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Numerical analysis of history-dependent hemivariational inequalities and applications to viscoelastic contact problems with normal penetration

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A B S T R A C T

In this paper numerical approximation of history-dependent hemivariational inequalities with constraint is considered, and corresponding Céa's type inequality is derived for error estimate. For a viscoelastic contact problem with normal penetration, an optimal order error estimate is obtained for the linear element method. A numerical experiment for the contact problem is reported which provides numerical evidence of the convergence order predicted by the theoretical analysis.

1. Introduction

Hemivariational inequalities are inequality problems characterized by nonconvexity and nonsmoothness, which may arise in engineering, economics, contact mechanics, etc. Techniques from nonsmooth analysis are applied to deal with difficulties caused by the nonsmoothness, which plays an important role in the mathematical theory of hemivariational inequality. In particular, properties of the generalized (Clarke) directional derivative and subdifferential of a locally Lipschitz function are frequently used; see [1–3].

Numerical methods for hemivariational inequalities have attracted much interest recently, in which the finite element approximation is widely used. An early comprehensive reference in this direction is [4]. In [5], an elliptic hemivariational inequality with convex constraint is considered, and two unconstrained problems can be regarded as special cases of a constrained problem. Finite element approximation is used to solve the problem, and the numerical solutions tend on subsequences to the solution of continuous problems. In [6], a variational–hemivariational inequality, which involves a convex functional and a nonconvex functional, is analyzed under appropriate assumptions. The existence and uniqueness of a solution are presented, and strong convergence for the finite element solution to the inequality is proved. Furthermore, an optimal first-order error estimate is derived for the linear element method. In [7], a general form of elliptic hemivariational inequality with or without convex constraint is considered, internal approximation of the discrete problem is studied with

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convergence result proved and Céa's type inequalities derived. The theoretical results are applied to some particular contact problems, and optimal error estimates are derived using the linear element method. A further study of the paper [7] is given in [8], where external approximation of the inequality is analyzed and new techniques are developed to derive optimal order error estimate. In [9], both internal and external approximations for a general stationary variational–hemivariational inequality are studied. The numerical solutions are proved to converge strongly under minimal solution regularity property available from the well-posedness analysis of the problem. Recently, semipermeable media is studied in the framework of hemivariational inequalities in [10]. The model covers both isotropic, homogeneous semipermeable media and non-isotropic, heterogeneous semipermeable media. Interior and boundary semipermeable terms are considered, and external approximation of an abstract hemivariational inequality is illustrated. Convergence and optimal order error estimate of numerical solutions are shown.

History-dependent hemivariational inequalities are documented in [11–18]. A hemivariational inequality defined on a whole space is studied theoretically in [13]. In [12], numerical methods of temporally semi-discrete and fully discrete schemes are introduced to approximate a quasivariational inequality. In [17], a discrete operator is introduced to approximate the history-dependent operator with specific form, and a fully discrete scheme is used to solve a variational–hemivariational inequality numerically. In [18], an existence and uniqueness result of a general variational–hemivariational inequality is proved. Its internal approximation is presented in [19], and an optimal order error bound is derived. In this paper we further study numerical methods of history-dependent hemivariational inequalities, including its external approximation. In Section 2, a general hemivariational inequality is introduced. Numerical method is used to solve the hemivariational inequality in Section 3. In Section 4, the theoretical results are applied to a contact model with normal penetration and an optimal error estimate is derived for the linear element method. Finally, a numerical example is reported providing numerical evidence of the convergence order predicted by the theoretical analysis in Section 5.

2. A general history-dependent hemivariational inequality

In this section we consider a general history-dependent hemivariational inequality and discuss the existence and uniqueness of the solution.

Let X, X_j be normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_{X_j}$, let X^* and X_j^* be the topological dual of X and X_j. The duality pairings are denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$ and $\langle\cdot,\cdot\rangle_{X_j^*\times X_j}$, respectively. When no confusion may arise, the notation $\langle\cdot,\cdot\rangle_{X^*\times X}$ may be replaced by $\langle\cdot,\cdot\rangle$ for simplicity. Let $\mathbb{R}_+ = [0,+\infty)$ and K be a nonempty subset of X. $C(\mathbb{R}_+;X)$ and $C(\mathbb{R}_+;K)$ denote the spaces of continuous functions from $\mathbb{R}_+$ to X and $\mathbb{R}_+$ to K respectively. Let there be given $A : X \to X^*$, $\mathcal{S} : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X^*)$, $\gamma_j : X \to X_j$ and function $j : X_j \to \mathbb{R}$ is locally Lipschitz. Then the general hemivariational inequality to be studied is the following.

Problem 1. Find a function $u \in C(\mathbb{R}_+;K)$ such that $\forall\, t \in \mathbb{R}_+$,

$$\begin{aligned}&\langle Au(t), v-u(t)\rangle + \langle(\mathcal{S}u)(t), v-u(t)\rangle + j^0(\gamma_j u(t);\gamma_j v-\gamma_j u(t))\\&\quad\geq \langle f(t), v-u(t)\rangle \quad \forall\, v\in K.\end{aligned} \tag{2.1}$$

The following assumptions are adopted in the analysis of Problem 1.

$$\begin{cases}X \text{ is a reflexive Banach space, } K \text{ is a closed and convex set in } X \text{ with}\\ 0\in K, \text{ and } X_j \text{ is a Banach space.}\end{cases} \tag{2.2}$$

$$\begin{cases}j : X_j \to \mathbb{R} \text{ is a locally Lipschitz function such that}\\ \quad \text{(a) } \|\partial j(z)\|_{X_j^*} \leq c_0 + c_1\|z\|_{X_j} \quad \forall\, z\in X_j \text{ with } c_0, c_1 \geq 0;\\ \quad \text{(b) there exists } \alpha_j > 0 \text{ such that}\\ \qquad j^0(z_1;z_2-z_1) + j^0(z_2;z_1-z_2) \leq \alpha_j\|z_1-z_2\|^2_{X_j} \quad \forall\, z_1,z_2\in X_j.\end{cases} \tag{2.3}$$

$$\mathcal{S} : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X^*) \text{ is a history-dependent operator.} \tag{2.4}$$

$$A : X \to X^* \text{ is pseudomonotone and strongly monotone.} \tag{2.5}$$

$$\gamma_j \in \mathcal{L}(X;X_j) \text{ with the norm bounded by } c_j > 0. \tag{2.6}$$

$$\alpha_j c_j^2 < m_A. \tag{2.7}$$

$$f \in C(\mathbb{R}_+;X^*). \tag{2.8}$$

We explain the assumptions as follows. For the locally Lipschitz function $j : X_j \to \mathbb{R}$, we use $j^0(w;z)$ to denote the generalized (Clarke) directional derivative of j at $w \in X_j$ in the direction $z \in X_j$, defined by

$$j^0(w;z) = \limsup_{y\to w,\lambda\downarrow 0} \frac{j(y+\lambda z)-j(y)}{\lambda}.$$

In applications to contact mechanics, the function j usually denotes an integral over a contact boundary and X_j represents the space of square integrable functions or vector-valued square integrable functions over the contact boundary.

The subdifferential (generalized gradient) of j at w, denoted as $\partial j(w)$, is a subset of the dual space X_j^* defined by

$$\partial j(w) = \{\xi \in X_j^* \mid j^0(w; z) \geq \langle \xi, z \rangle_{X_j^* \times X_j} \ \forall z \in X_j\}.$$

By the definition of the subdifferential, the generalized directional derivative can be calculated by the relation $j^0(w; z) = \max\{\langle \xi, z \rangle_{X_j^* \times X_j} \mid \xi \in \partial j(w)\}$. More properties about generalized directional derivative and generalized gradient can be found in [1–3].

The operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X^*)$ is called a history-dependent operator if for any $n \in \mathbb{N}$, there exists a constant $s_n > 0$ such that $\forall t \in [0, n]$,

$$\|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_{X^*} \leq s_n \int_0^t \|u_1(s) - u_2(s)\|_X ds \quad \forall u_1, u_2 \in C(\mathbb{R}_+; X). \tag{2.9}$$

History-dependence refers to the fact that the value $(\mathcal{S}u)(t)$ at a time t depends on the values of u in the entire interval $[0, t]$.

The operator $A : X \to X^*$ is called pseudomonotone if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u \rangle \leq 0$ imply

$$\langle Au, u - v \rangle \leq \liminf \langle Au_n, u_n - v \rangle \quad \forall v \in X.$$

A is strongly monotone if there exists $m_A > 0$ such that

$$\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \forall v_1, v_2 \in X.$$

The closed, convex set K reflects the constraint of the problem. For the contact problem we study, K usually denotes the admissible displacement field satisfying $v_\nu \leq g$ on the contact boundary, where v_ν is the normal component of the displacement $\boldsymbol{v}$ and $g \geq 0$ represents the upper bound of penetration.

Hypothesis (2.3)(a) means that the norm of the subdifferential of j is bounded by a linear function; (2.3)(b) describes constraint to the generalized directional derivative of j which guarantees the uniqueness of the solution to Problem 1. In addition, (2.3)(b) is equivalent to the relaxed monotone condition

$$\langle \partial j(z_1) - \partial j(z_2), z_1 - z_2 \rangle \geq -\alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall z_1, z_2 \in X_j;$$

see [3,7,14,17]. Using (2.3), $0 \in K$ and the strong monotonicity of A, we get that there exists a constant $c \geq 0$ such that $\langle Av, v \rangle \geq m_A \|v\|_X^2 - c\|v\|_X$ for all $v \in X$.

The following existence and uniqueness result for Problem 1 can be derived by taking $\varphi((\mathcal{S}u)(t), u(t), v) = \langle (\mathcal{S}u)(t), v \rangle$ in [18, Theorem 5] and modifying slightly its proof.

Theorem 2. *Assume that conditions (2.2)–(2.8) hold. Then Problem 1 has a unique solution $u \in C(\mathbb{R}_+; K)$.*

3. Numerical method

In this section a fully discrete method is introduced and studied for the numerical solution of Problem 1. We use finite dimensional set K^h to approximate the convex subset K and K^h is not necessarily a subset of K. In the case $K^h \subset K$, the approximation is called internal, such a numerical method is studied in solving a variational–hemivariational inequality in [19]. In the case $K^h \not\subseteq K$, the corresponding approximation is called external. A Céa's type inequality is derived for external approximation of Problem 1.

Next we describe the discrete scheme for Problem 1, in which the spatial variable, the time variable and the history-dependent operator are all discretized. On the spatial discretization, a regular family of finite element partition $\{T^h\}$ is introduced where h is the mesh parameter. Corresponding finite element space $X^h \subset X$ and $K^h \subset X^h$ are presented, and K^h is assumed to be convex and closed with $0 \in K^h$. On the temporal discretization, k is the time step size and uniform partition is used for the interval $I = [0, T]$. Denote $k = \frac{T}{N}$ and $t_n = nk$ where $0 \leq n \leq N$.

Now we describe the discretization for history-dependent operator. Let $\mathcal{L}(X)$ and $\mathcal{L}(X; X^*)$ be the spaces of bounded linear operators from X to X and X to X^* respectively. Let $C(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$ and $C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$ be the spaces of continuous and twice continuously differentiable functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$ with values in $\mathcal{L}(X)$. We focus on $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X^*)$ with a specific form

$$(\mathcal{S}v)(t) = \mathcal{R}\left(\int_0^t q(t, s)v(s)ds + a_S\right) \quad \forall v \in C(\mathbb{R}_+; X), \ \forall t \in \mathbb{R}_+, \tag{3.1}$$

where $\mathcal{R} \in \mathcal{L}(X; X^*)$, $q \in C(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $a_S \in X$. The above operator $\mathcal{S}$ is useful in dealing with history-dependent problems. For example, the well-known Volterra operator $\mathcal{S}$, defined by

$$(\mathcal{S}v)(t) = \int_0^t \mathcal{B}(t - s)v(s)ds \quad \forall v \in C(\mathbb{R}_+; X), \ \forall t \in \mathbb{R}_+,$$

is a history-dependent operator, where $\mathcal{B} \in C(0, T; \mathcal{L}(X; X^*))$. More discussions on $\mathcal{S}$ can be founded in [12,13,17,20]. For the integral in (3.1), we apply the trapezoidal rule

$$\int_0^{t_n} Z(s)ds \approx k \sum_{i=0}^{n}{}' Z(t_i),$$

where a prime indicates that the coefficients of the first and last terms of the summation are to be halved. For $n = 0$, the summation is defined to be 0. Then the fully discrete operator $\mathcal{S}_n^{kh}$ is defined by

$$\mathcal{S}_n^{kh} u^{kh} := \mathcal{R}\left(k \sum_{i=0}^{n}{}' q(t_n, t_i) u_i^{kh} + a_S\right), \quad u^{kh} = \{u_i^{kh}\}_{i=0}^N \subset X^h. \tag{3.2}$$

Below, we denote $\|\mathcal{R}\| = \|\mathcal{R}\|_{\mathcal{L}(X;X^*)}$ and $\|q\| = \|q\|_{C(I\times I;\mathcal{L}(X))}$. The space $C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$ is needed in bounding the difference between the history-dependent operator and its discrete analogue. In addition we introduce a constant C with possibly different values in different places which is independent of the mesh parameter h and the time step k. Now we illustrate the following fully discrete approximation of Problem 1.

Problem 3. The fully-discrete scheme for Problem 1 is to find the discrete solution $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset K^h$ such that for $0 \le n \le N$,

$$\begin{aligned} &\langle A u_n^{kh}, v^h - u_n^{kh}\rangle + \langle \mathcal{S}_n^{kh} u^{kh}, v^h - u_n^{kh}\rangle + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \\ &\quad \ge \langle f_n, v^h - u_n^{kh}\rangle \quad \forall\, v^h \in K^h. \end{aligned} \tag{3.3}$$

Under assumptions (2.2)–(2.3), (2.5)–(2.8), (3.1), 3.2 and a small time step

$$k < \frac{m_A - \alpha_j c_j^2}{\|\mathcal{R}\|\,\|q\|}, \tag{3.4}$$

then there exists a unique solution u^{kh} of Problem 3. The proof is similar to that found in [19] where the solution existence and uniqueness of a discrete scheme is proved for a history-dependent variational–hemivariational inequality.

As a preparation for error estimation, we prove that the sequence $\|u_n^{kh}\|_X$ is uniformly bounded.

