Computers and Mathematics with Applications 75 (2018) 2561–2573

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A penalty method for history-dependent variational–hemivariational inequalities[☆]

Mircea Sofonea [a], Stanisław Migórski [b,*], Weimin Han [c]

[a] Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France
[b] Faculty of Mathematics and Computer Science, Jagiellonian University, Institute of Computer Science, ul. Stanisława Łojasiewicza 6, 30348 Krakow, Poland
[c] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

ARTICLE INFO

Article history:
Received 1 July 2017
Received in revised form 10 October 2017
Accepted 22 December 2017
Available online 11 January 2018

Keywords:
History-dependent operator
Penalty method
Existence and uniqueness
Convergence
Contact Mechanics

ABSTRACT

Penalty methods approximate a constrained variational or hemivariational inequality problem through a sequence of unconstrained ones as the penalty parameter approaches zero. The methods are useful in the numerical solution of constrained problems, and they are also useful as a tool in proving solution existence of constrained problems. This paper is devoted to a theoretical analysis of penalty methods for a general class of variational–hemivariational inequalities with history-dependent operators. Unique solvability of penalized problems is shown, as well as the convergence of their solutions to the solution of the original history-dependent variational–hemivariational inequality as the penalty parameter tends to zero. The convergence result proved here generalizes several existing convergence results of penalty methods. Finally, the theoretical results are applied to examples of history-dependent variational–hemivariational inequalities in mathematical models describing the quasistatic contact between a viscoelastic rod and a reactive foundation.

1. Introduction

Inequality problems arise in a variety of nonlinear models in Physics, Mechanics and Engineering Sciences, cf. [1–9] on variational inequalities, and [10–13] on hemivariational inequalities. Studies of variational inequalities involve arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. Studies of hemivariational inequalities are based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions, which may be nonconvex. Variational–hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are present.

Recently, inequalities with history-dependent operators were introduced and studied in connection with contact problems for memory dependent materials. Abstract classes of quasivariational inequalities with history-dependent operators were considered in [14,15] where existence, uniqueness and regularity results were proved. In [16] and [17], a penalty method and numerical analysis of the corresponding inequalities were provided, respectively. Hemivariational inequalities with history-dependent operators were studied in [18–20]. The inequalities in these papers were formulated in the particular case of Sobolev spaces over a bounded domain in $\mathbb{R}^d$ and specific operators like the trace operator. A general existence and

☆ This work was supported by the National Science Center of Poland under the Maestro Advanced Project no. DEC-2012/06/A/ST1/00262. The third author was also supported by NSF under DMS-1521684.

* Corresponding author.
E-mail address: stanislaw.migorski@ii.uj.edu.pl (S. Migórski).

https://doi.org/10.1016/j.camwa.2017.12.018
0898-1221/

uniqueness result in the study of variational–hemivariational inequalities with history-dependent operators, on an abstract setting of reflexive Banach spaces, was carried out in the recent paper [21].

Our aim in the current paper is to present a penalty method in the study of history-dependent variational–hemivariational inequalities introduced in [21] and to apply it to a new model of contact. Penalty methods for variational inequalities have been studied by many authors, for numerical purposes and for proofs of solution existence. The reader is referred to [2,3,6] on penalty methods for variational inequalities. The main feature of the penalty methods is that constraints in a problem are enforced by penalty through a limiting procedure and the penalized problems are constraint free. The penalized problems have unique solutions which converge to the solution of the original problem, as the penalty parameter tends to zero. Penalty methods were also considered in [16] and [22] in the study of history-dependent variational inequalities and variational–hemivariational inequalities, respectively. The results in this paper generalize and extend the convergence results obtained in these two papers.

The remainder of the paper is structured as follows. In Section 2, we present the history-dependent variational–hemivariational inequalities, introduce the penalized problems, and state our main theoretical result on solution existence and convergence for the penalty method. The main result is proved in Section 3, in several steps. In Section 4 we present two mathematical models which describe the contact of a viscoelastic rod with a foundation. In the first model the contact is with normal compliance and unilateral constraint, whereas in the second one, the unilateral constraint is penalized. For each model we introduce a weak formulation as a history-dependent variational–hemivariational inequality for the displacement field. Then, we state in Theorem 18 the unique weak solvability of the contact problems and the convergence of the weak solution of the penalized problem to the weak solution of the original problem, as the stiffness coefficient of the foundation converges to infinity. The proof of Theorem 18 is given in Section 5, through an application of the abstract result provided by Theorems 2 and 5 of Section 2.

2. A history-dependent variational–hemivariational inequality

We use real spaces only. For a normed space X, its norm is denoted by $\|\cdot\|_X$. When no confusion may arise, the duality pairing between the dual space X^* and X, $\langle\cdot,\cdot\rangle_{X^*\times X}$, will be simply written as $\langle\cdot,\cdot\rangle$.

We review some definitions. Let $A : X \to X^*$. Then A is monotone if

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge 0 \quad \forall\, v_1, v_2 \in X.$$

It is strongly monotone with a constant $m_A > 0$ if

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A\|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{2.1}$$

The operator A is demicontinuous if

$$u_n \to u \text{ as } n \to \infty \quad \Longrightarrow \quad Au_n \to Au \text{ weakly as } n \to \infty.$$

It is hemicontinuous if the function $t \mapsto \langle A(u + t\,v), w\rangle$ is continuous on $[0, 1]$ for all $u, v, w \in X$. Finally, the operator A is pseudomonotone if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u\rangle_{X^*\times X} \le 0$ imply

$$\langle Au, u - v\rangle_{X^*\times X} \le \liminf \langle Au_n, u_n - v\rangle_{X^*\times X} \quad \forall\, v \in X.$$

A function $\varphi : X \to \mathbb{R}$ is said to be lower semicontinuous (l.s.c.) if for any sequence $\{x_n\} \subset X$, $x_n \to x$ in X implies $\varphi(x) \le \liminf \varphi(x_n)$.

Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. Its generalized (Clarke) directional derivative at $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

Its generalized gradient (subdifferential) at x is a subset of the dual space X^*:

$$\partial\varphi(x) = \left\{\zeta \in X^* \mid \varphi^0(x; v) \ge \langle\zeta, v\rangle_{X^*\times X}\ \forall\, v \in X\right\}.$$

It is said to be regular in the sense of Clarke at $x \in X$ if for all $v \in X$, the one-sided directional derivative $\varphi'(x; v)$ exists and $\varphi^0(x; v) = \varphi'(x; v)$.

We use the symbol $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+ = [0, +\infty)$ for the set of nonnegative real numbers. The notation $C(\mathbb{R}_+; X)$ stands for the space of X-valued continuous functions defined on $\mathbb{R}_+$. For a subset $K \subset X$, $C(\mathbb{R}_+; K) \subset C(\mathbb{R}_+; X)$ denotes the set of K-valued continuous functions defined on $\mathbb{R}_+$.

