On Planarity of Graphs in 3-manifolds *

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A graph Γ in a 3-manifold M is called *planar* if it is contained in an embedded 2-sphere in M. It is *abstractly planar* if it can be embedded into an abstract 2-sphere. In [3] Scharlemann and Thompson gave necessary and sufficient conditions for a graph Γ to be planar in S^3, (see Theorem 3 in section 3.) The special case that Γ has a single vertex was proved by Gordon [1], while the generic case was shown [2] to be equivalent to: An abstractly planar graph Γ in S^3 is planar if and only if both $\Gamma - e$ and Γ/e are planar, where e is a noncycle edge of Γ. Fix an embedding of Γ in a 2-sphere F. We say that the embedding of Γ in S^3 is F-planar if it can be extended to an embedding of F into S^3. It turns out that the above result is equivalent to: If both $\Gamma - e$ and Γ/e are F-planar, then Γ is also F-planar.

In this paper, we study the F-planarity of a graph Γ in a 3-manifold M, where F can be an arbitrary surface containing Γ, or more generally a 2-dimensional cell complex with Γ as 1-skeleton. An embedding of Γ in a 3-manifold M is called F-planar if it can be extended to an embedding of F in M. We are interested in the problem of whether the F-planarity of Γ is determined be that of $\Gamma - e$ and Γ/e. A statement parallel to the case of $F = S^2$ is not true in this general setting. However we will show it is true if Γ is a "regular" graph.

We first study the triviality of cycles. This can be considered a special case of the above problem, when the cell complex has only one 2-cell. A *cycle* of Γ is a subgraph C which is homeomorphic to a circle.

Definition. *Suppose Γ is embedded in a 3-manifold M. Then a cycle C of Γ is trivial (with respect to (M, Γ)) if it bounds a disk with interior disjoint from Γ.*

In section 2 we prove a theorem about triviality of simple cycles. Note that if C is a cycle of Γ, and e is an edge intersecting C at most once, then C remains a cycle in both $\Gamma - e$ and Γ/e. Therefore it makes sense talking about the triviality of C with respect to $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$.

Theorem 1 *Suppose Γ is a graph embedded in a 3-manifold M. Let C be a cycle in Γ, and let e be an edge of Γ with at most one end on C. If C is trivial with respect to both $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$, then it is trivial with respect to (M, Γ).*

A link L in S^3 is the unlink if each component of L is a trivial cycle. It turns out that this is also true for any abstractly planar graphs in a 3-manifold M:

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Theorem 2 *An abstractly planar graph* Γ *in* M *is planar if and only if all cycles of* Γ *are trivial.*

We will prove Theorem 2 in Section 3, and use these theorems to give an alternative proof of the Scharlemann-Thompson Theorem.

In Section 4, we study the F-planarity of graphs in arbitrary 3-manifolds M. Suppose Γ is a graph in a compact surface F. We assume that ∂F is either empty or a subgraph of Γ. An embedding of Γ into M is *F-planar* if it can be extended to an embedding of F into M. We call the closure of a component of $F - \Gamma$ a face of F. The graph Γ is called a *regular graph* in F if each face of F is a disk, and the intersection of any two faces is connected (or empty). Suppose e is an edge of Γ with at least one end in the interior of F. Then both $\Gamma - e$ and Γ/e can be considered as graphs in F in the natural way, so we can talk about the F-planarity of $\Gamma - e$ and Γ/e. The following theorem is proved in section 4.

Theorem 5 *Suppose* Γ *is a regular graph on a surface* F*, and suppose* Γ *is embedded in a 3-manifold* M*. Let* e *be an edge of* Γ *with at least one end in* $\text{Int}F$*. If both* Γ/e *and* $\Gamma - e$ *are* F*-planar, then* Γ *is* F*-planar.*

The regularity condition on Γ is necessary. We will give an example of a graph Γ on a torus F that can be embedded into S^3, so that both $\Gamma - e$ and Γ/e are F-planar, but Γ itself is not F-planar.

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1 Definitions and Preliminaries

Given a graph Γ in a 3-manifold M, choose a regular neighborhood for each vertex and each edge of Γ, so that the disks $\partial N(v) \cap N(e)$ are mutually disjoint for all v and e. The union of all such neighborhoods forms a regular neighborhood $N(\Gamma)$ of Γ and we define the exterior of Γ to be $E(\Gamma) = M - \text{Int}N(\Gamma)$. For each vertex v, denote by $\delta(v)$ the punctured sphere $\partial N(v) - \cup \text{Int}N(e)$; similarly, for each edge e, let $\delta(e)$ be the annulus $\partial N(e) - \cup \text{Int}N(v)$. Sometimes the graph may vary, in which case we use $\delta_\Gamma(e)$ and $\delta_\Gamma(v)$ to denote $\delta(e)$ and $\delta(v)$, respectively. If C is a cycle, or more generally a subgraph of Γ, we use $\delta(C)$ to denote the union of $\delta(t)$ with t ranges over all edges and vertices of C.

