Minimally Knotted Embeddings of Planar Graphs

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A graph Γ is *planar* if it can be embedded into a 2-sphere. An embedding φ of Γ into 3-sphere S^3 is *planar* or *unknotted* if there is a 2-sphere in S^3 containing $\varphi(\Gamma)$. Otherwise it is knotted. An embedding φ of Γ into S^3 is called *minimally knotted* if $\varphi(\Gamma)$ is knotted, but all proper subgraphs of $\varphi(\Gamma)$ are unknotted. A free edge of Γ is an edge e such that e has distinct endpoints, and one of its endpoints is incident to no other edges of Γ. Clearly, if Γ has a free edge, or if it has an isolated vertex, then it has no minimally knotted embeddings. It was conjectured [2, 3] that all other planar graphs admit minimally knotted embeddings. A proof of this conjecture was given by Kawauchi in [1], using "hyperbolic imitations".

Theorem 1 *If Γ is a planar graph with no free edges or isolated vertices, then it admits a minimally knotted embedding into S^3.*

Notice that the graph in the theorem is not necessarily connected. From the proof it is clear that there are actually infinitely many minimally knotted embeddings. Some special cases have also been proved by Simon and Wolcott [3]. The present note is to give an alternative proof of this theorem, which seems simpler and more natural. The idea is to consider some surfaces in the graph exterior, and show that they are incompressible. It then follows that the graph is unknotted only if each manifold piece obtained by cutting the graph exterior along these surfaces has free fundermental group. This will reduce the knottedness of the embedding to a "local" problem.

We need some basic facts about decomposable graphs. A graph G in S^3 is called decomposable if there is a 2-sphere S in S^3 such that $S \cap G$ is a vertex v_0 of G, and both sides of S contain edges of G incident to v_0. We call v_0 a decomposing vertex, and S a decomposing sphere of G. Choose a regular neighborhood $N(v)$ for each vertex v of G, and choose a regular neighborhood $N(e)$ for each edge, so that the disks $N(e) \cap \partial N(v)$ are mutually disjoint. Then $N(G) = (\cup N(e)) \cup (\cup N(v))$ is a regular neighborhood of G in S^3. Denote by $E(G) = S^3 - \mathrm{Int}N(G)$ the exterior of G, and by $\delta(v_0)$ the punctured sphere $\partial N(v_0) - \mathrm{Int}N(G)$ on the boundary of $E(G)$. A decomposing sphere of Γ at v_0 gives rise to a compressing disk of $\delta(v_0)$ in $E(G)$. Conversely, any compressing disk of $\delta(v_0)$ can be extended to a decomposing sphere of Γ at v_0. Therefore we have the following

Lemma 1 *v_0 is a decomposing vertex if and only if $\delta(v_0)$ is compressible in $E(G)$.*

Now suppose e_1, e_2 are two edges of G. They are called parallel if there is a map $\varphi : D^2 \to S^3$ such that $\varphi(\partial D^2) = e_1 \cup e_2$, and $\varphi|_{\mathrm{Int}D^2}$ is an embedding of $\mathrm{Int}D^2$ into $S^3 - G$. A subgraph H of G is a reduction of G if $G - H$ consists of some edges, each of which is parallel to some edge in H. We say that H is obtained from G by removing some parallel edges. An edge of G is a cycle if its two endpoints are on the same vertex.

Lemma 2 *Let H be a reduction of a graph G, and suppose every component of G has at least 2 vertices. If G is decomposable, then H is also decomposable.*

Proof. By induction we need only to prove the case that H is obtained from G by removing a single edge e' which is parallel to some edge e''. Let S be a decomposing sphere of G, and let B_1, B_2 be the 3-balls with $B_1 \cap B_2 = S$. Suppose $v_0 = S \cap G$. By definition, there are edges e_i in B_i incident to v_0. Since any component has at least two vertices, we can choose e_2, say, to be a noncycle edge. If $e' \neq e_i$, $(i = 1, 2)$, then it is clear that after removing e' the sphere S is still a decomposition sphere, and we are done. So suppose $e' = e_1$. (The other case is similar). Since e'' is parallel to e', by an isotopy of e'' we may assume that e'' also lies in B_1. Now e_1 and e_2 are not parallel because e_2 must have an endpoint in the interior of B_2, while e_1 has both endpoints in B_1. Thus $e_2 \neq e''$. The two edges e_2 and e'' of H lie on different sides of S, therefore S is a decomposing sphere of H. $\square$

Lemma 3 *Let e be a cycle edge of G bounding an embedded disk D in S^3 so that* $\mathrm{Int}D$ *intersects G transversely at a single point v on some edge e'. Let H be the graph obtained from $G \cup D$ by shrinking D into a single vertex. If G is decomposable, then so is H.*

Proof. Let S be a decomposing sphere of G, intersecting G at a vertex v_0, isotoped so that $S \cap D$ is a union of some circles, and different circles intersect only possibly at $v' = \partial e$. We choose S so that the number of circles in $S \cap D$ is minimal.

