |ad - bc|.$$

Fix two elements $\alpha, \beta \in \mathcal{S}$. Without loss of generality we may assume $\alpha = (0, 1)$, and $\beta = (a, b)$, where $a \geq 1$. Now if $\gamma = (c, d) \in \mathcal{S}$, then $\delta(\alpha, \gamma) = |c| \leq k$ implies $0 \leq c \leq k$, and $\Delta(\beta, \gamma) \leq k$ implies $-k \leq ad - bc \leq k$, or $-k + bc \leq d \leq k + bc$. So c has at most $k+1$ choices and d has less than $2(k+1)$ choices. Therefore $|\mathcal{S}| \leq 2(k+1)^2$. □

Suppose M is a hyperbolic 3-manifold. Let $\mathcal{T} = T_1 \cup \ldots \cup T_n$ be a union of tori on ∂M. Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$, where α_i is an essential curve on T_i. Recall from section 1 that we use $M[\alpha]$ to denote the manifold obtained from M by Dehn fillings along α, i.e $M[\alpha] = M[\alpha_1] \ldots [\alpha_n]$. The set α is called a degenerating set if $M[\alpha]$ is non hyperbolic, and it is a basic degenerating set if no proper subset of α is degenerating. The corresponding Dehn filling will be called degenerating or basic degenerating accordingly.

Proposition 2.4 *Suppose M is a hyperbolic manifold. If ∂M has other components than $\mathcal{T}$, then M has at most C_n basic degenerating Dehn fillings, where C_n is a constant depending only on n.*

Proof. We prove the theorem by induction. When $n = 1$, by Gordon's Theorem [G1], [G2 Thm 3.4], if both α_1 and β_1 are degenerating curves on T_1, then $\Delta(\alpha_1, \beta_1) \leq 8$. By Lemma 2.3 we may take $C_1 = 81$. So we assume $n \geq 2$, and suppose C_{n-1} has been defined to satisfy the theorem. Write

$$\mathcal{S} = \{\alpha \mid \alpha \text{ is a degenerating set}\}.$$

Let $\mathcal{S}'$ be a maximal subset of $\mathcal{S}$ such that if $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are both in $\mathcal{S}'$ then $\alpha_i \neq \beta_i$ for all i.

Lemma 2.5 $|\mathcal{S}| \le nC_{n-1}|\mathcal{S}'|$.

Proof. For each $\alpha \in \mathcal{S}'$, let $\mathcal{S}(\alpha_i) = \{\beta \in \mathcal{S} \mid \beta_i = \alpha_i\}$. Clearly, $\mathcal{S} = \cup\{\mathcal{S}(\alpha_i) \mid \alpha \in \mathcal{S}',\ i = 1, \ldots, n\}$, so we need only to show that $|\mathcal{S}(\alpha_i)| \le C_{n-1}$. Without loss of generality, we may assume $i = 1$. Then for any $\beta = (\alpha_1, \beta_2, \ldots, \beta_n) \in S(\alpha_i)$, we have $M[\beta] = M[\alpha_1][\beta']$, where $\beta' = (\beta_2, \ldots, \beta_n)$. Since α is basic, $M[\alpha_1]$ is hyperbolic, so by definition of C_{n-1} the set $\{\beta' \mid \beta \in \mathcal{S}(\alpha_1)\}$ has at most C_{n-1} elements. □

Since $M[\alpha]$ is a degenerating Dehn filling, and $\partial M - \mathcal{T}$ is nonempty, $M[\alpha]$ is either reducible, ∂-reducible, toroidal, or annular. In all cases, there is a properly embedded surface F in M, which is either a reducing sphere, a ∂-reducing disk, an essential annulus or an essential torus. We call F a *degenerating surface*. Let $P = F \cap M$. It is a punctured torus or a punctured sphere. When F is a sphere or torus, ∂P lies on $\mathcal{T}$; when F is a disk or annulus, ∂P has at most two components lying on $\partial M - \mathcal{T}$.