Given a reflexive Banach space X, $K \subset X$, a normed space Y, operators $A : X \to X^*$ and $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$, a function $\varphi : Y \times X \times X \to \mathbb{R}$, a locally Lipschitz function $j : X \to \mathbb{R}$, and $f : \mathbb{R}_+ \to X^*$, we consider the following problem.

Problem 1. Find $u \in C(\mathbb{R}_+; K)$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned}&\langle Au(t), v - u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t))\\ &\quad + j^0(u(t); v - u(t)) \ge \langle f(t), v - u(t)\rangle \quad \forall\, v \in K.\end{aligned} \tag{2.2}$$

We list conditions on the operators and functions that will be needed later.

$$A : X \to X^* \text{ is pseudomonotone and strongly monotone with constant } m_A. \tag{2.3}$$

$$\begin{cases} \mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y) \text{ and} \\ \text{for any } n \in \mathbb{N} \text{ there exists } s_n > 0 \text{ such that} \\ \qquad \|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_Y \le s_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \qquad\qquad \forall\, u_1, u_2 \in C(\mathbb{R}_+; X), \ \forall\, t \in [0, n]. \end{cases} \tag{2.4}$$

$$\begin{cases} \varphi : Y \times X \times X \to \mathbb{R} \text{ is a function such that} \\ \text{(a) } \varphi(y, u, \cdot) : X \to \mathbb{R} \text{ is convex and l.s.c., for all } y \in Y,\ u \in X. \\ \text{(b) there exists } \alpha_\varphi \ge 0 \text{ and } \beta_\varphi \ge 0 \text{ such that} \\ \qquad \varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \\ \qquad\quad \le \alpha_\varphi \|u_1 - u_2\|_X \, \|v_1 - v_2\|_X + \beta_\varphi \|y_1 - y_2\|_Y \, \|v_1 - v_2\|_X \\ \qquad\qquad \forall\, y_1, y_2 \in Y,\ u_1, u_2, v_1, v_2 \in X. \end{cases} \tag{2.5}$$

$$\begin{cases} j : X \to \mathbb{R} \text{ is a function such that} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) } \|\partial j(v)\|_{X^*} \le c_0 + c_1 \|v\|_X \ \forall\, v \in X \text{ with } c_0, c_1 \ge 0. \\ \text{(c) there exists } \alpha_j \ge 0 \text{ such that} \\ \qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2 \ \forall\, v_1, v_2 \in X. \end{cases} \tag{2.6}$$

The following result is proved in [21].

Theorem 2. *Let X be a reflexive Banach space, $K \subset X$ nonempty, closed and convex, and Y a normed space. Assume (2.3)–(2.6) and*

$$\alpha_\varphi + \alpha_j < m_A. \tag{2.7}$$

Then, for any $f \in C(\mathbb{R}_+; X^*)$, Problem 1 has a unique solution $u \in C(\mathbb{R}_+; K)$.

We comment that $\mathcal{S}$ represents a *history-dependent operator*. Examples of $\mathcal{S}$ satisfying the condition (2.4) can be found in [14,18], including Volterra-type operators and other integral-type operators. The hypothesis (2.6)(c) is equivalent to the condition

$$\langle \partial j(v_1) - \partial j(v_2), v_1 - v_2 \rangle_{X^* \times X} \ge -\alpha_j \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X, \tag{2.8}$$

known as the relaxed monotonicity condition. It is extensively used in the literature, see for instance [11] and the references therein. Examples of nonconvex functions which satisfy condition (2.6) can be found in [19]. If $j : X \to \mathbb{R}$ is a convex function, then (2.6)(c) and (2.8) are satisfied with $\alpha_j = 0$, due to the monotonicity of the (convex) subdifferential.

Note that the function φ is assumed to be convex with respect to its third argument while the function j is locally Lipschitz in its second argument and is allowed to be nonconvex. In addition, the inequality (2.2) involves a history-dependent operator $\mathcal{S}$. We refer to Problem 1 as a *history-dependent variational–hemivariational inequality*.

The condition $u(t) \in K$ in Problem 1 imposes a constraint. The main goal of this paper is to introduce and study a penalty method for approximating Problem 1 through a sequence of unconstrained penalized problems. In particular, we will investigate the convergence property of the penalty method. The penalty method is defined through a penalty operator $P : X \to X^*$ and a penalty parameter $\lambda > 0$. The penalty operator P has the following properties (cf. [22] for examples):

$$\begin{cases} P : X \to X^* \text{ is such that} \\ \text{(a) } P \text{ is bounded, demicontinuous and monotone.} \\ \text{(b) } Pu = 0 \text{ if and only if } u \in K. \end{cases} \tag{2.9}$$

For each $\lambda > 0$, we introduce the following penalized problem which is defined over the entire space X.

Problem 3. Find $u_\lambda \in C(\mathbb{R}_+; X)$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle Au_\lambda(t), v - u_\lambda(t) \rangle + \frac{1}{\lambda} \langle Pu_\lambda(t), v - u_\lambda(t) \rangle \\ &\quad + \varphi((\mathcal{S}u_\lambda)(t), u_\lambda(t), v) - \varphi((\mathcal{S}u_\lambda)(t), u_\lambda(t), u_\lambda(t)) \\ &\quad + j^0(u_\lambda(t); v - u_\lambda(t)) \ge \langle f(t), v - u_\lambda(t) \rangle \quad \forall\, v \in X. \end{aligned} \tag{2.10}$$

In the study of the penalty method, we need to impose further conditions on $j : X \to \mathbb{R}$ and $\varphi : Y \times X \times X \to \mathbb{R}$:

$$u_n \to u \text{ weakly in } X \implies \limsup j^0(u_n; v - u_n) \le j^0(u; v - u)\ \forall\, v \in X. \tag{2.11}$$

$$\begin{cases} \text{There exists a continuous function } c_\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \\ \quad \text{such that for all } y \in Y,\ u,\ v_1,\ v_2 \in X, \\ \quad\quad \varphi(y, u, v_1) - \varphi(y, u, v_2) \le c_\varphi(\|y\|_Y, \|u\|_X)\, \|v_1 - v_2\|_X. \end{cases} \tag{2.12}$$

The following result is proved in [22] on sufficient conditions for functions satisfying both (2.6) and (2.11).

