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Numerical analysis of history-dependent variational–hemivariational inequalities with applications in contact mechanics

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ABSTRACT

This paper is devoted to numerical analysis of history-dependent variational–hemivariational inequalities arising in contact problems for viscoelastic material. We introduce both temporally semi-discrete approximation and fully discrete approximation for the problem, where the temporal integration is approximated by a trapezoidal rule and the spatial variable is approximated by the finite element method. We analyze the discrete schemes and derive error bounds. The results are applied for the numerical solution of a quasistatic contact problem. For the linear finite element method, we prove that the error estimation for the numerical solution is of optimal order under appropriate solution regularity assumptions.

1. Introduction

Variational and hemivariational inequalities play an important role in the research of nonlinear problems arising in Contact Mechanics, Economics, Physics and Engineering. The study of variational inequalities is based on the properties of monotonicity, convexity and the subdifferential of a convex function; see e.g. [1,2]. The study of hemivariational inequalities is based on the subdifferential of locally Lipschitz functions in the sense of Clarke; see [3–5]. Variational–hemivariational inequalities have a characteristic structure that include both convex and nonconvex functions, and history-dependent inequalities are evolutionary inequality problems with history-dependent operators. The history-dependent inequality problems arise in various applications and have attracted the interest of many researchers. Some references on this topic are [6–9].

Numerical analysis of hemivariational inequalities has been an active research area recently. In the comprehensive reference [10], one finds applications of the finite element methods in solving hemivariational inequalities, and in particular, some convergence results of the numerical solutions were shown. Attempts were made by several authors to derive error estimates for numerical solutions of hemivariational inequalities. In [11], a sub-optimal order error estimate is derived for a static hemivariational inequality. The reference [12] represents the first paper that provides an optimal order error estimate for the linear finite element method in solving hemivariational inequalities. More recently, in [13], a general framework was established on convergence and optimal order error estimation for internal approximations of stationary hemivariational inequalities with or without convex constraints, and the theory has been extended to cover both internal

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and external approximations of general stationary hemivariational inequalities, including variational–hemivariational inequalities, in [14]. Numerical analysis of history-dependent quasivariational inequalities was presented in [15], both temporally semi-discrete scheme and fully discrete scheme were considered. In [16], a fully discrete method was studied for history-dependent variational–hemivariational inequalities, and error estimates were derived for a quasistatic fictional contact problem.

A new class of history-dependent variational–hemivariational inequalities with convex constraint is studied in [9]. There, the novel structure of the inequalities involve a history-dependent operator, unilateral constraint and two nondifferential functions, one of which is convex and the other may be nonconvex. The existence and uniqueness result and continuous dependence of data are proved. The abstract results are applied to a quasistatic frictionless contact problem, and the weak formulation with its unique solvability is presented.

In this paper we analyze numerical approximations for history-dependent variational–hemivariational inequalities, whose well-posedness follows from the results in [9]. The form of the inequality problems adopted in this paper is slightly different from that of [9], to facilitate error analysis of its numerical solutions. The structure of this paper is organized as follows. In Section 2 we review some preliminary material on functional analysis and present the history-dependent variational–hemivariational inequality problem. In Section 3 we consider temporally semi-discrete approximations for the problem, including the unique solvability and error estimation. In Section 4 we introduce fully discrete schemes for the inequality problems with or without convex constraints and derive Céa's type inequality for error estimation. Finally in Section 5 we utilize the abstract result to solve a viscoelastic contact problem and obtain an optimal order error estimation for linear element method under appropriate solution regularity assumptions.

2. Preliminaries

In this section we introduce the history-dependent variational–hemivariational inequalities and present an existence and uniqueness result. We first provide some notations, definitions and preliminary materials.

Let X be a normed space, X^* its topological dual, $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ are associated norms. The duality pairing of X and X^* is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$. For simplicity we use $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{X^*\times X}$.

Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of ϕ at x in the direction $v \in X$, denoted as $\phi^0(x; v)$, is defined by

$$\phi^0(x; v) = \limsup_{y\to x,\lambda\downarrow 0} \frac{\phi(y+\lambda v)-\phi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of ϕ at x, denoted as $\partial\phi(x)$, is a subset of the dual space X^* defined by

$$\partial\phi(x) = \{\xi \in X^* \mid \phi^0(x; v) \ge \langle\xi, v\rangle_{X^*\times X}\ \forall\, v \in X\}.$$

An operator $A : X \to X^*$ is called pseudomonotone, if it is bounded and $u_n \to u$ weakly in X together with $\limsup\langle Au_n, u_n - u\rangle_{X^*\times X} \le 0$ imply

$$\langle Au, u - v\rangle_{X^*\times X} \le \liminf\langle Au_n, u_n - u\rangle_{X^*\times X}\ \forall\, v \in X.$$

For a convex function $\varphi : X \to \mathbb{R}\cup\{+\infty\}$, the mapping $\partial\varphi(x)$ defined by

$$\partial\varphi(x) = \{x^* \in X^* \mid \varphi(v) - \varphi(x) \ge \langle x^*, v - x\rangle_{X^*\times X}\ \forall\, v \in X\}$$

is called the subdifferential of φ. An arbitrary element $x^* \in \partial\varphi$ is called a subgradient of φ at x.

Next we present some preliminary material on function spaces and properties on related operators. We denote by $\mathbb{N}$ as the set of positive integers, $\mathbb{R}_+ = [0, +\infty)$ as the set of nonnegative real numbers, $C(\mathbb{R}_+; X)$ and $C^1(\mathbb{R}_+; X)$ as the spaces of continuous and continuously differentiable functions from $\mathbb{R}_+$ to X respectively. Let K be a subset of X. We use the symbols $C(\mathbb{R}_+; K)$ and $C^1(\mathbb{R}_+; K)$ for the spaces of continuous and continuously differentiable functions from $\mathbb{R}_+$ to K. It is well known that if X is a Banach space, $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e., it is a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Furthermore we give an equivalent description for the convergence of a sequence $\{x_k\}_k$ to an element x in the space $C(\mathbb{R}_+; X)$:

$$\begin{cases} x_k \to x \text{ in } C(\mathbb{R}_+; X) \text{ as } k \to \infty \text{ if and only if} \\ \max\limits_{r\in[0,n]} \|x_k(r) - x(r)\|_X \to 0 \text{ as } k \to \infty,\ \ \forall\, n \in \mathbb{N}. \end{cases}$$

In fact the definition by the seminorms is equivalent to the definition by the following metrics (cf. [17, Proposition 2.1])

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\max_{r\in[0,n]} \|u(r) - v(r)\|_X}{1 + \max_{r\in[0,n]} \|u(r) - v(r)\|_X}.$$

It is worth noting that for any fixed $n \in \mathbb{N}$ the interval $[0, n]$ is compact and $C([0, n]; X)$ is the space of continuous functions defined on $[0, n]$ with values in X.

Let X and Y be normed spaces. Following [8], an operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ is called a history-dependent operator if it satisfies the following condition

$$\begin{cases} \forall\, n \in \mathbb{N} \text{ there exists } s_n > 0 \text{ such that} \\ \|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_Y \le s_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X ds \\ \quad \forall\, u_1, u_2 \in C(\mathbb{R}_+; X), \forall\, t \in [0, n]. \end{cases} \tag{2.1}$$

Now we introduce the variational–hemivariational inequalities. Let X, X_j, Y be normed spaces and $K \subset X$. Given operators $A : X \to X^*$, $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ and $\gamma_j : X \to X_j$, given functions $\varphi : Y \times K \times K \to \mathbb{R}$ and $j : X_j \to \mathbb{R}$.