Proposition 4. *There exist constants $M > 0$ and $k_0 > 0$ such that $\|u_n^{kh}\|_X \le M\ \forall\, 0 < k \le k_0, \forall\, h > 0$.*

Proof. Taking $v^h = 0$ in (3.3), we have

$$\langle A u_n^{kh}, u_n^{kh}\rangle \le \langle \mathcal{S}_n^{kh} u^{kh}, -u_n^{kh}\rangle + j^0(\gamma_j u_n^{kh}; -\gamma_j u_n^{kh}) + \langle f_n, u_n^{kh}\rangle. \tag{3.5}$$

Note that $0 \in K$, thus for all $v \in K$,

$$m_A \|v\|_X^2 \le \langle Av - A0, v\rangle \le \langle Av, v\rangle + \|A0\|_{X^*}\|v\|_X. \tag{3.6}$$

Taking $v = u_n^{kh}$, we get

$$\langle A u_n^{kh}, u_n^{kh}\rangle \ge m_A \|u_n^{kh}\|_X^2 - \|A0\|_{X^*}\|u_n^{kh}\|_X. \tag{3.7}$$

From the formula 3.2, we have

$$\begin{aligned} \langle \mathcal{S}_n^{kh} u^{kh}, -u_n^{kh}\rangle &\le \|\mathcal{S}_n^{kh} u^{kh}\|_{X^*}\|u_n^{kh}\|_X \\ &\le k\|\mathcal{R}\|\,\|q\|\left(\sum_{i=0}^{n} \|u_i^{kh}\|_X\right)\|u_n^{kh}\|_X + \|\mathcal{R}\|\,\|a_S\|_X\|u_n^{kh}\|_X. \end{aligned} \tag{3.8}$$

Take $z_1 = \gamma_j u_n^{kh}$, $z_2 = 0$ in (2.3)(b) and apply (2.3)(a) to obtain

$$\begin{aligned} j^0(\gamma_j u_n^{kh}; -\gamma_j u_n^{kh}) &\le \alpha_j \|\gamma_j u_n^{kh}\|_{X_j}^2 - j^0(0; \gamma_j u_n^{kh}) \\ &\le \alpha_j c_j^2 \|u_n^{kh}\|_X^2 + c_0 c_j \|u_n^{kh}\|_X. \end{aligned} \tag{3.9}$$

Obviously,

$$\langle f_n, u_n^{kh}\rangle \le \|f_n\|_{X^*}\|u_n^{kh}\|_X. \tag{3.10}$$

Combining (3.5) and (3.7)–(3.10), we have

$$\begin{aligned} \|u_n^{kh}\|_X \le{}& \frac{1}{m_A - \alpha_j c_j^2}\left(\|\mathcal{R}\|\,\|a_S\|_X + c_0 c_j + \|A0\|_{X^*} + \|f_n\|_{X^*}\right) \\ &+ \frac{k}{m_A - \alpha_j c_j^2}\|\mathcal{R}\|\,\|q\| \sum_{i=0}^{n} \|u_i^{kh}\|_X. \end{aligned} \tag{3.11}$$

We recall the discrete Gronwall's inequality [21]: let $\{e_n\}_{n=0}^N$ and $\{g_n\}_{n=0}^N$ be two sequences of non-negative numbers with

$$e_n \le C_1 g_n + C_2 k \sum_{i=0}^{n} e_i, \quad 0 \le n \le N,$$

where C_1, C_2 are two constants. Then there is a constant C_3 such that for $k > 0$ sufficiently small, we have

$$\max_{0\le n\le N} e_n \le C_3 \max_{0\le n\le N} g_n.$$

Thus we obtain

$$\max_{0\le n\le N} \|u_n^{kh}\|_X \le C\left(\max_{0\le n\le N} \|f_n\|_{X^*} + \|\mathcal{R}\|\|a_S\|_X + c_0 c_j + \|A0\|_{X^*}\right). \tag{3.12}$$

Since $f \in C(\mathbb{R}_+; X^*)$, the sequence $\{u_n^{kh}\}$ is bounded uniformly. □

To proceed further for error analysis, we assume the operator $A : X \to X^*$ is Lipschitz continuous, i.e., for a constant $L_A > 0$,

$$\|Au - Av\|_{X^*} \le L_A \|u - v\|_X \quad \forall\, u, v \in X. \tag{3.13}$$

Let $t = t_n$ in (2.1), we have

$$\langle Au_n, v - u_n\rangle + \langle \mathcal{S}_n u, v - u_n\rangle + j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n) \ge \langle f_n, v - u_n\rangle. \tag{3.14}$$

where $\mathcal{S}_n u = \mathcal{R}\left(\int_0^{t_n} q(t_n, s)u(s)ds + a_S\right)$ and $u_n = u(t_n)$. Define a residual term

$$E_n(v) = \langle Au_n, v\rangle + \langle \mathcal{S}_n u, v\rangle + j^0(\gamma_j u_n; \gamma_j v) - \langle f_n, v\rangle, \tag{3.15}$$

which play an important role in error estimation. Now we derive a Céa's type inequality.

Theorem 5. *Assume (2.2)–(2.3), (2.5)–(2.8), (3.1), 3.2, (3.4) and (3.13). Furthermore, assume regularity conditions $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$ and $u \in W^{2,\infty}_{loc}(\mathbb{R}_+; X)$. Then the following error estimate holds for the numerical solution of Problem 1 defined by Problem 3:*

$$\begin{aligned}\max_{0\le n\le N} \|u_n - u_n^{kh}\|_X \le C \max_{0\le n\le N} \Bigg\{ &\inf_{v^h\in K^h}\left(\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\right. \\ &\left. + |E_n(v^h - u_n)|^{\frac{1}{2}}\right) + \inf_{v\in K} |E_n(v - u_n^{kh})|^{\frac{1}{2}}\Bigg\} \\ &+ Ck^2\|u\|_{W^{2,\infty}(I;X)}.\end{aligned} \tag{3.16}$$

Proof. We begin with

$$\begin{aligned} m_A\|u_n - u_n^{kh}\|_X^2 &\le \langle Au_n - Au_n^{kh}, u_n - u_n^{kh}\rangle \\ &= \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \langle Au_n, v^h - u_n\rangle + \langle Au_n, u_n - v\rangle \\ &\quad + \langle Au_n, v - u_n^{kh}\rangle + \langle Au_n^{kh}, u_n^{kh} - v^h\rangle. \end{aligned} \tag{3.17}$$

Combining (3.17), (3.3) and (3.14), we have

$$\begin{aligned} m_A\|u_n - u_n^{kh}\|_X^2 &\le \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \langle Au_n, v^h - u_n\rangle + \langle Au_n, v - u_n^{kh}\rangle \\ &\quad + \langle \mathcal{S}_n u, v - u_n\rangle + j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n) - \langle f_n, v - u_n\rangle \\ &\quad + \langle \mathcal{S}_n^{kh} u^{kh}, v^h - u_n^{kh}\rangle + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \\ &\quad - \langle f_n, v^h - u_n^{kh}\rangle \\ &= \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + E_n(v^h - u_n) + E_n(v - u_n^{kh}) \\ &\quad + E_S(v, v^h, u_n, u_n^{kh}) + E_j(v, v^h, u_n, u_n^{kh}), \end{aligned} \tag{3.18}$$

where $E_n(v^h - u_n)$ and $E_n(v - u_n^{kh})$ are defined by (3.15), $E_S(v, v^h, u_n, u_n^{kh})$ and $E_j(v, v^h, u_n, u_n^{kh})$ are defined by

$$\begin{aligned} E_S(v, v^h, u_n, u_n^{kh}) &= \langle \mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}, u_n^{kh} - v^h\rangle \\ &= \langle \mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}, u_n^{kh} - u_n\rangle + \langle \mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}, u_n - v^h\rangle, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} E_j(v, v^h, u_n, u_n^{kh}) &= j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \\ &\quad - j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) - j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n^{kh}). \end{aligned} \tag{3.20}$$

Next we bound $E_S(v, v^h, u_n, u_n^{kh})$ and $E_j(v, v^h, u_n, u_n^{kh})$. For any small $\varepsilon > 0$, we have

$$\begin{aligned} E_S &\le \|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_{X^*} \|u_n - u_n^{kh}\|_X + \|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_{X^*} \|u_n - v^h\|_X \\ &\le \varepsilon \|u_n - u_n^{kh}\|_X^2 + \|u_n - v^h\|_X^2 + \frac{1}{4}(1 + \frac{1}{\varepsilon})\|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_{X^*}^2. \end{aligned} \tag{3.21}$$

