

# On the Reducibility of Surgered 3-Manifolds

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The reducibility of surgered manifolds has attracted much attention in the past years. (See for example [2, 3, 4, 5]). Let  $M$  be a connected orientable irreducible 3-manifold with torus  $T$  as a boundary component. Suppose that  $\gamma_1, \gamma_2$  are two slopes on  $T$  with geometric intersection number  $\Delta = \Delta(\gamma_1, \gamma_2)$ . Let  $M(\gamma_i)$  be the manifold obtained by gluing a solid torus  $J_i$  to  $M$  so that the boundary of a meridian disc has slope  $\gamma_i$ . An interesting unsolved problem is the *reducibility conjecture*, which says that if both  $M(\gamma_1)$  and  $M(\gamma_2)$  are reducible, then  $\Delta \leq 1$ . There is some strong evidence for this conjecture. For example, Gordon and Luecke observed that if  $\Delta \geq 2$ , then both  $M(\gamma_1)$  and  $M(\gamma_2)$  will be the connected sum of two lens spaces. Especially, the conjecture is true if either  $M$  is noncompact or it has more than one boundary component. In the general case, Gordon and Litherland [3] proved that  $\Delta$  can not be greater than 4. In this paper we will prove

**Theorem 1** *If  $M(\gamma_1)$  and  $M(\gamma_2)$  are both reducible, then  $\Delta \leq 2$ .*

This rules out the possibility of  $\Delta = 3, 4$ . A nontrivial example of  $\Delta = 1$  was presented in [3]. (Surgery on cable knots in reducible manifolds gives “trivial” examples). The possibility of  $\Delta = 2$  remains a challenging open problem.

The author is indebted to Cameron Gordon and John Luecke for many stimulating conversations and various kinds of help during his visit to Austin.

## 1 Notations and Definitions

Suppose  $M(\gamma_i)$  is reducible. Let  $S_i$  be a reducing sphere of  $M(\gamma_i)$ . Isotope  $S_1$  so that  $S_1 \cap J_1 = v_1 \cup \dots \cup v_{n_1}$  is a disjoint union of meridian discs, where  $v_i$  is subscripted so that  $v_1, \dots, v_{n_1}$  appear consecutively in  $J_1$ . Similarly, we have  $S_2 \cap J_2 = w_1 \cup \dots \cup w_{n_2}$ . We choose  $S_i$  so that  $n_i$  is minimal. By an isotopy of  $S_2$ , we may assume that  $\partial v_i$  intersects

$\partial w_j$  at  $\Delta$  points for all  $i, j$ . Thus when we travel around  $\partial v_i$ , we will consecutively meet  $\partial w_1, \partial w_2, \dots, \partial w_{n_2}, \dots, \partial w_1, \dots, \partial w_{n_2}$  (repeated  $\Delta$  times).

Let  $P_i = S_i \cap M$ . It is a planar surface with  $n_i$  boundary components. Since  $n_i$  is minimized,  $P_i$  is incompressible and boundary incompressible. By a further isotopy of  $S_2$  (fixing  $\cup w_i$ ), we may assume that  $P_1$  and  $P_2$  are in general position, and  $P_1 \cap P_2$  has the minimal number of components. A standard innermost disc argument then guarantees that no circle component of  $P_1 \cap P_2$  bounds a disc in either  $P_1$  or  $P_2$ . Since  $P_i$  is boundary incompressible, no arc in  $P_1 \cap P_2$  can be boundary parallel in  $P_i$ .

Let  $\Gamma_1 = (\cup v_i) \cup \{\text{arc components of } P_1 \cap P_2\}$ .  $\Gamma_1$  is considered a graph in  $S_1$ : It has the discs  $v_i$  as its ‘‘fat’’ vertices, and the arcs in  $P_1 \cap P_2$  as its edges. There are  $\Delta n_2$  edges incident to each vertex  $v_i$  of  $\Gamma_1$ . If  $e$  is such an edge, and an end of  $e$  is in  $\partial v_i \cap \partial w_j$ , then we give this end of  $e$  the label  $j$ . In this way, each end of each edge in  $\Gamma_1$  has a label. When we travel around  $\partial v_i$  in some direction, the labels appear in the order  $1, 2, \dots, n_2, \dots, 1, \dots, n_2$  (repeated  $\Delta$  times). The labels are considered to be a mod  $n_2$  number. Thus, for example,  $n_2 + 1$  is the same label as 1.