For an edge e in Γ, denote by $\Gamma - e$ the subgraph obtained from Γ by deleting the interior of the edge e. If e is not a loop, then Γ/e is a graph in M/e. Denote by $\bar{e}$ the image of e in Γ/e. The quotient map $\pi : M \to M/e$ sends $N(\Gamma)$ to a regular neighborhood $N(\Gamma/e)$ of Γ/e in M/e, so it induces a homeomorphism $E(\Gamma) \cong E(\Gamma/e) = M/e - \text{Int}N(\Gamma/e)$. We identify $E(\Gamma)$ with $E(\Gamma/e)$ by this homeomorphism. Note that $\delta_{\Gamma/e}(\bar{e}) = \delta_\Gamma(v) \cup \delta_\Gamma(e) \cup \delta_\Gamma(v')$ if $\partial e = v \cup v'$.

If X is a subset of M, denote the number of components in X by $|X|$.

We define a *simple disk* to be a disk D in M which is bounded by a cycle of Γ, and has interior disjoint from Γ. Thus a cycle of Γ is a trivial cycle if and only if it bounds a simple disk.

Define a *normal structure* on $N(\Gamma)$ to be a set of line segments $\{l_x \mid x \in \partial N(\Gamma)\}$ as follows: For any vertex $v \in \Gamma$ and any $x \in \delta(v)$, let l_x be the straight line in $D^3 = N(v)$ connecting x to v. If e is not a loop, the closure of $N(e) - \cup\{l_x \mid x \in \cup\delta(v)\}$ has a product structure $e \times D^2$ such that for $x \in \partial e \times \partial D^2$, the l_x defined above is the line between x and a point in $\partial e \times 0$. Now for any $x \in p \times \partial D^2$ with $p \in e$, let l_x be the line connecting x to $p \times 0$. If e is a loop, $N(e) - \cup\{l_x \mid x \in \cup\delta(v)\}$ is homeomorphic to $e \times D^2/\partial e \times 0$, so we can define l_x in the same way as above. For any $p \in e$, $p \times D^2$ is called a *meridian disk* of Γ (or e) at p, and $p \times \partial D^2$ is called a *meridian* of Γ.

Suppose P is a surface in $E(\Gamma)$. The normal extension D of P is the union of P and the lines l_x with $x \in P \cap \partial N(\Gamma)$. If P is a properly embedded disk in $E(\Gamma)$, and C is a cycle of Γ such that P intersects any meridian of C exactly once, and is disjoint from the other meridians of Γ, then D is a disc with $\partial D = C$. A surface S in M with ∂S in Γ is called in normal position if S is the normal extension of $S \cap E(\Gamma)$. The following lemma is useful in modifying disks to make their interiors disjoint.

Lemma 1.1 *Suppose $D_1, \ldots, D_n$ are simple disks in M with mutually disjoint interiors. Suppose C is a trivial cycle, and $C \cap D_i$ is connected for all i. Then C bounds a simple disk D with interior disjoint from D_i for all i.*

Proof. By an isotopy we may assume $D_1, \ldots, D_n$ are in normal position. Let $P_i = D_i \cap E(\Gamma)$. Choose a simple disk D in normal position and bounded by C so that $P = D \cap E(\Gamma)$ is transverse to P_i, and $|P \cap (\cup P_i)|$ is minimal. Let A be the closure of $\text{Int}D \cap (\cup D_i)$. Since $A \cap N(\Gamma)$ consists of lines l_x with $x \in \partial P \cap (\cup P_i)$, we know that A is the union of some circles which may intersect Γ at one point, and some arcs with different endpoints on Γ. These circles and arcs might intersect on Γ, but are otherwise disjoint. If A has some circles, choose a circle α which is innermost in some D_i, and let Δ and Δ_i be the disks it bounds in D and D_i respectively. Then $(D - \Delta) \cup \Delta_i$ can be rel ∂D isotoped into a disk D' with $|D' \cap (\cup P_i)| < |P \cap (\cup P_i)|$. If A has no circles but has some arcs, let β be an arc in A which is outermost in the sense that there is an arc γ in some $C \cap \partial D_i$, such that $\beta \cup \gamma$ bounds a disk Δ_i in D_i with $\text{Int}\Delta_i \cap D = \emptyset$. (This is possible because of the assumption that $C \cap \partial D_i$ is connected for all i). Let Δ be the disk in D with $\partial \Delta = \partial \Delta_i = \beta \cup \gamma$. Then a perturbation of $(D - \Delta) \cup \Delta_i$ produces a disk D' with $|D' \cap (\cup P_i)| < |P \cap (\cup P_i)|$. By the minimality of $|P \cap (\cup P_i)|$, neither case can happen. Therefore $A = \emptyset$. □

In section 3 we will need some handle addition lemmas. Let F be a surface on the boundary of a 3-manifold M, and let J be a simple loop on F. Denote by $\tau(M, J)$ the manifold obtained from M by attaching a 2-handle along J, that is, $\tau(M, J) = M \cup (D^1 \times D^2)$, where $D^1 \times \partial D^2$ is identified with a regular neighborhood $N(J)$ of J in F. Denote by $\sigma(F, J)$ the surface $(F - N(J)) \cup (\partial D^1 \times D^2)$. We have the following generalized handle addition lemma.