Using the subadditivity of the generalized directional derivative, we get

$$\begin{aligned} E_j &\le j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n^{kh}) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n) \\ &\quad + j^0(\gamma_j u_n^{kh}; \gamma_j u_n - \gamma_j u_n^{kh}) - j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) - j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n^{kh}) \\ &\le \alpha_j \|\gamma_j u_n - \gamma_j u_n^{kh}\|_{X_j}^2 + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n) - j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) \\ &\le \alpha_j c_j^2 \|u_n - u_n^{kh}\|_X^2 + (2c_0 + c_1 \|\gamma_j u_n^{kh}\|_{X_j} + c_1 \|\gamma_j u_n\|_{X_j}) \|\gamma_j u_n - \gamma_j v^h\|_{X_j}. \end{aligned}$$

Since A is Lipschitz continuous, we have

$$\begin{aligned} \langle A u_n - A u_n^{kh}, u_n - v^h \rangle &\le L_A \|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X \\ &\le \varepsilon \|u_n - u_n^{kh}\|_X^2 + \frac{L_A^2}{4\varepsilon} \|u_n - v^h\|_X^2. \end{aligned} \tag{3.22}$$

Thus, from (3.18), we obtain

$$\begin{aligned} m_A \|u_n - u_n^{kh}\|_X^2 &\le (2\varepsilon + \alpha_j c_j^2) \|u_n - u_n^{kh}\|_X^2 + (1 + \frac{L_A^2}{4\varepsilon}) \|u_n - v^h\|_X^2 \\ &\quad + (2c_0 + c_1 c_j \|u_n^{kh}\|_X + c_1 c_j \|u_n\|_X) \|\gamma_j u_n - \gamma_j v^h\|_{X_j} \\ &\quad + |E_n(v^h - u_n)| + |E_n(v - u_n^{kh})| + \frac{1}{4}(1 + \frac{1}{\varepsilon}) \|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_{X^*}^2. \end{aligned}$$

Using the conditions $u \in C(I; K)$ and (2.7) and applying Proposition 4, we find

$$\begin{aligned} \|u_n - u_n^{kh}\|_X &\le C \Bigg(\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E_n(v^h - u_n)|^{\frac{1}{2}} \\ &\quad + |E_n(v - u_n^{kh})|^{\frac{1}{2}} + \|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_{X^*} \Bigg). \end{aligned} \tag{3.23}$$

Recall the assumptions $u \in W_{loc}^{2,\infty}(\mathbb{R}_+; X)$, $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$. We have

$$\begin{aligned} \|\mathcal{S}_n^{kh} u^{kh} - \mathcal{S}_n u\|_{X^*} &\le \|\mathcal{R}(k \sum_{i=0}^{n}{}' q(t_n, t_i) u_i^{kh} + a_S) - \mathcal{R}(k \sum_{i=0}^{n}{}' q(t_n, t_i) u_i + a_S)\|_{X^*} \\ &\quad + \|\mathcal{R}(k \sum_{i=0}^{n}{}' q(t_n, t_i) u_i + a_S) - \mathcal{R}(\int_0^{t_n} q(t_n, s) u(s) ds + a_S)\|_{X^*} \\ &\le k \|\mathcal{R}\| \|q\| \left(\sum_{i=0}^{n} \|u_i^{kh} - u_i\|_X \right) + C k^2 \|u\|_{W^{2,\infty}(I;X)}. \end{aligned}$$

Thus, from (3.23),

$$\begin{aligned} \|u_n - u_n^{kh}\|_X &\le C \Bigg(\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E_n(v^h - u_n)|^{\frac{1}{2}} \\ &\quad + |E_n(v - u_n^{kh})|^{\frac{1}{2}} + k^2 \|u\|_{W^{2,\infty}(I;X)} \Bigg) + C k \sum_{i=0}^{n} \|u_i - u_i^{kh}\|_X. \end{aligned}$$

Using discrete Gronwall's inequality again, we obtain the error bound (3.17). □

In the case $K^h \subset K \cap X^h$, the approximation is internal and the residual term $\inf_{v \in K} |E_n(v - u_n^{kh})| = 0$. Then Theorem 5 reduces to the following result.

Theorem 6. *Assume (2.2)–(2.3), (2.5)–(2.8), (3.1), 3.2, (3.4) and (3.13). Furthermore, assume $K^h \subset K \cap X^h$ and regularities $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$ and $u \in W_{loc}^{2,\infty}(\mathbb{R}_+; X)$. Then we have the error estimate of the numerical solution of Problem 1 defined* *by Problem 3:*

$$\max_{0\le n\le N}\|u_n-u_n^{kh}\|_X\le C\max_{0\le n\le N}\left\{\inf_{v^h\in K^h}\left(\|u_n-v^h\|_X+\|\gamma_j u_n-\gamma_j v^h\|_{X_j}^{\frac12}\right.\right.$$
$$\left.\left.+|E_n(v^h-u_n)|^{\frac12}\right)\right\}+Ck^2\|u\|_{W^{2,\infty}(I;X)}. \tag{3.24}$$

A Céa's type inequality of form (3.24) is derived in [19] for the error of approximations of a history-dependent variational–hemivariational inequality.

4. Application to a viscoelastic contact problem

In this section a representative viscoelastic contact model with normal penetration is considered. The numerical method and theoretical results illustrated in Section 3 are applied to the contact problem, and an optimal order error estimate is derived for the linear finite element method.

We start with some standard notation in mechanics. Let $d=2\text{or}3$ be the spatial dimension. Let $\Omega\subset\mathbb{R}^d$ be a Lipschitz domain, representing the reference configuration of the deformable body. Let $\boldsymbol{u}=(u_i)$ be the displacement field, and $\boldsymbol{\sigma}=(\sigma_{ij})$ the stress tensor of second-order. They are the main unknowns of the problem. Symbols $\mathcal{A}=(a_{ijkl})$ and $\mathcal{B}=(b_{ijkl})$ both represent fourth-order tensors. All the subscripts i,j,k,l satisfy $1\le i,j,k,l\le d$. Since the boundary Γ of Ω is Lipschitz continuous, the unit outward normal $\boldsymbol{\nu}=(\nu_i)$ exists a.e. on Γ. For a vector field $\boldsymbol{v}$ defined on Γ, $v_\nu:=\boldsymbol{v}\cdot\boldsymbol{\nu}$ denotes its normal component and $\boldsymbol{v}_\tau:=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$ its tangential component. Similarly for a stress field $\boldsymbol{\sigma}$, $\sigma_\nu:=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ represents its normal component and $\boldsymbol{\sigma_\tau}:=\boldsymbol{\sigma\nu}-\boldsymbol{\sigma_\nu\nu}$ represents its tangential component. Let $\boldsymbol{\varepsilon}(\boldsymbol{u})=(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)/2$ be the linearized strain tensor. The classical formulation of the viscoelastic contact problem is as follows.

Problem 7. Find a displacement $\boldsymbol{u}:\Omega\times\mathbb{R}_+\to\mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}:\Omega\times\mathbb{R}_+\to\mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \text{ in } \Omega\times\mathbb{R}_+, \tag{4.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t)=\boldsymbol{0} \text{ in } \Omega\times\mathbb{R}_+, \tag{4.2}$$

$$\boldsymbol{u}(t)=\boldsymbol{0} \text{ on } \Gamma_1\times\mathbb{R}_+, \tag{4.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}=\boldsymbol{f}_2(t) \text{ on } \Gamma_2\times\mathbb{R}_+, \tag{4.4}$$

$$\left.\begin{array}{l} u_\nu(t)\le g,\ \sigma_\nu(t)+\xi_\nu(t)\le 0,\\ (\sigma_\nu(t)+\xi_\nu(t))(u_\nu(t)-g)=0,\\ \xi_\nu(t)\in\partial j_\nu(u_\nu(t))\end{array}\right\} \text{ on } \Gamma_3\times\mathbb{R}_+, \tag{4.5}$$

$$\boldsymbol{\sigma_\tau}(t)=\boldsymbol{0} \text{ on } \Gamma_3\times\mathbb{R}_+. \tag{4.6}$$

The model is illustrated graphically in Fig. 1 of Section 5. Note that a slight different model is studied in [18]. In the following we give the description of the physical setting. A layer of elastic material is laid on the rigid foundation, and a viscoelastic body which occupies Ω is in contact with the foundation. The boundary Γ of the body is divided into three measurable parts Γ_1, Γ_2 and Γ_3, with $\text{meas}(\Gamma_1)>0$. The whole contact process is quasi-static.