Lemma 4. *Let X and Y be reflexive Banach spaces, $\psi : Y \to \mathbb{R}$ satisfy (2.6) with X replaced by Y, and either ψ or $-\psi$ be regular. Let $M : X \to Y$ be given by $Mv = Lv + v_0$, where $L : X \to Y$ is a linear compact operator and $v_0 \in Y$. Then the function $j : X \to \mathbb{R}$ defined by $j(v) = \psi(Mv)$, $v \in X$, satisfies conditions (2.6) and (2.11).*

The main theoretical result of the paper is the following, and it will be proved in the next section.

Theorem 5. *Keep the assumptions stated in Theorem 2. Additionally, assume (2.9), (2.11) and (2.12).*

(i) *For each $\lambda > 0$, there exists a unique solution $u_\lambda \in C(\mathbb{R}_+; X)$ to Problem 3.*
(ii) *The solution u_λ of Problem 3 converges to the solution u of Problem 1, i.e.,*

$$\|u_\lambda(t) - u(t)\|_X \to 0 \quad \text{as } \lambda \to 0,\ t \in \mathbb{R}_+. \tag{2.13}$$

3. Proof of the main result

This section is devoted to a proof of Theorem 5. The proof is carried out in three steps. Throughout the section, we make the assumptions stated in Theorem 5, and recall that $u \in C(\mathbb{R}_+; K)$ is the solution of Problem 1.

In the first step, we prove the statement (i) of Theorem 5.

Lemma 6. *For each $\lambda > 0$, there exists a unique solution $u_\lambda \in C(\mathbb{R}_+; X)$ to Problem 3.*

Proof. By the properties (2.9) and Exercise I.9 in Section 1.9 of [23], it follows that the operator $P : X \to X^*$ is bounded, hemicontinuous and monotone. Thus, P is pseudomotone ([24, Proposition 27.6]). Define an operator $A_\lambda : X \to X^*$ by

$$A_\lambda v = Av + \frac{1}{\lambda} Pv, \quad v \in X. \tag{3.1}$$

It is easy to see that A_λ satisfies the condition (2.3) with the same constant m_A. Therefore, applying Theorem 2 with $K = X$ we know that Problem 3 has a unique solution $u_\lambda \in C(\mathbb{R}_+; X)$. □

In the second step, we define an auxiliary problem and show that its solution converges to the solution $u(t)$ of Problem 1 at all $t \in \mathbb{R}_+$.

For each $\lambda > 0$, the auxiliary problem is the following.

Problem 7. Find $\widetilde{u}_\lambda \in C(\mathbb{R}_+; X)$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle A\widetilde{u}_\lambda(t), v - \widetilde{u}_\lambda(t)\rangle + \frac{1}{\lambda}\langle P\widetilde{u}_\lambda(t), v - \widetilde{u}_\lambda(t)\rangle \\ &\quad + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_\lambda(t)) \\ &\quad + j^0(\widetilde{u}_\lambda(t); v - \widetilde{u}_\lambda(t)) \ge \langle f(t), v - \widetilde{u}_\lambda(t)\rangle \quad \forall\, v \in X. \end{aligned} \tag{3.2}$$

Note that while (2.10) is a history-dependent variational–hemivariational inequality, (3.2) is merely a time-dependent variational–hemivariational inequality. We will show, in turn, that Problem 7 admits a unique solution, the weak convergence and then the strong convergence of the solution to the solution of Problem 1 at each $t \in \mathbb{R}_+$, as the penalty parameter $\lambda \to 0$.

Lemma 8. *For each $\lambda > 0$, Problem 7 has a unique solution $\widetilde{u}_\lambda \in C(\mathbb{R}_+; X)$.*

Proof. For $t \in \mathbb{R}_+$ fixed but arbitrary, consider a function $\phi_t : X \times X \to \mathbb{R}$ defined by

$$\phi_t(w, v) = \varphi((\mathcal{S}u)(t), u(t), v), \quad w,\ v \in X. \tag{3.3}$$

Note that actually $\phi_t(w, v)$ is independent of w. By assumption (2.5), we see that for all $w \in X$, $\phi_t(w, \cdot) : X \to \mathbb{R}$ is convex and l.s.c., and

$$\phi_t(w_1, v_2) - \phi_t(w_1, v_1) + \phi_t(w_2, v_1) - \phi_t(w_2, v_2) = 0 \quad \forall\, w_1, w_2, v_1, v_2 \in X.$$

From the proof of Lemma 6, we know that the operator A_λ of (3.1) is pseudomonotone, coercive and strongly monotone with the same constants m_A and α_j. Due to the assumption (2.7), we can apply [22, Theorem 5] and conclude that there is a unique element $\widetilde{u}_\lambda(t) \in X$ satisfying

$$\begin{aligned} &\langle A\widetilde{u}_\lambda(t), v - \widetilde{u}_\lambda(t)\rangle + \frac{1}{\lambda}\langle P\widetilde{u}_\lambda(t), v - \widetilde{u}_\lambda(t)\rangle + \phi_t(\widetilde{u}_\lambda(t), v) \\ &\quad - \phi_t(\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t)) + j^0(\widetilde{u}_\lambda(t); v - \widetilde{u}_\lambda(t)) \geq \langle f, v - \widetilde{u}_\lambda(t)\rangle \quad \forall\, v \in X. \end{aligned} \tag{3.4}$$

Then, (3.2) holds.

We now show the continuity of the map $t \mapsto \widetilde{u}_\lambda(t) : \mathbb{R}_+ \to X$. Let $t_1, t_2 \in \mathbb{R}_+$ and denote $\widetilde{u}_\lambda(t_i) = \widetilde{u}_i$, $u(t_i) = u_i$, $(\mathcal{S}u)(t_i) = y_i$ and $f(t_i) = f_i$ for $i = 1, 2$. We take $v = u_2$ in (3.2) at $t = t_1$, take $v = u_1$ in (3.2) at $t = t_2$, and add the corresponding inequalities, to get

$$\begin{aligned} &\langle A\widetilde{u}_1 - A\widetilde{u}_2, \widetilde{u}_1 - \widetilde{u}_2\rangle + \frac{1}{\lambda}\langle P\widetilde{u}_1 - P\widetilde{u}_2, \widetilde{u}_1 - \widetilde{u}_2\rangle \\ &\quad \leq \varphi(y_1, u_1, \widetilde{u}_2) - \varphi(y_1, u_1, \widetilde{u}_1) + \varphi(y_2, u_2, \widetilde{u}_1) - \varphi(y_2, u_2, \widetilde{u}_2) \\ &\qquad + j^0(\widetilde{u}_1; \widetilde{u}_2 - \widetilde{u}_1) + j^0(\widetilde{u}_2; \widetilde{u}_1 - \widetilde{u}_2) + \langle f_1 - f_2, u_1 - u_2\rangle. \end{aligned}$$

