

Numerical Analysis of a Dynamic Contact Problem with History-Dependent Operators

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Abstract. In this paper, we study a dynamic contact model with long memory which allows both the convex potential and nonconvex superpotentials to depend on history-dependent operators. The deformable body consists of a viscoelastic material with long memory and the process is assumed to be dynamic. The contact involves a nonmonotone Clarke subdifferential boundary condition and the friction is modeled by a version of the Coulomb's law of dry friction with the friction bound depending on the total slip. We introduce and study a fully discrete scheme of the problem, and derive error estimates for numerical solutions. Under appropriate solution regularity assumptions, an optimal order error estimate is derived for the linear finite element method. This theoretical result is illustrated numerically.

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1. Introduction

Variational inequalities and hemivariational inequalities play an important role in the study of various nonlinear boundary value problems arising in Mechanics, Physics, Engineering Sciences and so on. For some comprehensive references, the reader is referred to [2,9–11,13,14,17,19,22] for variational inequalities, and to [16,20,21,23,25] for hemivariational inequalities. The analysis of variational inequalities is based on monotonicity arguments and convexity theory while the analysis of hemivariational inequalities uses properties of the subdifferential in the sense of Clarke defined for locally Lipschitz functions as main ingredient and allows nonconvex functionals in formulations. Variational-hemivariational inequalities represent a special class of inequalities, where both convex and nonconvex functions are present.

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This paper is devoted to the study on numerical approximation of a general evolutionary variational-hemivariational inequality involving history-dependent operators which models a dynamic contact problem with long memory. The model we consider here was first proposed in [12]. Existence and uniqueness result of the corresponding variational-hemivariational inequality are shown in [12]. In this paper, we consider numerical methods to solve the model in [12]. We derive optimal error estimates for the scheme. Since history-dependent operators appear at several places and the contact boundary conditions are of complex form, it is challenging to derive error estimates for numerical solutions of the model.

We first recall the model studied in [12]. Assume a viscoelastic body occupies a Lipschitz domain Ω in \mathbb{R}^d with $d = 2, 3$. We use the notation $\mathbf{x} = (x_i)_{i=1}^d$ for a generic point in $\bar{\Omega} = \Omega \cup \partial\Omega$ and we denote by $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$ the outward unit normal on $\partial\Omega$. We denote by $\mathbf{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ the displacement vector, the stress tensor, and linearized strain tensor, respectively. Sometimes we do not indicate explicitly the dependence of the variables on the spatial variable \mathbf{x} . Recall that the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ are $\varepsilon_{ij}(\mathbf{u}) = (u_{i,j} + u_{j,i})/2$, where $u_{i,j} = \partial u_i / \partial x_j$. The indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable \mathbf{x} . A superscript prime of a variable stands for the time derivative of the variable. Moreover, we use the notation v_ν and v_τ for the normal and tangential components of \mathbf{v} on $\partial\Omega$ given by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively. The symbol \mathbb{S}^d represents the space of second order symmetric tensors on \mathbb{R}^d .

The boundary $\partial\Omega$ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 and the measure of Γ_1 , denoted $m(\Gamma_1)$, is positive. The body is clamped on Γ_1 , so the displacement field vanishes there. Time-dependent surface tractions of density \mathbf{f}_2 act on Γ_2 and time-dependent volume forces of density \mathbf{f}_0 act in Ω . The part Γ_2 can be empty. The body is in permanent contact on Γ_3 with a device, say a piston. The contact is modeled with a nonmonotone normal damped response condition associated with a total slip-dependent version of Coulomb’s law of dry friction. We are interested in the evolutionary process of the mechanical state of the body in the time interval $(0, T)$ with $T > 0$. The mathematical model of the contact problem is stated as follows.

Problem 1.1. Find a displacement field $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \rightarrow \mathbb{S}^d$ such that for all $t \in (0, T)$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}'(s)) ds \quad \text{in } \Omega, \tag{1.1a}$$

$$\rho \mathbf{u}''(t) = \text{Div} \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \tag{1.1c}$$

$$\boldsymbol{\sigma}(t) \boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \tag{1.1d}$$

$$-\sigma_\nu(t) \in \phi(u_\nu(t))\partial j_\nu(u'_\nu(t)) \quad \text{on } \Gamma_3, \quad (1.1e)$$

$$\|\sigma_\tau(t)\| \leq F_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right),$$

$$-\sigma_\tau = F_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right) \frac{\mathbf{u}'_\tau(t)}{\|\mathbf{u}'_\tau(t)\|} \quad \text{if } \mathbf{u}'_\tau(t) \neq \mathbf{0} \quad \text{on } \Gamma_3, \quad (1.1f)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{w}_0 \quad \text{in } \Omega. \quad (1.1g)$$

Eq. (1.1a) is the constitutive law for viscoelastic materials in which \mathcal{A} represents the viscosity operator, \mathcal{B} represents the elasticity operator and \mathcal{C} is the relaxation tensor. Eq. (1.1b) is the equation of motion in which ρ denotes the density of mass. For simplicity, assume ρ is a constant which can be set to be 1 after scaling of the equation. We have the clamped boundary condition (1.1c) on Γ_1 and the surface traction boundary condition (1.1d) on Γ_2 .

Relation (1.1e) is the multivalued contact condition with normal damped response in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν and ϕ is a damper coefficient. Condition (1.1f) represents a version of Coulomb’s law of dry friction in which F_b is a given positive function, the friction bound. Details on such a frictional contact condition is found in [20] and some references therein. However, note that in contrast to the conditions used in the literature, in (1.1e) the damper coefficient is allowed to depend on the normal displacement $u_\nu(t)$. In (1.1f), the friction bound may depend on the the total slip $\int_0^t \|\mathbf{u}_\tau(\mathbf{x}, s)\| ds$ at the point $\mathbf{x} \in \Gamma_3$ over the time period $[0, t]$.

The quasistatic contact model with constitutive law (1.1a) was considered in [6]. In [7], the finite element method is used to approximate the model in [6] and an optimal order error estimate is derived. Recently, the number of publications on error estimate is growing rapidly. The reference [24] provides an error estimate for the numerical solution of a quasistatic viscoelastic problem with long memory, the reference [3] studies a fully dynamic viscoelastic contact model where the friction law is described by a nonmonotone relation between the tangential stress and the tangential velocity. Numerical analysis of history-dependent quasivariational inequalities was presented in [18], where both the temporally semi-discrete and fully discrete scheme are studied. More recently, a history-dependent hemivariational inequality with constraint is considered in [26] and the corresponding C ea’s type inequality is derived for error estimation. The reader is referred to [15] for a survey of recent results on the numerical solution of hemivariational inequalities of various kinds.

The paper is organized as follows. In Section 2, we first present preliminary material and list some assumptions on the data. And then, we formulate a dynamic history-dependent variational-hemivariational inequality corresponding to the contact model. In Section 3, we first introduce a discrete problem, present an existence and uniqueness result for it, and then derive error bounds for the fully discrete solutions and give an optimal order error estimate for finite element method. In the last section, we present numerical results in simulations of a two-dimensional contact problem and provide numerical evidence of optimal order convergence for the linear element solutions.

2. Notation and assumptions

In this section, we recall notation, basic definitions and unique solvability result of a dynamic history-dependent variational-hemivariational inequality. We start with the definitions of Clarke directional derivative and Clarke subdifferential. Let X be a Banach space, and X^* its dual. Denote by $\langle \cdot, \cdot \rangle_{X^* \times X}$ the duality pairing between X^* and X .

Definition 2.1. Let $\psi : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative, in the sense of Clarke, of ψ at $x \in X$ in the direction $v \in X$, denoted by $\psi^0(x; v)$, is defined by

$$\psi^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{\psi(y + \lambda v) - \psi(y)}{\lambda}$$

and the Clarke subdifferential of ψ at x , denoted by $\partial\psi(x)$, is a subset of a dual space X^* given by

$$\partial\psi(x) = \{ \zeta \in X^* \mid \psi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X}, \forall v \in X \}.$$

Definition 2.2. Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex functional. The convex subdifferential of φ at $x \in X$ is a subset of X^* given by

$$\partial_{Conv}\varphi(x) = \{ \xi \in X^* : \varphi(x + v) - \varphi(x) \geq \langle \xi, v \rangle_{X^* \times X}, \forall v \in X \}.$$

We use the standard notation for Lebesgue and Sobolev spaces. For $v \in H^1(\Omega; \mathbb{R}^d)$, we use the same symbol v for the trace of v on $\partial\Omega$ and we use the notation v_ν and v_τ for its normal and tangential traces. In addition, we introduce spaces V and Q as follows:

$$\begin{aligned} V &= \{ v = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid v = \mathbf{0} \text{ on } \Gamma_1 \}, \\ Q &= L^2(\Omega; \mathbb{S}^d), \\ H &= L^2(\Omega; \mathbb{R}^d). \end{aligned}$$

These are real Hilbert spaces with the canonical inner products in Q and H , and the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

in V . The associated norms are $\|\cdot\|_V$, $\|\cdot\|_Q$ and $\|\cdot\|_H$. By the Sobolev trace theorem,

$$\|v\|_{L^2(\Gamma_3; \mathbb{R}^d)} \leq \|\gamma\| \|v\|_V, \quad \forall v \in V, \tag{2.1}$$

where $\|\gamma\|$ represents the norm of the trace operator $\gamma : V \rightarrow L^2(\Gamma_3; \mathbb{R}^d)$.