In the same way we can define  $\Gamma_2 = (\cup w_j) \cup \{\text{arc components of } P_1 \cap P_2\}$  as a graph in  $S_2$ , and we label the ends of edges of  $\Gamma_2$  in a similar way. Since the edges in  $\Gamma_i$  are arcs in  $P_1 \cap P_2$ , each edge  $e$  in  $\Gamma_1$  can also be considered as an edge in  $\Gamma_2$ . Note that if in  $\Gamma_1$  an edge  $e$  is incident to  $v_i$  and has label  $j$  at that end, then in  $\Gamma_2$  it is incident to  $w_j$  and has label  $i$  at that end.

Given an orientation to  $S_i$  and  $K_i$  (the central curve of  $J_i$ ), we can refer to  $+$  or  $-$  vertex, according to the sign of its intersection with  $K_i$ . Two vertices are *parallel* if they have the same sign. Otherwise they are *antiparallel*. Since  $M$  is orientable, we have the following

*Parity rule: An edge  $e$  connects parallel vertices in  $\Gamma_1$  if and only if it connects antiparallel vertices in  $\Gamma_2$ .*

A pair of edges  $\{e_1, e_2\}$  in  $\Gamma_i$  is called an *S-cycle* if it is a Scharlemann cycle of length 2. That is,  $e_1, e_2$  are adjacent parallel edges connecting a pair of parallel vertices in  $\Gamma_i$ , and have the same two labels at their ends. Note that in this case the two labels are successive, and we call them the labels of the S-cycle.

A set of four parallel edges  $\{e_1, e_2, e_3, e_4\}$  is called an *extended S-cycle* if  $\{e_2, e_3\}$  is an S-cycle, and  $e_i$  is adjacent to  $e_{i+1}$ ,  $i = 1, 2, 3$ . (I.e,  $e_i$  and  $e_{i+1}$  are parallel, and there are

no edges between them.)

## 2 Proof of Theorem 1

By [3], Theorem 1 is true if  $n_1$  or  $n_2$  is less than 4. So we assume that  $n_i \geq 4$ . Suppose that  $\{e_1, e_2\}$  is an S-cycle in  $\Gamma_2$  with labels  $\{r, r+1\}$ . Then on the other graph  $\Gamma_1$ , the two edges  $e_1, e_2$  connect the vertex  $v_r$  to  $v_{r+1}$ .

**Lemma 2.1** *If there is a disc  $B$  in  $S_1$  such that  $e_1 \cup e_2 \cup v_r \cup v_{r+1} \subset B$ , then  $|B \cap J_1| \geq (n_1/2) + 1$ .*

*Proof* Let  $V$  be the part of  $J_1$  between  $v_r$  and  $v_{r+1}$ . Then a regular neighborhood  $N$  of  $V \cup B$  is a solid torus. Let  $D$  be the disc in  $P_2$  bounded by  $e_1, e_2$  and two arcs  $\alpha, \beta$  on  $\partial P_2$ . Then in  $M(\gamma_1)$ , the curve  $\partial D$  is contained in  $V \cup B$ , and intersects a meridian disc of  $N$  twice in the same direction. So a regular neighborhood of  $V \cup B \cup D$  is a projective space  $P$  (See [1, P280] for details). If  $B \cap \Gamma_1$  has  $k$  vertices, then  $P \cap J_1$  has  $k - 1$  components (since the vertices  $v_r$  and  $v_{r+1}$  are connected by  $V$ ). Thus  $\partial P$  is a reducing sphere, and  $\partial P \cap J_1$  has  $2(k - 1)$  components. By the minimality of  $n_1$ ,  $2(k - 1) \geq n_1$ . Therefore  $k \geq (n_1/2) + 1$  □

**Lemma 2.2** *If  $\Gamma_2$  has two S-cycles  $\{e_1, e_2\}$  and  $\{e'_1, e'_2\}$  with labels  $\{r, r+1\}$  and  $\{s, s+1\}$  respectively, then  $\{r, r+1\} = \{s, s+1\}$ .*