Lemma 1.2 *Suppose S is a surface on the boundary of a 3-manifold M. Let γ be a 1-manifold on S such that $S - \gamma$ is compressible, and let J be a circle in S disjoint from γ. If $\sigma(S, J)$ is compressible in $\tau(M, J)$ with D' a compressing disk, then $S - J$ has a compressing disk D such that $\partial D \cap \gamma \subset \partial D' \cap \gamma$.*

This was implied in the proof of [4, Thm 1]. It was shown that under the assumption we have $|\partial D \cap \gamma| \leq |\partial D' \cap \gamma|$, but the argument there has actually proved that $\partial D \cap \gamma \subset \partial D' \cap \gamma$.

2 Trivial cycles in a graph

Given a cycle C in $\Gamma \subset M$, and a noncycle edge e of Γ, if e does not have both endpoints on C, then C remains a cycle in $\Gamma - e$ and Γ/e. The following theorem shows that the triviality of C with respect to (M, Γ) is determined by that with respect to $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$.

Theorem 1 *Suppose Γ is a graph embedded in a 3-manifold M. Let C be a simple cycle in Γ, and let e be an edge of Γ with at most one end on C. If C is trivial with respect to both $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$, then it is trivial with respect to (M, Γ).*

Proof. The Theorem is simple when C is disjoint from e: Let $\pi : M \to M/e$ be the quotient map. By assumption C bounds a disk D in M/e with interior disjoint from Γ/e. Since e is disjoint from C, $\pi^{-1}(D)$ is a simple disk in M bounded by C.

Now we assume e has exactly one end on C. Since C is trivial with respect to $(M, \Gamma - e)$, there is a disk D in M such that $\partial D = C$, and $\text{Int}D \cap \Gamma = \text{Int}D \cap e$. Consider $E(\Gamma) = M - \text{Int}N(\Gamma)$. The surface $P = D \cap E(\Gamma)$ is a planar surface satisfying

(*1): ∂P consists of circles $\partial_0, \partial_1, \ldots, \partial_n$, where $\partial_1, \ldots, \partial_n$ are meridians of e on $\partial N(e)$, and ∂_0 is a curve on $\delta(C)$ intersecting each meridian of C at a single point.

Conversely, any planar surface P in $E(\Gamma)$ satisfying (*1) can be extended to a disk D in M such that $\partial D = C$ and $\text{Int}D \cap \Gamma = \text{Int}D \cap e$.

Now consider C as a cycle in Γ/e. Since C is trivial with respect to $(M/e, \Gamma/e)$, there is a disk D' in M/e bounded by C with $\text{Int}D'$ disjoint from Γ/e. The surface $Q = D' \cap E(\Gamma)$ is a disk satisfying

(*2): ∂Q is a curve on $\partial N(C \cup e)$, which intersects each meridian of C at a single point.

Conversely, any such disk Q can be extended to a disk D' in M/e with $\partial D' = C$ and $\text{Int}D' \cap (\Gamma/e) = \emptyset$.

We choose P and Q to satisfy (*1) and (*2), as well as the following general position and minimality conditions:

(*3): $n = |P \cap \delta(e)|$ is minimal, and $k = |Q \cap \delta(e)|$ is minimal.

(*4): P intersects Q transversely, and $|P \cap Q|$ is minimal subject to (*3).

(*5): $P \cap Q \cap \delta(e') = \emptyset$ for each edge e' in C.

(*5) is possible because by (*1) and (*2) each of $P \cap \delta(e')$ and $Q \cap \delta(e')$ is an essential arc in $\delta(e')$, so we can isotop Q to make them disjoint. Since k is minimal, $Q \cap \delta(e)$ consists of parallel essential arcs. So we may further assume

(*6): each component of $Q \cap \delta(e)$ intersects each ∂_j at a single point, $j = 1, \dots, n$.

If either $n = 0$ or $k = 0$, then an extension of P or Q is a disc D in M with $\partial D = C$ and $\text{Int}D \cap \Gamma = \emptyset$, so C is trivial with respect to (M, Γ), as required. Hence we assume both n and k are positive. Label the components of ∂P so that, beginning with a point on $\partial N(C)$, an arc of $\partial Q \cap \delta(e)$ intersects $\partial_1, \partial_2, \dots, \partial_n$ successively.