Then use (2.3), (2.6)(c), (2.5)(b) and (2.9)(a) to obtain

$$m_A\|\widetilde{u}_1 - \widetilde{u}_2\|_X \leq \alpha_\varphi\|u_1 - u_2\|_X + \beta_\varphi\|y_1 - y_2\|_Y + \alpha_j\|\widetilde{u}_1 - \widetilde{u}_2\|_X + \|f_1 - f_2\|_{X^*}.$$

Apply the condition (2.7) to get, for a constant $C > 0$,

$$\|\widetilde{u}_1 - \widetilde{u}_2\|_X \leq C\,(\|u_1 - u_2\|_X + \|y_1 - y_2\|_Y + \|f_1 - f_2\|_{X^*}). \tag{3.5}$$

Inequality (3.5) combined with the continuity of functions $\mathcal{S}u, f$ and u imply that $t \mapsto \widetilde{u}(t) : \mathbb{R}_+ \to X$ is continuous. This concludes the existence proof. The uniqueness is a direct consequence of that of the element $\widetilde{u}_\lambda(t)$ which solves the variational–hemivariational inequality (3.2) for each $t \in \mathbb{R}_+$. □

Next, we explore the weak convergence of the sequence $\{\widetilde{u}_\lambda(t)\}$, for $t \in \mathbb{R}_+$ fixed.

Lemma 9. *For each $t \in \mathbb{R}_+$, there exists a subsequence of the sequence $\{\widetilde{u}_\lambda(t)\}$, again denoted by $\{\widetilde{u}_\lambda(t)\}$, which converges weakly to an element $\widetilde{u}(t) \in X$, i.e.,*

$$\widetilde{u}_\lambda(t) \to \widetilde{u}(t) \quad \text{weakly in } X, \ \text{as } \lambda \to 0. \tag{3.6}$$

Proof. We start by proving the boundedness of the sequence $\{u_\lambda(t)\} \subset X$. Choose an element $u_0 \in K$. Write

$$j^0(\widetilde{u}_\lambda(t); u_0 - \widetilde{u}_\lambda(t)) = \left[j^0(\widetilde{u}_\lambda(t); u_0 - \widetilde{u}_\lambda(t)) + j^0(u_0; \widetilde{u}_\lambda(t) - u_0)\right] - j^0(u_0; \widetilde{u}_\lambda(t) - u_0).$$

Applying (2.6) and [11, Proposition 3.23(iii)], we have

$$\begin{aligned} j^0(\widetilde{u}_\lambda(t); u_0 - \widetilde{u}_\lambda(t)) &\leq \alpha_j\|\widetilde{u}_\lambda(t) - u_0\|_X^2 + |\max\{\langle \zeta, \widetilde{u}_\lambda(t) - u_0\rangle \mid \zeta \in \partial j(u_0)\}| \\ &\leq \alpha_j\|\widetilde{u}_\lambda(t) - u_0\|_X^2 + (c_0 + c_1\|u_0\|_X)\|\widetilde{u}_\lambda(t) - u_0\|_X. \end{aligned} \tag{3.7}$$

Take $v = u_0 \in K$ in (3.2), use inequality (3.7), the strong monotonicity of the operator A and equality $Pu_0 = 0$ to obtain

$$\begin{aligned} &m_A\,\|\widetilde{u}_\lambda(t) - u_0\|_X^2 \leq \langle A\widetilde{u}_\lambda(t) - Au_0, \widetilde{u}_\lambda(t) - u_0\rangle \\ &\quad = \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - u_0\rangle - \langle Au_0, \widetilde{u}_\lambda(t) - u_0\rangle \\ &\quad \leq \frac{1}{\lambda}\langle P\widetilde{u}_\lambda(t), u_0 - \widetilde{u}_\lambda(t)\rangle + \varphi(\mathcal{S}u(t), u(t), u_0) - \varphi(\mathcal{S}u(t), u(t), \widetilde{u}_\lambda(t)) \\ &\qquad + j^0(\widetilde{u}_\lambda(t); u_0 - \widetilde{u}_\lambda(t)) + \langle f(t) - Au_0, \widetilde{u}_\lambda(t) - u_0\rangle \\ &\quad \leq -\frac{1}{\lambda}\langle Pu_0 - P\widetilde{u}_\lambda(t), u_0 - \widetilde{u}_\lambda(t)\rangle + \varphi(\mathcal{S}u(t), u(t), u_0) \\ &\qquad - \varphi(\mathcal{S}u(t), u(t), \widetilde{u}_\lambda(t)) + \alpha_j\|\widetilde{u}_\lambda(t) - u_0\|_X^2 \\ &\qquad + (c_0 + c_1\|u_0\|_X)\|\widetilde{u}_\lambda(t) - u_0\|_X + \|f(t) - Au_0\|_{X^*}\|\widetilde{u}_\lambda(t) - u_0\|_X. \end{aligned}$$

Therefore, using the monotonicity of the operator P and assumption (2.12), we have

$$(m_A - \alpha_j)\,\|\widetilde{u}_\lambda(t) - u_0\|_X \leq c_\varphi(\|\mathcal{S}u(t)\|_Y, \|u(t)\|_X) + c_0 + c_1\|u_0\|_X + \|f(t) - Au_0\|_{X^*}.$$

Thus,

$$\|\widetilde{u}_\lambda(t) - u_0\|_X \leq C(t), \tag{3.8}$$

where

$$C(t)=\frac{1}{m_A-\alpha_j}\left[c_\varphi(\|\mathcal{S}u(t)\|_Y,\|u(t)\|_X)+c_0+c_1\|u_0\|_X+\|f(t)-Au_0\|_{X^*}\right].$$

The inequality (3.8) shows the boundedness of $\{\widetilde{u}_\lambda(t)\}_\lambda$ in X. Since X is reflexive, there exists an element $\widetilde{u}(t)\in X$ such that, passing to a subsequence if necessary, the weak convergence (3.6) holds. □

Let us show that the weak limit in Lemma 9 coincides with the solution of Problem 1.