Note that $V \subset H \subset V^*$ form an evolution triple of function spaces. Given $0 < T < +\infty$, we introduce spaces $\mathcal{V} = L^2(0, T; V)$ and $\mathcal{W} = \{w \in \mathcal{V} \mid w' \in \mathcal{V}^*\}$, where the time derivative $w' = \partial w / \partial t$ is understood in the sense of vector-valued distributions. The dual of \mathcal{V} is $\mathcal{V}^* = L^2(0, T; V^*)$. It is known that the space \mathcal{W} endowed with the graph

norm $\|\mathbf{w}\|_{\mathcal{W}} = \|\mathbf{w}\|_{\mathcal{V}} + \|\mathbf{w}'\|_{\mathcal{V}^*}$ is a separable and reflexive Banach space. Identifying $\mathcal{H} = L^2(0, T; H)$ with its dual, we obtain the continuous embeddings $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. The embedding $\mathcal{W} \subset C([0, T]; H)$ is continuous, $C([0, T]; H)$ being the space of continuous functions on $[0, T]$ with values in H . The duality pairing between \mathcal{V}^* and \mathcal{V} is

$$\langle \mathbf{w}, \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} = \int_0^T \langle \mathbf{w}(t), \mathbf{v}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt, \quad \mathbf{w} \in \mathcal{V}^*, \quad \mathbf{v} \in \mathcal{V}.$$

Define a space of fourth order tensor fields,

$$\mathbf{Q}_\infty = \{ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq d \}.$$

This is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \sum_{0 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Now we introduce assumptions on the data in the study of Problem 1.1. For the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$, we assume

$$\left\{ \begin{array}{l} (a) \text{ there exists } L_{\mathcal{A}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and a.e. } \mathbf{x} \in \Omega, \\ \quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|; \\ (b) \text{ there exists } m_{\mathcal{A}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and a.e. } \mathbf{x} \in \Omega, \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2; \\ (c) \text{ the mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ (d) \mathcal{A}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (2.2)$$

For the elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$, we assume

$$\left\{ \begin{array}{l} (a) \text{ there exists } L_{\mathcal{B}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and a.e. } \mathbf{x} \in \Omega, \\ \quad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|; \\ (b) \text{ the mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ (c) \mathcal{B}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (2.3)$$

For the relaxation operator \mathcal{C} , we assume

$$\mathcal{C} \in C(0, T; \mathbf{Q}_\infty). \quad (2.4)$$

For the potential function $j_\nu : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$, we assume

$$\left\{ \begin{array}{l} (a) j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad \text{exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3); \\ (b) j_\nu(\mathbf{x}, \cdot) \text{ is Lipschitz continuous on } \mathbb{R} \text{ for a.e. } \mathbf{x} \in \Gamma_3; \\ (c) |\partial j_\nu(\mathbf{x}, r)| \leq \bar{c}_0 \text{ for a.e. } \mathbf{x} \in \Gamma_3, \text{ for all } r \in \mathbb{R} \text{ with } \bar{c}_0 \geq 0; \\ (d) j_\nu^0(\mathbf{x}, r_1; r_2 - r_1) + j_\nu^0(\mathbf{x}, r_2; r_1 - r_2) \leq \bar{\beta} |r_1 - r_2|^2 \\ \quad \text{for a.e. } \mathbf{x} \in \Gamma_3, \text{ all } r_1, r_2 \in \mathbb{R} \text{ with } \bar{\beta} \geq 0. \end{array} \right. \quad (2.5)$$

For the damper coefficient $\phi : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$, we assume

$$\left\{ \begin{array}{l} (a) \text{ the mapping } \mathbf{x} \mapsto \phi(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}; \\ (b) \text{ there are constants } \phi_1, \phi_2 \text{ such that } 0 < \phi_1 \leq \phi(\mathbf{x}, r) \leq \phi_2 \\ \text{for all } r \in \mathbb{R} \text{ and a.e. } \mathbf{x} \in \Gamma_3; \\ (c) \text{ there exists } L_\phi > 0 \text{ such that for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } \mathbf{x} \in \Gamma_3, \\ |\phi(\mathbf{x}, r_1) - \phi(\mathbf{x}, r_2)| \leq L_\phi |r_1 - r_2|. \end{array} \right. \quad (2.6)$$

For the friction bound $F_b : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$, we assume

$$\left\{ \begin{array}{l} (a) \text{ the mapping } \mathbf{x} \mapsto F_b(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}; \\ (b) \text{ there exists } L_{F_b} > 0 \text{ such that for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } \mathbf{x} \in \Gamma_3, \\ |F_b(\mathbf{x}, r_1) - F_b(\mathbf{x}, r_2)| \leq L_{F_b} |r_1 - r_2|; \\ (c) \text{ the mapping } \mathbf{x} \mapsto F_b(\mathbf{x}, 0) \text{ belongs to } L^2(\Gamma_3). \end{array} \right. \quad (2.7)$$

Finally, for the densities of body forces, surface tractions and the initial data, we assume

$$\mathbf{f}_0 \in L^2(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \mathbf{f}_2 \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d)), \quad \mathbf{u}_0, \mathbf{w}_0 \in V. \quad (2.8)$$

Define a function $\mathbf{f} : (0, T) \rightarrow V^*$ by

$$\langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} = (\mathbf{f}_0(t), \mathbf{v})_H + (\mathbf{f}_2(t), \mathbf{v})_{L^2(\Gamma_2; \mathbb{R}^d)}, \quad \mathbf{v} \in V, \text{ a.e. } t \in (0, T). \quad (2.9)$$

Through a standard derivation, we have the following variational formulation of Problem 1.1 (cf. [20]).

Problem 2.1. Find a displacement field $\mathbf{u} : (0, T) \rightarrow V$ such that for a.e. $t \in (0, T)$,

$$\begin{aligned} & \langle \mathbf{u}''(t), \mathbf{v} - \mathbf{u}'(t) \rangle_{V^* \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}'(t)))_Q \\ & + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}'(t)))_Q + \left(\int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}'(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}'(t)) \right)_Q \\ & + \int_{\Gamma_3} F_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right) (\|\mathbf{v}_\tau\| - \|\mathbf{u}'_\tau(t)\|) d\Gamma \\ & + \int_{\Gamma_3} \phi(u_\nu(t)) j_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t)) d\Gamma \\ & \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{u}'(t) \rangle_{V^* \times V}, \quad \forall \mathbf{v} \in V, \end{aligned} \quad (2.10)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{w}_0. \quad (2.11)$$

The unique solvability of Problem 2.1 is provided in the following result ([12, Theorem 13]).

Theorem 2.1. Assume (2.2)-(2.8). If

$$m_{\mathcal{A}} > \bar{\beta}\phi_2\|\gamma\|^2, \quad (2.12)$$

then Problem 2.1 has a unique solution with regularity $\mathbf{u} \in \mathcal{V}$, $\mathbf{u}' \in \mathcal{W}$.

For the numerical approximation of the problem, it is convenient to reformulate the dynamic history-dependent variational-hemivariational inequality in terms of the velocity variable

$$\mathbf{w} = \mathbf{u}' \tag{2.13}$$

By using the initial value condition (2.11), we can recover \mathbf{u} from \mathbf{w} as follows:

$$\mathbf{u}(t) = \mathbf{u}_0 + (I\mathbf{w})(t), \tag{2.14}$$

where

$$(I\mathbf{w})(t) = \int_0^t \mathbf{w}(s) ds. \tag{2.15}$$

Then Problem 2.1 can be equivalently stated as follows.

Problem 2.2. Find a velocity field $\mathbf{w} \in \mathcal{W}$ such that for a.e. $t \in (0, T)$,

$$\begin{aligned} & \langle \mathbf{w}'(t), \mathbf{v} - \mathbf{w}(t) \rangle_{V^* \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{w}(t)))_Q \\ & + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_0 + (I\mathbf{w})(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{w}(t)))_Q + \left(\int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{w}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{w}(t)) \right)_Q \\ & + \int_{\Gamma_3} F_b \left(\int_0^t \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds \right) (\|\mathbf{v}_\tau\| - \|\mathbf{w}_\tau(t)\|) d\Gamma \\ & + \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})(t)_\nu) j_\nu^0(w_\nu(t); v_\nu - w_\nu(t)) d\Gamma \\ & \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{w}(t) \rangle_{V^* \times V}, \quad \forall \mathbf{v} \in V, \end{aligned} \tag{2.16}$$

and

$$\mathbf{w}(0) = \mathbf{w}_0. \tag{2.17}$$

3. A fully discrete scheme and error estimate

In this section, we introduce a fully discrete scheme for the variational-hemivariational inequality formulated in Problem 2.2 and provide a result on error estimate. First, we recall a discrete Gronwall inequality (cf. [14, Lemma 7.26]) that will be used later in error analysis.