Suppose $S \cap \text{Int}D \neq \emptyset$. Let C be a circle of $S \cap D$ which is innermost on the sphere S, i.e. it bounds a disk D' in S with interior disjoint from D. Denote by D'' the disk bounded by C in D.

If D'' intersects e', then after shrinking D to a point $D' \cup D''$ becomes a sphere $\bar{S}$ such that the two edges of H coming from e' are on different sides of $\bar{S}$, so $\bar{S}$ is a decomposing sphere of H in S^3, and the result follows. If D'' does not intersect e', then $D' \cup D'' = S'$ is a sphere in S^3 which is either disjoint from G or intersects G at v_0. We get contradictions as follows:

If on both sides of S' there are edges incident to v_0, then S' is a decomposing sphere of G which can be isotoped off the interior of D, contradicting the choice of S. So we assume S' bounds a 3-ball B such that no edges of G in B is incident to v_0. Using B one can find an isotopy of S in S^3 which fixes v_0 and pushes the rest of $S \cap B$ (including D') out of B through the disk D'', resulting in a sphere $\bar{S}$ whose intersection with D has fewer circles than S does. This isotopy may intersect the components of G lying in B. But since none of these components has edges incident to v_0, any two edges of G lying on different sides of S are still on different sides of $\bar{S}$, so $\bar{S}$ is a decomposing sphere of G. It contradicts the minimality of the number of circles in $S \cap D$.

We now suppose $S \cap \text{Int}D = \emptyset$. If S is also disjoint from $v' = \partial e$, then $S \cap D = \emptyset$, and it is obvious that S remains a decomposing sphere of H. If S intersects v', then $v' = v_0 = S \cap G$. Let B_1, B_2 be the 3-balls such that $B_1 \cap B_2 = S$, and suppose D lies in B_1. By assumption there are edges of G in B_2 which are incident to v_0. These edges are also edges of H. After shrinking D into a point, the edge e' becomes two edges, both lie in B_1 and are incident to v_0. Since on both sides of S there are edges of H incident to v_0, S is a decomposing sphere of H. □

An n-strand tangle is a pair (B, A), where $B = D^2 \times I$ is a 3-ball, and A is the union of n properly embedded, mutually disjoint arcs with $\partial A \subset \partial D^2 \times \{1/2\}$. Usually we present a diagram of a tangle by drawing the graph $A \cup (\partial D^2 \times \{1/2\})$ so that $\partial D^2 \times \{1/2\}$ is the outside circle of the diagram. Two tangles (B, A) and (B, A') are equivalent if A is isotopic to A' rel ∂A. A tangle (B, A) is trivial if it has a diagram with no crossings, i.e. A can be isotoped (rel ∂A) to lie in $D^2 \times \{1/2\}$.

Let T_n be a tangle (B, A), where A consists of 2^n parallel arcs, each of which is a "trefoil" arc in B, i.e. an arc whose exterior in B is homeomorphic to the trefoil knot exterior in S^3. See Figure 1 for a diagram of T_2. Let $M_n = B - \text{Int}N(A)$, and let F_n be the punctured sphere $\partial B \cap M_n$.