Problem 1. Find a function $u \in C(\mathbb{R}_+; K)$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle Au(t), v - u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t)) \\ &\quad + j^0(\gamma_j u(t); \gamma_j v - \gamma_j u(t)) \ge \langle f(t), v - u(t)\rangle \quad \forall\, v \in K. \end{aligned} \tag{2.2}$$

In the study of Problem 1, we assume the following hypotheses:

$$X \text{ is a reflexive Banach space, } K \subset X \text{ is nonempty, closed and convex.} \tag{2.3}$$

$$\begin{cases} X_j \text{ is a Banach space, } \gamma_j \in \mathcal{L}(X; X_j), \text{ there exists } c_j > 0 \text{ such that} \\ \qquad \|\gamma_j v\|_{X_j} \le c_j \|v\|_X \quad \forall\, v \in X. \end{cases} \tag{2.4}$$

$$\begin{cases} A : X \to X^* \text{ is an operator such that} \\ \quad (a)\, A \text{ is pseudomonotone.} \\ \quad (b)\, A \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \qquad \langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \end{cases} \tag{2.5}$$

$$\begin{cases} \mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y) \text{ is a history-dependent operator,} \\ Y \text{ is a normed space.} \end{cases} \tag{2.6}$$

$$\begin{cases} \varphi : Y \times K \times K \to \mathbb{R} \text{ is a function such that} \\ \quad \text{(a) } \varphi(y, u, \cdot) : K \to \mathbb{R} \text{ is convex and l.s.c. on } K, \forall\, y \in Y, \forall\, u \in K. \\ \quad \text{(b) there exists } \alpha_\varphi > 0 \text{ and } \beta_\varphi > 0 \text{ such that} \\ \qquad \varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \\ \qquad\quad \le \alpha_\varphi \|u_1 - u_2\|_X \|v_1 - v_2\|_X + \beta_\varphi \|y_1 - y_2\|_Y \|v_1 - v_2\|_X, \\ \qquad \forall\, y_1, y_2 \in Y, \forall\, u_1, u_2, v_1, v_2 \in K. \end{cases} \tag{2.7}$$

$$\begin{cases} j : X_j \to \mathbb{R} \text{ is a function such that} \\ \quad \text{(a) } j \text{ is locally Lipschitz;} \\ \quad \text{(b) } \|\partial j(z)\|_{X_j^*} \le c_0 + c_1 \|z\|_{X_j} \,\forall\, z \in X_j \text{ with } c_0, c_1 \ge 0; \\ \quad \text{(c) there exists } \alpha_j > 0 \text{ such that} \\ \qquad j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \le \alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j. \end{cases} \tag{2.8}$$

$$\alpha_\varphi + \alpha_j c_j^2 < m_A. \tag{2.9}$$

$$f \in C(\mathbb{R}_+; X^*) \tag{2.10}$$

Note that the function j in (2.8) is an integral over part of the boundary in a contact problem, and it may be nonconvex. Assumption (2.8) (c) is equivalent to the following relaxed monotonicity condition

$$\langle \partial j(z_1) - \partial j(z_2), z_1 - z_2\rangle \ge -\alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j,$$

which is used to guarantee the uniqueness of the solution in hemivariational inequalities; see [7,13,16].

Remark 2. Assume X is a Hilbert space. Following Proposition 2.29 in [18], if

$$\langle \partial j(u) - \partial j(v), u - v\rangle \ge -\alpha \|u - v\|_X^2 \quad \forall\, u, v \in X,$$

then

$$\langle \partial(j(u) + \frac{\alpha}{2}\|u\|_X^2) - \partial(j(v) + \frac{\alpha}{2}\|v\|_X^2), u - v\rangle \ge 0$$

and the function

$$h(u) = j(u) + \frac{\alpha}{2}\|u\|_X^2$$

is convex, and its convex subdifferential is given by

$$\partial_{convex} h = \partial\left(j(u) + \frac{\alpha}{2}\|u\|_X^2\right) = \partial j(u) + \alpha u.$$

So the Clarke subdifferential ∂j in the formulation of the problem can be replaced by $\partial_{convex} h - \alpha u$, and the variational–hemivariational inequality is equivalent to a certain variational inequality with a linear perturbation term.

Remark 3. Remark 2 implies that $j(u)$ can be expressed as

$$j(u) = j_1(u) - j_2(u),$$

where both j_i are convex functions, and moreover $j_2(u)$ can be assumed to be quadratic.

The space X_j is helpful in error estimation of the discrete problems. For a spatial dimension d, X_j can be chosen to be $L^2(\Gamma_3)$ or $L^2(\Gamma_3; \mathbb{R}^d)$ and $\gamma_j : X \to X_j$ can be the trace operator. Now we present an existence and uniqueness result by modifying slightly the proof in [9, Theorem 5].

Theorem 4. *Assume (2.3)–(2.10). Then Problem 1 has a unique solution $u \in C(\mathbb{R}_+; K)$.*

We study numerical methods for solving inequality problem (2.2) in this paper. We focus on the special case where the operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ has the following form [8,15,16]:

$$(\mathcal{S}v)(t) = R(\int_0^t q(t,s)v(s)ds + a_S) \quad \forall\, v \in C(\mathbb{R}_+; X),\ \forall\, t \in \mathbb{R}_+, \tag{2.11}$$

where $R \in \mathcal{L}(X; Y)$, $q \in C(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $a_S \in X$. It can be shown that the operator $\mathcal{S}$ given in (2.11) is a history-dependent operator.

In the next two sections we consider temporally semi-discrete approximation and fully discrete approximation for the problem respectively. Below, C represents a constant independent of the time step k and the mesh parameter h whose values may change from place to place.

3. Temporally semi-discrete approximation

In this section we study temporally discrete method for solving Problem 1 and derive error bound. For an arbitrarily fixed $T \in \mathbb{R}_+$, denote the time interval $I = [0, T]$. We consider Problem 1 on $I = [0, T]$ for computation. The interval I is divided uniformly here; however, the numerical method can be directly extended to the case of non-uniform partitions. For a positive integer N, let $k = T/N$ be the time step-size, $t_n = nk$, $0 \le n \le N$. For a continuous function $v(t)$ with values in a function space, we write $v_j = v(t_j)$, $0 \le j \le N$. For the operator $\mathcal{S}$ in the form (2.11), we use the trapezoidal rule to approximate the integral $\int_0^t q(t,s)v(s)ds$. We denote $\|R\| = \|R\|_{\mathcal{L}(X;Y)}$ and $\|q\| = \|q\|_{C(I\times I;\mathcal{L}(X))}$ for simplicity. Recall the trapezoidal rule

$$\int_0^{t_n} Z(s)ds \approx k \sum_{j=0}^{n}{}' Z(t_j), \tag{3.1}$$

where a prime denotes that in the summation the first and the last terms are to be halved. Let $\mathcal{S}_n := \mathcal{S}(t_n)$ be approximated by $\mathcal{S}_n^k$ defined as follows:

$$\mathcal{S}_n^k v := R(k \sum_{j=0}^{n}{}' q(t_n, t_j)v_j + a_S). \tag{3.2}$$

Problem 5. The temporally semi-discrete scheme for Problem 1 is to find the discrete solution $u^k := \{u_n^k\}_{n=0}^N \subset K$ such that

$$\begin{aligned} &\langle Au_n^k, v - u_n^k\rangle + \varphi(\mathcal{S}_n^k u^k, u_n^k, v) - \varphi(\mathcal{S}_n^k u^k, u_n^k, u_n^k) \\ &\quad + j^0(\gamma_j u_n^k; \gamma_j v - \gamma_j u_n^k) \ge \langle f_n, v - u_n^k\rangle \quad \forall\, v \in K. \end{aligned} \tag{3.3}$$

We present an existence and uniqueness result for Problem 5.