Lemma 3.1. *Let $T > 0$ be given. For a positive integer N , define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying*

$$e_n \leq \bar{c}g_n + \bar{c} \sum_{j=1}^n ke_j, \quad n = 1, \dots, N,$$

for a positive constant \bar{c} independent of N or k . Then, there exists a positive constant c , independent of N or k , such that

$$\max_{1 \leq n \leq N} e_n \leq c \max_{1 \leq n \leq N} g_n.$$

We will also make use of the modified Cauchy-Schwarz inequality with an arbitrary $\epsilon > 0$:

$$ab \leq \epsilon a^2 + cb^2, \quad a, b \in \mathbb{R}, \tag{3.1}$$

where the constant $c > 0$ depends on ϵ ; indeed, we may simply take $c = 1/(4\epsilon)$.

Let V^h be a finite dimensional subspace of V , where $h > 0$ denotes a spatial discretization parameter. In addition, we consider an equidistant time grid with abscissae $t_n = nk$, where $n = 0, 1, \dots, N$, $N \in \mathbb{N}$, and the constant step-size $k = T/N$. For a time continuous function $g = g(t)$, we write $g_n = g(t_n)$ for $n = 0, 1, \dots, N$. For convenience, we assume

$$\mathbf{f}_0 \in C([0, T]; L^2(\Omega; \mathbb{R}^d)), \quad \mathbf{f}_2 \in C([0, T]; L^2(\Gamma_2; \mathbb{R}^d)). \tag{3.2}$$

Then $\mathbf{f} \in C([0, T]; V^*)$.

We use the left rectangle formula in each subinterval $[t_j, t_{j+1}]$ of $[0, T]$ to approximate the history-dependent integral term in Problem 2.2:

$$\int_0^{t_n} \mathcal{C}(t_n - s) \boldsymbol{\varepsilon}(\mathbf{w}(s)) ds \approx k \sum_{j=0}^{n-1} \mathcal{C}(t_n - t_j) \boldsymbol{\varepsilon}(\mathbf{w}_j).$$

The initial values \mathbf{u}_0 and \mathbf{w}_0 will be approximated by their V -orthogonal projections into V^h : $\mathbf{u}_0^h, \mathbf{w}_0^h \in V^h$ such that

$$(\mathbf{u}_0^h - \mathbf{u}_0, \mathbf{v}^h)_V = 0, \quad \forall \mathbf{v}^h \in V^h, \tag{3.3a}$$

$$(\mathbf{w}_0^h - \mathbf{w}_0, \mathbf{v}^h)_V = 0, \quad \forall \mathbf{v}^h \in V^h. \tag{3.3b}$$

The integration operator I of (2.15) will be approximated by the discrete operator I^k , defined by the formula

$$(I^k \mathbf{w})_n = k \sum_{j=0}^{n-1} \mathbf{w}_j \tag{3.4}$$

for a continuous function \mathbf{w} on $[0, T]$, where $\mathbf{w}_j = \mathbf{w}(t_j)$, and by the formula

$$(I^k \mathbf{w}^{hk})_n = k \sum_{j=0}^{n-1} \mathbf{w}_j^{hk} \tag{3.5}$$

for a discrete function

$$\mathbf{w}^{hk} = \{\mathbf{w}_0^{hk}, \dots, \mathbf{w}_N^{hk}\}.$$

Then the numerical scheme we consider is the following.

Problem 3.1. Find a discrete velocity field $\mathbf{w}^{hk} = \{\mathbf{w}_0^{hk}, \dots, \mathbf{w}_N^{hk}\}$ such that for $1 \leq n \leq N$,

$$\begin{aligned} & \left(\frac{\mathbf{w}_n^{hk} - \mathbf{w}_{n-1}^{hk}}{k}, \mathbf{v}^h - \mathbf{w}_n^{hk} \right)_H + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q \\ & + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_0^h + (I^k \mathbf{w}^{hk})_n), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q + k \sum_{j=0}^{n-1} \left(\mathcal{C}(t_n - t_j) \boldsymbol{\varepsilon}(\mathbf{w}_j^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}) \right)_Q \\ & + \int_{\Gamma_3} F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) (\|\mathbf{v}_\tau^h\| - \|\mathbf{w}_{n,\tau}^{hk}\|) d\Gamma \\ & + \int_{\Gamma_3} \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_\nu) j_\nu^0(w_{n,\nu}^{hk}; v_\nu^h - w_{n,\nu}^{hk}) d\Gamma \\ & \geq \langle \mathbf{f}_n, \mathbf{v}^h - \mathbf{w}_n^{hk} \rangle_{V^* \times V}, \quad \forall \mathbf{v}^h \in V^h, \end{aligned} \tag{3.6}$$

and

$$\mathbf{w}_0^{hk} = \mathbf{w}_0^h. \tag{3.7}$$

In the rest of the paper, we use c for a generic positive constant whose value may change in various inequalities, but it is independent of h and k . Let us first show the existence of a unique solution to Problem 3.1.

Theorem 3.1. *Keep the assumptions of Theorem 2.1 and assume (3.2), Problem 3.1 has a unique solution.*

Proof. We rewrite Problem 3.1 as

$$\begin{aligned} & \left(\frac{\mathbf{w}_n^{hk}}{k}, \mathbf{v}^h - \mathbf{w}_n^{hk} \right)_H + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q \\ & + \int_{\Gamma_3} F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) (\|\mathbf{v}_\tau^h\| - \|\mathbf{w}_{n,\tau}^{hk}\|) d\Gamma \\ & + \int_{\Gamma_3} \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) j_\nu^0(w_{n,\nu}^{hk}; v_\nu^h - w_{n,\nu}^{hk}) d\Gamma \\ & \geq \ell_n(\mathbf{v}^h - \mathbf{w}_n^{hk}), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \ell_n(\mathbf{v}^h) = & \left(\frac{\mathbf{w}_{n-1}^{hk}}{k}, \mathbf{v}^h \right)_H - (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_0^h + (I^k \mathbf{w}^{hk})_n), \boldsymbol{\varepsilon}(\mathbf{v}^h))_Q \\ & - k \sum_{j=0}^{n-1} \left(\mathcal{C}(t_n - t_j) \boldsymbol{\varepsilon}(\mathbf{w}_j^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}^h) \right)_Q + \langle \mathbf{f}_n, \mathbf{v}^h \rangle_{V^* \times V}. \end{aligned}$$

We conduct an induction argument. The initial value \mathbf{w}_0^{hk} is given by (3.7). Suppose $\mathbf{w}_0^{hk}, \dots, \mathbf{w}_{n-1}^{hk}$ are known. Let us show that (3.8) has a unique solution \mathbf{w}_n^{hk} . For this

purpose, introduce an operator $A^h : V^h \rightarrow (V^h)^*$ by

$$\langle A^h \mathbf{w}^h, \mathbf{v}^h \rangle_{(V^h)^* \times V^h} = (\mathcal{A}\varepsilon(\mathbf{w}^h), \varepsilon(\mathbf{v}^h))_Q + \left(\frac{\mathbf{w}^h}{k}, \mathbf{v}^h \right)_H, \quad \mathbf{w}^h, \mathbf{v}^h \in V^h,$$

and consider the following variational-hemivariational inequality

$$\begin{aligned} & \langle A^h \mathbf{w}_n^{hk}, \mathbf{v}^h - \mathbf{w}_n^{hk} \rangle_{(V^h)^* \times V^h} + \int_{\Gamma_3} F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) (\|\mathbf{v}_\tau^h\| - \|\mathbf{w}_{n,\tau}^{hk}\|) d\Gamma \\ & + \int_{\Gamma_3} \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) j_\nu^0(w_{n,\nu}^{hk}; v_\nu^h - w_{n,\nu}^{hk}) d\Gamma \\ & \geq \ell_n(\mathbf{v}^h - \mathbf{w}_n^{hk}), \quad \forall \mathbf{v}^h \in V^h. \end{aligned} \quad (3.9)$$

Under the assumptions (2.2)-(2.8) and (2.12), it follows from a well-posedness result on elliptic variational-hemivariational inequalities (cf. [25, Theorem 84]) for $K = V^h$ that there exists a unique solution $\mathbf{w}_n^{hk} \in V^h$ of the Problem (3.9). \square

For an error analysis of the numerical method, we will assume the smoothness

$$\mathbf{w} \in H^1(0, T; V) \cap H^2(0, T; V^*), \quad (3.10)$$

which implies that

$$\mathbf{w} \in C([0, T]; V).$$

Moreover, the first term in (2.16) can be replaced by

$$(\mathbf{w}'(t), \mathbf{v} - \mathbf{w}(t))_H.$$

We first set $t = t_n$ and $\mathbf{v} = \mathbf{w}_n^{hk}$ in (2.16) and deduce that

$$\begin{aligned} & (\mathcal{A}\varepsilon(\mathbf{w}_n), \varepsilon(\mathbf{w}_n - \mathbf{w}_n^{hk}))_Q \\ & \leq (\mathbf{w}'_n, \mathbf{w}_n^{hk} - \mathbf{w}_n)_H + (\mathcal{B}\varepsilon(\mathbf{u}_0 + (I\mathbf{w})_n), \varepsilon(\mathbf{w}_n^{hk} - \mathbf{w}_n))_Q \\ & + \left(\int_0^{t_n} \mathcal{C}(t_n - s) \varepsilon(\mathbf{w}(s)) ds, \varepsilon(\mathbf{w}_n^{hk} - \mathbf{w}_n) \right)_Q \\ & + \int_{\Gamma_3} F_b \left(\int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds \right) (\|\mathbf{w}_{n,\tau}^{hk}\| - \|\mathbf{w}_{n,\tau}\|) d\Gamma \\ & + \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - w_{n,\nu}) d\Gamma - \langle \mathbf{f}_n, \mathbf{w}_n^{hk} - \mathbf{w}_n \rangle_{V^* \times V}. \end{aligned} \quad (3.11)$$