*Proof* If  $\{r, r+1\} \neq \{s, s+1\}$ , then on  $\Gamma_1$ , the complement of  $(e_1 \cup e_2 \cup v_r \cup v_{r+1}) \cup (e'_1 \cup e'_2 \cup v_s \cup v_{s+1})$  has three components  $D_1, D_2$  and  $A$  such that  $\partial \overline{D}_1 \subset e_1 \cup e_2 \cup v_r \cup v_{r+1}$ , and  $\partial \overline{D}_2 \subset e'_1 \cup e'_2 \cup v_s \cup v_{s+1}$ , where  $\overline{D}_i$  is the closure of  $D_i$ . ( $A$  is an annulus or a disc, depending on whether  $\{r, r+1\} \cap \{s, s+1\} = \emptyset$ ). Since  $D_1 \cup D_2$  does not contain  $\{v_r, v_{r+1}, v_s, v_{s+1}\}$ , it contains at most  $n_1 - 3$  vertices of  $\Gamma_1$ . Thus one of the  $D_i$ , say  $D_1$ , contains at most  $(n_1 - 3)/2$  vertices of  $\Gamma_1$ . Let  $D = D_1 \cup e_1 \cup e_2 \cup v_r \cup v_{r+1}$ . Then  $|D \cap J_1| \leq (n_1 + 1)/2$ , contradicting Lemma 2.1. □

**Lemma 2.3**  $\Gamma_2$  has no extended S-cycles.

*Proof* Let  $\{e_1, e_2, e_3, e_4\}$  be an extended S-cycle on  $\Gamma_2$  such that  $\{e_2, e_3\}$  has labels  $\{r, r+1\}$ . Then, since the labels on  $\partial w_i$  are successive, both  $e_1$  and  $e_4$  have labels  $r - 1$

and  $r + 2$  on their ends. Thus on  $\Gamma_1$ ,  $e_2, e_3$  connect  $v_r$  to  $v_{r+1}$ , and  $e_1, e_4$  connect  $v_{r-1}$  to  $v_{r+2}$ . Since we have assumed that  $n_i \geq 4$ , the two sets  $C_1 = e_1 \cup e_4 \cup v_{r-1} \cup v_{r+2}$  and  $C_2 = e_2 \cup e_3 \cup v_r \cup v_{r+1}$  are disjoint in  $S_1$ . So we can choose disjoint discs  $B_1, B_2$  on  $S_1$  such that  $B_i \supset C_i$ , and  $B_i$  has boundary disjoint from the vertices of  $\Gamma_1$ . By Lemma 2.1,  $|B_2 \cap J_1| > n_1/2$ . Hence  $|B_1 \cap J_1| < n_1/2$ . The rest of the proof is similar to that of Lemma 2.1: Choose  $V$  to be the part of  $J_1$  which is between  $v_{r-1}$  and  $v_{r+2}$ , and contains  $v_r$  and  $v_{r+1}$ . Let  $D$  be the disc on  $P_2$  bounded by  $e_1, e_4$  and two other arcs  $\alpha$  and  $\beta$  on  $\partial P_2$ . Then  $\partial D \subset \partial V \cup B_1$ . Note that  $\text{Int}D \cap P_1 \subset e_2 \cup e_3 \subset B_2$ . Thus  $\text{Int}D \cap (V \cup B_1) = \emptyset$ . Now it is easy to see that a regular neighborhood  $N$  of  $B_1 \cup V \cup D$  is a projective space, and  $|\partial N \cap J_1| < n_1$ , contradicting the minimality of  $n_1$ .  $\square$

**Lemma 2.4**  $\Gamma_2$  can not have more than  $(n_1/2) + 1$  parallel edges connecting a pair of parallel vertices.

*Proof* Suppose  $e_1, \dots, e_t$  are parallel edges connecting  $w_a$  to  $w_b$ , where  $t = n_1/2 + 2$  if  $n_1$  is even, and  $t = (n_1 - 1)/2 + 2$  if  $n_1$  is odd. Then there is an S-cycle within them. (See [1, Cor.2.6.7]). Suppose that  $\{e_i, e_{i+1}\}$  is an S-cycle. If  $i \neq 1, t - 1$ , then  $\{e_{i-1}, e_i, e_{i+1}, e_{i+2}\}$  would be an extended S-cycle, contradicting lemma 2.3.

Now suppose  $i = 1$ . By relabeling  $v_i$  if necessary, we may assume that  $e_i$  has label  $i$  at  $w_a$ . Since  $\{e_1, e_2\}$  form an S-cycle,  $e_1$  has label 2 at  $w_b$ ,  $e_2$  has label 1 at  $w_b$ . Thus  $e_i$  has label  $n_1 - i + 3$  at  $w_b$  for  $i \geq 3$ . If  $n_1$  were odd,  $e_t$  would have label  $t$  at both ends, contradicting the parity rule. If  $n_1$  were even,  $e_{t-1}$  would have label  $n_1 - ((t - 1) - 3) = t$  at  $w_b$ , and  $e_t$  would have label  $t - 1$  at  $w_b$ . Thus  $\{e_{t-1}, e_t\}$  would be an S-cycle with labels  $\{t - 1, t\}$ . So we have two S-cycles with different set of labels, contradicting lemma 2.2. The proof of the case  $i = t - 1$  is similar.  $\square$

**Lemma 2.5** Suppose that  $e', e''$  are two parallel edges connecting a pair of parallel vertices. If they have a label  $r$  in common, then they form an S-cycle.