Eq. (4.1) is a constitutive law for viscoelastic material with long memory (cf. [14,17,18,22]), where $\mathcal{A}$ is a nonlinear elasticity tensor and $\mathcal{B}$ is a relaxation tensor. Relation (4.2) is the equilibrium equation, where Div is the divergence operator, and $\boldsymbol{f_0}$ is the density of volume forces acting on the body Ω. Relations (4.3) and (4.4) mean that the body is fixed on Γ_1 and the density of surface forces $\boldsymbol{f_2}$ acts on Γ_2. Eq. (4.5) describes the state of the contact process on Γ_3. Since the penetration is restricted by $v_\nu\le g$, the set of admissible displacements is a convex and closed subset of $H^1(\Omega;\mathbb{R}^d)$. The normal stress is described by a Clarke subdifferential of a function j_ν. Relation (4.6) means the contact process is frictionless on the boundary Γ_3.

To study Problem 7, we introduce the space $V=\{\boldsymbol{v}=(v_i)\in H^1(\Omega;\mathbb{R}^d)\,|\,\boldsymbol{v}=0 \text{ a.e. on } \Gamma_1\}$ for the displacement field, and the space $\mathcal{H}=\{\boldsymbol{\tau}=(\tau_{ij})\in L^2(\Omega;\mathbb{S}^d)\mid\tau_{ij}=\tau_{ji},\ 1\le i,j\le d\}$ for the stress field, where $\mathbb{S}^d$ denotes the space of symmetric tensor of second-order. Denote $Q_\infty=\{\mathcal{E}=(\mathcal{E}_{ijkl})|\ \mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d\}$ with the associated norm defined by $\|\mathcal{E}\|_{Q_\infty}=\sum_{1\le i,j,k,l\le d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}$. The canonical inner products in the spaces $\mathbb{R}^d$, $\mathbb{S}^d$ and $\mathcal{H}$ are defined by $(\boldsymbol{u},\boldsymbol{v})_{\mathbb{R}^d}=\sum_{i=1}^d u_iv_i$ for all $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d$, $(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathbb{S}^d}=\sum_{1\le i,j\le d}\sigma_{ij}\tau_{ij}$ for all $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d$ and $(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}=\int_\Omega\sum_{1\le i,j\le d}\sigma_{ij}(x)\tau_{ij}(x)dx$ for all $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H}$ respectively. Since $\text{meas}(\Gamma_1)>0$, V is a Hilbert space with the inner product $(\boldsymbol{u},\boldsymbol{v})_V=(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$ for all $\boldsymbol{u},\boldsymbol{v}\in V$.

For any $\boldsymbol{v}\in V$, the trace of $\boldsymbol{v}$ on the boundary Γ is still denoted by $\boldsymbol{v}$. The normal trace is $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ and tangential trace $\boldsymbol{v_\tau}=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$.

Let $K=\{\boldsymbol{v}\in V|\ v_\nu\le g \text{ a.e. on } \Gamma_3\}$, where the thickness function g is nonnegative. It is clear that K is convex and closed in V and $0\in K$.

The function $j_\nu:\Gamma_3\times\mathbb{R}\to\mathbb{R}$ satisfies the following assumptions: $j_\nu(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\boldsymbol{x}\in\Gamma_3$; $j_\nu(\cdot,r)$ is measurable on Γ_3 for all $r\in\mathbb{R}$, and there exists $\overline{e}\in L^2(\Gamma_3)$ such that $j_\nu(\cdot,\overline{e}(\cdot))\in L^1(\Gamma_3)$; there exist constants $\overline{c}_0,\overline{c}_1>0$ such that $|\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1 |r|$ for a.e. $\boldsymbol{x} \in \Gamma_3$, for all $r \in \mathbb{R}$; there exists a constant $\overline{\alpha}_\nu \ge 0$ such that the sum of $j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1)$ and $j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2)$ is less than or equal to the quantity $\overline{\alpha}_\nu |r_1 - r_2|^2$ for all $r_1, r_2 \in \mathbb{R}$ for a.e. $\boldsymbol{x} \in \Gamma_3$.

For the elasticity tensor $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, we assume $\mathcal{A}(\cdot, \boldsymbol{\varepsilon})$ is measurable on Ω, $\mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0}$, $\mathcal{A}(\boldsymbol{x}, \cdot)$ is strongly monotone and Lipschitz continuous. We denote by $m_\mathcal{A} > 0$ the constant in the strong monotonicity, and by $L_\mathcal{A} > 0$ the Lipschitz constant. The fourth-order tensor $\mathcal{B}$ satisfies $\mathcal{B} \in C(\mathbb{R}_+; \mathbf{Q}_\infty)$.

The density of body force $\boldsymbol{f_0}$ and the density of surface traction $\boldsymbol{f_2}$ satisfy $\boldsymbol{f_0} \in C(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^d))$ and $\boldsymbol{f_2} \in C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d))$ respectively. Using Riesz representation theorem, we can define a function $\boldsymbol{f} : \mathbb{R}_+ \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f_0}(t), \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f_2}(t), \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+.$$

Then the weak formulation of Problem 7 is the following.

Problem 8. Find a displacement $\boldsymbol{u} : \mathbb{R}_+ \to K$ such that $\forall\, t \in \mathbb{R}_+$,

$$\begin{aligned} &\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)_\mathcal{H} + \left(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\right)_\mathcal{H} \\ &\quad + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))d\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle_{V^* \times V}\ \forall\, \boldsymbol{v} \in K. \end{aligned} \tag{4.7}$$

We define the operators $A : V \to V^*$, $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V^*)$ and function $j : X_j \to \mathbb{R}$ as follows:

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} = \Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_\mathcal{H} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4.8}$$

$$(\mathcal{S}\boldsymbol{u})(t) = \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V). \tag{4.9}$$

Let $X = V$, $X_j = L^2(\Gamma_3)$, $\gamma_j \boldsymbol{v} = v_\nu\ \forall\, \boldsymbol{v} \in V$. Define

$$j(v) = \int_{\Gamma_3} j_\nu(v)d\Gamma \quad \forall\, v \in X_j. \tag{4.10}$$

Thus we have

$$j(\gamma_j \boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)d\Gamma \quad \forall\, \boldsymbol{v} \in V. \tag{4.11}$$

Using the argument in [18, Section 6] and Theorem 2, under the hypotheses for Problem 7 and the smallness condition $\overline{\alpha}_\nu \|\gamma_j\|^2 < m_\mathcal{A}$, we can show that Problem 8 has a unique solution $\boldsymbol{u} \in C(\mathbb{R}_+; K)$.

Next we consider a fully discrete method to solve Problem 8. For simplicity, assume Ω is a polygonal/polyhedral. Let $\{\mathcal{T}^h\}$ be a regular family of partition of Ω into triangles/tetrahedrons, and the boundary Γ is divided into $\Gamma_{i,l}$, $1 \le l \le l_i$, $1 \le i \le 3$. If there is intersection between the side/face of an element and the set $\Gamma_{i,l}$, and the measurement of the intersection is positive with respect to $\Gamma_{i,l}$, then the side/face locates entirely in $\Gamma_{i,l}$. Denote $V^h = \{\boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d$ for $T \in \mathcal{T}^h, \boldsymbol{v}^h = \boldsymbol{0}$ on $\Gamma_1\}$ be the linear element spaces for $\{\mathcal{T}^h\}$, and K^h be the subsets of V^h defined by $K^h = \{\boldsymbol{v}^h \in V^h \mid v_\nu^h \le g$ at node points on $\Gamma_3\}$. Thus $0 \in K^h$ is satisfied automatically. If the thickness function g is concave, then $K^h \subset K$. The numerical method for this case is studied in [19]. We consider the general case $K^h \not\subseteq K$ in this paper.