By (2.2)(b), we have for any $\mathbf{v}_n^h \in V^h$

$$\begin{aligned} & m_{\mathcal{A}} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \\ & \leq (\mathcal{A}\varepsilon(\mathbf{w}_n) - \mathcal{A}\varepsilon(\mathbf{w}_n^{hk}), \varepsilon(\mathbf{w}_n - \mathbf{v}_n^h))_Q + (\mathcal{A}\varepsilon(\mathbf{w}_n), \varepsilon(\mathbf{v}_n^h - \mathbf{w}_n))_Q \\ & + (\mathcal{A}\varepsilon(\mathbf{w}_n), \varepsilon(\mathbf{w}_n - \mathbf{w}_n^{hk}))_Q + (\mathcal{A}\varepsilon(\mathbf{w}_n^{hk}), \varepsilon(\mathbf{w}_n^{hk} - \mathbf{v}_n^h))_Q. \end{aligned} \quad (3.12)$$

From (3.6),

$$\begin{aligned}
 & (\mathcal{A}\varepsilon(\mathbf{w}_n^{hk}), \varepsilon(\mathbf{w}_n^{hk} - \mathbf{v}_n^h))_Q \\
 \leq & \frac{1}{k} \left(\mathbf{w}_n^{hk} - \mathbf{w}_{n-1}^{hk}, \mathbf{v}_n^h - \mathbf{w}_n^{hk} \right)_H + (\mathcal{B}\varepsilon(\mathbf{u}_0^h + (I^k \mathbf{w}^{hk})_n), \varepsilon(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \\
 & + k \sum_{j=0}^{n-1} \left(\mathcal{C}(t_n - t_j) \varepsilon(\mathbf{w}_j^{hk}), \varepsilon(\mathbf{v}_n^h - \mathbf{w}_n^{hk}) \right)_Q \\
 & + \int_{\Gamma_3} F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) (\|\mathbf{v}_{n,\tau}^h\| - \|\mathbf{w}_{n,\tau}^{hk}\|) d\Gamma \\
 & + \int_{\Gamma_3} \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) j_\nu^0(w_{n,\nu}^{hk}; v_{n,\nu}^h - w_{n,\nu}^{hk}) d\Gamma - \langle \mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n^{hk} \rangle_{V^* \times V}. \quad (3.13)
 \end{aligned}$$

Denote the error

$$\mathbf{e}_n = \mathbf{w}_n - \mathbf{w}_n^{hk}.$$

By using (3.11) and (3.13) in (3.12), we obtain

$$\begin{aligned}
 m_{\mathcal{A}} \|\mathbf{e}_n\|_V^2 \leq & (\mathcal{A}\varepsilon(\mathbf{w}_n) - \mathcal{A}\varepsilon(\mathbf{w}_n^{hk}), \varepsilon(\mathbf{w}_n - \mathbf{v}_n^h))_Q + \left(\mathbf{w}'_n - \frac{\mathbf{w}_n - \mathbf{w}_{n-1}}{k}, \mathbf{w}_n^{hk} - \mathbf{v}_n^h \right)_H \\
 & - \frac{1}{k} (\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_H + \frac{1}{k} \left(\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{w}_n - \mathbf{v}_n^h \right)_H + R_n(\mathbf{v}_n^h) \\
 & + I_1 + I_2 + I_3, \quad (3.14)
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 = & (\mathcal{B}\varepsilon(\mathbf{u}_0 + (I\mathbf{w})_n) - \mathcal{B}\varepsilon(\mathbf{u}_0^h + (I^k \mathbf{w}^{hk})_n), \varepsilon(\mathbf{w}_n^{hk} - \mathbf{v}_n^h))_Q \\
 & + \left(\int_0^{t_n} \mathcal{C}(t_n - s) \varepsilon(\mathbf{w}(s)) ds - k \sum_{j=0}^{n-1} \mathcal{C}(t_n - t_j) \varepsilon(\mathbf{w}_j^{hk}), \varepsilon(\mathbf{w}_n^{hk} - \mathbf{v}_n^h) \right)_Q, \\
 I_2 = & \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - w_{n,\nu}) d\Gamma \\
 & + \int_{\Gamma_3} \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) j_\nu^0(w_{n,\nu}^{hk}; v_{n,\nu}^h - w_{n,\nu}^{hk}) d\Gamma \\
 & - \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) j_\nu^0(w_{n,\nu}; v_{n,\nu}^h - w_{n,\nu}) d\Gamma, \\
 I_3 = & \int_{\Gamma_3} F_b \left(\int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds \right) (\|\mathbf{w}_{n,\tau}^{hk}\| - \|\mathbf{v}_{n,\tau}^h\|) d\Gamma \\
 & + \int_{\Gamma_3} F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) (\|\mathbf{v}_{n,\tau}^h\| - \|\mathbf{w}_{n,\tau}^{hk}\|) d\Gamma,
 \end{aligned}$$

and the residual term is defined as

$$\begin{aligned}
 R_n(\mathbf{v}_n^h) &= \left(\mathbf{w}'_n, \mathbf{v}_n^h - \mathbf{w}_n \right)_H + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n), \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q \\
 &\quad + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_0 + (I\mathbf{w})_n), \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q \\
 &\quad + \left(\int_0^{t_n} \mathcal{C}(t_n - s) \boldsymbol{\varepsilon}(\mathbf{w}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n) \right)_Q \\
 &\quad + \int_{\Gamma_3} F_b \left(\int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds \right) (\|\mathbf{v}_{n,\tau}^h\| - \|\mathbf{w}_{n,\tau}\|) d\Gamma \\
 &\quad + \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) j_\nu^0(w_{n,\nu}; v_{n,\nu}^h - w_{n,\nu}) d\Gamma \\
 &\quad - \langle \mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n \rangle_{V^* \times V}.
 \end{aligned} \tag{3.15}$$

Lemma 3.2. *Let \mathbf{w} and \mathbf{w}^{hk} be solutions to Problems 2.2 and 3.1, respectively. We have the following bound for $n = 1, \dots, N$:*

$$\|(I\mathbf{w})_n - (I^k \mathbf{w}^{hk})_n\|_V \leq c k \|\mathbf{w}\|_{H^1(0,T;V)} + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V. \tag{3.16}$$

Proof. From the definitions of $(I\mathbf{w})_n$ and $(I^k \mathbf{w}^{hk})_n$, we have

$$\begin{aligned}
 \|(I\mathbf{w})_n - (I^k \mathbf{w}^{hk})_n\|_V &\leq \left\| \int_0^{t_n} \mathbf{w}(s) ds - k \sum_{j=0}^{n-1} \mathbf{w}_j \right\|_V + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V \\
 &= \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\mathbf{w}(s) - \mathbf{w}_j) ds \right\|_V + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V \\
 &= \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^s \frac{d}{d\tau} (\mathbf{w}(\tau)) d\tau ds \right\|_V + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V \\
 &\leq k \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|\mathbf{w}'(\tau)\|_V d\tau + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V,
 \end{aligned}$$

i.e., (3.16) holds. □

The next lemma provides a result on the estimate for three history-dependent terms in (3.14).

Lemma 3.3. *Let \mathbf{w} and \mathbf{w}_n^{hk} be solutions to Problems 2.2 and 3.1. Under the regularity assumption $\mathcal{C} \in H^1(0, T; \mathcal{L}(\mathbb{S}^d))$, we have the following inequality:*

$$\begin{aligned}
 I_1 + I_2 + I_3 &\leq c \|\mathbf{w}_n^{hk} - \mathbf{v}_n^h\|_V \left(k \|\mathbf{w}\|_{H^1(0,T;V)} + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V + \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V \right) \\
 &\quad + \bar{\beta} \phi_2 \|\gamma\|^2 \|\mathbf{e}_n\|_V^2 + c \|\mathbf{w}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}
 \end{aligned} \tag{3.17}$$

for all $\mathbf{v}_n^h \in V^h$, $n = 1, \dots, N$.