Theorem 6. *Assume (2.3)–(2.5), (2.7)–(2.11) and (3.2). For step-size*

$$k < \frac{m_A - \alpha_\varphi - \alpha_j c_j^2}{\beta_\varphi \|R\| \|q\|}, \tag{3.4}$$

Problem 5 has a unique solution.

Proof. We prove the result through an induction. With $\{u_j^k\}_{j\le n-1}$ known, we prove that $u_n^k \in K$ is uniquely determined by inequality (3.3). The proof is divided into three steps.

Step 1. Let $\eta \in X$ be given. Denote

$$u^{k\eta} = \{u_0^k, \ldots, u_{n-1}^k, \eta\}$$

and

$$y_\eta = \mathcal{S}_n^k u^{k\eta}. \tag{3.5}$$

We first consider the auxiliary problem of finding $u_\eta \in K$ such that

$$\begin{aligned}&\langle Au_\eta, v-u_\eta\rangle + \varphi(y_\eta, u_\eta, v) - \varphi(y_\eta, u_\eta, u_\eta)\\ &\quad + j^0(\gamma_j u_\eta; \gamma_j v - \gamma_j u_\eta) \ge \langle f_n, v-u_\eta\rangle \quad \forall\, v\in K,\end{aligned} \tag{3.6}$$

where $f_n = f(t_n)$. Applying [9, Lemma 6] we get there exists unique $u_\eta \in K$ satisfying (3.6) under the assumptions in Theorem 6.

Step 2. Define an operator $\Lambda : X \to K$ by

$$\Lambda\eta = u_\eta. \tag{3.7}$$

We verify that the operator Λ has a unique fixed point $\eta^* \in K$. Let $\eta_1, \eta_2 \in X$ be given and denote $y_i = y_{\eta_i}$, $u_i = u_{\eta_i}$, $u^{k\eta_i} = \{u_0^k, \dots, u_{n-1}^k, \eta_i\}$, $i = 1, 2$. Thus $u_1, u_2 \in K$ satisfy

$$\begin{aligned}&\langle Au_1, v-u_1\rangle + \varphi(y_1, u_1, v) - \varphi(y_1, u_1, u_1)\\ &\quad + j^0(\gamma_j u_1; \gamma_j v - \gamma_j u_1) \ge \langle f_n, v-u_1\rangle \quad \forall\, v\in K,\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\langle Au_2, v-u_2\rangle + \varphi(y_2, u_2, v) - \varphi(y_2, u_2, u_2)\\ &\quad + j^0(\gamma_j u_2; \gamma_j v - \gamma_j u_2) \ge \langle f_n, v-u_2\rangle \quad \forall\, v\in K\end{aligned} \tag{3.9}$$

respectively. Taking $v = u_2$ in (3.8) and $v = u_1$ in (3.9), we obtain

$$\begin{aligned}&\langle Au_1, u_2-u_1\rangle + \varphi(y_1, u_1, u_2) - \varphi(y_1, u_1, u_1)\\ &\quad + j^0(\gamma_j u_1; \gamma_j u_2 - \gamma_j u_1) \ge \langle f_n, u_2-u_1\rangle,\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\langle Au_2, u_1-u_2\rangle + \varphi(y_2, u_2, u_1) - \varphi(y_2, u_2, u_2)\\ &\quad + j^0(\gamma_j u_2; \gamma_j u_1 - \gamma_j u_2) \ge \langle f_n, u_1-u_2\rangle.\end{aligned} \tag{3.11}$$

Adding (3.10) and (3.11), we get

$$\begin{aligned}\langle Au_1 - Au_2, u_1-u_2\rangle \le{}& \varphi(y_1, u_1, u_2) - \varphi(y_1, u_1, u_1) + \varphi(y_2, u_2, u_1) - \varphi(y_2, u_2, u_2)\\ &+ j^0(\gamma_j u_1; \gamma_j u_2 - \gamma_j u_1) + j^0(\gamma_j u_2; \gamma_j u_1 - \gamma_j u_2).\end{aligned}$$

We use assumptions (2.5)(b), (2.7)(b) and (2.8)(c) to obtain

$$(m_A - \alpha_\varphi - \alpha_j c_j^2)\|u_1-u_2\|_X \le \beta_\varphi \|y_1-y_2\|_Y. \tag{3.12}$$

By Eq. (3.7), we have

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_X = \|u_1-u_2\|_X.$$

Together with (3.12), we get

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_X \le \frac{\beta_\varphi}{m_A - \alpha_\varphi - \alpha_j c_j^2}\|y_1-y_2\|_Y. \tag{3.13}$$

Utilizing the hypotheses on operators R and q, we obtain

$$\begin{aligned}\|y_1-y_2\|_Y &= \|\mathcal{S}_n^k u^{k\eta_1} - \mathcal{S}_n^k u^{k\eta_2}\|_Y\\ &= \|R(k\sum_{j=0}^{n}{}' q(t_n, t_j)u_j^{k\eta_1} + a_S) - R(k\sum_{j=0}^{n}{}' q(t_n, t_j)u_j^{k\eta_2} + a_S)\|_Y\\ &\le k\|R\|\|q\|\|\eta_1-\eta_2\|_X.\end{aligned} \tag{3.14}$$

Therefore,

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_X \le \frac{\beta_\varphi}{m_A - \alpha_\varphi - \alpha_j c_j^2}k\|R\|\|q\|\|\eta_1-\eta_2\|_X. \tag{3.15}$$

By the smallness condition (3.4), we have $\frac{\beta_\varphi}{m_A-\alpha_\varphi-\alpha_j c_j^2}k\|R\|\|q\| < 1$. According to the Banach fixed point theorem (e.g. [19, p. 209]), we conclude that the operator Λ has a unique fixed point $\eta^* \in K$.

Step 3. Let η^* be the fixed point of the operator Λ. Eqs. (3.5) and (3.7) imply that

$$y_{\eta^*} = \mathcal{S}_n^k u^{k\eta^*},\ u_{\eta^*} = \eta^*. \tag{3.16}$$

Let $\eta = \eta^*$ in the inequality (3.6), we draw the conclusion that $u_n^k = \eta^* \in K$ is a solution of (3.3). The uniqueness of the solution is a direct consequence of Banach fixed point theorem. □

Next we derive an error bound for semi-discrete solution from Problem 5.

