Incompressibility of Surfaces in Surgered 3-Manifolds

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The problem we consider in this paper was raised in [3]. Suppose T is a torus on the boundary of an orientable 3-manifold X, and S is a surface on $\partial X - T$ which is incompressible in X. A slope γ is the isotopy class of a nontrivial simple closed curve on T. Denote by $X(\gamma)$ the manifold obtained by attaching a solid torus to X so that γ is the slope of the boundary of a meridian disc. Given two slopes γ_1 and γ_2, we denote their (minimal) geometric intersection number by $\Delta(\gamma_1, \gamma_2)$.

Theorem 1 *Suppose X contains no incompressible annulus with one boundary component in S and the other in T. If S compresses in $X(\gamma_1)$ and $X(\gamma_2)$, then $\Delta(\gamma_1, \gamma_2) \leq 1$. Hence, there are at most three slopes γ such that S is compressible in $X(\gamma)$.*

This proves Conjecture 2.4.1 of [3]. Besides its own interest, the theorem will also (at least conceptually) simplify part of the proof of the Cyclic Surgery Theorem. (See the remark in [3, P275]). In that paper the authors proved some results on the above theorem: If the conditions are satisfied, then $\Delta(\gamma_1, \gamma_2) \leq 2$. (They also proved that the theorem is true when S is a torus). Our task is to rule out the possibility of $\Delta(\gamma_1, \gamma_2) = 2$.

Suppose S is compressible in $X(\gamma_1)$, then the central curve K of the attached solid torus will be a knot in $X(\gamma_1)$. Thus Theorem 1 follows immediately from the (equivalent) theorem bellow, which is more suitable to our method of proof:

Theorem 2 *Let S be a surface on the boundary of a 3-manifold M. Let K be a knot in M which is not isotopic to a simple closed curve on S. If S is compressible in M and is incompressible in $M - K$, then S is incompressible in $(M, K; \gamma)$ unless $\Delta(m, \gamma) \leq 1$, where m is the meridian slope of K.*

In the Theorem, $(M, K; \gamma)$ denotes the manifold obtained by surgery on K with slope γ. In the above notation, $(M, K; \gamma) = E(K)(\gamma)$, where $E(K) = M - \text{Int} N(K)$ is the exterior of K. Note that in Theorem 2 we can not conclude that there are at most three surgeries to make S compressible: There may still be some annulus with one boundary on ∂M and the other a meridian curve on $\partial N(K)$.

In Theorem 1, if X contains some incompressible annulus with one boundary in S and the other in T, then S may be compressible in $X(\gamma)$ for infinitely many γ. (See [3, Thm 2.4.3]). Examples are presented in [1, 2, 4] where ∂M is compressible in $(M, K; \gamma)$ for three different slopes, and yet there is no incompressible annulus between ∂M and $\partial N(K)$. So the above theorems are the best possible. Nevertheless, we notice that the theorems are still true if S is a properly embedded surface: Cutting the manifold along S, we will get back to the situation in the theorems.

In section 1 we will prove Theorem 2 under the additional assumption that K can be isotoped to $\alpha \cup \beta$, where α is an arc in ∂M, and β is an arc properly embedded in M so that $\partial M - \partial\beta$ is compressible in $M - \beta$. Any compressing disc of ∂M in M will intersect K. The above hypothesis means that K is not "very knotted". The intersection can be arranged to be on a boundary arc α. Examples of such knots are presented by the 1-bridge knots, in which case the arc β can be isotoped *rel* $\partial\beta$ to an arc β' on ∂M, but no such β' can be disjoint from α.

The first proof of Theorem 2 was then completed by a result of Gordon and Luecke (unpublished). Using the "representing all type" techniques developed in [5], they were able to prove that if $\Delta(m, \gamma) = 2$, then there exist compressing discs of S in M and $(M, K; \gamma)$ such that one of the intersection graphs Γ_1, Γ_2 contains at least n parallel boundary edges. It is easy to see that in this case K can be isotoped to $\alpha \cup \beta$ satisfying the hypothesis of Proposition 1, and the result follows.

Sections 2 and 3 play the same role as the above result of Gordon and Luecke, but the proof is more elementary in the sense that we only use the results of [3], and the argument is simpler. In section 2, we reduce the proof of Theorem 2 to the existence of some bands in M with certain "nice" properties. Then in section 3 we apply some results of [3] to show that such nice bands exist if $\Delta(m, \gamma) \geq 2$. This will complete the proof.