Proof. First, we derive an upper bound on I_1 . The term containing the operator \mathcal{C} can be bounded as follows:

$$\begin{aligned} & \left\| \int_0^{t_n} \mathcal{C}(t_n - s)\varepsilon(\mathbf{w}(s)) ds - k \sum_{j=0}^{n-1} \mathcal{C}(t_n - t_j)\varepsilon(\mathbf{w}_j^{hk}) \right\|_Q \\ & \leq \left\| \int_0^{t_n} \mathcal{C}(t_n - s)\varepsilon(\mathbf{w}(s)) ds - k \sum_{j=0}^{n-1} \mathcal{C}(t_n - t_j)\varepsilon(\mathbf{w}_j) \right\|_Q \\ & \quad + \left\| k \sum_{j=0}^{n-1} \mathcal{C}(t_n - t_j)\varepsilon(\mathbf{w}_j) - k \sum_{j=0}^{n-1} \mathcal{C}(t_n - t_j)\varepsilon(\mathbf{w}_j^{hk}) \right\|_Q \\ & \leq c \left(k \|\mathbf{w}\|_{H^1(0,T;V)} + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V \right). \end{aligned}$$

For the term containing operator \mathcal{B} , by using (2.3)(a), we obtain

$$\begin{aligned} & \left(\mathcal{B}\varepsilon(\mathbf{u}_0 + (I\mathbf{w})_n) - \mathcal{B}\varepsilon(\mathbf{u}_0^h + (I^k\mathbf{w}^{hk})_n) \right) \\ & \leq L_{\mathcal{B}}(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V + \|(I\mathbf{w})_n - (I^k\mathbf{w}^{hk})_n\|_V). \end{aligned}$$

Using (3.16) we conclude that there is a constant c such that

$$I_1 \leq c \|\mathbf{w}_n^{hk} - \mathbf{v}_n^h\|_V \left(k \|\mathbf{w}\|_{H^1(0,T;V)} + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V + \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V \right). \quad (3.18)$$

Next, we bound I_2 . By (2.5)(c), $j_\nu^0(\mathbf{x}, t, \xi; \eta) \leq \bar{c}_0 |\eta|$. Using the subadditivity of the generalized directional derivative, (2.5)(d) and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} & \int_{\Gamma_3} [j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - v_{n,\nu}^h) + j_\nu^0(w_{n,\nu}^{hk}; v_{n,\nu}^h - w_{n,\nu}^{hk})] d\Gamma \\ & \leq \int_{\Gamma_3} [j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - w_{n,\nu}) + j_\nu^0(w_{n,\nu}^{hk}; w_{n,\nu} - w_{n,\nu}^{hk})] d\Gamma \\ & \quad + \int_{\Gamma_3} [j_\nu^0(w_{n,\nu}; w_{n,\nu} - v_{n,\nu}^h) + j_\nu^0(w_{n,\nu}^{hk}; v_{n,\nu}^h - w_{n,\nu})] d\Gamma \\ & \leq \bar{\beta} \|\gamma\|^2 \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 + 2 \bar{c}_0 \sqrt{m(\Gamma_3)} \|\mathbf{w}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}. \end{aligned}$$

Moreover, using again the subadditivity of the generalized directional derivative, we have

$$\begin{aligned} & \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - w_{n,\nu}) d\Gamma \\ & \leq \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) \left[j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - v_{n,\nu}^h) + j_\nu^0(w_{n,\nu}; v_{n,\nu}^h - w_{n,\nu}) \right] d\Gamma. \end{aligned}$$

Consequently, from (2.5) and (2.6) we have

$$\begin{aligned}
 I_2 &\leq \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - v_{n,\nu}^h) d\Gamma \\
 &\quad + \int_{\Gamma_3} \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) j_\nu^0(w_{n,\nu}^{hk}; v_{n,\nu}^h - w_{n,\nu}^{hk}) d\Gamma \\
 &\leq \int_{\Gamma_3} \left[\phi(u_{0,\nu} + (I\mathbf{w})_{n,\nu}) - \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) \right] j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - v_{n,\nu}^h) \\
 &\quad + \phi(u_{0,\nu}^h + (I^k \mathbf{w}^{hk})_{n,\nu}) \left[j_\nu^0(w_{n,\nu}; w_{n,\nu}^{hk} - v_{n,\nu}^h) + j_\nu^0(w_{n,\nu}^{hk}; v_{n,\nu}^h - w_{n,\nu}^{hk}) \right] d\Gamma \\
 &\leq \bar{c}_0 L_k \|\gamma\|^2 (\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V + \|(I\mathbf{w})_n - (I^k \mathbf{w}^{hk})_n\|_V) \|\mathbf{w}_n^{hk} - \mathbf{v}_n^h\|_V \\
 &\quad + \phi_2(\bar{\beta} \|\gamma\|^2 \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 + 2\bar{c}_0 \sqrt{m(\Gamma_3)} \|\mathbf{w}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}). \tag{3.19}
 \end{aligned}$$

Finally, we bound I_3 , which is expressed as

$$\begin{aligned}
 I_3 &= \int_{\Gamma_3} \left[F_b \left(\int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds \right) - F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) \right] \\
 &\quad \cdot (\|\mathbf{w}_{n,\tau}^{hk}\| - \|\mathbf{v}_{n,\tau}^h\|) d\Gamma.
 \end{aligned}$$

By (2.7),

$$\begin{aligned}
 &\left| F_b \left(\int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds \right) - F_b \left(k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right) \right| \\
 &\leq c \left| \int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds - k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right| \\
 &\leq c(I_4 + I_5),
 \end{aligned}$$

where

$$\begin{aligned}
 I_4 &= \left| \int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds - k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau} + (I^k \mathbf{w})_{j,\tau}\| \right|, \\
 I_5 &= k \left| \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau} + (I^k \mathbf{w})_{j,\tau}\| - \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau}^h + (I^k \mathbf{w}^{hk})_{j,\tau}\| \right|.
 \end{aligned}$$

Write

$$\begin{aligned}
 &\int_0^{t_n} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| ds - k \sum_{j=0}^{n-1} \|\mathbf{u}_{0,\tau} + (I^k \mathbf{w})_{j,\tau}\| \\
 &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| - \|\mathbf{u}_{0,\tau} + (I^k \mathbf{w})_{j,\tau}\| \right] ds.
 \end{aligned}$$

Then,

$$\begin{aligned}
 I_4 &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left| \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_\tau\| - \|\mathbf{u}_{0,\tau} + (I^k\mathbf{w})_{j,\tau}\| \right| ds \\
 &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|(I\mathbf{w})(s)_\tau - (I^k\mathbf{w})_{j,\tau}\| ds \\
 &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\| \int_{t_j}^t \mathbf{w}'_\tau(s) ds \right\| dt \\
 &\leq k \int_0^{t_n} \|\mathbf{w}'_\tau(s)\| ds.
 \end{aligned}$$

Easily,

$$I_5 \leq k \sum_{j=0}^{n-1} \left[\|\mathbf{u}_{0,\tau} - \mathbf{u}_{0,\tau}^h\| + \left\| k \sum_{i=0}^{j-1} \mathbf{w}_{i,\tau} - k \sum_{i=0}^{j-1} \mathbf{w}_{i,\tau}^{hk} \right\| \right].$$

Then,

$$I_5 \leq c \|\mathbf{u}_{0,\tau} - \mathbf{u}_{0,\tau}^h\| + ck \sum_{j=0}^{n-1} \|\mathbf{w}_{j,\tau} - \mathbf{w}_{j,\tau}^{hk}\|.$$

Summarizing,

$$|I_3| \leq c \int_{\Gamma_3} (I_4 + I_5) \|\mathbf{v}_n^h - \mathbf{w}_n^{hk}\| d\Gamma,$$

and then,

$$I_3 \leq c \|\mathbf{v}_n^h - \mathbf{w}_n^{hk}\|_V \left(k \|\mathbf{w}\|_{H^1(0,T;V)} + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V + \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V \right). \quad (3.20)$$

From (3.18)-(3.20), we have (3.17). This finishes the proof. \square

Now, we are ready to bound the other terms on the right side of (3.14). First,

$$\begin{aligned}
 (\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_H &= \frac{1}{2} (\|\mathbf{e}_n\|_H^2 - \|\mathbf{e}_{n-1}\|_H^2 + \|\mathbf{e}_n - \mathbf{e}_{n-1}\|_H^2) \\
 &\geq \frac{1}{2} (\|\mathbf{e}_n\|_H^2 - \|\mathbf{e}_{n-1}\|_H^2).
 \end{aligned}$$

Thus,

$$-\frac{1}{k} (\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_H \leq -\frac{1}{2k} (\|\mathbf{e}_n\|_H^2 - \|\mathbf{e}_{n-1}\|_H^2). \quad (3.21)$$

Let

$$\mathbf{E}_n = \mathbf{w}'_n - \frac{\mathbf{w}_n - \mathbf{w}_{n-1}}{k}.$$

We note that

$$\langle \mathbf{E}_n, \mathbf{w}_n^{hk} - \mathbf{v}_n^h \rangle_{V^* \times V} \leq \|\mathbf{E}_n\|_{V^*} \|\mathbf{w}_n^{hk} - \mathbf{v}_n^h\|_V.$$

For any $\epsilon > 0$, by applying the modified Cauchy-Schwarz inequality (3.1), we obtain that

$$\begin{aligned} \langle \mathbf{E}_n, \mathbf{w}_n^{hk} - \mathbf{v}_n^h \rangle_{V^* \times V} &\leq \|\mathbf{E}_n\|_{V^*} (\|\mathbf{e}_n\|_V + \|\mathbf{w}_n - \mathbf{v}_n^h\|_V) \\ &\leq \epsilon \|\mathbf{e}_n\|_V^2 + c \|\mathbf{E}_n\|_{V^*}^2 + c \|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2. \end{aligned} \tag{3.22}$$