Lemma 10. *For each $t\in\mathbb{R}_+$,*

$$\widetilde{u}(t)=u(t). \tag{3.9}$$

Proof. We first prove that $\widetilde{u}(t)\in K$. Use (3.2), assumptions (2.3), (2.6) and (2.12) to see that, for all $v\in X$,

$$\begin{aligned}
\frac{1}{\lambda}\langle P\widetilde{u}_\lambda(t),\widetilde{u}_\lambda(t)-v\rangle &\le \langle A\widetilde{u}_\lambda(t),v-\widetilde{u}_\lambda(t)\rangle\\
&\quad+\varphi(\mathcal{S}u(t),u(t),v)-\varphi(\mathcal{S}u(t),u(t),\widetilde{u}_\lambda(t))\\
&\quad+j^0(\widetilde{u}_\lambda(t);v-\widetilde{u}_\lambda(t))+\langle f(t),\widetilde{u}_\lambda(t)-v\rangle\\
&\le -\langle A\widetilde{u}_\lambda(t)-Av,\widetilde{u}_\lambda(t)-v\rangle-\langle Av-f(t),\widetilde{u}_\lambda(t)-v\rangle\\
&\quad+\Big(c_\varphi(\|\mathcal{S}u(t)\|_Y,\|u(t)\|_X)+c_0+c_1\|\widetilde{u}_\lambda(t)\|_X\Big)\|\widetilde{u}_\lambda(t)-v\|_X\\
&\le\Big(\|Av-f(t)\|_{X^*}+c_\varphi(\|\mathcal{S}u(t)\|_Y,\|u(t)\|_X)+c_0+c_1\|\widetilde{u}_\lambda(t)\|_X\Big)\|\widetilde{u}_\lambda(t)-v\|_X.
\end{aligned}$$

Using the bound (3.8), we infer that

$$\frac{1}{\lambda}\langle P\widetilde{u}_\lambda(t),\widetilde{u}_\lambda(t)-v\rangle\le C(t,v)\quad\forall\, v\in X, \tag{3.10}$$

where the constant $C(t,v)>0$ is independent of λ. Choosing $v=\widetilde{u}(t)$ in (3.10), we have

$$\limsup_{\lambda\to 0}\,\langle P\widetilde{u}_\lambda(t),\widetilde{u}_\lambda(t)-\widetilde{u}(t)\rangle\le 0. \tag{3.11}$$

We now use (3.6), (3.11) and the pseudomonotonicity of the operator P to obtain

$$\langle P\widetilde{u}(t),\widetilde{u}(t)-v\rangle\le\liminf_{\lambda\to 0}\,\langle P\widetilde{u}_\lambda(t),\widetilde{u}_\lambda(t)-v\rangle\quad\forall\, v\in X. \tag{3.12}$$

The inequalities (3.10) and (3.12) together imply that

$$\langle P\widetilde{u}(t),\widetilde{u}(t)-v\rangle\le 0\quad\forall\, v\in X,$$

which leads to

$$\langle P\widetilde{u}(t),w\rangle=0\quad\forall\, w\in X.$$

Thus, $P\widetilde{u}(t)=0$, and by (2.9)(b), $\widetilde{u}(t)\in K$.

Let $v\in K$. Then, using (3.2) and (2.9)(b), we have

$$\begin{aligned}
\langle A\widetilde{u}_\lambda(t),\widetilde{u}_\lambda(t)-v\rangle\le-\frac{1}{\lambda}\langle Pv-P\widetilde{u}_\lambda(t),v-\widetilde{u}_\lambda(t)\rangle&\\
+\,\varphi((\mathcal{S}u)(t),u(t),v)-\varphi((\mathcal{S}u)(t),u(t),\widetilde{u}_\lambda(t))&\\
+\,j^0(\widetilde{u}_\lambda(t);v-\widetilde{u}_\lambda(t))+\langle f(t),\widetilde{u}_\lambda(t)-v\rangle.&
\end{aligned}$$

Therefore, the monotonicity of P yields

$$\begin{aligned}
\langle A\widetilde{u}_\lambda(t),\widetilde{u}_\lambda(t)-v\rangle\le\varphi((\mathcal{S}u)(t),u(t),v)-\varphi((\mathcal{S}u)(t),u(t),\widetilde{u}_\lambda(t))&\\
+\,j^0(\widetilde{u}_\lambda(t);v-\widetilde{u}_\lambda(t))+\langle f(t),\widetilde{u}_\lambda(t)-v\rangle.&
\end{aligned} \tag{3.13}$$

By the weak convergence (3.6) and the weak lower semicontinuity of φ, cf. (2.5)(a),

$$\limsup_{\lambda\to 0}(\varphi((\mathcal{S}u)(t),u(t),\widetilde{u}(t))-\varphi((\mathcal{S}u)(t),u(t),\widetilde{u}_\lambda(t)))\le 0. \tag{3.14}$$

From hypothesis (2.11) and (3.6),

$$\limsup_{\lambda\to 0}j^0(\widetilde{u}_\lambda(t);\widetilde{u}(t)-\widetilde{u}_\lambda(t))\le 0. \tag{3.15}$$

Take $v = \widetilde{u}(t) \in K$ in (3.13), then use the weak convergence (3.6), and inequalities (3.14) and (3.15) to obtain that

$$\limsup_{\lambda\to 0} \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - \widetilde{u}(t)\rangle \le 0.$$

From this inequality, (3.6) and the pseudomonotonicity of the operator A, we have

$$\langle A\widetilde{u}(t), \widetilde{u}(t) - v\rangle \le \liminf_{\lambda\to 0} \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - v\rangle \quad \forall\, v \in X. \tag{3.16}$$

We now pass to the upper limit in (3.13). Using (3.6), the weak lower semicontinuity of φ with respect to the third argument and (2.11), we obtain, for all $v \in K$,

$$\limsup_{\lambda\to 0} \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - v\rangle \le \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)) + j^0(\widetilde{u}(t); v - \widetilde{u}(t)) + \langle f(t), \widetilde{u}(t) - v\rangle. \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\langle A\widetilde{u}(t), \widetilde{u}(t) - v\rangle \le \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)) + j^0(\widetilde{u}(t); v - \widetilde{u}(t)) + \langle f(t), \widetilde{u}(t) - v\rangle \quad \forall\, v \in K.$$

Therefore,

$$\langle A\widetilde{u}(t), v - \widetilde{u}(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)) + j^0(\widetilde{u}(t); v - \widetilde{u}(t)) \ge \langle f(t), v - \widetilde{u}(t)\rangle \quad \forall\, v \in K. \tag{3.18}$$

Take $v = \widetilde{u}(t)$ in (2.2), $v = u(t)$ in (3.18), both being in K, add the corresponding inequalities and use assumptions (2.3)(b), (2.6)(c) to find that

$$(m_A - \alpha_j)\|\widetilde{u}(t) - u(t)\|_X \le 0.$$

Due to the assumption (2.7), we deduce from the above inequality that $\widetilde{u}(t) - u(t) = 0$, i.e., (3.9) holds. □

Since the weak limit in (3.6) is independent of the subsequence in the proof of Lemma 9, the entire family $\{\widetilde{u}_\lambda(t)\}$ converges weakly to the same limit.

Lemma 11. *For each* $t \in \mathbb{R}_+$,

$$\widetilde{u}_\lambda(t) \to u(t) \text{ weakly in } X, \text{ as } \lambda \to 0.$$

We proceed to prove the strong convergence.