A point of $\partial P \cap \partial Q$ is labeled i if it is a point on ∂_i. Thus any arc on $P \cap Q$ has a label on each of its end points.

Lemma 2.1 *A component of $P \cap Q$ in P is an arc which is either essential or has both ends on ∂_0.*

Proof. If $P \cap Q$ has some circle components, a 2-surgery of P along some disk in Q bounded by an innermost circle will reduce $|P \cap Q|$. Therefore $P \cap Q$ consists of arcs only.

If $P \cap Q$ has some arc which is inessential in P and has both ends on some ∂_j with $j \neq 0$, let α be an outermost one, so there is an arc β on ∂_j such that $\alpha \cup \beta$ bounds a disk Δ in P with interior disjoint from Q. A boundary compression of Q along Δ produces two disks, one of which satisfies (*2), but has less components of intersection with $\delta(e)$, contradicting the minimality of k. □

Lemma 2.2 *There is a label $i_0 > 0$ such that no arc of $P \cap Q$ has both ends labeled i_0.*

Proof. Otherwise choose an α_i for each $i = 1, \dots, n$, with $\partial \alpha_i$ on ∂_i. Then the innermost such α_i will be an inessential arc on P. □

Examine the order in which the indices appear on ∂Q. By (*6), if we delete all the 0 indices, the sequence is $1, 2, \dots, n, n, \dots, 2, 1$ repeated $k/2$ times. The 0 indices appear only possibly between two successive 1's.

Lemma 2.3 *An arc α of $P \cap Q$ which is outermost in Q is of one of the following types.*

Type (i)*: α has both ends labeled 1 or both ends labeled n.*

Type (ii)*: α has one end labeled* 1 *and the other labeled* 0*.*

Proof. Note that if i, j are successive labels on ∂Q, then $|i - j| \leq 1$. Therefore if α is not of Type (i) or (ii), then the labels of α are either $\{0, 0\}$ or $\{i, i+1\}$ for some $i > 0$. Let β be the arc on ∂Q so that $\alpha \cup \beta$ bounds a disk Δ in Q with interior disjoint from P.

Suppose α has label 0 on both endpoints. Then $\partial\alpha$ divides $\partial_0 \subset \partial P$ into two arcs ∂_0' and ∂_0'', one of which, say ∂_0'', has the property that it intersects a meridian of C if and only if β does. So $\partial_0' \cup \beta$ intersects any meridian of C at a single point. Let P_1 be the part of P bounded by $\partial_0' \cup \alpha$. Then $P' = P_1 \cup \Delta$ satisfies (*1). Moreover, $|\partial P'| \leq |\partial P|$, and a perturbation of P' has less components of intersection with Q than P does. This is impossible by (*4).

Now suppose α has labels $\{i, i+1\}$ for some $i > 0$. Then the normal extension of Δ is a disk Δ' in M such that $\partial\Delta' = \alpha' \cup \beta'$, where $\alpha' \subset D$, $\beta' \subset e$, and $\text{Int}\Delta' \cap \Gamma = \emptyset$. So we can isotop β' through Δ' to reduce $|D \cap e|$. This contradicts the minimality of n. □

Note that the proof does not apply to the case when the labels of α are $\{0, 1\}$, since part of β' may be on C.

Lemma 2.4 *There are at least two outermost edges α_1, α_2 of Type* (ii).

Proof. By Lemma 2.2, there is an index i_0 such that no arc in $P \cap Q$ has both ends labeled i_0. Let A be the set of arcs in $P \cap Q$ with one end labeled i_0. Let Δ be a disk in Q such that $\gamma = \partial\Delta - \partial Q$ is an arc in A, and Δ contains no other arcs in A. Note that there are at least two such Δ's. So we need only to show that there is at least one type (ii) outermost edge in Δ.

Suppose there is no outermost arc of type (ii) in Δ. Then by Lemma 2.3, each outermost arc in Δ is of type (i), so the labels of the arc are either $\{1, 1\}$ or $\{n, n\}$. If there are two such outermost arcs, then the index i_0 appears between them, which is impossible by the definition of Δ. So there is only one outermost arc on Δ. This implies that the arcs of $P \cap Q$ are all parallel in Δ. It is now clear that every arc in Δ has the same index on both ends. Especially, both ends of γ are labeled i_0, contradicting the choice of i_0. □