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1 A Special Case

Let K be a knot in a 3-manifold M such that its exterior $E(K) = M - \mathrm{Int}N(K)$ is irreducible, and K is not isotopic to a simple closed curve on ∂M. Suppose that ∂M is compressible in M and is incompressible in $M - K$. In this section we will prove the following special case of Theorem 1.

Proposition 1 *If K can be isotoped to $\alpha \cup \beta$, where α is an arc on ∂M, and β is properly embedded in M such that $\partial(M - \beta)$ is compressible in $M - \beta$, then ∂M is incompressible in $(M, K; \gamma)$ unless $\Delta(\gamma, m) \leq 1$.*

Proof. We construct a new version of the pair (M, K) as follows:

Let Y be the manifold obtained by adding a 1-handle H to M such that $\partial\beta' = \partial\beta$, where β' denotes the central curve of H. Let δ be an arc on ∂H with $\partial\delta = \partial\alpha'$, where $\alpha' = \alpha \cap \partial Y$. Then $c = \delta \cup \alpha'$ is a simple closed curve on ∂Y. Attaching a 2-handle to Y along the curve c, we get a manifold M' which, in notation of [3], can be written as $M' = \tau(Y, c)$. Note that c intersects once with a meridian disc of H. So the 2-handle will cancel the 1-handle H, and we have a homeomorphism $M' = M$. Let $K' = \beta \cup \beta'$. The pair (M', K') is our new version of (M, K) because one can easily choose the above homeomorphism to map K' to K, getting a homeomorphism of pairs $(M', K') \cong (M, K)$. Let γ' be the slope in $\partial N(K')$ corresponding to γ under the above homeomorphism. We have an induced homeomorphism $(M', K'; \gamma') \cong (M, K; \gamma)$.

From now on, we will denote by Q the manifold $(Y, K'; \gamma')$. Then $(M', K'; \gamma') = \tau((Y, K'; \gamma'), c) = \tau(Q, c)$. In other words, the surgered manifold $(M', K'; \gamma')$ can be obtained by first doing surgery on (Y, K') to get Q, and then attaching the 2-handle.

(Figure 1)

Lemma 1.1 *$\partial N(K)$ is incompressible in $E(K) = M - \mathrm{Int}N(K)$.*

Proof. If D were a compressing disc of $\partial N(K)$ in $E(K)$, then $\partial(N(K) \cup N(D))$ would be an essential sphere in $E(K)$, (where $N(D)$ denotes a regular neighborhood of D), contradicting the assumption that $E(K)$ is irreducible. $\square$

Lemma 1.2 *∂Q is compressible in Q.*

Proof. A compressing disc of $\partial(M-\beta)$ in $M-\beta$ gives rise to a compressing disc of $\partial Y = \partial Q$ in $Y - K'$, which compresses ∂Q in the surgered manifold Q. □

Lemma 1.3 *If $\Delta(\gamma, m) \geq 2$, then $\partial Q - c$ is incompressible in Q.*

Since we have $(M, K; \gamma) \cong (M', K'; \gamma') \cong \tau(Q, c)$, the incompressibility of $\partial(M, K; \gamma)$ follows from these lemmas and the following Handle Addition Theorem of Jaco's [6, Thm 2]. This completes the proof of Proposition 1 modulo Lemma 1.3.

Lemma 1.4 [6]. *Let c be a simple closed curse on the boundary ∂X of a 3-manifold X. If ∂X is compressible and $\partial X - c$ is incompressible in X, then $\tau(X, c)$ has incompressible boundary.*

Proof of Lemma 1.3. We need another version of Q.

Let $H_1 = D^2 \times I$ be a regular neighborhood of β in M, such that $D^2 \times \partial I$ is the attaching region of H. Then $V = H \cup H_1$ is a solid torus with K' as its central curve. Denote by $W = \overline{M - H_1}$. Let A be the annulus $V \cap W$. We have $Y = V \cup_A W$, $Q = W \cup (V, K'; \gamma') = W \cup_A V_1$, where $V_1 = (V, K'; \gamma')$ is again a solid torus because K' is the central curve of V. The meridian of V_1 is γ' pushing to the boundary, and hence intersects $\Delta(\gamma, m)$ times with the central curve of A. In other words, we have $Q = W \cup_A V_1$, where A, when viewed as an annulus on ∂V_1, runs $\Delta(\gamma, m)$ times around the longitude direction.