Lemma 12. *For each* $t \in \mathbb{R}_+$,

$$\|\widetilde{u}_\lambda(t) - u(t)\|_X \to 0 \quad \text{as } \lambda \to 0. \tag{3.19}$$

Proof. We take $v = \widetilde{u}(t) \in K$ in both (3.16) and (3.17) to obtain

$$0 \le \liminf_{\lambda\to 0} \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - \widetilde{u}(t)\rangle \text{ and } \limsup_{\lambda\to 0} \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - \widetilde{u}(t)\rangle \le 0,$$

respectively. These inequalities combined with (3.9) imply that

$$\langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - u(t)\rangle \to 0 \quad \text{as } \lambda \to 0. \tag{3.20}$$

By the weak convergence of the sequence $\{\widetilde{u}_\lambda(t)\}$ to $u(t)$ from Lemma 11,

$$\langle Au(t), \widetilde{u}_\lambda(t) - u(t)\rangle \to 0 \quad \text{as } \lambda \to 0. \tag{3.21}$$

Using the strong monotonicity of A, (3.20) and (3.21), we have

$$m_A\|\widetilde{u}_\lambda(t) - u(t)\|_X^2 \le \langle A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - u(t)\rangle - \langle Au(t), \widetilde{u}_\lambda(t) - u(t)\rangle \to 0. \tag{3.22}$$

Hence, the convergence (3.19) follows. □

In the last step, we prove the strong convergence (2.13).

Lemma 13. *For each* $t \in \mathbb{R}_+$,

$$\|u_\lambda(t) - u(t)\|_X \to 0 \quad \text{as } \lambda \to 0. \tag{3.23}$$

Proof. Choose $n \in \mathbb{N}$ large enough so that $t \in [0, n]$. We take $v = u_\lambda(t)$ in (3.2) and $v = \widetilde{u}_\lambda(t)$ in (2.10), and add the two resulting inequalities,

$$\begin{aligned}
&\langle Au_\lambda(t) - A\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - u_\lambda(t)\rangle + \frac{1}{\lambda}\langle Pu_\lambda(t) - P\widetilde{u}_\lambda(t), \widetilde{u}_\lambda(t) - u_\lambda(t)\rangle\\
&\quad + \varphi((\mathcal{S}u_\lambda)(t), u_\lambda(t), \widetilde{u}_\lambda(t)) - \varphi((\mathcal{S}u_\lambda)(t), u_\lambda(t), u_\lambda(t))\\
&\quad + \varphi((\mathcal{S}u)(t), u(t), u_\lambda(t)) - \varphi((\mathcal{S}u)(t), u, \widetilde{u}_\lambda(t))\\
&\quad + j^0(u_\lambda(t), \widetilde{u}_\lambda(t) - u_\lambda(t)) + j^0(\widetilde{u}_\lambda(t), u_\lambda(t) - \widetilde{u}_\lambda(t)) \geq 0.
\end{aligned}$$

We use assumptions (2.3)(b), (2.5)(b), (2.6)(c) and the monotonicity of the operator P, (2.9)(a), to obtain that

$$\begin{aligned}
m_A\,\|\widetilde{u}_\lambda(t) - u_\lambda(t)\|_X^2 \leq\ & \alpha_\varphi\|u_\lambda(t) - u(t)\|_X\|\widetilde{u}_\lambda(t) - u_\lambda\|_X\\
& + \beta_\varphi\|(\mathcal{S}u_\lambda)(t) - (\mathcal{S}u)(t)\|_Y\|\widetilde{u}_\lambda(t) - u_\lambda(t)\|_X\\
& + \alpha_j\|\widetilde{u}_\lambda(t) - u_\lambda(t)\|_X^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\widetilde{u}_\lambda(t) - u_\lambda(t)\|_X \leq\ & \frac{\alpha_\varphi}{m_A - \alpha_j}\,\|u_\lambda(t) - u(t)\|_X\\
& + \frac{\beta_\varphi}{m_A - \alpha_j}\,\|(\mathcal{S}u_\lambda)(t) - (\mathcal{S}u)(t)\|_Y.
\end{aligned} \tag{3.24}$$

Next, we write

$$\|u_\lambda(t) - u(t)\|_X \leq \|u_\lambda(t) - \widetilde{u}_\lambda(t)\|_X + \|\widetilde{u}_\lambda(t) - u(t)\|_X,$$

and then by (3.24),

$$\begin{aligned}
\|u_\lambda(t) - u(t)\|_X \leq\ & \|\widetilde{u}_\lambda(t) - u(t)\|_X + \frac{\alpha_\varphi}{m_A - \alpha_j}\,\|u_\lambda(t) - u(t)\|_X\\
& + \frac{\beta_\varphi}{m_A - \alpha_j}\,\|(\mathcal{S}u_\lambda)(t) - (\mathcal{S}u)(t)\|_Y.
\end{aligned} \tag{3.25}$$

Use assumptions (2.7) and (2.4) to deduce that the existence of two positive constants c and s_n, independent of λ and t, such that

$$\|u_\lambda(t) - u(t)\|_X \leq c\,\|\widetilde{u}_\lambda(t) - u(t)\|_X + s_n\int_0^t \|u_\lambda(s) - u(s)\|_X\,ds.$$

Apply the Gronwall inequality,

$$\|u_\lambda(t) - u(t)\|_X \leq c\,\|\widetilde{u}_\lambda(t) - u(t)\|_X + s_n c\int_0^t e^{s_n(t-s)}\|\widetilde{u}_\lambda(s) - u(s)\|_X\,ds. \tag{3.26}$$

Note that $e^{s_n(t-s)} \leq e^{s_n t} \leq e^{n s_n}$ for all $s \in [0, t]$; thus, (3.26) yields

$$\|u_\lambda(t) - u(t)\|_X \leq c\,\|\widetilde{u}_\lambda(t) - u(t)\|_X + s_n c e^{n s_n}\int_0^t \|\widetilde{u}_\lambda(s) - u(s)\|_X\,ds. \tag{3.27}$$

By the boundedness (3.8), Lemma 12 and Lebesgue's convergence theorem, we have

$$\int_0^t \|\widetilde{u}_\lambda(s) - u(s)\|_X\,ds \to 0 \quad \text{as } \lambda \to 0. \tag{3.28}$$

The convergence (3.23) then follows from (3.19), (3.27) and (3.28). □

4. Two contact models for rods

The abstract results presented in Section 2 can be applied in the study of various mathematical models which describe the contact between a deformable body and a foundation. In this section, we consider two quasistatic problems describing the contact of a rod with a foundation. The rod occupies the interval $[0, L]$ on the Ox axis, is fixed at one end $x = 0$, is acted by body forces of density f_0, and its other end $x = L$ is in contact with a reactive obstacle.