Lemma 2.6 *F can be chosen so that $P = F \cap M$ is an essential surface.*

Proof. Among all degenerating surfaces, choose F so that $|F \cap \mathcal{T}|$ is minimal. If P is compressible, let D be a compressing disk. Since F is incompressible, ∂D is inessential in F, so a 2-surgery of F along D yields two surfaces, one of which must be a degenerating surface with fewer intersections with $\mathcal{T}$, contradicting the choice of F.

Suppose P is ∂-compressible. Let D be a ∂-compressing disk. ∂D consists of two arcs e_1 and e_2, with e_1 on P and e_2 on ∂M. If e_2 is on $\mathcal{T}$, part of F can be isotoped through D and the attached solid torus to reduce $|F \cap \mathcal{T}|$. If e_2 is on $\partial M - \mathcal{T}$, then F must be either a compressing disk or an essential annulus, so e_2 is an inessential arc in F. Hence 2-surgery of F along D yields two surfaces, one of which is a degenerating surface having fewer intersections with $\mathcal{T}$. □

Fix a degenerating surface F_α for each α, so that $P_\alpha = F_\alpha \cap M$ is incompressible and ∂-incompressible. Let

$$\mathcal{S}_i = \{\alpha \mid |\partial P_\alpha \cap T_i| \ge |\partial P_\alpha \cap T_j| \text{ for all } j\}.$$

Then $\mathcal{S}' = \cup_{i=1}^n \mathcal{S}_i$. We need to show that each $\mathcal{S}_i$ is a finite set.

Suppose α, β are two elements in $\mathcal{S}_i$. By definition $P_\alpha \cap T_i$ has no fewer components than $P_\alpha \cap T_j$ for all j, so $|P_\alpha \cap \mathcal{T}| \le n|P_\alpha \cap T_i|$. Also, ∂P has at most two components not on $\mathcal{T}$. Therefore,

$$|\partial P| \le 2 + n|\partial P \cap T_i| \le 2n|\partial P \cap T_i|.$$

A similar formula holds for Q. Since M is hyperbolic, by Lemma 2.2 we have

$$\Delta(\alpha_i, \beta_i) \leq 18|\partial P||\partial Q|/|\partial P \cap T_i||\partial Q \cap T_i| \leq 72n^2.$$

Recall that if α, β are different elements in $\mathcal{S}_i$, then α_i and β_i are different, so the set $\{\alpha_i \mid \alpha \in \mathcal{S}_i\}$ has the same cardinality as that of $\mathcal{S}_i$. Therefore by Lemma 2.3 we have $|\mathcal{S}_i| \leq 2(72n^2+1)^2$.

Combining with Lemma 2.5, this gives

$$|\mathcal{S}| \leq nC_{n-1}|\mathcal{S}'| \leq nC_{n-1}(|\mathcal{S}_1| + \ldots + |\mathcal{S}_n|) \leq 2n^2(72n^2+1)^2 C_{n-1}.$$

The right hand side depends only on n, the number of components in $\mathcal{T}$. This completes the proof of Theorem 2.4. □

3 Basic degenerating handle additions

In this section we apply the techniques in the previous section to show that a hyperbolic manifold admits only finitely many basic degenerating handle additions. The proof is quite similar to that of Theorem 2.4, but some difference arises in finding the essential surfaces. Actually, it is generally impossible to find an essential degenerating surface with most boundary components parallel to the degenerating curves. This is amended by considering the "major boundary slope" instead (see Proof of Theorem 3.4 for definition).

The following lemma and its corollary show that on a given surface there are only finitely many curves with mutual intersection number bounded above. This is easy for a given set, but it is not so simple to find a universal bound. Note that the constant $C(g,k)$ in Corollary 3.2 is independent of the set $\mathcal{C}$.

Definition. For S a compact orientable surface, and Λ a set of points in ∂S, a Λ-*curve* Γ in S is a compact 1-manifold properly imbedded in S so that $\partial\Gamma \subset \Lambda$, every closed component is essential, and no two closed components are parallel.

If Γ and Γ' are two Λ-curves in S, define $\Delta(\Gamma, \Gamma') = \max\{\Delta(\gamma, \gamma') \mid \gamma$ and γ' are components of Γ and Γ' respectively$\}$.