Suppose $\partial Q - c$ were compressible. Let D be a compressing disc such that $|D \cap A|$, the number of components in $D \cap A$, is minimal.

Claim 1. *$D \cap A$ is nonempty.*

D cannot lie in V_1 because $\partial V_1 - A$ is incompressible. So if $D \cap A = \emptyset$, then D must be a compressing disc of $\partial W - (A \cup \alpha')$ in W.

Consider D as a properly embedded disc in $E(K') = M' - \mathrm{Int} N(K')$. As $\partial M'$ is incompressible in $E(K')$, ∂D bounds a disc D' in $\partial M'$. Notice that D' must contain the boundary part of H, otherwise D' would lie in $\partial W - (A \cup \alpha')$, and D would not be a compressing disc. It follows that if $D \cup D'$ is a separating 2-sphere in X', then the manifold it bounds will contain the 1-handle H, and hence contain $\partial N(K')$. Therefore $D \cup D'$ does not bound a 3-ball in $E(K')$. This is impossible because $E(K')$ is assumed to be irreducible.

Claim 2. *No component of $D \cap A$ is a circle or an arc b which is rel ∂b isotopic to an arc in ∂A disjoint from c.*

The annulus A is incompressible in V_1 when $\gamma \neq m$. It is also incompressible in W, otherwise $\partial N(K)$ would be compressible in $E(K)$, contradicting Lemma 1.1. Therefore A is incompressible in Q. If b is a circle component of $D \cap A$, then b bounds a disc on A. Surger D along an innermost such disc will reduce $|D \cap A|$.

If b is an arc of $D \cap A$ such that it is rel ∂b isotopic in A to an arc b' in ∂A disjoint from c, then an outermost such b will bound with b' a disc Δ which can be used to surger D into two discs, one of which must be a compressing disc of $\partial Q - c$ having less components of intersection with A, contradicting the choice of D.

Now let b be an arc of $|D \cap A|$ which is outermost in D. There is an arc d on ∂D such that $b \cup d$ bounds a disc B in D with interior disjoint from A.

Claim 3. *B is not contained in V_1.*

Since $\Delta(m, \gamma) \geq 2$, any compressing disc of ∂V_1 intersects A at least twice. So if $B \subset V_1$, then ∂B bounds a disc B' on ∂V_1. Note that $b' = B' \cap \partial A$ is an arc on ∂A. Since $\partial B' = \partial B$ is disjoint from c, and since the ends of $\delta = c \cap V_1$ lie on different components of ∂A, B' must be disjoint from δ. It follows that b is rel ∂b isotopic to the arc b' which is disjoint from c, contradicting Claim 2.

Claim 4. *b is not an essential arc on A.*

By Claim 3, B in contained in W. If b is essential, there is a disc B'' in $M = W \cup N(\beta)$ which is a union of B and a disc B' in $N(\beta)$, such that $\partial B'' = \beta \cup \beta'$, where β' is a simple arc in ∂M with $\beta' \cap \alpha = \partial \beta'$. It follows that β is rel $\partial \beta$ isotopic to β', and therefore K is isotopic to the simple closed curve $\alpha \cup \beta'$ on ∂M. This contradicts the assumption at the beginning of this section.

Claim5. *b is not an inessential arc on A.*

If b is inessential in A, there is an arc b' in ∂A such that $b \cup b'$ bounds a disc B' in A. By Claim 1, $b' \cap c \neq \emptyset$, therefore b' intersects c at one point (because each component of ∂A only intersects c once). Since b' is on ∂A, $b' \cap c = b' \cap \alpha$. It follows that $B'' = B \cup B'$ is a disc in $M = W \cup N(\beta)$ that intersects $\alpha \cup \beta$, and hence K, at one point. So the intersection of B'' with $E(K) = M - \mathrm{Int} N(K)$ is an annulus, and its boundary on $\partial N(K)$ is the meridian slope. From [3, Thm 2.4.3], we know that ∂M being compressible in $(M, K; \gamma)$ with $\Delta(m, \gamma) \geq 2$ implies that $E(K)$ is homeomorphic to (torus)$\times I$, and therefore K is isotopic to a simple closed curve on ∂M, again a contradiction.

These claims show that there is no compressing disc for $\partial Q - c$, and hence complete the proofs of Lemma 1.3 and Proposition 1. $\square$

2 Nice Bands and Nice Arcs

Let D be a compressing disk of ∂M, and let $K(1), \ldots, K(n)$ be the consecutive points of $K \cap D$ in K. There are two arcs in K going from $K(r)$ to $K(s)$. We denote by $K[r,s]$ the one which passes through the $K(i)'$s for i between r and s.