We denote by $u : [0, L] \times \mathbb{R}_+ \to \mathbb{R}$ the displacement field and by $\sigma : [0, L] \times \mathbb{R}_+ \to \mathbb{R}$ the stress field. These are functions of $x \in [0, L]$ and $t \in \mathbb{R}_+$. We use a prime to denote the derivative with respect to x; thus, u' represents the linearized strain field. We model the material's behavior with a viscoelastic constitutive law with long memory, i.e.,

$$\sigma(x, t) = \mathcal{F}(x)\,u'(x, t) + \int_0^t \mathcal{R}(t - s)u'(x, s)\,ds \quad \text{for } (x, t) \in (0, L) \times \mathbb{R}_+. \tag{4.1}$$

Here $\mathcal{F}$ and $\mathcal{R}$ represent given constitutive functions, which describe the elasticity and the relaxation properties of the material. We neglect the inertial term in the equation of motion. Therefore, we have the equilibrium equation

$$\sigma'(x,t)+f_0(x,t)=0 \quad \text{for } (x,t)\in(0,L)\times\mathbb{R}_+. \tag{4.2}$$

Since the rod is fixed at $x=0$,

$$u(0,t)=0 \quad \text{for } t\in\mathbb{R}_+. \tag{4.3}$$

In the first model we consider the following boundary condition at $x=L$:

$$\left.\begin{array}{l} u(L,t)\le g,\\ \sigma(x,t)+k(u(L,t))=0 \quad \text{if} \quad u(L,t)<g\\ \sigma(x,t)+k(u(L,t))\le 0 \quad \text{if} \quad u(L,t)=g \end{array}\right\} \quad \text{for } t\in\mathbb{R}_+. \tag{4.4}$$

The condition (4.4) represents a contact condition with normal compliance and unilateral constraint. It models the contact with a rigid foundation covered by a layer of elastic material, see for instance [9]. Here k is a given positive function vanishing for a negative argument and $g>0$ represents the thickness of the elastic layer. Let $t\in\mathbb{R}_+$ be given. Then, a detailed description of this condition is the following.

(a) First, note that the unilateral constraint $u(L,t)\le g$ represents a bound for the displacement at $x=L$, which is imposed since the rigid body does not allow penetration.

(b) Second, when $u(L,t)<0$ we have $\sigma(L,t)=0$. This shows that when there is separation between the rod and the foundation then there is no reaction on the point $x=L$.

(c) Next, when $0<u(L,t)<g$ we have $-\sigma(L,t)=k(u(L,t))$. In this case there is partial penetration into the elastic layer; moreover, the reaction of the foundation is towards the rod and depends on the penetration.

(d) Finally, when $u(L,t)=g$ we have $-\sigma(L)\ge k(u(t,L))$. This situation arises when the elastic layer is completely penetrated and, therefore, the tip $x=L$ reached the rigid body. In this case the magnitude of the reaction at this point is larger than $k(u(L,t))$, since at the reaction of the elastic layer we add the reaction of the rigid body, which now is active.

We gather the above equations and condition to obtain the following model of contact.

Problem 14. Find a displacement field $u:[0,L]\times\mathbb{R}_+\to\mathbb{R}$ and a stress field $\sigma:[0,L]\times\mathbb{R}_+\to\mathbb{R}$ such that (4.1)–(4.4) hold.

In the second model we assume that the stress in $x=L$ satisfies the condition

$$-\sigma(x,t)=k(u(L,t))+\frac{1}{\lambda}\,p(u(L,t)-g) \quad \text{for } t\in\mathbb{R}_+. \tag{4.5}$$

This is a normal compliance condition modeling the contact with an elastic foundation, as explained in [5,25]. Here p is a given positive function vanishing for a negative argument. The penalty parameter $\lambda>0$ can be interpreted as a deformability coefficient of the foundation [5,25]. Note that in this condition the displacement is not restricted by the bound g. Nevertheless, the penetration is penalized by the second term in (4.5).

The second contact model is based on the use of the contact condition (4.5).

Problem 15. Find a displacement field $u:[0,L]\times\mathbb{R}_+\to\mathbb{R}$ and a stress field $\sigma:[0,L]\times\mathbb{R}_+\to\mathbb{R}$ such that (4.1)–(4.3) and (4.5) hold.

Note that the solution of Problem 15 depends on λ and, therefore, we shall denote it below by $(u_\lambda,\sigma_\lambda)$. In the study of Problems 14 and 15 we use the space

$$V=\{\,v\in H^1(0,L)\mid v(0)=0\,\}.$$

It is a real Hilbert space with the inner product

$$(u,v)_V=\int_0^L u'\,v'\,dx, \quad u,v\in V.$$

The Sobolev trace theorem states that there exists a positive constant k_0 such that

$$|v(L)|\le k_0\|v\|_V \quad \forall\, v\in V. \tag{4.6}$$

We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V.

We assume that the elasticity operator $\mathcal{F}$, the relaxation function $\mathcal{R}$ and the normal compliance functions k, p satisfy the following conditions.

$$\begin{cases} \text{(a) } \mathcal{F} : (0, L) \times \mathbb{R} \to \mathbb{R}. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \qquad |\mathcal{F}(x, \varepsilon_1) - \mathcal{F}(x, \varepsilon_2)| \le L_{\mathcal{F}} |\varepsilon_1 - \varepsilon_2| \\ \qquad\qquad \forall\, \varepsilon_1,\ \varepsilon_2 \in \mathbb{R},\ \text{a.e. } x \in (0, L). \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \qquad (\mathcal{F}(x, \varepsilon_1) - \mathcal{F}(x, \varepsilon_2))(\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{F}} |\varepsilon_1 - \varepsilon_2|^2 \\ \qquad\qquad \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{R},\ \text{a.e. } x \in (0, L). \\ \text{(d) The mapping } x \mapsto \mathcal{F}(x, \varepsilon) \text{ is measurable on } (0, L), \\ \qquad\qquad \text{for any } \varepsilon \in \mathbb{R}. \\ \text{(e) The mapping } x \mapsto \mathcal{F}(x, 0) \text{ belongs to } L^2(0, L). \end{cases} \tag{4.7}$$

$$\mathcal{R} : \mathbb{R}_+ \to \mathbb{R} \quad \text{is continuous.} \tag{4.8}$$

$$\begin{cases} \text{(a) } k : \mathbb{R} \to \mathbb{R}. \\ \text{(b) } k \text{ is continuous.} \\ \text{(c) There exists } c_0 > 0,\ c_1 > 0 \text{ such that} \\ \qquad |k(r)| \le c_0 + c_1 |r| \ \ \forall\, r \in \mathbb{R}. \\ \text{(d) There exists } \alpha_k > 0 \text{ such that} \\ \qquad r \mapsto \alpha_k r + k(r) \text{ is nondecreasing.} \\ \text{(e) } k(r) \ge 0 \ \text{ if } \ r > 0 \ \text{ and } \ k(r) = 0 \ \text{ if } \ r \le 0. \end{cases} \tag{4.9}$$

$$\begin{cases} \text{(a) } p : \mathbb{R} \to \mathbb{R}. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \ \ \forall\, r_1,\ r_2 \in \mathbb{R}. \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \ \ \forall\, r_1,\ r_2 \in \mathbb{R}. \\ \text{(d) } p(r) \ge 0 \ \text{ if and only if } \ r \le 0. \end{cases} \tag{4.10}$$