Moreover, from (2.2)(a) and applying the modified Cauchy-Schwarz inequality (3.1) again, we have

$$\begin{aligned} &(\mathcal{A}\varepsilon(\mathbf{w}_n) - \mathcal{A}\varepsilon(\mathbf{w}_n^{hk}), \varepsilon(\mathbf{w}_n - \mathbf{v}_n^h))_Q \\ &\leq L_{\mathcal{A}} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V \|\mathbf{w}_n - \mathbf{v}_n^h\|_V \\ &\leq \epsilon \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 + c \|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2. \end{aligned} \tag{3.23}$$

Then, by using inequalities (3.17), (3.21)-(3.23) on the right side of the inequality (3.14) and taking $\epsilon > 0$ sufficiently small, under assumption (2.12), we obtain the following result

$$\begin{aligned} &k \|\mathbf{e}_n\|_V^2 + \|\mathbf{e}_n\|_H^2 - \|\mathbf{e}_{n-1}\|_H^2 \\ &\leq ck (\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + \|\mathbf{w}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_3; \mathbb{R}^d)} + \|\mathbf{E}_n\|_{V^*}^2) \\ &\quad + ck \|\mathbf{w}_n^{hk} - \mathbf{v}_n^h\|_V \left(k \|\mathbf{w}\|_{H^1(0,T;V)} + k \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V + \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V \right) \\ &\quad + \left(\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{w}_n - \mathbf{v}_n^h \right)_H + ck |R_n(\mathbf{v}_n^h)|. \end{aligned}$$

Since

$$\|\mathbf{w}_n^{hk} - \mathbf{v}_n^h\|_V^2 \leq 2(\|\mathbf{w}_n^{hk} - \mathbf{w}_n\|_V^2 + \|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2),$$

it follows that

$$\begin{aligned} &k \|\mathbf{e}_n\|_V^2 + \|\mathbf{e}_n\|_H^2 - \|\mathbf{e}_{n-1}\|_H^2 \\ &\leq ck (\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + \|\mathbf{w}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_3; \mathbb{R}^d)} + \|\mathbf{E}_n\|_{V^*}^2) \\ &\quad + ck^3 \|\mathbf{w}\|_{H^1(0,T;V)}^2 + ck^2 \sum_{j=0}^{n-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V^2 \\ &\quad + ck \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 + c \left(\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{w}_n - \mathbf{v}_n^h \right)_H + ck |R_n(\mathbf{v}_n^h)|. \end{aligned} \tag{3.24}$$

We replace n by l in (3.24) and make a summation over l from 1 to n ,

$$\begin{aligned} & k \sum_{l=1}^n \|e_l\|_V^2 + \|e_n\|_H^2 - \|e_0\|_H^2 \\ & \leq ck \sum_{l=1}^n (\|\mathbf{w}_l - \mathbf{v}_l^h\|_V^2 + \|\mathbf{w}_l - \mathbf{v}_l^h\|_{L^2(\Gamma_3; \mathbb{R}^d)} + \|\mathbf{E}_l\|_{V^*}^2 + |R_l(\mathbf{v}_l^h)|) \\ & \quad + ck^2 \|\mathbf{w}\|_{H^1(0,T;V)}^2 + ck^2 \sum_{l=1}^n \sum_{j=0}^{l-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V^2 + c\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 \\ & \quad + c \sum_{l=1}^n \left(e_l - e_{l-1}, \mathbf{w}_l - \mathbf{v}_l^h \right)_H. \end{aligned}$$

For the term \mathbf{E}_l , we have

$$\mathbf{E}_l = \frac{1}{k} \int_{t_{l-1}}^{t_l} (t - t_{l-1}) \mathbf{w}''(t) dt.$$

It follows that

$$\|\mathbf{E}_l\|_{V^*}^2 \leq \frac{1}{k^2} \int_{t_{l-1}}^{t_l} (t - t_{l-1})^2 dt \int_{t_{l-1}}^{t_l} \|\mathbf{w}''(t)\|_{V^*}^2 dt = \frac{k}{3} \int_{t_{l-1}}^{t_l} \|\mathbf{w}''(t)\|_{V^*}^2 dt.$$

And then, we have

$$k \sum_{l=1}^n \|\mathbf{E}_l\|_{V^*}^2 \leq \frac{k^2}{3} \|\mathbf{w}''\|_{L^2(0,T;V^*)}^2.$$

For the term $\sum_{l=1}^n (e_l - e_{l-1}, \mathbf{w}_l - \mathbf{v}_l^h)_H$, according to [14], we have

$$\begin{aligned} & \sum_{l=1}^n \left(e_l - e_{l-1}, \mathbf{w}_l - \mathbf{v}_l^h \right)_H \\ & \leq \frac{1}{2} (\|e_n\|_H^2 + \|\mathbf{w}_n - \mathbf{v}_n^h\|_H^2) \\ & \quad + \frac{k}{2} \sum_{l=1}^{n-1} (\|e_l\|_H^2 + k^{-2} \|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2) \\ & \quad + \frac{1}{2} (\|e_0\|_H^2 + \|\mathbf{w}_1 - \mathbf{v}_1^h\|_H^2). \end{aligned}$$

Finally, we obtain the following inequality.

$$\begin{aligned} & k \sum_{l=1}^n \|e_l\|_V^2 + \|e_n\|_H^2 \\ & \leq ck \sum_{l=1}^n (\|\mathbf{w}_l - \mathbf{v}_l^h\|_V^2 + |R_l(\mathbf{v}_l^h)| + \|\mathbf{w}_l - \mathbf{v}_l^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}) \end{aligned}$$

$$\begin{aligned}
 &+ ck^2 \|\mathbf{w}\|_{H^1(0,T;V)}^2 + c\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 + ck^2 \|\mathbf{w}\|_{H^2(0,T;V^*)}^2 + ck\|\mathbf{w}_0 - \mathbf{w}_0^h\|_V^2 \\
 &+ c\|\mathbf{w}_n - \mathbf{v}_n^h\|_H^2 + ck^{-1} \sum_{l=1}^{n-1} \|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2 \\
 &+ c\|\mathbf{e}_0\|_H^2 + c\|\mathbf{w}_1 - \mathbf{v}_1^h\|_H^2 + ck \sum_{l=0}^{n-1} \left(\|\mathbf{e}_l\|_H^2 + k \sum_{j=1}^l \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V^2 \right). \tag{3.25}
 \end{aligned}$$

For the discrete displacement,

$$\mathbf{u}_n^{hk} = \mathbf{u}_0^h + k \sum_{j=0}^{n-1} \mathbf{w}_j,$$

we have

$$\begin{aligned}
 k \sum_{l=1}^n \|\mathbf{u}_l - \mathbf{u}_l^{hk}\|_V^2 &\leq c(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 + k^2 \|\mathbf{w}\|_{H^1(0,T;V)}^2) \\
 &+ ck \sum_{l=1}^n k \sum_{j=0}^{l-1} \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V^2. \tag{3.26}
 \end{aligned}$$

Combining (3.25) and (3.26), we have

$$\begin{aligned}
 &k \sum_{l=1}^n \|\mathbf{u}_l - \mathbf{u}_l^{hk}\|_V^2 + k \sum_{l=1}^n \|\mathbf{e}_l\|_V^2 + \|\mathbf{e}_n\|_H^2 \\
 &\leq ck \sum_{l=1}^n (\|\mathbf{w}_l - \mathbf{v}_l^h\|_V^2 + |R_l(\mathbf{v}_l^h)| + \|\mathbf{w}_l - \mathbf{v}_l^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}) + ck\|\mathbf{e}_0\|_V^2 \\
 &+ ck^2 \|\mathbf{w}\|_{H^1(0,T;V)}^2 + c\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 + ck^2 \|\mathbf{w}\|_{H^2(0,T;V^*)}^2 + c\|\mathbf{w}_1 - \mathbf{v}_1^h\|_H^2 \\
 &+ c\|\mathbf{w}_n - \mathbf{v}_n^h\|_H^2 + ck^{-1} \sum_{l=1}^{n-1} \|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2 + c\|\mathbf{e}_0\|_H^2 \\
 &+ ck \sum_{l=0}^{n-1} \left(\|\mathbf{e}_l\|_H^2 + k \sum_{j=1}^l \|\mathbf{w}_j - \mathbf{w}_j^{hk}\|_V^2 + k \sum_{j=1}^l \|\mathbf{u}_j - \mathbf{u}_j^{hk}\|_V^2 \right).
 \end{aligned}$$

Applying Gronwall lemma, we deduce that

$$\begin{aligned}
 &\max_{1 \leq n \leq N} \|\mathbf{e}_n\|_H^2 + k \sum_{n=1}^N \|\mathbf{e}_n\|_V^2 + k \sum_{n=1}^N \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 \\
 &\leq ck^2 (\|\mathbf{w}\|_{H^2(0,T;V^*)}^2 + \|\mathbf{w}\|_{H^1(0,T;V)}^2) \\
 &+ c(\|\mathbf{e}_0\|_H^2 + \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 + k\|\mathbf{e}_0\|_V^2) + c \max_{1 \leq n \leq N} \tilde{E}_n,
 \end{aligned}$$

where

$$\begin{aligned} \tilde{E}_n = \inf_{\mathbf{v}_l^h \in V^h} \left\{ k \sum_{l=1}^n (\|\mathbf{w}_l - \mathbf{v}_l^h\|_V^2 + |R_l(\mathbf{v}_l^h)| + \|\mathbf{w}_l - \mathbf{v}_l^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}) \right. \\ \left. + \|\mathbf{w}_n - \mathbf{v}_n^h\|_H^2 + k^{-1} \sum_{l=1}^{n-1} \|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2 \right. \\ \left. + \|\mathbf{w}_1 - \mathbf{v}_1^h\|_H^2 \right\}. \end{aligned} \tag{3.27}$$

Summarizing the above arguments, we have the following theorem.