Theorem 7. *Keep the assumptions stated in Theorem 6. Moreover, assume the regularity* $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $u \in W^{2,\infty}_{loc}(\mathbb{R}_+; X)$. *Then for the semi-discrete solution of Problem 5, the following error bound holds*

$$\max_{0\le n\le N} \|u_n - u_n^k\|_X \le Ck^2. \tag{3.17}$$

Proof. We take $t = t_n$ in the inequality (2.2) to get

$$\begin{aligned}&\langle Au_n, v-u_n\rangle + \varphi(\mathcal{S}_n u, u_n, v) - \varphi(\mathcal{S}_n u, u_n, u_n)\\&\quad + j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n) \ge \langle f_n, v-u_n\rangle \quad \forall\, v \in K,\end{aligned} \tag{3.18}$$

where $\mathcal{S}_n u = R(\int_0^{t_n} q(t_n,s)u(s)ds + a_S)$. Let $v = u_n^k$ in (3.18),

$$\begin{aligned}&\langle Au_n, u_n^k-u_n\rangle + \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n)\\&\quad + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n) \ge \langle f_n, u_n^k-u_n\rangle.\end{aligned} \tag{3.19}$$

Taking $v = u_n$ in (3.3), we have

$$\begin{aligned}&\langle Au_n^k, u_n-u_n^k\rangle + \varphi(\mathcal{S}_n^k u^k, u_n^k, u_n) - \varphi(\mathcal{S}_n^k u^k, u_n^k, u_n^k)\\&\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) \ge \langle f_n, u_n-u_n^k\rangle.\end{aligned} \tag{3.20}$$

We add (3.19) and (3.20) to obtain

$$\begin{aligned}\langle Au_n - Au_n^k, u_n-u_n^k\rangle &\le \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n) + \varphi(\mathcal{S}_n^k u^k, u_n^k, u_n)\\&\quad - \varphi(\mathcal{S}_n^k u^k, u_n^k, u_n^k) + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n)\\&\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k)\\&\le \alpha_\varphi \|u_n - u_n^k\|_X^2 + \beta_\varphi \|\mathcal{S}_n u - \mathcal{S}_n^k u^k\|_Y \|u_n - u_n^k\|_X\\&\quad + \alpha_j c_j^2 \|u_n - u_n^k\|_X^2.\end{aligned} \tag{3.21}$$

Together with assumptions $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $u \in W^{2,\infty}_{loc}(\mathbb{R}_+; X)$ and arguments in [15, Section 3], we have

$$\|\mathcal{S}_n u - \mathcal{S}_n^k u\|_Y \le Ck^2 \|u\|_{W^{2,\infty}(I;X)}. \tag{3.22}$$

Combining with the hypotheses on R, we obtain

$$\begin{aligned}\|\mathcal{S}_n u - \mathcal{S}_n^k u^k\|_Y &\le \|\mathcal{S}_n u - \mathcal{S}_n^k u\|_Y + \|S_n^k u - S_n^k u^k\|_Y\\&\le Ck^2 \|u\|_{W^{2,\infty}(I;X)} + k\|R\|\,\|q\| \sum_{j=0}^n \|u_n - u_n^k\|_X.\end{aligned} \tag{3.23}$$

Together with (2.5)(b), (3.21) and (3.23), we get

$$\|u_n - u_n^k\|_X \le k^2 \frac{C\beta_\varphi \|u\|_{W^{2,\infty}(I;X)}}{m_A - \alpha_\varphi - \alpha_j c_j^2} + k \frac{\beta_\varphi \|R\|\,\|q\|}{m_A - \alpha_\varphi - \alpha_j c_j^2} \sum_{j=0}^n \|u_n - u_n^k\|_X. \tag{3.24}$$

Recall the discrete Gronwall's inequality ([20, p. 164]): Let $\{e_n\}_{n=0}^N$ and $\{g_n\}_{n=0}^N$ be two sequences of non-negative numbers with

$$e_n \le C_1 g_n + C_2 k \sum_{j=0}^n e_j, \quad 0 \le n \le N,$$

where C_1, C_2 are two constants. Then for some constant C_3, we have

$$\max_{0\le n\le N} e_n \le C_3 \max_{0\le n\le N} g_n.$$

Thus we derive the error bound (3.17) from (3.24). □

4. Fully discrete approximation

In this section we consider fully discrete approximations of Problem 1 with or without constraints. Here, both the temporal and spatial variables are discretized. The symbols and assumptions in Section 3 are still used for the temporal discretization, and a regular family of finite element partitions $\{T^h\}$ is introduced for the spatial discretization. We use the finite element space $X^h \subset X$ that matches the partition $\{T^h\}$. Considering the internal approximation we assume nonempty, convex and close subsets $K^h \subset X^h$ and $K^h \subset K$. For the discrete problem with constraint, we derive a Céa's type inequality for the error estimation. For the discrete problem without constraint, we assume the function φ further satisfies the following condition

$$\begin{cases} \varphi : Y \times K \times K \to \mathbb{R} \text{ is a function such that} \\ \quad \text{there exists a constant } c_\varphi > 0 \text{ satisfies} \\ \quad \varphi(y, u, v_1) + \varphi(y, u, v_2) - 2\varphi(y, u, \dfrac{v_1 + v_2}{2}) \leq c_\varphi \|v_1 - v_2\|_X^2 \\ \qquad \forall y \in Y,\ \forall u, v_1, v_2 \in K. \end{cases} \tag{4.1}$$

Remark 8. If the function $\varphi : Y \times K \times K \to \mathbb{R}$ is linear with its third argument, for example [9, (6.4)], then it satisfies condition (4.1) trivially with $c_\varphi = 0$ and the following equation is satisfied

$$\varphi(y, u, v_1) + \varphi(y, u, v_2) = 2\varphi(y, u, \frac{v_1 + v_2}{2}).$$

Define an operator

$$S_n^{kh} u^{kh} := R(k \sum_{j=0}^{n}{}' q(t_n, t_j) u_j^{kh} + a_S), \quad u^{kh} = \{u_j^{kh}\}_{j=0}^N \subset X^h. \tag{4.2}$$

Now we consider the following fully discrete approximation of Problem 1.

Problem 9. Find $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset K^h$ such that

$$\begin{aligned} &\langle A u_n^{kh}, v^h - u_n^{kh} \rangle + \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, v^h) - \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, u_n^{kh}) \\ &\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \geq \langle f_n, v^h - u_n^{kh} \rangle \quad \forall v^h \in K^h. \end{aligned} \tag{4.3}$$

Using arguments similar to that in the proof of Theorem 6, we can show that under the conditions (2.3)–(2.5), (2.7)–(2.11), (3.4) and (4.2), Problem 9 has a unique solution. Next we derive an error bound for the fully discrete scheme in solving Problem 9.

Theorem 10. *Assume (2.3)–(2.5), (2.7)–(2.11), (3.4) and (4.2). Moreover, assume $A : X \to X^*$ is Lipschitz continuous with a Lipschitz constant $L_A > 0$. Under the regularity assumptions $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $u \in W^{2,\infty}_{loc}(\mathbb{R}_+; X)$, we have the error bound*

$$\begin{aligned} \max_{0 \leq n \leq N} \|u_n - u_n^{kh}\|_X \leq C \max_{0 \leq n \leq N} \inf_{v^h \in K^h} \{ &\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} \\ &+ |E_n(v^h, u_n)|^{\frac{1}{2}} \} + Ck^2, \end{aligned} \tag{4.4}$$

where the term $E(v^h, u_n)$ is defined by

$$\begin{aligned} E_n(v^h, u_n) = &\langle A u_n, v^h - u_n \rangle + \varphi(\mathcal{S}_n u, u_n, v^h) - \varphi(\mathcal{S}_n u, u_n, u_n) \\ &+ j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) - \langle f_n, v^h - u_n \rangle, \quad v^h \in K^h. \end{aligned} \tag{4.5}$$

Proof. We take $t = t_n$ and $v = u_n^{kh}$ in (2.2) to get

$$\begin{aligned} &\langle A u_n, u_n^{kh} - u_n \rangle + \varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - \varphi(\mathcal{S}_n u, u_n, u_n) \\ &\quad + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \geq \langle f_n, u_n^{kh} - u_n \rangle. \end{aligned} \tag{4.6}$$

Since $A : X \to X^*$ is strongly monotone, we have

$$m_A \|u_n - u_n^{kh}\|_X^2 \leq \langle A u_n - A u_n^{kh}, u_n - u_n^{kh} \rangle, \tag{4.7}$$