Lemma 3.1 *Given a compact orientable surface S, a set Λ of points in ∂S, and $k > 0$, there is an integer $L(S, \Lambda, k)$ such that, if $\{\Gamma_i \mid 1 \leq i \leq n\}$ is a set of mutually non-isotopic Λ-curves in S so that each $\Delta(\Gamma_i, \Gamma_j) \leq k$, , then $n \leq L(S, \Lambda, k)$.*

Proof. It suffices to prove the lemma for S connected. The proof is by induction on the complexity of S, as measured, for example, by $3genus(S) + |\partial S|$.

If S is a disk D, then k is irrelevant, for $\Delta(\Gamma_i, \Gamma_j) \leq 1$. Moreover, since a pair of points in Λ bounds a unique arc in D, the set of possible isotopy classes for Λ-curves is clearly bounded above by the number of ways of dividing up subsets of Λ into pairs of points.

If D is altered by removing a small disk in the interior to create an annulus A, then each pair of points in Λ bounds two arcs in A, and A contains a single essential simple closed curve. Thus, again, there is an obvious bound $b(|\Lambda|)$ to the number of possible Λ-curves in A.

More generally, if S is an arbitrary compact connected orientable surface, and each component of each Γ_i is ∂-parallel, then all the Γ_i's lie in an annular neighborhood of ∂S. This implies $n \leq b(|\Lambda|)^{|\partial S|}$.

Thus the lemma is proven unless some component of some Γ_i is not ∂-parallel in S. If there is such a component γ, let κ be a set of k points in γ. Let S' be the surface obtained by removing an open regular neighborhood $\eta(\gamma)$ of γ from S. Then $\partial S'$ is obtained from ∂S by removing a neighborhood of $\partial\gamma$ and adding two copies $\gamma_\pm$ of γ. Let $\Lambda' \subset \partial S'$ be $\Lambda - \partial\gamma$ together with a copy $\kappa_\pm$ of κ in each of $\gamma_\pm$.

Define $\Gamma'_1 \subset S'$ to be $\Gamma_1 - \gamma$. For each $i > 1$, isotope Γ_i to intersect γ minimally and define $\Gamma'_i \subset S'$ to be $\Gamma_i - \eta(\gamma)$. Note that each component of Γ'_i lies in a component of Γ_i, so each component of Γ'_i intersects each component of Γ'_j in at most k points. Each Γ'_i has at most k boundary points on each of $\gamma_\pm \subset \partial S'$, but of course they do not automatically lie in $\kappa_\pm$. This can be fixed by ambiently isotoping the set of points $(\Gamma_i \cap \gamma)$, $i > 1$, into a subset of κ. Even when γ is a circle, this isotopy can be chosen so no point passes more than once through the same point of κ. In particular, this isotopy introduces at most one new intersection between an end of an arc in Γ'_i and an end of an arc in Γ'_i, so $\Delta(\Gamma'_i, \Gamma'_j) \leq k + 4$.

Now an isotopy from Γ'_i to Γ'_j in S' is the restriction of an isotopy of Γ_i to Γ_j, rel their identical intersection with γ. Hence we know that the $\{\Gamma'_j\}$ are mutually non-isotopic in S'. Each component of S' is simpler than S, so, by inductive hypothesis, there is a number $L(S', \Lambda', k + 4)$ so that $n \leq L(S', \Lambda', k + 4)$. The proof is then completed by observing that there are only a finite number of surfaces of at most two components, each simpler than S, and in each of these there are only a finite number of ways of distributing $|\Lambda| + 2k$ points into the boundary. □

Corollary 3.2 *There is a constant $C(g, k)$ such that if $\mathcal{C}$ is a set of mutually non-isotopic* *simple closed curves on a genus g closed orientable surface F with $\Delta(\alpha, \beta) \leq k$ for all $\alpha, \beta \in \mathcal{C}$, then $|\mathcal{C}| \leq C(g, k)$.*

Proof. Regard each simple closed curve as a Λ-curve, $\Lambda = \emptyset$, on the genus g surface S, and define $C(g, k) = L(S, \emptyset, k)$. □

Lemma 3.3 *Suppose α is a degenerating curve on F. Then there is a punctured sphere or torus P in M such that P is essential, and all but possibly two boundary components of P are coplanar with α.*

Proof. Suppose P is a surface in M. If ∂ is a component of ∂P and is coplanar with α, then ∂ bounds a disk in $M[\alpha]$. Capping off all such components by mutually disjoint disks in $M[\alpha]$, we get a surface $\hat{P}$ in $M[\alpha]$. The surface P is called a presurface if $\hat{P}$ is a degenerating surface. We assume α is nonseparating. The proof of the other case is similar and simpler.