Theorem 3.2. *Let \mathbf{w} and \mathbf{w}^{hk} be solutions to Problems 2.2 and 3.1, respectively. Assume (2.2)-(2.8), (2.12) and (3.2). Then under the regularity assumptions in Lemma 3.3 and (3.10), we have the following inequality*

$$\begin{aligned} \max_{1 \leq n \leq N} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_H^2 + k \sum_{n=1}^N \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 + k \sum_{n=1}^N \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 \\ \leq ck^2 (\|\mathbf{w}\|_{H^2(0,T;V^*)}^2 + \|\mathbf{w}\|_{H^1(0,T;V)}^2) \\ + c (\|\mathbf{w}_0 - \mathbf{w}_0^h\|_H^2 + \|\mathbf{u}_0 - \mathbf{u}_0^h\|_V^2 + k \|\mathbf{w}_0 - \mathbf{w}_0^h\|_V^2) + c \max_{1 \leq n \leq N} \tilde{E}_n, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} \tilde{E}_n = \inf_{\mathbf{v}_l^h \in V^h} \left\{ k \sum_{l=1}^n (\|\mathbf{w}_l - \mathbf{v}_l^h\|_V^2 + |R_l(\mathbf{v}_l^h)| + \|\mathbf{w}_l - \mathbf{v}_l^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}) \right. \\ \left. + \|\mathbf{w}_n - \mathbf{v}_n^h\|_H^2 + k^{-1} \sum_{l=1}^{n-1} \|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2 \right. \\ \left. + \|\mathbf{w}_1 - \mathbf{v}_1^h\|_H^2 \right\}. \end{aligned}$$

Theorem 3.2 is valid for any finite dimensional subspace V^h of V . We will apply the finite element space. For simplicity, we assume that Ω is a polygonal or polyhedral domain and let \mathcal{T}^h be a regular family of finite element triangulations of $\bar{\Omega}$ into triangles or tetrahedrons. For an element $T \in \mathcal{T}^h$, denote by $P_1(T; \mathbb{R}^d)$ the space of polynomials of a total degree less than or equal to one in T . Then we can use the linear element space of piecewise continuous affine functions

$$V^h = \{\mathbf{v}^h \in C(\bar{\Omega}; \mathbb{R}^d) : \mathbf{v}^h|_T \in P_1(T; \mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_1\}. \tag{3.29}$$

Corollary 3.1. *Under the assumptions stated in Theorem 3.2. Assume Ω is a polygonal or polyhedral domain, and let $\{V^h\}$ be the family of linear element spaces defined by (3.29), corresponding to a regular family of finite element triangulations of $\bar{\Omega}$ into triangles or tetrahedrons. Assume further that*

$$\mathbf{w} \in C([0, T]; H^2(\Omega; \mathbb{R}^d)), \quad \mathbf{w}|_{\Gamma_3} \in C([0, T]; H^2(\Gamma_3; \mathbb{R}^d)).$$

Then we have the following optimal order error estimate:

$$\begin{aligned} & \max_{1 \leq n \leq N} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_H^2 + k \sum_{n=1}^N \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \\ & + k \sum_{n=1}^N \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 \leq c(k^2 + h^2). \end{aligned} \tag{3.30}$$

Proof. We apply the standard finite element interpolation error estimates ([1,5,8]). Take $\mathbf{v}_l^h \in V^h$ to be the finite element interpolation of \mathbf{w}_l . Then, we have

$$\|\mathbf{w}_l - \mathbf{v}_l^h\|_V \leq ch \|\mathbf{w}_l\|_{H^2(\Omega; \mathbb{R}^d)}, \quad 0 \leq l \leq N. \tag{3.31}$$

Notice that \mathbf{v}_l^h interpolates \mathbf{v}_l on Γ_3 . We have the following error estimate [14]:

$$\|\mathbf{w}_l - \mathbf{v}_l^h\|_{L^2(\Gamma_3)^d} \leq ch^2 \|\mathbf{w}_l\|_{H^2(\Gamma_3; \mathbb{R}^d)}.$$

Moreover, note that $(\mathbf{v}_l^h - \mathbf{v}_{l+1}^h)$ is the finite element interpolation of $(\mathbf{w}_l - \mathbf{w}_{l+1})$, then

$$\|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2 \leq ch^2 \|\mathbf{w}_l - \mathbf{w}_{l+1}\|_V^2 \leq ch^2 k \int_{t_l}^{t_{l+1}} \|\mathbf{w}'(t)\|_V^2 dt,$$

and then, we have

$$\begin{aligned} & k^{-1} \sum_{l=1}^{n-1} \|(\mathbf{w}_l - \mathbf{v}_l^h) - (\mathbf{w}_{l+1} - \mathbf{v}_{l+1}^h)\|_H^2 \\ & \leq ch^2 \|\mathbf{w}\|_{H^1(0,T;H^2(\Omega; \mathbb{R}^d))}^2, \quad 1 \leq n \leq N. \end{aligned} \tag{3.32}$$

It remains to consider $R_l(\mathbf{v}_l^h)$. Similar to [14, Section 8.1], it can be shown that under the stated regularity assumptions, we have the equality (1.1b) in V^* . Apply both sides of (1.1b) to an arbitrary $\mathbf{v} \in V$, integrate over Ω and integrate by parts to get

$$\langle \mathbf{w}'(t), \mathbf{v} \rangle_{V^* \times V} + \int_{\Omega} \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx = \langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} + \int_{\Gamma_3} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \mathbf{v} da.$$

Thus, the term $R_l(\mathbf{v}_l^h)$ is simplified to

$$\begin{aligned} R_l(\mathbf{v}_l^h) &= \int_{\Gamma_3} \boldsymbol{\sigma}_l \boldsymbol{\nu} \cdot (\mathbf{v}_l^h - \mathbf{w}_l) da + \int_{\Gamma_3} \phi(u_{0,\nu} + (I\mathbf{w})_{l,\nu}) j_{\nu}^0(\mathbf{w}_{l,\nu}; \mathbf{v}_{l,\nu}^h - \mathbf{w}_{l,\nu}) d\Gamma \\ &+ \int_{\Gamma_3} F_b \left(\int_0^{t_l} \|\mathbf{u}_{0,\tau} + (I\mathbf{w})(s)_{\tau}\| ds \right) (\|\mathbf{v}_{l,\tau}^h\| - \|\mathbf{w}_{l,\tau}\|) d\Gamma. \end{aligned}$$

Therefore,

$$|R_l(\mathbf{v}_l^h)| \leq c \|\mathbf{v}_l^h - \mathbf{w}_l\|_{L^2(\Gamma_3)^d}. \tag{3.33}$$

Finally, we have

$$\max_{1 \leq n \leq N} \|\mathbf{w}_n - \mathbf{v}_n^h\|_H \leq ch^2 \|\mathbf{w}\|_{C([0,T];H^2(\Omega;\mathbb{R}^d))}. \tag{3.34}$$

Combining (3.31)-(3.34) and (3.28), we obtain the optimal order error estimate

$$\max_{1 \leq n \leq N} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_H^2 + k \sum_{n=1}^N \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 + k \sum_{n=1}^N \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 \leq c(k^2 + h^2).$$

This concludes the proof of Corollary 3.1. □

Finally, we note that for the error in the displacement,

$$\mathbf{u}_n - \mathbf{u}_n^{hk} = (\mathbf{u}_0 - \mathbf{u}_0^{hk}) + \left[(I\mathbf{w})_n - (I^k\mathbf{w}^{hk})_n \right].$$

So from Corollary 3.1 and Lemma 3.2, we have the next result.

Corollary 3.2. *Keep the assumptions stated in Corollary 3.1. Then we have the optimal order error estimate*

$$\max_{1 \leq n \leq N} \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V \leq c(k + h). \tag{3.35}$$

4. Numerical example

In this section, some numerical results are presented to illustrate the behavior of the solution of the history-dependent frictional contact problem Problem 1.1. Particular attention is paid on the numerical convergence orders.