(Figure 1)

Lemma 4 *F_n is incompressible in M_n, and $\pi_1 M_n$ is not a free group.*

Proof. Let X be the exterior of a trefoil knot in S^3, let $Y = P_n \times I$, where P_n is a disk with 2^n holes, and I is the unit interval. Denote by C the annulus (*outer circle of* P_n) $\times I$. Then M_n is homeomorphic to $X \cup_C Y$, where C is identified with a regular neighborhood of a meridian on ∂X. By an innermost circle outermost arc argument it can be shown that any compressing disk of F_n in M_n can be isotoped off C, so it lies in either X or Y. But clearly $F_n \cap X$ and $F_n \cap Y$ are both incompressible. Therefore F_n is incompressible in M_n. Also, since C is incompressible in M_n, the trefoil knot group $\pi_1 X$ is a subgroup of $\pi_1 M_n$. Since $\pi_1 X$ is not a free group, $\pi_1 M_n$ is not either. $\square$

Construct a tangle $Q_n = (B_n, A_n) = (B_n, a_1 \cup \ldots \cup a_n)$ as in Figure 2, where V_k is the tangle T_k above when $k < n$, and $V_n = T_{n-1}$. Notice that if we replace a_1 by a trivial arc a_1' near ∂B_n with $\partial a_1' = a_1$, we get a trivial tangle $U_n = (B_n, A_n') = (B_n, a_1' \cup a_2 \cup \ldots \cup a_n)$.

(Figure 2)

Lemma 5 *The tangle $(B_n, A_n - a_i)$ is equivalent to the trivial tangle $(B_n, A_n' - a_i)$ for all $i \geq 2$.*

Proof. This is obvious when $n = 2$. In general we need to show that when removing a_i from B_n, the arc a_1 can be isotoped into a_1' without intersecting the other a_j's $(j \neq 1, i)$.

We notice the following relation between Q_n and Q_{n-1}: Let $b_1, a_3, \ldots a_n$ be the arcs of Q_{n-1}, where b_1 is the one with endpoints on the left hand side of B_{n-1}. Replacing b_1 by two parallel arcs c_1, c_2, we get the tangle $Q_n' = (B_n', C_n) = (B_n', c_1 \cup c_2 \cup a_3 \cup \ldots a_n)$ in Figure 2. So Q_n is obtained by gluing the tangle A in Figure 2 to the left hand side of Q_n'.

By induction, after removing a_i from Q_n' $(i \geq 3)$, the arcs c_1, c_2 are isotopic to some trivial arcs c_1', c_2' near the left hand side boundary of B_n'. By another induction, it can be shown that c_1', c_2' may be chosen to have no intersection in the projection diagram, i.e. $(B'_n, C_n - a_i)$ is a trivial tangle. So a_1 can be further isotoped off the tangle V_1, and becomes a trivial arc in Q_n. Also, since c_1, c_2 are parallel, there is a disk D in Q'_n with $\partial D = c_1 \cup c_2 \cup \gamma_1 \cup \gamma_2$, where γ_1, γ_2 are two small arcs on $\partial Q'_n$ connecting the ends of c_1 to that of c_2, and $\mathrm{Int} D \cap a_j = \emptyset$, $j \neq 1$. Hence when deleting a_2 from Q_n, the arc a_1 can also be isotoped through D so that it is disjoint from Q'_n, and a further isotopy will modify it into a trivial arc near the left hand side of B_n. □

Now let Γ be a planar graph in S^3 with no free edges. We want to find another embedding of Γ which is minimally knotted. If some component of Γ has only one vertex, we can add a vertex to the middle of an edge in that component, dividing it into two edges. This will not affect any planarity, and a minimally knotted embedding of this new graph corresponds to a minimally knotted embedding of Γ. Therefore, without loss of generality, we may further make the following

Assumption: *Every component of Γ has at least two vertices and two edges.*

Label the edges as $e_1, \ldots, e_n$, so that e_1 and e_n have a common endpoint. Let p_i be the middle point of e_i. Choose an arc C embedded in S^3, so that C passes through $p_1, \ldots, p_n$ successively with endpoints on p_1 and p_n, and has no other intersections with Γ. We consider $\Gamma \cup C$ as a graph in the natural way.

Lemma 6 *If Γ has no free edges, then $\Gamma \cup C$ is indecomposable.*

Proof. Since $\partial C = \{p_1, p_n\}$ are midpoints of e_1, e_n. Since e_1 and e_n have a common vertex, we can connect p_1 and p_n by an arc in $e_1 \cup e_n$ to get an embedded circle J in $\Gamma \cup C$ containing C. Let S be a decomposing sphere of $\Gamma \cup C$ containing a vertex v_0 of $\Gamma \cup C$, let B_1, B_2 be 3-balls such that $B_1 \cap B_2 = S$. Since J is a circle, it is contained in B_1, say. Therefore $(\Gamma \cup C) \cap B_2 = \Gamma \cap B_2$ is a subset of Γ since the vertices of $\Gamma \cup C$ lie in Γ.