For the time interval $[0, T]$ the uniform partition is used for the temporal discretization, and the notations k, t_n still denotes the time step and time nodes in the partition. The trapezoidal rule is applied to approximate the integral in (4.9).

Thus the numerical method for Problem 8 is defined as follows.

Problem 9. Find a discrete displacement $\boldsymbol{u}^{kh} := \{\boldsymbol{u}_n^{kh}\}_{n=0}^N \subset K^h$ such that for $0 \le n \le N$,

$$\begin{aligned} &\Big(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh})\Big)_\mathcal{H} + \left(k \sum_{i=0}^{n}{}' \mathcal{B}(t_n - t_i)\boldsymbol{\varepsilon}(\mathbf{u}_i^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh})\right)_\mathcal{H} \\ &\quad + \int_{\Gamma_3} j_\nu^0(u_{n,\nu}^{kh}; v_\nu^h - u_{n,\nu}^{kh})d\Gamma \ge \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{kh} \rangle_{V^* \times V}\ \forall\, \boldsymbol{v}^h \in K^h. \end{aligned} \tag{4.12}$$

By the same kind of derivation used in proving Theorem 5, we have

$$\begin{aligned} \max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \le C \max_{0\le n\le N} \Bigg\{ &\inf_{\boldsymbol{v}^h\in K^h} \Big(\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_V + \|u_{n,\nu} - v_\nu^h\|_{L^2(\Gamma_3)}^{\frac{1}{2}} \\ &+ |E_n(\boldsymbol{v}^h - \boldsymbol{u}_n)|^{\frac{1}{2}} \Big) + \inf_{\boldsymbol{v}\in K} |E_n(\boldsymbol{v} - \boldsymbol{u}_n^{kh})|^{\frac{1}{2}} \Bigg\} \\ &+ Ck^2 \|\boldsymbol{u}\|_{W^{2,\infty}(I;V)}, \end{aligned} \tag{4.13}$$

where

$$E_n(\boldsymbol{v}) = \langle \mathcal{A}\boldsymbol{u}_n, \boldsymbol{v}\rangle + \langle \mathcal{S}_n\boldsymbol{u}, \boldsymbol{v}\rangle + \int_{\Gamma_3} j_\nu^0(u_{n,\nu}; v_\nu)d\Gamma - \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle. \tag{4.14}$$

For solution regularity, we assume $\boldsymbol{u} \in W^{2,\infty}_{loc}(\mathbb{R}_+; V) \cap C(\mathbb{R}_+; H^2(\Omega; \mathbb{R}^d))$, $\boldsymbol{\sigma\nu} \in C(\mathbb{R}_+; L^2(\Gamma_3; \mathbb{R}^d))$, $\boldsymbol{u}|_{\Gamma_{3,l}} \in C(\mathbb{R}_+; H^2(\Gamma_{3,l}; \mathbb{R}^d))$ and $g|_{\Gamma_{3,l}} \in H^2(\Gamma_{3,l})$, $1 \le l \le l_3$.

Note that if $\mathcal{A}$ and $\mathcal{B}$ are suitably smooth, then we can derive $\boldsymbol{\sigma} \in C(\mathbb{R}_+; H^1(\Omega; \mathbb{S}^d))$ from $C(\mathbb{R}_+; H^2(\Omega; \mathbb{R}^d))$. In addition, the condition $\boldsymbol{\sigma} \in C(\mathbb{R}_+; H^1(\Omega; \mathbb{S}^d))$ implies $\boldsymbol{\sigma\nu} \in C(\mathbb{R}_+; L^2(\Gamma_3; \mathbb{R}^d))$.

Now we bound the residual term $E_n(\boldsymbol{v})$. Define a subset of K,

$$\widetilde{K} = \{\boldsymbol{v} \in C^\infty(\overline{\Omega})^d |\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \cup \Gamma_3\}.$$

Pointwise equations will be shown from the weak formulation (4.7) and the arguments in [21, Section 8.1]. Define

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \text{ in } \Omega.$$

Let $\widetilde{\boldsymbol{v}}$ be an arbitrary function from the subset $\widetilde{K}$, and take $\boldsymbol{v} = \boldsymbol{u} \pm \widetilde{\boldsymbol{v}}$ in (4.7) to get

$$\Big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}})\Big)_{\mathcal{H}} = \langle \boldsymbol{f}(t), \widetilde{\boldsymbol{v}}\rangle_{V^*\times V}\ \forall\ \widetilde{\boldsymbol{v}} \in \widetilde{K}.$$

The following two equations are derived from the above relation:

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \text{ a.e. in } \Omega, \tag{4.15}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \text{ a.e. on } \Gamma_2. \tag{4.16}$$

Taking $t = t_n$ in (4.15) and (4.16), we get

$$\mathrm{Div}\,\boldsymbol{\sigma}(t_n) + \boldsymbol{f}_0(t_n) = \boldsymbol{0} \text{ a.e. in } \Omega, \tag{4.17}$$

$$\boldsymbol{\sigma}(t_n)\boldsymbol{\nu} = \boldsymbol{f}_2(t_n) \text{ a.e. on } \Gamma_2. \tag{4.18}$$

Multiplying (4.17) by $\boldsymbol{v}$ and integrating over Ω, we obtain

$$\int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}(t_n)\cdot\boldsymbol{v}dx + \int_\Omega \boldsymbol{f}_0(t_n)\cdot\boldsymbol{v}dx = 0. \tag{4.19}$$

Performing an integration by parts, we get

$$\int_\Omega \boldsymbol{\sigma}(t_n)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})dx = \int_\Omega \boldsymbol{f}_0(t_n)\cdot\boldsymbol{v}dx + \int_\Gamma \boldsymbol{\sigma}(t_n)\boldsymbol{\nu}\cdot\boldsymbol{v}d\Gamma. \tag{4.20}$$

Therefore

$$\begin{aligned}\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u_n})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})dx + \int_\Omega\int_0^{t_n} \mathcal{B}(t_n-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})dx \\ = \int_\Omega \boldsymbol{f}_0(t_n)\cdot\boldsymbol{v}dx + \int_{\Gamma_2}\boldsymbol{f}_2(t_n)\cdot\boldsymbol{v}d\Gamma + \int_{\Gamma_3}\boldsymbol{\sigma}(t_n)\boldsymbol{\nu}\cdot\boldsymbol{v}d\Gamma,\end{aligned}$$

i.e.,

$$\langle \mathcal{A}\boldsymbol{u}_n, \boldsymbol{v}\rangle + \langle \mathcal{S}_n\boldsymbol{u}, \boldsymbol{v}\rangle - \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle = \int_{\Gamma_3}\boldsymbol{\sigma}(t_n)\boldsymbol{\nu}\cdot\boldsymbol{v}d\Gamma. \tag{4.21}$$

Therefore

$$E_n(\boldsymbol{v}) = \int_{\Gamma_3}\boldsymbol{\sigma}(t_n)\boldsymbol{\nu}\cdot\boldsymbol{v}d\Gamma + \int_{\Gamma_3} j_\nu^0(u_{n,\nu}; v_\nu)d\Gamma. \tag{4.22}$$

Thus

$$|E_n(\boldsymbol{v})| \le |\int_{\Gamma_3}\boldsymbol{\sigma}(t_n)\boldsymbol{\nu}\cdot\boldsymbol{v}d\Gamma| + |\int_{\Gamma_3} j_\nu^0(u_{n,\nu}; v_\nu)d\Gamma|.$$

Combining with $\boldsymbol{\sigma\nu} \in C(\mathbb{R}_+; L^2(\Gamma_3; \mathbb{R}^d))$, we have

$$|E_n(\boldsymbol{v})| \le C\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)}. \tag{4.23}$$

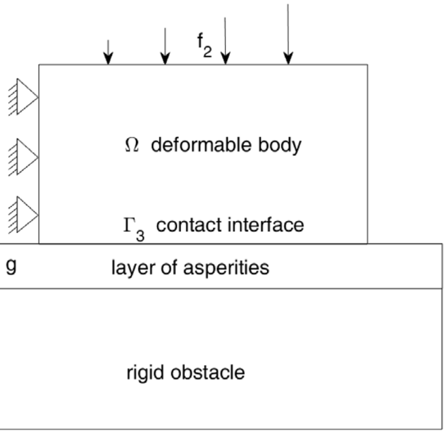

Fig. 1. Initial configuration of the contact problem.