Now let Δ_1, Δ_2 be two disks in Q such that $\partial\Delta_i - \partial Q$ is an outermost arc of type (ii). Then the normal extension of Δ_i is a disk Δ_i' in M with $\partial\Delta_i' = \alpha_i \cup \beta_i \cup \gamma_i$, where α_i is an arc in D connecting a vertex v_i of C to the first intersection x of e with $\text{Int}D$, β_i is an arc on e connecting x to $v_0 = e \cap C$, and γ_i is an arc on C connecting v_0 to v_i. (γ_i may degenerate to a single point). Since ∂Q intersects a meridian of C at a single point, the two arcs γ_1 and γ_2 cannot have an edge in common, and hence intersect only at v_0. Thus $\Delta_1' \cap \Delta_2' = \beta_1 = \beta_2$, so $\Delta = \Delta_1' \cup \Delta_2'$ is a disk in M. Let D_1 be the part of D bounded by $\partial\Delta$, and let D_2 be $(D - D_1) \cup \Delta$ pushed off $\beta_1 - v_0$. Then D_2 is a disk in M with $\partial D_2 = C$, and $|\text{Int}D_2 \cap e| \leq n - 1$. This contradicts the minimality of $n = |\text{Int}D \cap e|$. □

3 Planar graphs in manifolds

In this section we will discuss the planarity of graphs in a 3-manifold. Suppose Γ is a graph embedded in M. An edge e of Γ is called a free edge if it is not a cycle, and one of its endpoints is not incident to any other edges. Clearly, if e is a free edge, then Γ is planar in M if and only if $\Gamma - e$ is planar. Therefore, without loss of generality we will always assume that Γ has no free edges.

We need the following definitions: A graph Γ in M is called *split* if there is a 2-sphere S in M which is disjoint from Γ, and separates M into M_1 and M_2, such that both M_i contains part of Γ. It is called *decomposable* if there is a vertex $v \in \Gamma$ such that $\delta(v)$ has a compressing disk D in $E(\Gamma)$ which is separating. The following lemma and its proof is similar to that of [3, Lemma 1.3].

Lemma 3.1 *Let Γ be a split or decomposable graph in a 3-manifold. If all proper subgraphs of Γ are planar, then Γ is planar.*

Proof. First assume Γ is split. Let S be a 2-sphere disjoint from Γ, separating M into M_1 and M_2, such that $\Gamma_i = M_i \cap \Gamma$ are proper subgraphs of Γ. By assumption, there are 2-spheres $S_i \subset M$ such that $\Gamma_i \subset S_i$. By 2-surgery along disks bounded by innermost circles of $S_i \cap S$, we can delete all intersections of S_i with S, and get $S_i \subset M_i$. Tubing S_1 to S_2 gives a 2-sphere containing Γ.

Now suppose Γ is decomposable, and let D be a separating compressing disk of $\delta(v)$ in $E(\Gamma)$. It can be extended to a 2-sphere S in M so that $S \cap \Gamma = \{v\}$, and S separates M into M_1 and M_2. Let $\Gamma_i = \Gamma \cap M_i$. Since Γ_i is planar, there is a 2-disk D_i in $E = M - \text{Int}N(v)$ which contains $\Gamma_i \cap E$. By surgery along disks bounded by innermost circles or outermost arcs of $D \cap D_i$ in D, we can assume $D_i \cap D = \emptyset$. Gluing a band on $\delta(v)$ to $D_1 \cup D_2$ produces a single disk containing $\Gamma \cap E$, which can be extended to a sphere in M containing Γ. □

Define a cut point of Γ to be a vertex v such that $\Gamma - v$ has more components than Γ. Let $\{v_1, \ldots, v_k\}$ be the cut points of Γ. Then there is a component X of $\Gamma - \{v_1, \ldots, v_k\}$ which has the property that $\Gamma_1 =$ (the closure of X) contains at most one of these v_i; for otherwise one can find a simple loop in Γ passing through some of the v_j's, contradicting the definition of cutting points. This subgraph Γ_1 is connected, and has no cut point of its own. (It is possible that $\Gamma_1 = \Gamma$.)

Suppose Γ is abstractly planar. Embed Γ_1 into a 2-sphere S. Since Γ_1 is connected and has no cut points of its own, the closure of each components of $S - \Gamma_1$ is a disk. Let $D_0, D_1, \ldots, D_n$ be these disks. If Γ_1 contains a cut point v of Γ, choose D_0 to contain v. Let $D = S - \text{Int}D_0$.

Denote by Γ_1^c the closure of $\Gamma - \Gamma_1$. Then $\Gamma_1^c \cap \Gamma_1 = \{v\}$ or $\emptyset$, depending on whether Γ_1 contains a cut point v of Γ. Embed Γ_1^c into a disk D' so that $\partial D' \cap \Gamma_1^c = \{v\}$ or $\emptyset$ accordingly. Glue D and D' together, we get an embedding of Γ into $S^2 = D \cup D'$. We fix this embedding.

Recall that $D_1, \ldots, D_n$ are the closures of the components of $D - \Gamma_1$.