A *band* is an embedding $B: I \times I \longrightarrow M$. We shall also use the same B to denote the image of B. A band is called a *nice band* if $B \cap D$ is a set of vertical lines $B(\{\text{points}\} \times I)$, and $(\text{Int } B) \cap (K \cup \partial M) = \emptyset$. We write $B(L) = B(0 \times I), B(R) = B(1 \times I), B(B) = B(I \times 0), B(T) = B(I \times 1)$, and call them the left, right, bottom and top edge of B, respectively. By a subband of B, we mean a band $B(I' \times I)$, where I' is a subarc of I.

Lemma 2.1 *If there is a nice band B in M such that $B(B) = K[1,n]$, and $B(T) \subset \partial M$, then K can be isotoped to $\alpha \cup \beta$ satisfying the conditions of Proposition 1.*

Proof. Let $\alpha = B(T)$, and let $\beta = (K - B(B)) \cup B(L) \cup B(R)$. It is clear that K is isotopic to $\alpha \cup \beta$. The disk D can be perturbed off β, giving a compressing disk of $\partial(M - \beta)$ in $M - \beta$. $\square$

The remaining part of the paper is focused on finding a band B satisfying the conditions of Lemma 2.1.

Let Γ be a graph in D. An arc e in $\text{Int}\, D$ is called a *nice arc* in (D, Γ) if there are edges e' and e'' of Γ connecting ∂e to ∂D, such that $e' \cup e \cup e''$, together with some arc in ∂D, bounds a disk with interior disjoint from Γ. If e is an edge of Γ, we also call it a *nice edge*.

Lemma 2.2 *Let B_1, B_2 be nice bands in M with disjoint interior, such that $B_1(B) = K[1,r], B_1(T) \subset \partial M, B_2(B) = K[r,m], B_2(T) = K[s,t]$, and $1 \leq s < r \leq m$. Suppose $B_2(L)$ is a nice arc in $(D, (B_1 \cup K) \cap D)$. Then*

(1) *there is a nice band B_1' with $B_1'(B) = K[1,m], B_1'(T) \subset \partial M$; and*

(2) *the arc $B_2(R)$ is a nice arc in $(D, (B_1' \cup K) \cap D)$.*

Sublemma. *The lemma is true when $t \le r$.*

Proof. In this case $K[s,t] \subset K[1,r]$. (Note that we may have $s > t$). So there is a nice subband B_3 of B_1 such that $B_3(B) = B_2(T) = K[s,t]$. Since $B_2(L)$ is a nice arc, and since $B_3(L)$ and $B_1(R)$ are the only arcs in $(B_1 \cup K) \cap D$ connecting $\partial B_2(L)$ to ∂D, there is an arc γ in ∂D such that $\gamma \cup B_1(R) \cup B_2(L) \cup B_3(L)$ bounds a disc D' in D with $\text{Int} D' \cap K = \emptyset$. Note that $B_2 \cup D' \cup B_3$ is a disk, which can be isotoped (rel $(B_2(B) \cup B_1(R))$) to a nice band B_2' such that (Int B_2') $\cap B_1 = \emptyset$, and $B_2'(T) \subset \partial M$. (See Figure 2). It is clear that $B_1' = B_1 \cup B_2'$ is a nice band satisfying conclusion (1). To prove (2), we notice that $B_1'(R)$ is obtained by perturbing the arc $B_2(R) \cup B_3(R)$ off itself. So the interior of the disk bounded by $B_3(R) \cup B_2(R) \cup B_1'(R)$ and a small arc in ∂D is disjoint from $B_1' \cup K$.

(Figure 2)

Proof of Lemma 2.1. Let B_2' be the nice subband of B_2 with $B_2'(B) = K[r, r+1]$ and let $B_2'' = \overline{B_2 - B_2'}$. The sublemma covers the case that $s > t$, so we may assume $s < t$. Then $B_2'(T) = K[s, s+1]$, and $s+1 \le r$. Applying the sublemma to B_1 and B_2' , we get a nice band B_1'' such that $B_1''(B) = K[1, r+1], B_1''(T) \subset \partial M$, and $B_2'(R) = B_2''(L)$ is a nice arc in $(D, (B_1'' \cup K) \cap D)$. It follows that B_1'' and B_2'' satisfy the hypotheses of the Lemma, while the numbers r and s have been increased to $r+1$ and $s+1$, respectively. The proof is now completed by induction on $t - r$. □