*Proof* Let  $e' = e_1, e_2, \dots, e_k = e''$  be the successive parallel edges between  $e'$  and  $e''$ . The edges  $e', e''$  cannot both have label  $r$  at the same vertex, otherwise  $k \geq n_1 + 1$ , contradicting Lemma 2.4. So suppose  $e'$  (resp.  $e''$ ) has label  $r$  at  $w_a$  (resp.  $w_b$ ). We assume that  $e_2$  has label  $r + 1$  at  $w_a$ . (The other case is similar). Then  $e_i$  has label  $r + i - 1$  at  $w_a$ . Since the vertices are parallel, the label of  $e_{k-i}$  at  $w_b$  is  $r + i$ . Now  $k$  must be even,

otherwise the edge  $e_{(k+1)/2}$  has the same label  $r + (k - 1)/2$  at both ends, contradicting the parity rule. Let  $t = k/2$ . Then  $\{e_t, e_{t+1}\}$  is an S-cycle, because the two edges are adjacent and both have labels  $\{r + t - 1, r + t\}$  at their ends. If  $k > 2$ , we would have an extended S-cycle  $\{e_{t-1}, e_t, e_{t+1}, e_{t+2}\}$ , contradicting Lemma 2.3. Therefore,  $k = 2$ , and  $\{e', e''\}$  is an S-cycle.  $\square$

**Lemma 2.6** *One of  $\Gamma_1$  and  $\Gamma_2$  satisfies*

Property (\*): *Each vertex is incident to an edge connecting it to an antiparallel vertex.*

*Proof* If  $\Gamma_1$  does not have property (\*), then there is a vertex  $v_r$  such that each edge incident to it will connect it to a parallel vertex. By the parity rule, for each vertex  $w_i$  of  $\Gamma_2$ , all the edges incident to  $w_i$  with label  $r$  will connect  $w_i$  to antiparallel vertices. Thus  $\Gamma_2$  has Property (\*).  $\square$

Now we suppose that  $\Gamma_2$  has Property (\*). Let  $\Gamma'_2$  be the subgraph of  $\Gamma_2$  consisting of edges connecting parallel vertices. A component  $F'$  of  $\Gamma'_2$  is called an extremal component if there is a disc  $D$  such that  $D \cap \Gamma'_2 = F'$ . In this case  $F = D \cap \Gamma_2$  is a graph in  $D$ . If  $e$  is an edge in  $\Gamma_2$  connecting a vertex of  $F'$  to an antiparallel vertex, then  $e \cap D$  is an edge of  $F$  connecting that vertex to  $\partial D$ . Such an edge is called a boundary edge of  $F$ . Property (\*) means that each vertex of  $F$  belongs to a boundary edge.

The reduced graph  $\overline{F}$  of  $F$  is defined to be the graph obtained from  $F$  by choosing one edge from each family of parallel edges. Define the *valency* of a vertex to be the number of edges incident to it.

**Lemma 2.7** *Let  $\Gamma$  be a graph in a disk  $D$  with no trivial loops or parallel edges, such that every vertex of  $\Gamma$  belongs to a boundary edge. Then either  $\Gamma$  has only one vertex, or there are at least two vertices of valency at most 3, each of which belongs to a single boundary edge.*

This follows immediately from the proof of [1, Lemma 2.6.5]. In particular, it is true for the graph  $\overline{F}$  in  $D$ .