Lemma 3.2 *We can number the disks so that $B_k = D_1 \cup \ldots \cup D_k$ is a disk for all k.*

Proof. If D_i is a disk such that $D_1 \cap D_i$ is not connected, then $D_1 \cup D_i$ is not simply connected, so there is a region Ω in D bounded by a boundary component of $D_1 \cup D_i$. Ω is the union of some D_j's. Choose i so that Ω contains a minimal number of these disks. Since D_i is a disk, $\partial\Omega$ is not completely contained in ∂D_i, so there is a disk D_j in Ω which has an edge in common with D_1. If $D_1 \cup D_j$ is not simply connected, then it bounds a region $\Omega' \subset \Omega$, contradicting the choice of D_i. Hence we can name this D_j as D_2. Generally, if $B_k = D_1 \cup \ldots \cup D_k$ is a disk, then by the same argument we can find D_{k+1} so that $B_k \cap D_{k+1}$ is an arc, and hence $B_k \cup D_{k+1}$ is a disk. The Lemma now follows by induction. □

A link in S^3 is a trivial link if and only if all of its components are trivial. The following theorem shows that this is also true for graphs in 3-manifolds.

Theorem 2 *An abstractly planar graph Γ in M is planar if and only if all cycles of Γ are trivial.*

Proof. We want to show that the inclusion $\Gamma \to M$ can be extended to an embedding of $\Gamma \cup B_n$ into M. This is done by induction. By assumption, ∂D_1 bounds a disk with interior disjoint from Γ, so we have an embedding of $\Gamma \cup D_1$ into M. Generally, suppose we have extended $\Gamma \to M$ to an embedding $i_k : \Gamma \cup B_k \to M$. By lemma 3.2, B_k is a disk, and $B_k \cap D_{k+1}$ is an arc. Consider the graph $\Gamma' = \Gamma - \mathrm{Int} B_k \subset M$. Then ∂B_k and ∂D_{k+1} are cycles in Γ' which are trivial with respect to (M, Γ'). So by Lemma 1.1, ∂D_{k+1} bounds a disk Δ_{k+1} which has interior disjoint from $\Gamma' \cup B_k$. Now we can define $i_{k+1} : \Gamma \cup D_1 \cup \ldots \cup D_{k+1} \to M$ so that D_{k+1} is mapped to Δ_{k+1}. This completes the induction.

It follows that the image of B_n is an embedded disk Δ in M so that $\Delta \cap \Gamma = \Gamma_1$, and $\partial \Delta \subset \Gamma_1$. When $\Gamma = \Gamma_1$ this implies Γ is planar. When $\Gamma \neq \Gamma_1$, the set $\Gamma_1 \cup \Gamma_1^c$ is either empty or a cut point, which implies Γ is split or decomposable. By induction we may assume that all proper subgraphs of Γ are planar. The theorem now follows from Lemma 3.1. □

As an application of the above theorems, we give an alternative proof of a theorem of Scharlemann and Thompson [3].

Theorem 3 *A finite graph $\Gamma \subset S^3$ is planar if and only if*

(a) Γ *is abstractly planar;*

(b) *every graph properly contained in* Γ *is planar;*

(c) $\pi_1(E(\Gamma))$ *is a free group.*

Proof. Since $\pi_1(E(\Gamma))$ is free, $E(\Gamma)$ is the connected sum of some handlebodies. If Γ is not connected, then it is split, and the theorem follows from Lemma 3.1. So we assume Γ is connected. When Γ has only one vertex, the theorem was proved in [1], so we assume Γ has some noncycle edge e. By induction on the number of edges in Γ, we may assume that Γ/e is planar for all such e.

According to Theorem 2, we need only to show that each cycle of Γ is trivial. Let C be a cycle in Γ. There are several cases.

CASE 1. *C does not contain all vertices of Γ.*

In this case there is some noncycle edge e which has at most one endpoint on C. Since both $\Gamma - e$ and Γ/e are planar, C is trivial with respect to both $(S^3, \Gamma - e)$ and $(S^3/e, \Gamma/e)$. By Theorem 1, C is also trivial with respect to (S^3, Γ).

CASE 2. *Γ has some cycle edges.*

A cycle edge cannot contain all vertices of Γ because Γ has more than one vertex. By Case 1, a cycle edge is a trivial cycle, so it bounds a simple disk. It follows that Γ is decomposable, and the Theorem follows from Lemma 3.1.

In the remaining cases, all edges not in C are noncycle edges with both ends on C. Let e be such an edge. Its endpoints divide C into two arcs C_1 and C_2.

CASE 3. *There is an edge e' which has one endpoint on each of* $\text{Int}C_i$.