Write $\partial P = \hat{\partial} \cup \partial_1 \cup \ldots \cup \partial_s \cup \partial'_1 \cup \ldots \cup \partial'_t$, where $\hat{\partial} = \partial \hat{P}$, ∂_i are parallel to α, and ∂'_j are coplanar but not parallel to α. We label the components so that ∂_i is adjacent to ∂_{i+1}, ∂'_j is adjacent to ∂'_{j+1}, and ∂'_1 is the closest one to the ∂_i's. Define

$$b(\hat{P}) = \begin{cases} 1 & \text{if } \hat{P} \text{ is a disk;} \\ 2 & \text{if } \hat{P} \text{ is an annulus;} \\ 3 & \text{if } \hat{P} \text{ is a torus.} \end{cases}$$

Define the complexity of P to be $c(P) = (b(\hat{P}), |\partial P|)$ in lexicographic order. Since α is degenerating, presurfaces do exist. Let P be one with least complexity. Clearly, P is incompressible, for a compression of P would produce a presurface of less complexity.

Now suppose P is ∂-compressible, with D a ∂-compressing disk. Write $\partial D = u \cup v$, where u is an arc in F, and v is an arc in P. Since P is incompressible, u cannot be rel ∂u isotoped into ∂P. In other words, $(u, \partial u)$ is essential in $(F, \partial P)$. There are several cases.

(1) u has endpoints on different components of ∂P;

(2) $t > 0$, $s = 1$, and $\partial u \subset \partial_1$;

(3) $\partial u \subset \hat{\partial}$;

(4) $\partial u \subset \partial'_1$;

(5) $\partial u \subset \partial'_t$, or $t = 0$, $\partial u \subset \partial_1$ or ∂_s.

∂-compressing P along D, we get a new surface, which has one or two new boundary components, depending on whether the two ends of u lie on different components of ∂P. If a new boundary component is trivial in F, we cap off the component by a disk. In this way we get a new surface, denoted by P'. We will show that Case (2) is impossible, and in all other cases P' has a component which is a presurface with less complexity than P.

In Case (1), since $\hat{P}$ is ∂-compressible, u can not have both ends on $\hat{\partial}$. For all the other possibilities one can see that $\hat{P}' = \hat{P}$, and $|\partial P'| < |\partial P|$.

In Case (2), a regular neighborhood of v and a disk bounded by ∂_1 would form a Mobius band in $\hat{P}$, which is absurd.

In Case (3), since $\hat{P}$ is ∂-incompressible, $\hat{P}'$ must have a component isotopic to $\hat{P}$, so the corresponding component of P' is a presurface of less complexity.

In Case (4), the two new components of P' are parallel to α. They bounds disks in $M[\alpha]$. Since $\hat{P}$ is incompressible, they are inessential in $\hat{P}$, so v is a separating arc. Therefore, P' has two components, one of which is the required surface.

In Case (5), the new boundary components are essential curves on $\partial M[\alpha]$, so $|\partial \hat{P}'| = |\partial \hat{P}| + 2$. Thus either $\hat{P}'$ has a compressing disk component for $\partial M[\alpha]$, in which case the corresponding component of P' is a presurface of less complexity, or $\hat{P}'$ is one or two annuli. One can see that $\hat{P}'$ can be obtained from $\hat{P}$ by 2-surgery along an annulus, so one component of $\hat{P}'$ is essential. It follows that the corresponding component of P' is a presurface with less complexity. □

Theorem 3.4 *Suppose M is a hyperbolic 3-manifold. Let $\mathcal{S}$ be the set of basic degenerating curves on a genus $g > 1$ boundary component F of M. Then $|\mathcal{S}| \leq k_g$, where k_g is a constant depending only on g.*