Then we derive the bounds

$$|E_n(\boldsymbol{v}^h - \boldsymbol{u}_n)| \le C\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}, \tag{4.24}$$

$$|E_n(\boldsymbol{v} - \boldsymbol{u}_n^{kh})| \le C\|\boldsymbol{v} - \boldsymbol{u}_n^{kh}\|_{L^2(\Gamma_3;\mathbb{R}^d)}. \tag{4.25}$$

Then from (4.13),

$$\begin{aligned}\max_{0\le n\le N}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \le C\max_{0\le n\le N}\Bigg\{&\inf_{\boldsymbol{v}^h\in K^h}\left(\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_V + \|\boldsymbol{u}_n - \boldsymbol{v}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{\frac{1}{2}}\right)\\ &+\inf_{\boldsymbol{v}\in K}\|\boldsymbol{v} - \boldsymbol{u}_n^{kh}\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{\frac{1}{2}}\Bigg\} + Ck^2\|\boldsymbol{u}\|_{W^{2,\infty}(I;V)}.\end{aligned} \tag{4.26}$$

Now we focus on the estimate of the third term on the right hand of (4.26), which can be solved using the arguments presented in [8, Section 4].

Suppose the contact boundary Γ_3 is smooth. For $\boldsymbol{x} \in \Gamma_3$, the unit outward normal $\boldsymbol{\nu}(\boldsymbol{x})$ is extended to $\overline{\boldsymbol{\nu}}$, where $\overline{\boldsymbol{\nu}} \in H^1(\Omega)$ and $\overline{\boldsymbol{\nu}}|_{\Gamma_3} = \boldsymbol{\nu}(\boldsymbol{x})$. Let $\boldsymbol{\phi}(\boldsymbol{x})$ be a smooth cut-off function defined by $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0}$ outside the small neighborhood of Γ_3 and $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{1}$ near Γ_3. Let $\overline{\boldsymbol{\nu}}$ multiplying $\boldsymbol{\phi}(\boldsymbol{x})$, we obtain a function $\widetilde{\boldsymbol{\nu}} \in H^1(\Omega;\mathbb{R}^d)$ with $\widetilde{\boldsymbol{\nu}}|_{\Gamma_3} = \boldsymbol{\nu}$. Assume $\widetilde{g} \in H^1(\Omega;\mathbb{R}^d)$, $\widetilde{g}|_{\Gamma_3} = g$ and g is continuous. Define

$$\boldsymbol{v} = \boldsymbol{u}_n^{kh} + (\min\{\widetilde{g}, \boldsymbol{u}_n^{kh}\cdot\widetilde{\boldsymbol{\nu}}\} - \boldsymbol{u}_n^{kh}\cdot\widetilde{\boldsymbol{\nu}})\widetilde{\boldsymbol{\nu}}.$$

Thus on Γ_3, we have $\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{u}_{n,\boldsymbol{\tau}}^{kh}$, $v_\nu = \min\{g, u_{n,\nu}^{kh}\}$. Therefor we get $v_\nu \le g$ on Γ_3, i.e., $\boldsymbol{v} \in K$. We observe that

$$\|\boldsymbol{v} - \boldsymbol{u}_n^{kh}\|_{L^2(\Gamma_3;\mathbb{R}^d)} = \|v_\nu - u_{n,\nu}^{kh}\|_{L^2(\Gamma_3)}. \tag{4.27}$$

Note that $v_\nu = \min\{g, u_{n,\nu}^{kh}\}$, thus on Γ_3, $0 \le u_{n,\nu}^{kh}(\boldsymbol{x}) - v_\nu(\boldsymbol{x})$. Since $u_{n,\nu}^{kh}(\boldsymbol{x}) \le g(\boldsymbol{x})$ at nodes $\boldsymbol{x}$ on $\overline{\Gamma_3}$, we obtain $u_{n,\nu}^{kh}(\boldsymbol{x}) \le \Pi^h g(\boldsymbol{x})$, $\boldsymbol{x} \in \Gamma_3$, where $\Pi^h g(\boldsymbol{x})$ is the linear finite element interpolate of g on $\overline{\Gamma_3}$.

Then we consider two cases on Γ_3: if $u_{n,\nu}^{kh} \le g$, then $v_\nu = u_{n,\nu}^{kh}$; if $u_{n,\nu}^{kh} > g$, then $v_\nu = g$, and thus $0 \le u_{n,\nu}^{kh}(\boldsymbol{x}) - v_\nu(\boldsymbol{x}) \le \Pi^h g(\boldsymbol{x}) - g(\boldsymbol{x})$. Therefore

$$\|v_\nu - u_{n,\nu}^{kh}\|_{L^2(\Gamma_3)} \le \|\Pi^h g - g\|_{L^2(\Gamma_3)}. \tag{4.28}$$

Combining (4.27) and (4.28), we have

$$\|\boldsymbol{v} - \boldsymbol{u}_n^{kh}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\Pi^h g - g\|_{L^2(\Gamma_3)}. \tag{4.29}$$

Then applying finite element interpolation theory [23,24], we have the optimal order error bound for Problem 9

$$\max_{0\le n\le N}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \le C(h + k^2). \tag{4.30}$$

Thus in spatial mesh size the method is of first-order, and in the time step it is of second-order.

5. A numerical example

In this section we report a numerical example for Problem 7. The physical setting of the problem is depicted in Fig. 1. Let $\Omega = (0, L_1) \times (0, L_2)$ be a rectangle with a boundary Γ which is divided into three parts

$$\Gamma_1 = \{0\} \times (0, L_2),\ \ \Gamma_2 = \{L_1\} \times (0, L_2) \cup [0, L_1] \times \{L_2\},\ \ \Gamma_3 = [0, L_1] \times \{0\}.$$