Consider the cycle $C_i \cup e$. It is incident to just one endpoint of e'. By Case 1, $C_i \cup e$ bounds a disk D_i with interior disjoint from Γ. By Lemma 1.1, we can choose the D_i to have disjoint interiors. Thus $D = D_1 \cup D_2$ can be modified off e to become a simple disk bounded by C.

CASE 4. *No such edges e' as in Case 3 exist.*

Note that in this case $\bar{e}$, the image of e in Γ/e, is a cut point of Γ/e, and hence a decomposing point because Γ/e is planar. We want to apply Lemma 1.2 to our situation. To do this, let $M = E(\Gamma)$, and let $F = \partial N(C \cup e) - \text{Int}N(\Gamma)$. This is a punctured genus 2 surface, with one hole for each end of each edge which is not in $C \cup e$. Let $e_1, \ldots, e_k$ be the edges and $v_1, \ldots, v_k$ the vertices of C. Denote by m_i a meridian of e_i, and by J a meridian of e. Let $\gamma = m_1 \cup \ldots \cup m_k$.

$F - \gamma$ is isotopic to $\delta(e) \cup \delta(v_1) \cup \ldots \cup \delta(v_k) = \delta_{\Gamma/e}(\bar{e}) \cup (\cup\{\delta(v_i) \mid v_i \notin \partial e\})$. Since $\bar{e}$ is a decomposing point of Γ/e, $\delta_{\Gamma/e}(\bar{e})$ is compressible in $E(\Gamma)$. So $F - \gamma$ is compressible.

Consider $\tau(E(\Gamma), J)$. This is the manifold obtained from $E(\Gamma)$ by attaching a 2-handle along a meridian of e, so it is actually the exterior of $\Gamma - e$. The surface $\sigma(F, J)$ is the punctured torus $\partial N(C) - \text{Int}N(\Gamma - e)$. Since $\Gamma - e$ is planar, C bounds a disk in M, which gives rise to a compressing disk D' of $\sigma(F, J)$ in $E(\Gamma - e)$, so that D' intersects each m_j at a single point. By Lemma 1.2, $F - J$ has a compressing disk D in $E(\Gamma)$ intersecting each m_j at most once. Since $F - J$ is a punctured torus, and m_j are meridians, if D is disjoint from some m_j, it is disjoint from all m_j, so it will be a compressing disk of some $\delta(v_i)$, which implies Γ is decomposable, and the Theorem follows. So we assume D intersects each m_j at one point. Then we can modify D so that ∂D intersects any meridian of C at a single point. The normal extension Δ of D is now a simple disk bounded by C. □

4 F-planarity of graphs

Let F be a finite 2-dimensional cell complex with a connected graph Γ as its 1-skeleton. Γ is called a regular graph in F if the attaching map of each face (i.e. 2-cell) is a cycle in Γ, and the intersection of any two faces is connected. Suppose Γ is embedded in a 3-manifold M. Then Γ is called F-planar if it can be extended to an embedding of F in M. Suppose e is an edge of Γ which is not contained in the boundary of any faces of F. Then $F - \text{Int}e$ has $\Gamma - e$ as 1-skeleton, and F/e has Γ/e as 1-skeleton. To simplify notations, we call Γ/e (resp. $\Gamma - e$) F-planar if it is (F/e)-planar (resp. $(F - \text{Int}e)$-planar). The following is a generalization of Theorem 1.

Theorem 4 *Suppose F is a regular 2-complex with Γ as its 1-skeleton, and suppose Γ is embedded in a 3-manifold M. Let e be a noncycle edge of Γ such that both $\Gamma - e$ and Γ/e are F-planar. If e intersects each face of F at most at one of its endpoints, then Γ is F-planar.*

Proof. This follows from Theorem 1 and Lemma 1.1 by induction on the number of faces in F. □

The most interesting case of F-planarity is when F is a surface. It was shown in [2] that Theorem 3 is equivalent to the following:

Theorem $3'$ *Let* Γ *be an abstractly planar graph in* S^3 *(or* R^3*). If* Γ *has a noncycle edge* e *such that both* $\Gamma - e$ *and* Γ/e *are planar, then* Γ *is planar.*

The following is a similar result for regular graphs in an arbitrary compact surface F. Suppose Γ is such a graph, and e is a noncycle edge which has at least one endpoint in the interior of F. Since $F/e \cong F$, both $\Gamma - e$ and Γ/e can be considered naturally as a graph in F.

Theorem 5 *Suppose* Γ *is a regular graph on a surface* F*, and suppose* Γ *is embedded in a 3-manifold* M*. Let* e *be a noncycle edge of* Γ *with at least one end in* $\text{Int}F$*. If both* Γ/e *and* $\Gamma - e$ *are* F*-planar, then* Γ *is* F*-planar.*

Proof. We may assume that each end of e has valence at least 3, otherwise Γ is homeomorphic to Γ/e, and the planarity of Γ follows from that of Γ/e. Especially, an end of e in $\text{Int}F$ is incident to at least 3 faces of F.