3 The Existence of Nice Bands and Nice Arcs

In this section we assume that ∂M is compressible in both $M = (M, K; m)$ and $(M, K; \gamma)$, where γ is a slope with $\Delta(m, \gamma) \ge 2$. Let D_1, D_2 be compressing discs of ∂M in M and $(M, K; \gamma)$ respectively, and let P_i be the planar surface $D_i - \text{Int} N(K)$. The intersection of P_1 and P_2 induces a graph Γ_i on each D_i. We choose D_i to minimize the number of inner boundary components of P_i, and denote by n the number of inner boundary components of P_1. Note that n equals the number of points of $K \cap D_1$.

Let $e_1, \ldots, e_k$ be some edges of Γ_2. They form a great x-cycle in the sense of [3] if

(1) they bound a disk in D_2 whose interior contains no vertices of Γ_2;

(2) they can be oriented so that $\partial_+ e_i = \partial_- e_{i+1}$ and $\partial_- e_i$ has label x for all $i = 1, \ldots, k$; and

(3) all the vertices $\partial_\pm e_i$ are parallel.

We quote the following two facts, and refer the readers to Chapter 2 of [3] for notations and definitions that are not given here.

Lemma 3.1 (See Lemma 2.6.2. and Lemma 2.5.2. of [3]). *The graphs* Γ_1 *and* Γ_2 *have no great cycles.*

Lemma 3.2 (See the proofs in section 2.6 of [3], especially page 296–297). *Either* Γ_2 *has* n *parallel boundary edges, or it has a vertex* x *such that the subgraph* G_2 *of* Γ_2 *consisting of all edges incident to* x *has the form illustrated in Figure 3, with the following properties:*

(1) *The vertices* u *and* v *are antiparallel.*

(2) *Each interior edge has different labels at its two ends.*

(3) *If* u *and* x *are parallel, each label will appear at most once among the ends of the edges connecting* x *to* u.

(4) *There are no* n *parallel edges.*

(Figure 3)

We label the points $K \cap D_1$ so that the labels of G_2 look like that in Figure 3. Let G_1 be the subgraph of Γ_1 corresponding to G_2. It is the subgraph consisting of all edges with one end labeled x.

Lemma 3.3 *If* Γ_2 *has no* n *parallel boundary edges, then* G_1 *has a nice edge.*

Assuming this lemma, we proceed to prove

Lemma 3.4 *If* Γ_2 *has no* n *parallel boundary edges, then there exist nice bands* B_1, B_2, *in* M, *satisfying the hypotheses of Lemma 2.2 with* $m = n$. *That is,* $\text{Int}B_1 \cap \text{Int}B_2 = \emptyset$, $B_1(B) = K[1, r]$, $B_1(T) \subset \partial M$, $B_2(B) = K[r, n]$, $B_2(T) = K[s, t]$, $1 \leq s < r$, *and* $B_2(L)$ *is a nice arc in* $(D_1, (B_1 \cup K) \cap D_1)$.

Proof. Let e_0 be a nice edge of G_1. Without loss of generality, we may assume that e_0, when viewed as an edge in G_2, connects x to u. (Otherwise reflect G_2 and relabel it). Suppose the labels of ∂e_0 are r and s. By Lemma 3.2.(2), these lables are different, so we may assume $r > s$. Since e_0 is a nice edge, r and s are labels of some boundary edges of G_2. So we have $1 \leq s < r \leq p$. (See Figure 3). There are two cases.

Case 1. The label of e_0 at x is r.

Take the two bands B_1' and B_2' as shown in Figure 3. Let B_1'' be a nice band in $N(K)$ such that $B_1''(B) = K[1,r]$, $B_1''(T) = B_1' \cap N(K)$. Then $B_1 = B_1' \cup B_1''$ is a nice band with $B_1(B) = K[1,r]$ and $B_1(T) \subset \partial M$. Similarly we can extend B_2' to a nice band B_2 so that $\mathrm{Int}B_2 \cap \mathrm{Int}B_1 = \emptyset, B_2(B) = K[r,n]$, and $B_2(T)$ is an arc of K running from $K(s)$ to $K(r)$. We claim that $B(T) = K[s,t]$. This follows from the fact that the label n does not appear on $B_2'(T)$, which in tern follows from Lemma 3.2.(3) when x and u are parallel, and from the fact $r > s$ when x and u are antiparallel. Finally, since $B_2'(L) = e_0$ is a nice arc in (D_1, G_1), we see that $B_2(L)$ is a nice arc in $(D_1, (B_1 \cup K) \cap D_1)$.