Proof. For each α in $\mathcal{S}$, fix an essential surface P_α as in Lemma 3.3. Let $\partial_1, \ldots, \partial_s$ be those components of ∂P_α that are parallel to α, and let $\partial'_1, \ldots, \partial'_t$ be those that are coplanar to α but not parallel to α. Since each ∂'_i bounds a punctured torus containing α, and since these ∂'_i are mutually disjoint, it is easy to see that they must be parallel to each other. Use α' to denote a curve parallel to ∂'_i. Define

$$\tilde{\alpha} = \begin{cases} \alpha & \text{if } s \geq t; \\ \alpha' & \text{if } s < t. \end{cases}$$

Call $\tilde{\alpha}$ the *major boundary slope* of P_α.

Lemma 3.5 *If $\alpha_1, \ldots, \alpha_k$ are elements in $\mathcal{S}$ so that $\tilde{\alpha}_i$ are isotopic to $\tilde{\alpha}_j$ for all i, j, then $k \leq 2C_1$.*

Proof. By isotopies we may assume without loss of generality $\tilde{\alpha}_i = \tilde{\alpha}_j$ for all i, j. Write $\tilde{\alpha} = \tilde{\alpha}_i$. If $\tilde{\alpha}$ is nonseparating, then by the definition of $\tilde{\alpha}_j$ we must have $\tilde{\alpha} = \tilde{\alpha}_j = \alpha_j$

for all j. (Note that α' in the definition of $\tilde{\alpha}$ is separating.) If $\tilde{\alpha}$ is separating and also degenerating, then since α_j are basic degenerating curves and are coplanar to $\tilde{\alpha}$, we also have $\tilde{\alpha} = \alpha_j$ for all j. In these cases $k = 1$.

Now assume $\tilde{\alpha}$ is a separating nondegenerating curve. Then α_i must be a nonseparating curve in a punctured torus bounded by $\tilde{\alpha}$. By Lemma 1.1 we have $M[\alpha_i] = M[\tilde{\alpha}][\alpha_i]$, so α_i is a degenerating curve on a torus boundary component of the hyperbolic manifold $M[\tilde{\alpha}]$. Since $\tilde{\alpha}$ bounds at most two punctured torus, these α_i lie in at most two tori. Therefore by Theorem 2.4 we have $k \leq 2C_1$. □

Let $\mathcal{S}' = \{\tilde{\alpha}_i\}$ be the set of different major boundary slopes on F. By Lemma 3.5 we have

$$|\mathcal{S}| \leq 2C_1|\mathcal{S}'|.$$

Consider two elements $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{S}'$. Let $P_i = P_{\alpha_i}$ be the corresponding essential surfaces. Let s_i be the number of boundary components of P_i which are parallel to $\tilde{\alpha}_i$. Since $\tilde{\alpha}_i$ is the major boundary slope, the number of boundary components of P_i which are coplanar to $\tilde{\alpha}_i$ is at most $2s_i$. Since P_i has at most two components not coplanar to $\tilde{\alpha}_i$, we have

$$|\partial P_i|/s_i \leq (2 + 2s_i)/s_i \leq 4.$$

Now by Lemma 2.2 we have

$$\Delta(\tilde{\alpha}_1, \tilde{\alpha}_2) \leq 18|\partial P_1||bddP_2|/s_1 s_2 \leq 288.$$

Applying Corollary 3.2 to S', we get $|\mathcal{S}'| \leq C(g, 288)$. Therefore, $|\mathcal{S}| \leq 2C_1C(g, 288)$. The right hand side depends only on the genus of F. This completes the proof of the theorem. □

4 Reducing or ∂-reducing handle additions

Somehow the study of the handle addition problems is inspired by results about Dehn fillings. There are some known results about what happens if two Dehn fillings degenerate the manifold in certain sense, see for example [G2, GL, Sch, W]. It is an interesting problem whether similar results hold for handle additions. The following result is a special case of [Sch, Thm 6.1].

Theorem 4.1 *Suppose X is a ∂-reducible manifold. Let K be a knot in X such that $X - K$ is irreducible and ∂-irreducible. Then a Dehn surgery on K produces a reducible* *manifold if and only if K is a cable knot, and the surgery slope is that of the cabling annulus.*

Let $M = X - \eta(K)$. Then a Dehn surgery on K is the same as a Dehn filling of M along $\partial N(K)$. The filling along a meridian of K produces a ∂-reducible manifold and the filling along the cabling slope yields a reducible one. In particular, the geometric intersection number of the two degenerating slopes is one.