On the other hand,

$$\begin{aligned} \langle A u_n - A u_n^{kh}, u_n - u_n^{kh} \rangle = &\langle A u_n, u_n - u_n^{kh} \rangle + \langle A u_n^{kh}, u_n^{kh} - v^h \rangle \\ &+ \langle A u_n^{kh}, v^h - u_n \rangle. \end{aligned} \tag{4.8}$$

Combined with (4.3) and (4.6), we get

$$\begin{aligned} \langle A u_n - A u_n^{kh}, u_n - u_n^{kh} \rangle \leq\ &\varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - \varphi(\mathcal{S}_n u, u_n, u_n) \\ &+ \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, v^h) - \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, u_n^{kh}) \\ &+ \langle A u_n^{kh}, v^h - u_n \rangle + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \\ &+ j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) - \langle f_n, v^h - u_n \rangle \\ =\ &E_{\varphi_1} + E_{\varphi_2} + E_j + E_A + E_n(v^h, u_n), \end{aligned} \tag{4.9}$$

where

$$E_{\varphi_1} = \varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - \varphi(\mathcal{S}_n u, u_n, u_n) + \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, u_n) - \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, u_n^{kh}), \tag{4.10}$$

$$E_{\varphi_2} = \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, v^h) - \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, u_n) + \varphi(\mathcal{S}_n u, u_n, u_n) - \varphi(\mathcal{S}_n u, u_n, v^h), \tag{4.11}$$

$$E_j = j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) - j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n), \tag{4.12}$$

$$E_A = \langle A u_n^{kh}, v^h - u_n \rangle - \langle A u_n, v^h - u_n \rangle. \tag{4.13}$$

In the following we bound E_{φ_1}, E_{φ_2}, E_j and E_A in turn.

$$E_{\varphi_1} \le \alpha_\varphi \|u_n - u_n^{kh}\|_X^2 + \beta_\varphi \|S_n u - S_n^{kh} u^{kh}\|_Y \|u_n - u_n^{kh}\|_X, \tag{4.14}$$

$$E_{\varphi_2} \le \alpha_\varphi \|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X + \beta_\varphi \|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_Y \|u_n - v^h\|_X. \tag{4.15}$$

Utilizing sub-additive property of generalized directional derivative [3,18], we have

$$\begin{aligned} E_j &\le j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j v^h) \\ &\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) - j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) \\ &\le j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \\ &\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j u_n - \gamma_j u_n^{kh}) \\ &\le (2c_0 + c_1 \|\gamma_j u_n\|_{X_j} + c_1 \|\gamma_j u_n^{kh}\|_{X_j}) \|\gamma_j u_n - \gamma_j v^h\|_{X_j} \\ &\quad + \alpha_j \|\gamma_j u_n - \gamma_j u_n^{kh}\|_{X_j}^2 \\ &\le (2c_0 + 2c_1 c_j \|u_n\|_X) \|\gamma_j u_n - \gamma_j v^h\|_{X_j} + \alpha_j c_j^2 \|u_n - u_n^{kh}\|_X^2 \\ &\quad + c_1 c_j^2 \|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X. \end{aligned} \tag{4.16}$$

Note that A is Lipschitz continuous with Lipschitz constant $L_A > 0$, we get

$$E_A \le L_A \|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X. \tag{4.17}$$

Together with (4.9), (4.14)–(4.17) and (2.9), we derive that

$$\|u_n - u_n^{kh}\|_X \le C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E_n(v^h, u_n)|^{\frac{1}{2}}\} + C\|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_Y. \tag{4.18}$$

The terms (3.22) and (4.2) imply that

$$\begin{aligned} \|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_Y &\le \|\mathcal{S}_n u - \mathcal{S}_n^k u\|_Y + \|\mathcal{S}_n^k u - \mathcal{S}_n^{kh} u^{kh}\|_Y \\ &\le Ck^2 \|u\|_{W^{2,\infty}(I;X)} + k\|R\|\|q\| \sum_{j=0}^n \|u_j - u_j^{kh}\|_X. \end{aligned} \tag{4.19}$$

We combine (4.18) and (4.19) to get

$$\begin{aligned} \|u_n - u_n^{kh}\|_X &\le C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E_n(v^h, u_n)|^{\frac{1}{2}} \\ &\quad + k^2 \|u\|_{W^{2,\infty}(I;X)}\} + Ck \sum_{j=0}^n \|u_j - u_j^{kh}\|_X. \end{aligned} \tag{4.20}$$

Applying the discrete Gronwall's inequality again, we have the error bound (4.4). □

Now we consider the error estimation for numerical solution of the discrete problem without constraint.

Theorem 11. *Keep the assumptions stated in Theorem 10. Moreover, let $K = X$ and assume the function φ satisfies the assumption (4.1). Then the following error bound holds for the fully discrete Problem*

$$\max_{0\le n\le N} \|u_n - u_n^{kh}\|_X \le C \max_{0\le n\le N} \inf_{v^h\in K^h} \{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\} + Ck^2. \tag{4.21}$$

Proof. We start with

$$\begin{aligned}\langle Au_n - Au_n^{kh}, u_n - u_n^{kh}\rangle &= \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \langle Au_n - Au_n^{kh}, v^h - u_n^{kh}\rangle \\ &= \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \langle Au_n, v^h - u_n\rangle \\ &\quad + \langle Au_n, u_n - u_n^{kh}\rangle + \langle Au_n^{kh}, u_n^{kh} - v^h\rangle.\end{aligned} \tag{4.22}$$

Let $t = t_n$ in (2.2),

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + \varphi(\mathcal{S}_n u, u_n, v) - \varphi(\mathcal{S}_n u, u_n, u_n) \\ &\quad + j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n) \geq \langle f_n, v - u_n\rangle \quad \forall\, v \in X.\end{aligned} \tag{4.23}$$

Taking $v = u_n^{kh}$ in (4.23) we obtain (4.6). Replacing v with $2u_n - v$ in (4.23), we get

$$\begin{aligned}&\langle Au_n, u_n - v\rangle + \varphi(\mathcal{S}_n u, u_n, 2u_n - v) - \varphi(\mathcal{S}_n u, u_n, u_n) \\ &\quad + j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v) \geq \langle f_n, u_n - v\rangle \quad \forall\, v \in X.\end{aligned} \tag{4.24}$$

We take $v = v^h$ in (4.24) to get

$$\begin{aligned}&\langle Au_n, u_n - v^h\rangle + \varphi(\mathcal{S}_n u, u_n, 2u_n - v^h) - \varphi(\mathcal{S}_n u, u_n, u_n) \\ &\quad + j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) \geq \langle f_n, u_n - v^h\rangle.\end{aligned} \tag{4.25}$$

Together with (4.22), (4.6), (4.25) and (4.3), we have

$$\begin{aligned}\langle Au_n - Au_n^{kh}, u_n - u_n^{kh}\rangle &\leq \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \varphi(\mathcal{S}_n u, u_n, 2u_n - v^h) \\ &\quad + \varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - 2\varphi(\mathcal{S}_n u, u_n, u_n) \\ &\quad + \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, v^h) - \varphi(\mathcal{S}_n^{kh} u^{kh}, u_n^{kh}, u_n^{kh}) \\ &\quad + j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \\ &\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \\ &= E_{\varphi_1} + E_{\varphi_2} + E_{\varphi_3} + E_{\tilde{j}} + E_A,\end{aligned} \tag{4.26}$$

where $E_{\varphi_1}, E_{\varphi_2}, E_A$ are the same in (4.10), (4.11), (4.13) respectively with their bounds (4.14), (4.15), (4.17). Here

$$E_{\varphi_3} = \varphi(\mathcal{S}_n u, u_n, 2u_n - v^h) + \varphi(\mathcal{S}_n u, u_n, v^h) - 2\varphi(\mathcal{S}_n u, u_n, u_n), \tag{4.27}$$

$$\begin{aligned}E_{\tilde{j}} = &\, j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \\ &+ j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}).\end{aligned} \tag{4.28}$$