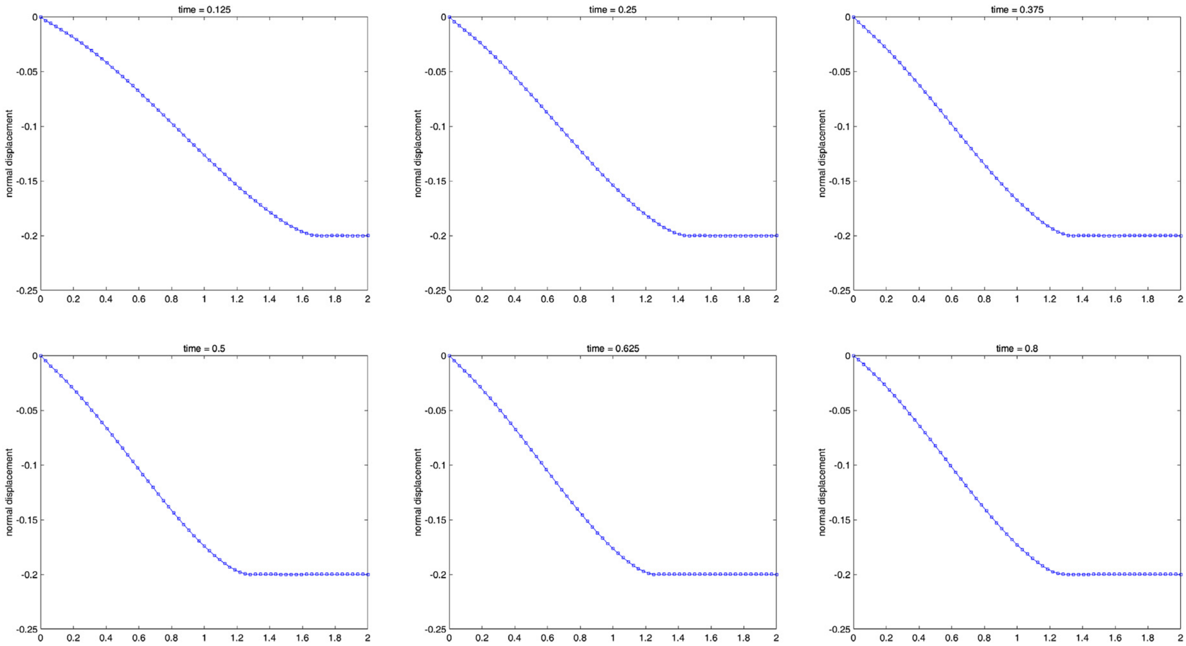

Fig. 2. Normal displacements on Γ_3 at different time steps.

The elasticity tensor $\mathcal{A}$ satisfies

$$(\mathcal{A}\tau)_{ij} = \frac{E\kappa}{1-\kappa^2}(\tau_{11}+\tau_{22})\delta_{ij} + \frac{E}{1+\kappa}\tau_{ij},\ \ 1 \le i, j \le 2. \tag{5.1}$$

The notation E denotes the Young modulus, κ denotes the Poisson ratio of the material, and δ_{ij} denotes the Kronecker symbol. The relaxation tensor $\mathcal{B}(s) = (0.5+s)^3\mathbb{I}$, where $\mathbb{I}$ is identity matrix. For a given value $S \ge 0$, the function $j_\nu(\cdot)$ is defined by

$$j_\nu(u_\nu) = S\int_0^{|u_\nu|}\mu(s)ds, \tag{5.2}$$

and the function $\mu(\cdot)$ defined by

$$\mu(s) = (a-b)e^{-\alpha s} + b, \tag{5.3}$$

with $a \ge b > 0$ and $\alpha > 0$. The following parameters are used in numerical experiments:

$$L_1 = 2\text{ m},\quad L_2 = 1\text{ m},\quad E = 1\text{ N/m}^2,\quad \kappa = 0.3,$$
$$\alpha = 100,\quad a = 0.04,\quad b = 0.02,\quad S = 0.1\text{ N},\quad g = 0.2\text{ m},$$

and

$$\boldsymbol{f}_2 = \begin{cases} (0,0)\ \text{N/m} & \text{on } \{L_1\}\times(0,L_2) \\ \left(0, -0.1(1-e^{-2t})x\right)\ \text{N/m} & \text{on } [0,L_1]\times\{L_2\}. \end{cases} \tag{5.4}$$

To numerically solve the sample problem, uniform rectangular finite element partitions are introduced, with the intervals $[0, L_1]$ and $[0, L_2]$ being divided into $1/h$ equal parts, and the corresponding continuous bilinear rectangular finite element space for V^h is used.

The normal displacements of the deformable body on Γ_3 at different time steps are showed in Fig. 2. To compute the numerical solution errors, the numerical solution with $h = k = \frac{1}{256}$ is used as the "reference" solution. The numerical convergence orders in H^1 norm are reported in Tables 1 and 2 which show the first order convergence for spatial discretization and second order for time discretization, and confirm the previous theoretical results.

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Table 1
Convergence orders of the errors with fixed time step size.

h	k	$\left\| \boldsymbol{u}(\cdot, T) - \boldsymbol{u}_N^{kh} \right\|_1$	Order
1/8	1/256	4.049067247799612e−03	–
1/16	1/256	9.899921240195822e−04	1.31
1/32	1/256	4.290570202713140e−04	1.19
1/64	1/256	2.416076212233180e−04	1.13

Table 2
Convergence orders of the errors with fixed spatial step size.

h	k	$\left\| \boldsymbol{u}(\cdot, T) - \boldsymbol{u}_N^{kh} \right\|_1$	Order
1/256	1/4	7.044095264494761e−03	–
1/256	1/8	2.831608699747937e−03	2.03
1/256	1/12	1.244870264157188e−03	2.06
1/256	1/16	5.692251463968669e−04	2.00

References

[1] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, Interscience, New York, 1983.
[2] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[3] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, in: Advances in Mechanics and Mathematics, vol. 26, Springer, New York, 2013.
[4] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.
[5] M. Miettinen, J. Haslinger, Finite element approximation of vector-valued hemivariational problems, J. Global Optim. 10 (1997) 17–35.
[6] W. Han, S. Migórski, M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, SIAM J. Math. Anal. 46 (2014) 3891–3912.
[7] W. Han, M. Sofonea, M. Barboteu, Numerical analysis of elliptic hemivariational inequalities, SIAM J. Math. Anal. 55 (2017) 640–663.
[8] W. Han, Error analysis for internal and external approximations of elliptic hemivariational inequalities, preprint.
[9] W. Han, Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics, Math. Mech. Solids 23 (2018) 279–293.
[10] W. Han, Z. Huang, C. Wang, W. Xu, Numerical analysis for elliptic hemivariational inequalities for semipermeable media, J. Comput. Math. 351 (2019) 364–377.
[11] J.F. Han, S. Migórski, A quasistatic viscoelastic frictional contact problem with multivalued normal compliance, unilateral constraint and material damage, J. Math. Anal. Appl. 443 (2016) 57–80.
[12] K. Kamran, M. Barboteu, W. Han, M. Sofonea, Numerical analysis of history-dependent quasivariational inequalities with applications in contact mechanics, Math. Model. Numer. Anal. 48 (2014) 919–942.
[13] S. Migórski, A. Ochal, M. Sofonea, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, Nonlinear Anal. RWA 12 (2011) 3384–3396.
[14] S. Migórski, A. Ochal, M. Sofonea, History-dependent variational-hemivariational inequalities in contact mechanics, Nonlinear Anal. RWA 22 (2015) 604–618.
[15] J. Ogorzaly, Quasistatic contact problem with unilateral constraint for elastic-viscoplastic materials, 2016, arXiv:1608.04660v1.
[16] M. Sofonea, C. Avramescu, A. Matei, A fixed point result with applications in the study of viscoplastic frictionless contact problems, Commun. Pure Appl. Anal. 7 (2008) 645–658.
[17] M. Sofonea, W. Han, S. Migórski, Numerical analysis of history-dependent variational-hemivariational inequalities with applications to contact problems, European J. Appl. Math. 26 (2015) 427–452.
[18] M. Sofonea, S. Migórski, A class of history-dependent variational-hemivariational inequalities, Nonlinear Differential Equations Appl. 23 (3) (2016) 1–23.
[19] W. Xu, Z. Huang, W. Han, W. Chen, C. Wang, Numerical analysis of history-dependent variational-hemivariational inequalities with applications in Contact Mechanics, J. Comput. Appl. Math. 351 (2019) 364–377.
[20] M. Sofonea, A. Matei, History-dependent quasivariational inequalities arising in Contact Mechanics, European J. Appl. Math. 22 (2011) 471–491.
[21] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, in: Studies in Advanced Mathematics, vol. 30, Americal Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.
[22] S. Migórski, A. Ochal, Weak solvability of two quasistatic viscoelastic contact problems, Math. Mech. Solids 18 (2012) 745–759.
[23] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, third ed., Springer-Verlag, New York, 2009.
[24] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.