We are interested in the problem about what could happen if two degenerating handle additions produce reducible and ∂-reducible manifolds respectively. One way to obtain manifolds allowing such handle additions is to drill a hole in the above manifolds, see Example 1.2 for details. In that example, the two degenerating curves intersect at a single point. Generally this may not be true. For example, let W be a reducible and ∂-reducible manifold; let K be a trivial knot, and let γ be an arc from K to ∂W so that $W - \eta(K \cup \gamma)$ is hyperbolic. By the same method as in Examples 1.1 and 1.2 one can show that infinitely many handle additions produce reducible or ∂-reducible manifolds. However, in all these examples, if two degenerating curves have nontrivial intersections, then they are contained in a punctured torus. It turns out that this always happens.

Theorem 4.2 *Suppose M is a hyperbolic manifold, α, β are essential simple closed curves on a nontorus boundary component of M, such that $M[\alpha]$ is reducible and $M[\beta]$ is ∂-reducible. Then either $\Delta(\alpha, \beta) = 0$, or there is a once punctured torus P in ∂M containing both α and β.*

Proof. Let F be the component of ∂M containing α and β. We can write M as $M' \cup F \times I$, where $F \times I$ is a regular neighborhood of F in M, and M' is the closure of $M - F \times I$. Let K be the curve $\alpha \times 1$ on the surface $F' = F \times 1$. We consider K as a knot in M. Denote by X the manifold obtained from M by deleting a regular neighborhood of K, then attaching a 2-handle along the curve β on F. In our earlier notation, $X = (M - \eta(K))[\beta]$.

Denote by S the surface $F' \cap X = F' - \eta(K)$. It divides X into two parts X_1 and X_2, where $X_1 = M' - \eta(K)$ is homeomorphic to both M' and M, and $X_2 = (F \times I)[\beta] - \eta(K)$ is homeomorphic to $(F \times I)[\beta]$. Let m be a meridian curve of K on $\partial N(K)$, and let l be a longitude isotopic to the boundaries of S. Clearly, the Dehn filled manifold $X[m]$ is exactly equal to $M[\beta]$, so by our assumption it is ∂-reducible.

Now consider $X[l]$. The surface S has two boundary components on the torus $\partial N(K)$. They bound disjoint disks D_1 and D_2 in the attached solid torus, dividing the solid torus into two 2-handles. Thus instead of attaching the solid torus, we can attach the 2-handles to X_1 and X_2 respectively, then glue the manifolds along the surface $\hat{S} = S \cup D_1 \cup D_2$. Since $(X_1, l) \cong (M', K) \cong (M, \alpha)$, and $(X_2, l) \cong ((F \times I)[\beta], K)$, the two pieces of $X[l]$ cutting along $\hat{S}$ are homeomorphic to $M[\alpha]$ and $(F \times I)[\beta][K]$, respectively. By assumption, $M[\alpha]$ is reducible. Since $(F \times I)[\beta][K]$ has two boundary components, gluing it to a reducible manifold along one of its boundary components will never produce an irreducible manifold. Hence $X[l]$ is reducible.

In summary, we have two Dehn fillings of X along a torus boundary component $\partial N(K)$, one produces a ∂-reducible manifold $X[m]$, the other produces a reducible manifold $X[l]$. Therefore we can use Theorem 4.1 to conclude that either

(1) X is reducible or ∂-reducible, or

(2) X contains an essential annulus A with both boundary components on $\partial N(K)$ and parallel to l.

Consider the manifold $X_2 \cong (F \times I)[\beta]$. The compressing disks of ∂X_2 are well understood: An essential curve γ on $F \times 1$ bounds a disk in X_2 if and only if either (i) γ is parallel to $\beta \times 1$, or (ii) γ bounds a once punctured torus T on $F \times 1$ which contains $\beta \times 1$ as a nonseparating curve.