By the assumption (4.1), we obtain

$$E_{\varphi_3} \leq c_{\varphi_3} \|u_n - v^h\|_X^2. \tag{4.29}$$

Using the sub-additive property again, we obtain

$$E_{\tilde{j}} \leq C\|\gamma_j u_n - \gamma_j v^h\|_{X_j} + \alpha_j c_j^2 \|u_n - u_n^{kh}\|_X^2. \tag{4.30}$$

Therefore we immediately get

$$\|u_n - u_n^{kh}\|_X \leq C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\} + C\|\mathcal{S}_n u - \mathcal{S}_n^{kh} u^{kh}\|_Y. \tag{4.31}$$

Using (4.19) and the discrete Gronwall's inequality, we get the error bound (4.21). □

Note that the square root involved in the error bounds in (4.4) and (4.21) results from the inequality problem, and further estimation of the residual term $E_n(v^h, u_n)$ depends on the specific problems in applications. In Section 5, we illustrate an example of contact problem to bound $E_n(v^h, u_n)$. In Theorem 11 the inequality problem is defined on the whole space X, thus there is no convex constraint. Through a modified proof, the residual term $E_n(v^h, u_n)$ vanishes.

5. Application to a contact problem

In this section we illustrate an application of the numerical analysis results developed in the previous sections to a history-dependent variational–hemivariational inequality of the form (2.2) that arises in contact mechanics. We recall a viscoelastic frictionless contact model, the notation for function spaces and its weak formulation studied in [9]. Numerical analysis for the discrete problem of contact model is shown, and an optimal order error estimation for the linear finite element method is proved.

We first describe the physical setting. For a spatial dimension $d = 2$ or 3, a viscoelastic body is in contact with a foundation. The contact model is frictionless and quasistatic. The body occupies a regular domain $\Omega \subset \mathbb{R}^d$ with boundary Γ in its reference configuration, and the boundary Γ is divided into three parts Γ_1, Γ_2 and Γ_3 such that each Γ_i, $1 \leq i \leq 3$ is measurable with $m(\Gamma_1) > 0$. The body is fixed on Γ_1, thus no displacement arises here. A volume force of density $\boldsymbol{f}_0$ acting on Ω and a surface traction of density $\boldsymbol{f}_2$ acting on Γ_2 are both time-dependent. In addition the time interval we consider is $\mathbb{R}_+$. On the contact surface Γ_3, a unilateral constraint condition with a nonmonotone normal compliance condition is considered. Then we present the mathematical model.

Problem 12. Find a displacement $\mathbf{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\\ + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \text{ in } \Omega,\end{aligned} \tag{5.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \ \text{ in } \Omega, \tag{5.2}$$

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \text{ on } \Gamma_2, \tag{5.4}$$

$$\begin{cases} u_\nu(t) \leq g, \ \sigma_\nu(t) + \xi_\nu(t) \leq 0, \\ (\sigma_\nu(t) + \xi_\nu(t))(u_\nu(t) - g) = 0, \quad \text{on } \Gamma_3, \\ \xi_\nu(t) \in \partial j_\nu(u_\nu(t)) \end{cases} \tag{5.5}$$

$$\boldsymbol{\sigma}_{\boldsymbol{\tau}}(t) = \mathbf{0} \text{ on } \Gamma_3, \tag{5.6}$$

for all $t \in \mathbb{R}_+$.

Next we list some standard notation in mechanics and function spaces. Let $\boldsymbol{u} = (u_i)$, $\boldsymbol{\nu} = (\nu_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$ be the displacement field, the outward unit normal, the stress tensor and the linearized strain tensor respectively. In addition $v_\nu := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ stand for the normal and tangential components of a vector field $\boldsymbol{v}$, $\sigma_\nu := (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ represent the normal and tangential components of the stress field $\boldsymbol{\sigma}$ respectively. In Eq. (5.1) $P_{M(\kappa(\cdot))}$ denotes the projection on the Von Mises convex, μ is a fixed constant, $\mathcal{A}$ and $\mathcal{B}$ are the elastic and relaxation tensor. For any $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$, the trace of $\boldsymbol{v}$ on the boundary Γ is still denoted by $\boldsymbol{v}$, and the normal and tangential traces are denoted by v_ν and $\boldsymbol{v}_{\boldsymbol{\tau}}$ respectively. The function spaces V and $\mathcal{H}$ are defined by

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = 0 \text{ a.e.on } \Gamma_1\},$$
$$\mathcal{H} = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) \mid \tau_{ij} = \tau_{ji}, \ 1 \leq i, j \leq d\}.$$

The inner product in the Hilbert space $\mathcal{H}$ is

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(x)\tau_{ij}(x)dx,$$

and the associated norm is denoted by $\|\cdot\|_{\mathcal{H}}$. The space V is a Hilbert space since $m(\Gamma_1) > 0$, which is equipped with inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}, \ \boldsymbol{u}, \boldsymbol{v} \in V$$

and norm $\|\cdot\|_V$. The space of fourth order tensor field Q_∞ is defined by

$$\mathrm{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \ 1 \leq i, j, k, l \leq d\},$$

and the associated norm is denoted by

$$\|\mathcal{E}\|_{\mathrm{Q}_\infty} = \sum_{1 \leq i,j,k,l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

We now list hypotheses for Problem 12. The elasticity tensor $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is of fourth order, positive with coefficient $m_{\mathcal{A}} > 0$ and symmetric about subscripts. The relaxation tensor $\mathcal{B}$ belongs to $C(\mathbb{R}_+; \mathrm{Q}_\infty)$ and the bound function $\kappa : \mathbb{R} \to \mathbb{R}_+$ is Lipschitz continuous with a constant $L_\kappa > 0$. Furthermore the potential function $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is measurable for the first variable on Γ_3 for all $r \in \mathbb{R}$ and locally Lipschitz for the second variable on $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Gamma_3$, and $j_\nu(\cdot, \bar{e}(\cdot))$ belongs to $L^1(\Gamma_3)$ for some $\bar{e} \in L^2(\Gamma_3)$. There also exist $\bar{c}_0, \bar{c}_1 > 0$ such that the absolute value of the subdifferential $\partial j_\nu(\boldsymbol{x}, r)$ is controlled by $\bar{c}_0 + \bar{c}_1|r|$ for a.e. $\boldsymbol{x} \in \Gamma_3$, for all $r \in \mathbb{R}$. In addition, there exists $\overline{\alpha}_\nu \geq 0$ such that for a.e. $\boldsymbol{x} \in \Gamma_3$

$$j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \leq \overline{\alpha}_\nu |r_1 - r_2|^2 \quad \forall r_1, r_2 \in \mathbb{R}.$$

The densities of body forces $\boldsymbol{f}_0$ belongs to $C(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^d))$ and the surface traction $\boldsymbol{f}_2$ belongs to $C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d))$. The description of the physical setting and the problem data with formulation can be found in [9, Section 5].

Let

$$U = \{\boldsymbol{v} \in V \mid v_\nu \leq g \text{ a.e. on } \Gamma_3\}$$

be the set of admissible displacements. Define the function $\boldsymbol{F} : \mathbb{R}_+ \to V^*$ by

$$\langle \boldsymbol{F}(t), \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall \boldsymbol{v} \in V,\ \forall t \in \mathbb{R}_+.$$

Then we state the weak formulation of Problem 12 and its existence and uniqueness result.