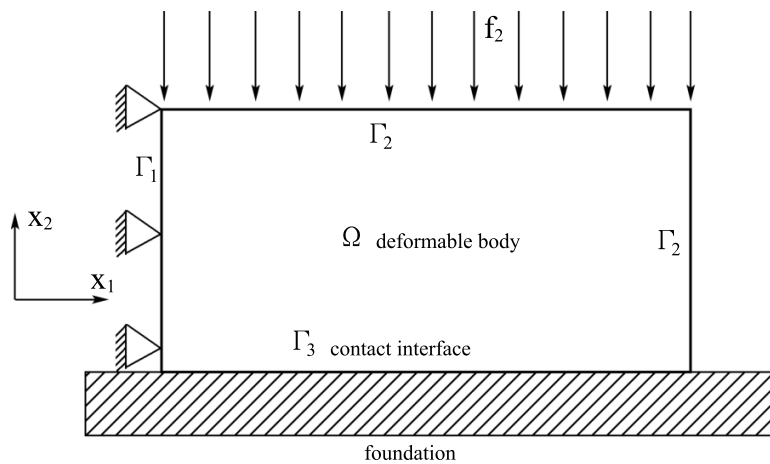


Figure 1: Reference configuration of the two-dimensional example.

We consider the physical setting shown in Fig. 1. Here, the domain $\Omega = (0, 3) \times (0, 1.5)$, and its boundary is split into $\Gamma_1 = \{0\} \times (0, 1.5)$, $\Gamma_2 = ((0, 3) \times \{1.5\}) \cup (\{3\} \times (0, 1.5))$, and $\Gamma_3 = (0, 3) \times \{0\}$. The time interval of interest is $[0, T]$ with $T = 1s$. For the force densities, we take $\mathbf{f}_0 = (0, 0)N/m^2$ in Ω , $\mathbf{f}_2 = (0, -30)N/m$ on $(0, 3) \times \{1.5\}$. The compressible material response is governed by a linearly viscoelastic constitutive law with history-dependent operator in which the viscosity tensor \mathcal{A} and the elasticity tensor \mathcal{B} are given by

$$\begin{aligned}
 (\mathcal{A}\tau)_{\alpha\beta} &= \mu_1(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \mu_2\tau_{\alpha\beta}, & 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2, \\
 (\mathcal{B}\tau)_{\alpha\beta} &= \frac{E}{1 + \kappa}\tau_{\alpha\beta} + \frac{E\kappa}{(1 - \kappa)(1 - 2\kappa)}(\tau_{11} + \tau_{22})\delta_{\alpha\beta}, & 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2,
 \end{aligned}$$

where μ_1 and μ_2 are viscosity constants, E and κ are Young’s modulus and Poisson’s ratio of the material, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. In the numerical example, we take $\mu_1 = 25$, $\mu_2 = 50$, $E = 1000N/m$, and $\kappa = 0.3$. The relaxation tensor $\mathcal{C}(s) = (0.5 + s)^3I$, where I is the identity matrix.

We assume $\phi(u_\nu(t)) = 1$. Let $j_\nu : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$j_\nu(r) = \begin{cases} 0, & \text{if } r < 0, \\ -\frac{1}{2}r^2 + 2r, & \text{if } 0 \leq r \leq 1, \\ r + \frac{1}{2}, & \text{if } r > 1. \end{cases} \tag{4.1}$$

The Clarke subdifferential of this function is given by

$$\partial j_\nu(r) = \begin{cases} 0, & \text{if } r < 0, \\ [0, 2], & \text{if } r = 0, \\ -r + 2, & \text{if } 0 < r \leq 1, \\ 1, & \text{if } r > 1. \end{cases} \tag{4.2}$$

For the friction law, we take

$$F_b(z) = (a - b)e^{-\alpha z} + b. \tag{4.3}$$

Then,

$$-\sigma_\tau = \begin{cases} (b - a)e^{-\alpha z} - b, & \text{if } \mathbf{u}'_\tau < 0, \\ [(b - a)e^{-\alpha z} - b, (a - b)e^{-\alpha z} + b], & \text{if } \mathbf{u}'_\tau = 0, \\ (a - b)e^{-\alpha z} + b, & \text{if } \mathbf{u}'_\tau > 0, \end{cases} \tag{4.4}$$

where

$$z = \int_0^t \|\mathbf{u}_\tau\| ds.$$

We use a Primal-Dual Active Set Strategy to solve the discrete problem. We can see the details in paper [4].

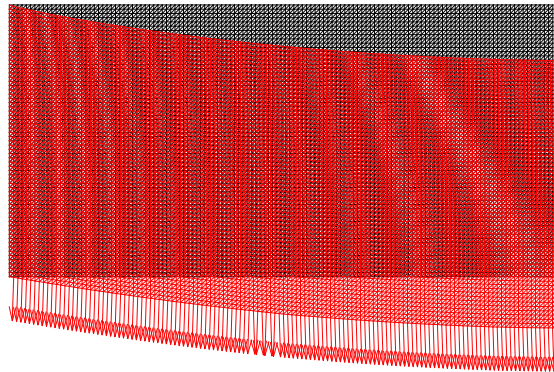


Figure 2: Deformed configuration and interface forces on Γ_3 at $t = 1s$.

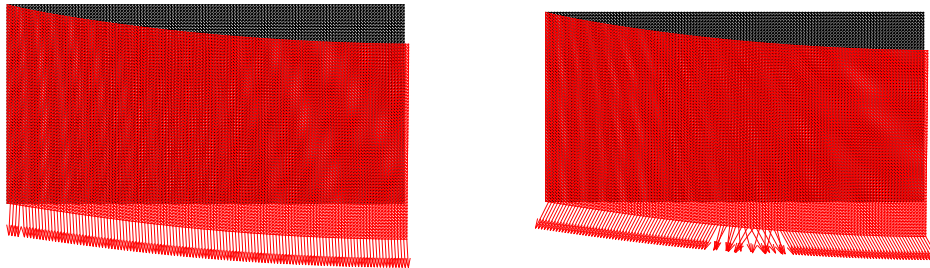


Figure 3: Results with $a = 0.1$ and $b = 0.1$ (left), and with $a = 1$ and $b = 1$ (right).

In Fig. 2, we plot the deformed configuration as well as the interface forces on Γ_3 during the dynamic compression process at time $t = 1s$. We plot the deformed meshes and the interface forces on Γ_3 for coefficient $a = 1$ and $b = 0.01$. In Fig. 3 we plot the deformed meshes and the interface forces on Γ_3 for two group of the same value of the coefficients $a = b = 0.1$ and $a = b = 1$, respectively. In the case $a = b = 0.1$, we note that the contact nodes are in slip contact since, there, the friction bound is low and, therefore, is reached. In contrast, in the case $a = b = 1$ the friction bound is higher and, as a consequence, all the contact nodes are in stick status.

Table 1: H^1 -norm errors.

| $h + k$ | 0.03125 | 0.0625 | 0.125 | 0.25 | 0.5 | 1 |
|------------------|-------------------------|-------------------------|-------------------------|-------------------------|--------|--------|
| $\ u - u^{hk}\ $ | 0.8628×10^{-2} | 0.2024×10^{-1} | 0.4072×10^{-1} | 0.8689×10^{-1} | 0.1600 | 0.3525 |

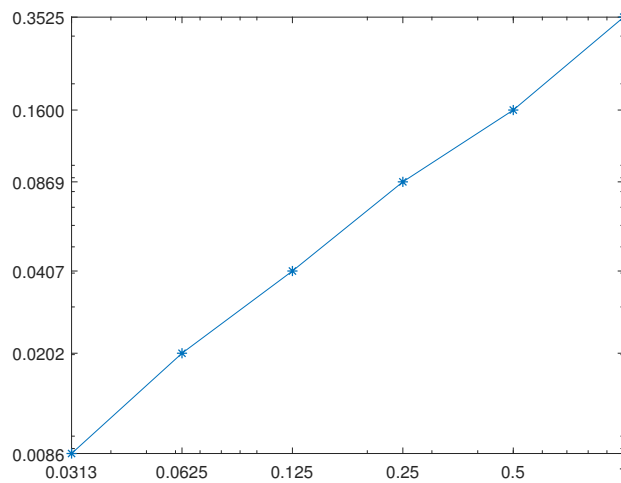


Figure 4: H^1 -norm errors.

To examine the numerical convergence orders, we computed a sequence of numerical solutions by using uniform discretizations of the problem domain according to the spatial discretization parameter h and time step k . For instance, the deformed configuration and the interface forces plotted in Fig. 2 correspond to the choices $h = 3/128$ and $k = 1/128$. The numerical error $\|\mathbf{u} - \mathbf{u}^{hk}\|_V$ is computed for several discretization parameters of h and k . Here, the boundary Γ_3 of Ω is divided into $3/h$ equal parts. We start with $h = 3/4$ and $k = 1/4$, which are successively halved. The numerical solution corresponding to $h = 3/256$ and $k = 1/256$ was taken as the “exact” solution, used to compute the errors of the numerical solutions. The numerical results are presented in Table 1 and Fig. 4. Observe that the error $\|\mathbf{u} - \mathbf{u}^{hk}\|_V$ clearly displays a linear convergence pattern; This matches well the theoretical prediction from (3.35).

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