We first assume $S = F \times I - \eta(K)$ is compressible in X_2, and let γ be the boundary of a compressing disk. In case (i), γ is parallel to $\beta \times 1$. Since γ is disjoint from $K = \alpha \times 1$, we see that α and β are disjoint. In case (ii), if K is not in T, then α is disjoint from β, while if K is in T, both α and β are in a once punctured torus. In all cases, the conclusion of the theorem follows.

We now assume S is incompressible in X_2. Notice that since α is essential and ∂M is incompressible, $\partial M - \alpha$ is incompressible in M, so S is also incompressible in X_1. By assumption $X_1 \cong M$ is irreducible. As X_2 is obtained from $F \times I$ by attaching a single 2-handle, it is easy to see that X_2 is also irreducible. Therefore by a standard innermost circle argument one can show that $X = X_1 \cup X_2$ is irreducible and boundary irreducible. This complete the proof of Case (1) above.

Now assume Case (2) and let A be an annulus such that ∂A lies on $\partial N(K)$ and is disjoint from S. Since S is incompressible, and X_1, X_2 irreducible, by an isotopy we may assume $A \cap S$ consists of essential circles in A, so all components of $A \cap X_i$ are annuli. We may assume that A has been isotoped so that $A \cap S$ is minimal. Since $X_1 \cong M$ is hyperbolic, and $X_2 = (F \times I)[\beta]$ is a compression body, neither of them contains essential annuli. In particular $A \cap S \neq \emptyset$. Let A' be a component of $A \cap X_i$ which has one boundary component on ∂A. Since A' is incompressible (otherwise S would be compressible), it is

∂-compressible. But a ∂-compression would yield an embedded disk with boundary on S. Since S is incompressible and X_i are irreducible, the disk is parallel to a disk in S. Therefore, A' is ∂-parallel. By an isotopy of A, we can reduce the number of components in $A \cap S$, contradicting the choice of A. □

The second conclusion of the theorem can be further clarified. Actually from the proof we have the following Corollary, which shows that Example 1.2 and the construction given prior to Theorem 4.2 have produced all such manifolds.

Corollary 4.3 *Let M, α, β be as in the theorem. If α, β are contained in a once punctured torus P, then either $M[\partial P]$ is reducible or ∂-reducible, or $M[\partial P]$ contains a cable space. Hence if $\Delta(\alpha, \beta) > 1$, then $M[\partial P]$ must be either reducible or ∂-reducible.*

Corollary 4.4 *Let M, α, β be as in the theorem. If either α or β is a basic degenerating curve, then $\Delta(\alpha, \beta) = 0$.*

In particular if one of α and β is a separating curve, then $\Delta(\alpha, \beta) = 0$.

Proof. By Corollary 4.3, if $\Delta(\alpha, \beta) \neq 0$, then $M[\partial P]$ is reducible, ∂-reducible, or contains a cable space. In all cases, $M[\partial P]$ is non hyperbolic. □

We suspect that a result similar to Theorem 4.2 might be true for all degenerating handle additions:

Conjecture 1 *If α and β are both degenerating curves, then either $\Delta(\alpha, \beta) = 0$, or they can be isotoped into a once punctured torus.*

Conjecture 1 seems still far from being solved. A consequence of it would be the following. If α and β are both in a punctured torus P, then $M[\alpha] = M[\partial P][\alpha]$, and $M[\beta] = M[\partial P][\beta]$ by Lemma 1. If $M[\partial P]$ is not hyperbolic, then ∂P is a separating degenerating curve, so neither α nor β is basic. If $M[\partial P]$ is hyperbolic, then by Gordon's Theorem [G1, Thm 3.4, 3.5], $\Delta(\alpha, \beta) \leq 5$. So Conjecture 1 implies the following

Conjecture 2 *If α and β are basic degenerating curves on ∂M, then $\Delta(\alpha, \beta) \leq 5$.*

If the conjecture is false, it would still be interesting to know if a universal upper bound for $\Delta(\alpha, \beta)$ exists and to determine the least upper bound. By similar method as in the proof of Theorem 3.4, we can show that if α, β are *separating* degenerating curves, then $\Delta(\alpha, \beta) \leq 14$. As this is not expected to be the best possible, we omit the proof.

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