Problem 13. Find a displacement $\mathbf{u} : \mathbb{R}_+ \to U$ and such that the following inequality holds:

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\
&\quad - \mu(P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\
&\quad + \left(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\right)_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))d\Gamma \geq \langle \mathbf{F}(t), \mathbf{v} - \mathbf{u}(t)\rangle_{V^*\times V} \quad \forall \mathbf{v} \in U,\ \forall t \in \mathbb{R}_+.
\end{aligned} \tag{5.7}$$

We define the operator $A : V \to V^*$, the space Y, the function $\varphi : Y \times V \times V \to \mathbb{R}$, the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Y)$ and the function $j : X_j \to \mathbb{R}$ as follows:

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{5.8}$$

The space $Y = \mathbb{R} \times \mathcal{H}$ is endowed with the norm

$$\|y\|_Y = |r| + \|\boldsymbol{\theta}\|_{\mathcal{H}} \quad \forall y = (r, \boldsymbol{\theta}) \in Y, \tag{5.9}$$

and

$$\begin{aligned}
\varphi(y, \boldsymbol{u}, \boldsymbol{v}) = &-\mu(P_{M(\kappa(r))}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\theta}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\
&\forall y = (r, \boldsymbol{\theta}) \in Y,\ \forall \boldsymbol{u}, \boldsymbol{v} \in V.
\end{aligned} \tag{5.10}$$

Note that the function φ defined by (5.10) is linear with respect to the last variable, according to Remark 8 the function φ satisfies (4.1) with $c_\varphi = 0$.

Denote

$$(\mathcal{S}\boldsymbol{u})(t) = \left(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\|_{\mathcal{H}} ds, \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds\right) \quad \forall \boldsymbol{u} \in C(\mathbb{R}_+; V). \tag{5.11}$$

Let $X = V$, $K = U$, $X_j = L^2(\Gamma_3)$, $\gamma_j\boldsymbol{v} = \boldsymbol{v}_\nu$ for $\boldsymbol{v} \in V$. Define

$$j(v) = \int_{\Gamma_3} j_\nu(v)d\Gamma \quad \forall v \in X_j. \tag{5.12}$$

Thus we have

$$j(\gamma_j\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)d\Gamma \quad \forall \boldsymbol{v} \in V. \tag{5.13}$$

Using the arguments in [9, Section 6] and Theorem 4, under the hypotheses for Problem 12 and the smallness condition $\overline{\alpha}_\nu\|\gamma_j\|^2 < m_{\mathcal{A}}$, we can show that Problem 13 has a unique solution $\boldsymbol{u} \in C(\mathbb{R}_+; U)$. Note that the coefficients $\alpha_\varphi = \mu$, $m_A = m_{\mathcal{A}} + \mu$, thus $\overline{\alpha}_\nu\|\gamma_j\|^2 < m_{\mathcal{A}}$ implies $\alpha_\varphi + \overline{\alpha}_\nu\|\gamma_j\|^2 < m_A$, i.e., the smallness condition (2.9) is satisfied.

Next we introduce a fully discrete method described in Section 4 to solve Problem 13. For the temporal discretization, we still use the uniform partition for the interval $[0, T]$ and the associate notations. We assume the domain Ω is a polygonal/polyhedral for simplicity and introduce a regular family of partitions $\{\mathcal{T}^h\}$ for the spatial discretization, in which the domain $\overline{\Omega}$ is divided into triangles/tetrahedrons and the whole boundary Γ is divided into $\Gamma_{k,i}$, $1 \leq i \leq i_k$, $1 \leq k \leq 3$. Notice that if the side/face of an element has intersection with set $\Gamma_{k,i}$ and the measure of the intersection is positive respect to $\Gamma_{k,i}$, then the side/face locates entirely in $\Gamma_{k,i}$. Thus the linear element spaces respect to $\{\mathcal{T}^h\}$ is constructed as follows:

$$V^h = \{\boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d \text{ for } T \in \mathcal{T}^h, \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1\}.$$

Define

$$U^h = \{\boldsymbol{v}^h \in V^h \mid v_\nu^h \leq g \text{ at node points on } \Gamma_3\}.$$

Assume g is concave, we obtain $U^h \subset U$. Thus the approximation is internal and the numerical method for Problem 13 is defined as follows:

Problem 14. Find a discrete displacement $\mathbf{u}^{kh} := \{\mathbf{u}_n^{kh}\}_{n=0}^N \subset U^h$ such that

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}), \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}), \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}))_{\mathcal{H}} \\ &\quad - \mu(P_{M(\kappa(\widetilde{\zeta}(t_n)))}\boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}), \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}_n^{kh}))_{\mathcal{H}} \\ &\quad + \left(k\sum_{j=0}^{n}{}'\mathcal{B}(t_n - t_j)\boldsymbol{\varepsilon}(\mathbf{u}_j^{kh}), \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}_n^{kh})\right)_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_\nu^0(u_{n,\nu}^{kh}; v_\nu^h - u_{n,\nu}^{kh})d\Gamma \geq \langle \boldsymbol{F}_n, \boldsymbol{v}^h - \mathbf{u}_n^{kh}\rangle_{V^*\times V}\ \forall \mathbf{v}^h \in U^h, \forall\, 0 \leq n \leq N,\end{aligned} \tag{5.14}$$

where

$$\widetilde{\zeta}(t_n) = k\sum_{j=0}^{n}{}'\|\boldsymbol{\varepsilon}(\mathbf{u}_j^{kh})\|_{\mathcal{H}}.$$

It is easy to see that the arguments used in the proof of Theorem 10 can be applied to the numerical solution defined by Problem 14, and assuming

$$\boldsymbol{u} \in W_{loc}^{2,\infty}(\mathbb{R}_+; V), \tag{5.15}$$

we have the error bound

$$\begin{aligned}\max_{0\leq n\leq N}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \leq C\max_{0\leq n\leq N}\inf_{\boldsymbol{v}^h\in U^h}\{&\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_V + \|u_{n,\nu} - v_\nu^h\|_{L^2(\Gamma_3)}^{\frac{1}{2}} \\ &+ |E_n(\boldsymbol{v}^h, \boldsymbol{u}_n)|^{\frac{1}{2}}\} + C\,k^2,\end{aligned} \tag{5.16}$$

where

$$\begin{aligned}E_n(\boldsymbol{v}^h, \boldsymbol{u}_n) = &\langle A\boldsymbol{u}_n, \boldsymbol{v}^h - \boldsymbol{u}_n\rangle + \varphi(\mathcal{S}_n\boldsymbol{u}, \boldsymbol{u}_n, \boldsymbol{v}^h) - \varphi(\mathcal{S}_n\boldsymbol{u}, \boldsymbol{u}_n, \boldsymbol{u}_n) \\ &+ \int_{\Gamma_3} j_\nu^0(u_{n,\nu}; v_\nu^h - u_{n,\nu})\,d\Gamma - \langle \boldsymbol{F}_n, \boldsymbol{v}^h - \boldsymbol{u}_n\rangle_{V^*\times V}.\end{aligned} \tag{5.17}$$

Let us bound the residual term defined in (5.17). For this purpose, in addition to (5.15), we assume the following solution regularity property:

$$\boldsymbol{\sigma} \in C(\mathbb{R}_+; H^1(\Omega; \mathbb{S}^d)), \quad \boldsymbol{\sigma\nu} \in C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d)), \tag{5.18}$$

where

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \quad \text{in } \Omega.$$

Define a subset of U,

$$\widetilde{U} = \{\boldsymbol{v} \in C^\infty(\overline{\Omega})^d \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1, v_\nu = 0 \text{ on } \Gamma_3\}.$$