Denote by D', D'' the two disks incident to e . Consider the 2-complex $G = F - \text{Int}D' \cup \text{Int}D''$. First suppose e has both ends on some face D of G, then D contains $\partial D' - e$ because $D \cap D'$ is connected. Similarly, D contains $\partial D'' - e$. By assumption ∂D is a cycle, so $\partial D = \partial(D' \cup D'')$. This is now a very special case: Γ has 3 edges and 2 vertices, and F is a 2-sphere. Since Γ/e is F-planar, Γ/e, and hence Γ, is contained in a 3-ball. Therefore the theorem follows from Theorem $3'$.

Now we assume e has at most one end on any face of G. Applying Theorem 4, we see that Γ is G-planar, so it can be extended to an embedding of G in M. We want to further extend this to an embedding of F.

Let $D_1, \ldots, D_n$ be the faces of G and consider G as a subset of M. Since D_i intersects e at most once, it remains a disk in M/e. By assumption Γ/e is F-planar in M/e, so $\partial D'/e$ bounds a disk Δ in M/e with $\text{Int}\Delta \cap \Gamma/e = \emptyset$. Since $\partial D' \cap D_i$ is connected, $\partial\Delta \cap D_i$ is connected for all $i = 1, \ldots, n$. By Lemma 1.1 we can choose Δ so that $\text{Int}\Delta \cap D_i = \emptyset$ for $i = 1, \ldots, n$. Let Q be the disk $\Delta \cap E(\Gamma/e) = \Delta \cap E(\Gamma)$ in $E(\Gamma)$. Q is disjoint from $\cup D_i$, and ∂Q intersects each meridian of $\partial D' - e$ at a single point. Let v be an end of e in $\text{Int}F$. Isotop Q so that $|\partial Q \cap \delta(e)|$ is minimal. Then $A = \partial Q \cap \delta(v)$ consists of arcs on the punctured sphere $\delta(v)$ which are all essential. As the circle ∂Q intersects a meridian of $\partial D' - e$ at a single point, there is an arc $\alpha \in A$ with exactly one end on the circle $J = \delta(v) \cap \delta(e)$, while all the other arcs in A have both ends on J. The arcs in A, being part of ∂Q, are disjoint from the disks $D_1, \ldots, D_n$. Because F is a surface, and v is in $\text{Int}F$, these disks cut $\delta(v)$ into an annulus. It follows that all arcs in $A - \{\alpha\}$ are inessential, which is absurd unless α is the only arc in A. Therefore ∂Q intersects a meridian of e at a single point. The normal extension Δ' of Q is now a disk bounded by $\partial D'$, with interior disjoint from

G. Similarly, there is a disk Δ'' bounded by $\partial D''$, such that $\text{Int}\Delta'' \cap G = \emptyset$, and by Lemma 1.1, it can be chosen so that $\Delta' \cap \Delta'' = e$. The surface $G \cup \Delta' \cup \Delta''$ is now an embedding of F in M. □

The regularity condition in Theorem 4 is necessary. Consider the graph Γ on a torus F as shown in Figure 1. Embedding F into S^3 in the trivial way, we get a graph Γ_1 which is F-planar in S^3. Let Γ_2 be the embedding of Γ in S^3 as shown in Figure 2, obtained from Γ_1 by interchanging a crossing in Figure 1. Let e the the edge shown in the figure. It is easy to see that $\Gamma_2 - e$ and Γ_2/e are isotopic to $\Gamma_1 - e$ and Γ_1/e respectively, so they are F-planar in S^3. One can also isotop Γ_2 so that it lies on the trivial torus. But Γ_2 is not F-planar. To see this, one may need the following fact.

Lemma 4.1 *Suppose Γ is a graph in S^3, and C is a trivial cycle with respect to (S^3, Γ). If $\Gamma \cap E(C)$ is connected, then the simple disk D bounded by C is unique up to ambient isotopy fixing Γ.*

Label the vertices of Γ_2 as in Figure 2. Denote by $C(i_1, \ldots, i_k)$ the cycle successively passing through the vertices labeled $i_1, \ldots, i_k$. Suppose Γ_2 is F-planar. Then $C(1,2,3,4)$ and $C(1,5,3,6)$ should bound simple disks with disjoint interiors. By Lemma 4.1, the disks are unique up to isotopy, so we can take the disk D bounded by $C(1,2,3,4)$ to be the shaded region in Figure 2. Now $C(1,5,3,6)$ cannot bound a disk with interior disjoint from $\Gamma_2 \cup D$, because it has linking number 1 with some curve in $\Gamma_2 \cup D - C(1,5,3,6)$.

(Figure 1)

(Figure 2)

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