Case 2. The label of E_0 at x is s.

Since $s < r \leq n$, by Lemma 3.2.(2), u must be antiparallel to x (because the label r appears twice among the edges between u and x). Let e_1 be the edge connecting x to u with label $t = n - r + s$ at x. Then it must have label n at x . Now take B_1' as in case 1, and let B_2' be the band between the arc e_0 and e_1 in P_2 (see Figure 4). Extending them to B_1 and B_2 as in case 1, we get the required nice bands. □

(Figure 4)

Before proving Lemma 3.3, we need some properties of the graph G_1.

Fact 1: No edge of G_1 has two ends incident to a single vertex.

This is because by Lemma 3.2.(2), G_2 has no edge with same labels at its two ends.

Fact 2: G_1 has no parallel edges.

By Lemma 3.2.(4), each vertex of of G_1 is incident to at most one boundary edge. So there are no parallel boundary edges. Suppose e_1, e_2 were parallel edges connecting (say) v_s and v_t. If v_s and v_t were antiparallel, then e_1 and e_2 would connect x to a parallel vertex, say u, in G_2. But then the label s would appear twice among the edges connecting x to u, contradicting Lemma 3.2.(3). If v_s, v_t were parallel, and e_1, e_2 both had label x at (say) v_s, then e_1 and e_2 would be edges in G_2 connecting x to v and having the same label s at x. In this case there are at least $n+1$ edges between x and v, contradicting Lemma

3.2.(4). The remaining case is: v_s and v_t are parallel, e_1 and e_2 have label x at different vertex. But now they would form a great x-cycle in Γ_1, contradicting Lemma 3.1.

Fact 3: No three edges of G_1 bound a face.

If e_1, e_2, e_3 bound a face, and v_p, v_q, v_r are the vertices, then by the same argument as above, we see that v_p, v_q, v_r cannot be parallel. But if v_p is antiparallel to v_q and v_r, we can show as above that the label p would appear twice among the edges connecting x to the parallel vertex in G_2, which contradicts Lemma 3.2.(3).

Proof of Lemma 3.3. Shrinking ∂D_1 to a point , G_1 becomes a graph G_1' in S^2. It has $2n$ edges, and $n+1$ vertices. Let $f_1, \ldots, f_k$ be the faces of G_1' in S^2, and let c_i be the number of edges bounding f_i. Suppose G_1' has j components. Then we have

$(n+1) - 2n + k = \chi(S^2) + (j-1) \geq 2$, therefore $n \leq k-1$;

$(c_1 + \cdots + c_k)/2 =$ number of edges $= 2n$.

These together imply that some face, say f_1, is bounded by at most 3 edges. Facts 1 and 2 imply that f_1 can not be bounded by 1 or 2 edges, and Fact 3 asserts that the disc D' in D_1 corresponding to f_1 can not lie in the interior. Let e_0, e_1, e_2 be the edges bounding f_1, and let e_0 be the interior edge. Then $e_1 \cup e_0 \cup e_2$, together with some arc in ∂D_1, will bound the disc D' which has interior disjoint from G_1. It follows that e_0 is a nice edge in (D_1, G_1). □

Proof of Theorem 2. The theorem is stated in a general form, but we may assume that $S = \partial M$. Otherwise, consider the manifold $M' = M - (\partial M - S)$. We can also assume that the exterior of K is prime (and hence irreducible). The reason is: If S is compressible in $(M, K; \gamma)$, and W is a connected sum factor of $(M, K; \gamma)$ containing S, then S must be compressible in W. So if $E(K)$ is not prime, we may consider the prime factor containing S instead, and get the result $\Delta(m, \gamma) \leq 1$.

By Lemma 3.4, either Γ_2 has n parallel boundary edges or there are nice bands B_1, B_2 satisfying hypotheses of Lemma 2.2 (with $m = n$). In the first case, the band in P_2 containing n parallel boundary edges can be extended to a nice band B satisfying the conditions in Lemma 2.1, while in the second case we can also construct such a nice band by Lemma 2.2. By the above remark, we may assume that $S = \partial M$ and $E(K)$ is irreducible. The theorem now follows from Lemma 2.1 and Proposition 1. □

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