Then we show the pointwise equality relations using the weak formulation (5.7) and the arguments in [20, Section 8.1]. We take $\boldsymbol{v} = \boldsymbol{u} \pm \widetilde{\boldsymbol{v}}$ in (5.7), where $\widetilde{\boldsymbol{v}}$ is an arbitrary function from the subset $\widetilde{U}$, which implies

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}}))_{\mathcal{H}} = \langle \boldsymbol{F}(t), \widetilde{\boldsymbol{v}}\rangle_{V^*\times V} \quad \forall \widetilde{\boldsymbol{v}} \in \widetilde{U}.$$

From the above relation we obtain the following equalities:

$$\mathrm{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \text{ a.e. in } \Omega, \tag{5.19}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \text{ a.e. on } \Gamma_2, \quad \boldsymbol{\sigma}_{\boldsymbol{\tau}} = \mathbf{0} \text{ a.e. on } \Gamma_3. \tag{5.20}$$

Multiplying (5.19) by $\boldsymbol{v} - \boldsymbol{u}$ with $\boldsymbol{v} \in U$, and integrating over Ω, we apply Green's formula to get

$$-\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{u}(t))d\Gamma + \int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v} - \boldsymbol{u}(t))dx = 0.$$

Thus,

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}\\ &\quad - \mu(P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}\\ &\quad + \left(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\right)_{\mathcal{H}}\\ &\quad = \langle \boldsymbol{F}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} + \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))d\Gamma, \quad \forall \boldsymbol{v} \in U,\ \forall t \in \mathbb{R}_+.\end{aligned} \tag{5.21}$$

Taking $t = t_n$ and $\boldsymbol{v} = \boldsymbol{v}^h$ in (5.21), we obtain

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n))_{\mathcal{H}}\\ &\quad - \mu(P_{M(\kappa(\zeta(t_n)))}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n))_{\mathcal{H}}\\ &\quad + \left(\int_0^t \mathcal{B}(t_n-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n)\right)_{\mathcal{H}}\\ &\quad = \langle \boldsymbol{F}_n, \boldsymbol{v}^h - \boldsymbol{u}_n\rangle_{V^*\times V} + \int_{\Gamma_3} \sigma_{n,\nu}(v_{n,\nu} - u_{n,\nu})d\Gamma, \quad \forall \boldsymbol{v}^h \in U^h.\end{aligned} \tag{5.22}$$

Together with (5.8), (5.10) and (5.11), Eq. (5.22) can be rewritten as

$$\langle A\boldsymbol{u}_n, \boldsymbol{v}^h - \boldsymbol{u}_n\rangle + \varphi(\mathcal{S}_n\boldsymbol{u}, \boldsymbol{u}_n, \boldsymbol{v}^h) - \varphi(\mathcal{S}_n\boldsymbol{u}, \boldsymbol{u}_n, \boldsymbol{u}_n) = \langle \boldsymbol{F}_n, \boldsymbol{v}^h - \boldsymbol{u}_n\rangle + \int_{\Gamma_3} \sigma_{n,\nu}(v_\nu^h - u_{n,\nu})\,d\Gamma.$$

Thus,

$$E_n(\boldsymbol{v}^h, \boldsymbol{u}_n) = \int_{\Gamma_3} [\sigma_{n,\nu}(v_\nu^h - u_{n,\nu}) + j_\nu^0(u_{n,\nu}; v_\nu^h - u_{n,\nu})]d\Gamma.$$

Using the solution regularity assumptions (5.15) and (5.18), we have

$$|E_n(\boldsymbol{v}^h, \boldsymbol{u}_n)| \le C\|u_{n,\nu} - v_\nu^h\|_{L^2(\Gamma_3)}. \tag{5.23}$$

Then from (5.16),

$$\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \le C \max_{0\le n\le N} \inf_{\boldsymbol{v}^h\in V^h} (\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_V + \|u_{n,\nu} - v_\nu^h\|_{L^2(\Gamma_3)}^{\frac{1}{2}}) + C\,k^2. \tag{5.24}$$

To proceed further, we further assume

$$\boldsymbol{u} \in C(\mathbb{R}_+; H^2(\Omega; \mathbb{R}^d)), \quad u_\nu \in C(\mathbb{R}_+; \widetilde{H}^2(\Gamma_3)). \tag{5.25}$$

Here, $v \in \widetilde{H}^2(\Gamma_3)$ means that $v \in \widetilde{H}^2(\Gamma_{3,i})$ for $1 \le i \le i_3$; recall that $\overline{\Gamma_3} = \bigcup_{1\le i\le i_3} \Gamma_{3,i}$. We comment that if $\mathcal{A}$, κ and $\mathcal{B}$ are suitable smooth, then (5.18) follows from the first part of (5.25). Utilizing the finite element interpolation theory [19,21], we can then derive the following optimal order error bound

$$\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \le C(h + k^2). \tag{5.26}$$

Thus in spatial mesh size the method is of first-order, and in the time step it is of second-order.

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References

[1] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, in: Classics Appl. Math., vol. 31, SIAM, Philadelphia, 2000.
[2] J.L. Lions, G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math. 20 (1967) 493–519.
[3] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Springer, New York, 2013.
[4] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, New York, 1995.
[5] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag, Berlin, 1993.

[6] S. Migórski, A. Ochal, M. Sofonea, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, Nonlinear Anal. RWA 12 (2011) 3384–3396.
[7] S. Migórski, A. Ochal, M. Sofonea, History-dependent variational-hemivariational inequalities in contact mechanics, Nonlinear Anal. Real World Appl. 22 (2015) 604–618.
[8] M. Sofonea, A. Matei, History-dependent quasi-variational inequalities arising in contact mechanics, European J. Appl. Math. 22 (2011) 471–491.
[9] M. Sofonea, S. Migórski, A class of history-dependent variational-hemivariational inequalities, Nonlinear Differential Equations Appl. 23 (3) (2016) 1–23.
[10] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications, Kluwer, Boston, 1999.
[11] M. Bartoteu, K. Bartosz, P. Kalita, An analytical and numerical approach to a bilateral contact problem with nonmonotone friction, Int. J. Appl. Math. Comput. Sci. 23 (2013) 263–276.
[12] W. Han, S. Migórski, M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, SIAM J. Math. Anal. 46 (2014) 3891–3912.
[13] W. Han, M. Sofonea, M. Barboteu, Numerical analysis of elliptic hemivariational inequalities, SIAM J. Math. Anal. 55 (2017) 640–663.
[14] W. Han, Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics, Math. Mech. Solids 23 (2018) 279–293.
[15] K. Kazmi, M. Barboteu, W. Han, M. Sofonea, Numerical analysis of history-dependent quasivariational inequalities with applications to contact mechanics, ESAIM Math. Model. Numer. Anal. 48 (2014) 919–942.
[16] M. Sofonea, W. Han, S. Migórski, Numerical analysis of history-dependent variational-hemivariational inequalities with applications to contact problems, European J. Appl. Math. 26 (2015) 427–452.
[17] J.B. Conway, A Coures in Functional Analysis, second ed., Spring-Verlag, New York, 1990.
[18] F.H. Clarke, Optimization and Nonsmooth Analysis, John Wiley, New York, 1983.
[19] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, third ed., Springer-Verlag, New York, 2009.
[20] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, AMS/IP, Providence, RI, 2002.
[21] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.