If v_0 is a vertex of Γ, then v_0 is not a point of C, so C is disjoint from any edge of Γ in B_2, contradicting the construction of C. If v_0 is not a vertex of Γ, then it is the middle point of some edge e of Γ because these are the only vertices of $\Gamma \cup C$ which are not vertices of Γ. Since C is disjoint from $\mathrm{Int} B_2$, there are no other edges in B_2. So if e is not completely contained in B_2, then it is a free edge because the end of e in $\mathrm{Int} B_2$ can not be incident to any other edges. This is impossible by the hypothesis of the lemma. If e lies completely in B_2, then e is a component of Γ because v_0 is not a vertex of Γ. This contradicts the above assumption that every component of Γ has at least two edges. □

Proof of Theorem 1: Choose an arc C as above. Let $B = N(C)$ be a regular neighborhood of C, and let b_i be the arc $e_i \cap B$. Then $(B, b_1 \cup \ldots b_n)$ is a trivial tangle. Recall the trivial tangle $U_n = (B_n, A'_n)$ defined before Lemma 5. Choose a homeomorphism $\varphi : B \to B_n$ so that $\varphi(b_i) = a_i$ for $i \geq 2$, and $\varphi(b_1) = a'_1$. Replacing b_1 by $\varphi^{-1}(a_1)$, we get another embedding G of Γ in S^3. We want to show that G is a minimally knotted embedding of Γ in S^3.

Identity $(B, G \cap B)$ with Q_n by the map φ. Denote by $e'_1 = (e_1 - b_1) \cup a_1$ the edge in G corresponding to e_1 in Γ. Clearly, $G - e'_1 = \Gamma - e_1$ is unknotted. Also, by Lemma 5, when deleting any edge e_i $(i > 1)$ from G, the edge e'_1 can be isotoped to e_1, so $G - \text{Int}\, e_i$ is isotopic to $\Gamma - \text{Int}\, e_i$ in S^3 for all i. Hence any proper subgraph of G is unknotted in S^3. It remains to show that G is knotted.

Consider $E(G) = S^3 - \text{Int}N(G)$, the exterior of G. It suffices to show that $E(G)$ is not a handlebody. Recall the tangles V_i in B_n, (see Figure 2). We use the same notation V_i for the underlying 3-ball of V_i. Let M_i be the manifold $V_i \cap E(G)$, and let $X = E(G) - \cup \text{Int} M_i$. We have $E(G) = X \cup (\cup M_i)$. The surface $F_i = X \cap M_i$ is a punctured sphere, which by Lemma 4 is incompressible in M_i. We want to show that it is also incompressible in X.

Assume F_i were compressible in X. Shrinking each 3-ball V_i into a single vertex v_i, we get a new graph G_1. It is clear that $X = E(G_1)$, and the surface F_i is exactly the punctured sphere $\delta(v_i) = \partial N(v_i) \cap E(G_1)$. By Lemma 1, F_i is compressible in X if and only if v_i is a decomposing vertex of G_1, so G_1 would be decomposable.

Denote by G_2 the graph obtained from G_1 by successively removing parallel edges of G_1 in the 3-ball B_n until there are no parallel edges in B_n. The part of G_2 in B_n is shown in Figure 3. There are $n-1$ cycles $r_1, \ldots, r_{n-1}$ in G_2, each of which bounds a disk D_i in B intersecting G_2 at a single point. Let G_3 be obtained from G_2 by shrinking every D_i into a single vertex. By further deleting parallel edges of G_3 in B, we get a graph G_4. The part of G_4 in B is shown in Figure 4.

(Figure 3)

(Figure 4)

It is now clear that $G_4 = \Gamma \cup C$. By Lemma 2 and Lemma 3, if G_1 is decomposable then so is G_4. On the other hand, Lemma 6 says that $\Gamma \cup C$ is indecomposable. This is a contradiction, therefore F_i is incompressible in X.

The manifold $E(G)$ is the union of X and the M_i's along the surfaces F_i. Since every F_i is incompressible in both M_i and X, $\pi_1 M_i$ is a subgroup of $\pi_1 E(G)$. By Lemma 4, $\pi_1 M_i$ are not free groups, so $\pi_1 E(G)$ is not a free group. But this certainly implies that $E(G)$ is not a handlebody, and so G is a knotted graph.

Therefore, G is a minimally knotted embedding of Γ. □

References

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