**Lemma 2.8** *Suppose  $\Delta \geq 3$ . If  $w_r$  is a vertex of  $\overline{F}$  which has valency at most 3, then  $\Gamma_1$  has an S-cycle with  $r$  as one of its labels.*

*Proof* The hypothesis implies that in  $\Gamma_2$ , there are at most two families of parallel edges connecting  $w_r$  to parallel vertices, and if there are two, then they are successive. By Lemma 5, each family has at most  $(n_1/2) + 1$  edges. So there are at most  $n_1 + 2$  successive edges connecting  $w_r$  to parallel vertices, and all the others connect  $w_r$  to antiparallel vertices. On  $\Gamma_1$  it means

(\*\*): *Except for at most two vertices, each vertex  $v_i$  is incident to at most one edge that has label  $r$  at  $v_i$  and connects  $v_i$  to an antiparallel vertex. For each exceptional vertex, there are at most two such edges.*

Denote by  $K$  the subgraph of  $\Gamma_1$  consisting of edges which connects parallel vertices and has one end labeled  $r$ . Let  $E$  be an extremal component of  $K$  and let  $D$  be a disc in  $S_1$  such that  $D \cap K = E$ . Let  $E_1$  be the subgraph of  $\Gamma_1$  consisting of edges which has one end at a vertex of  $E$  and has label  $r$  at that vertex. Restricting this graph to  $D$ , we get a graph  $H = E_1 \cap D$  in  $D$ . The disc  $D$  can be chosen so that no edges of  $H$  have both ends in  $\partial D$ . The interior edges are just the edges of  $E$ , while a boundary edge corresponds to an edge in  $\Gamma_1$  which connects a vertex of  $E$  to an antiparallel vertex, and has label  $r$  at the end in  $E$ . By (\*\*) above, at most two vertices of  $H$  have two boundary edges. We want to show that  $H$  has a pair of parallel inner edges.

To prove this, we shrink the boundary of  $D$  into one point. Then  $H$  becomes a connected graph  $H'$  in a sphere  $S^2$ . Note that among all the edges connecting to a certain vertex of  $H$ , just  $\Delta$  of them have label  $r$  at that vertex. Thus if  $H$  has  $v$  vertices, then  $H'$  has  $v + 1$  vertices and  $\Delta v$  edges. Denote by  $f_1$  the number of faces bounded by 2 edges, and by  $f_2$  the number of the other faces (which must be bounded by at least 3 edges because  $H'$  has no trivial loops). Then  $2f_1 + 3f_2 \leq 2(\text{number of edges}) = 2\Delta v$ . Thus,  $f_2 \leq (2\Delta v - 2f_1)/3$ . So we have

$$\begin{aligned} 2 &= \chi(S^2) = (v + 1) - (\Delta v) + (f_1 + f_2) \\ &\leq 1 + (1 - \Delta/3)v + f_1/3 \leq 1 + f_1/3 \end{aligned}$$

It follows that  $f_1 \geq 3$ . Since  $H$  has at most two pairs of parallel boundary edges, it must have at least one pair of parallel interior edges. The conclusion now follows from Lemma 2.5.  $\square$

*Proof of Theorem 1.* Suppose  $\Delta \geq 3$ . By Lemma 2.8, for each extremal component  $F'$  of  $\Gamma'_2$ , and each vertex  $w_r$  of valency at most 3 in the corresponding graph  $\overline{F}$ , there is an S-cycle in  $\Gamma_1$  with  $r$  as a label. By Property (\*),  $\Gamma'_2$  is disconnected. So there are at least

two extremal components. By Lemma 2.7, if  $F'$  has more than one vertex, then there are two vertices of valency at most 3 in  $\overline{F}$ . Therefore, if either  $\Gamma'_2$  has more than 2 extremal components, or one of the extremal components has more than one vertex, then we can find 3 different labels, each of which is a label of some S-cycles in  $\Gamma_1$ . But since an S-cycle has only two labels, this would imply that there exist two S-cycles with different pair of labels, contradicting Lemma 2.2. So we suppose that  $\{w_r\}, \{w_s\}$  are the only extremal components of  $\Gamma'_2$ . Since we have assumed  $n_i \geq 4$ , there must be some other components in  $\Gamma'_2$ . Each of these components will separate  $\{w_r\}$  from  $\{w_s\}$ , for otherwise we could find another extremal component. It follows that in  $\Gamma_2$ , there is no arc connecting  $w_r$  to  $w_s$ . Now suppose  $\{e_1, e_2\}$  (resp.  $\{e'_1, e'_2\}$ ) is an S-cycle in  $\Gamma_1$  with  $r$  (resp.  $s$ ) as a label. Since  $e_i$  can not connect  $w_r$  to  $w_s$  in  $\Gamma_2$ , we conclude that  $s$  is not a label of  $e_i$ . Thus these two S-cycles have different sets of labels, again a contradiction to Lemma 2.2. Therefore we have  $\Delta \leq 2$ , and the theorem follows.  $\square$

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