For the density of body forces, we assume

$$f_0 \in L^2(0, L). \tag{4.11}$$

Finally, we assume the smallness condition

$$\alpha_k < \frac{m_{\mathcal{F}}}{k_0^2}, \tag{4.12}$$

where k_0, $m_{\mathcal{F}}$ and α_k are the constants in (4.6), (4.7) and (4.9), respectively.

Introduce the set of admissible displacement fields

$$U = \{\, v \in V \mid v(L) \le g \,\}. \tag{4.13}$$

Following a standard approach, we can derive the following weak formulation for Problem 14.

Problem 16. Find a displacement field $u : \mathbb{R}_+ \to U$ such that, for all $t \in \mathbb{R}_+$,

$$\int_0^L \mathcal{F}u'(t)(v' - u'(t))\, dx + \int_0^L \Big(\int_0^t \mathcal{R}(t-s)u'(s)\, ds\Big)(v' - u'(t))\, dx$$
$$+ k(u(L))(v(L) - u(L)) \ge \int_0^T f(t)(v - u(t))\, dx \quad \forall\, v \in U.$$

Here and below, for simplicity, we do not indicate the dependence of various functions on the spatial variable x. The weak formulation for Problem 15 is the following.

Problem 17. Find a displacement field $u_\lambda : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$,

$$\int_0^L \mathcal{F}u_\lambda'(t)(v' - u_\lambda'(t))\,dx + \int_0^L \Big(\int_0^t \mathcal{R}(t-s)u_\lambda'(s)\,ds\Big)(v' - u_\lambda'(t))\,dx$$
$$+ k(u_\lambda(L))(v(L) - u_\lambda(L)) + \frac{1}{\lambda}\,p(u_\lambda(L))(v(L) - u_\lambda(L))$$
$$= \int_0^T f_0(t)(v - u_\lambda(t))\,dx \quad \forall\, v \in V.$$

In the study of these problems, we have the following existence, uniqueness and convergence result; the proof of the result is given in the next section.

Theorem 18. *Assume (4.7)–(4.11). Then the following statements hold.*

(i) *There exists a unique solution $u \in C(\mathbb{R}_+; U)$ to Problem 16.*

(ii) *For each $\lambda > 0$, there exists a unique solution $u_\lambda \in C(\mathbb{R}_+; V)$ to Problem 17.*

(iii) *The solution u_λ of Problem 17 converges to the solution u of Problem 16, that is, for all $t \in \mathbb{R}_+$,*

$$\|u_\lambda(t) - u(t)\|_V \to 0 \quad \text{as } \lambda \to 0. \tag{4.14}$$

Theorem 18(i), (ii) provides the unique weak solvability of Problems 14 and 15, respectively, in terms of displacement. Once the displacement field is obtained, the corresponding stress field is uniquely determined by using the constitutive law (4.1). The convergence result in Theorem 18(iii) is important from the mechanical point of view, since it shows that the weak solution of the viscoelastic contact problem with a deformable foundation approaches, as closely as one wishes, the solution of an viscoelastic contact problem with a rigid–deformable foundation, with a sufficiently small deformability coefficient.

5. Proof of Theorem 18

The proof of Theorem 18 will be done in several steps, based on the abstract existence, uniqueness and convergence results provided by Theorems 2 and 5. Throughout this section, we assume (4.7)–(4.12). Introduce the operators $A : V \to V^*$, $P : V \to V^*$, $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(0,L))$ and the function $f : \mathbb{R}_+ \to V^*$ defined by

$$\langle Au, v\rangle = \int_0^L \mathcal{F}(u')v'\,dx, \quad u, v \in V, \tag{5.1}$$

$$\langle Pu, v\rangle = p(u(L) - g)v(L), \quad u, v \in V, \tag{5.2}$$

$$(\mathcal{S}u)(t) = \int_0^t \mathcal{R}(t-s)u'(s)\,ds, \quad u \in C(\mathbb{R}_+; V), \tag{5.3}$$

$$\langle f(t), v\rangle = \int_0^L f_0(t)v\,dx, \quad v \in V,\ t \in \mathbb{R}_+. \tag{5.4}$$

Denote by $q : \mathbb{R} \to \mathbb{R}$ and $j : V \to \mathbb{R}$ the functions defined by

$$q(r) = \int_0^r k(s)\,ds, \quad r \in \mathbb{R}, \tag{5.5}$$

$$j(v) = q(v(L)), \quad v \in V. \tag{5.6}$$

Consider the following problems.

Problem 19. Find a function $u \in C(\mathbb{R}_+; U)$ such that, for all $t \in \mathbb{R}_+$,

$$\langle Au(t), v - u(t)\rangle + ((\mathcal{S}u)(t), v' - u'(t))_{L^2(0,L)}$$
$$+ j^0(u(t); v - u(t)) \geq \langle f(t), v - u(t)\rangle \quad \forall\, v \in U. \tag{5.7}$$

Problem 20. Find a function $u_\lambda \in C(\mathbb{R}_+; V)$ such that, for all $t \in \mathbb{R}_+$,

$$\langle Au_\lambda(t), v - u(t)\rangle + \frac{1}{\lambda}\langle Pu_\lambda(t), v - u_\lambda(t)\rangle + ((\mathcal{S}u_\lambda)(t), v' - u_\lambda'(t))_{L^2(0,L)}$$
$$+ j^0(u_\lambda(t); v - u_\lambda(t)) = \langle f(t), v - u_\lambda(t)\rangle \quad \forall\, v \in V. \tag{5.8}$$

We have the following equivalence result.

Lemma 21. *Let $u : \mathbb{R}_+ \to U$ and $u_\lambda : \mathbb{R}_+ \to V$. Then:*

(1) *The function u is a solution to Problem 16 with regularity $u \in C(\mathbb{R}_+; U)$ if and only if u is a solution to Problem 19.*

(2) *The function u_λ is a solution to Problem 17 with regularity $u_\lambda \in C(\mathbb{R}_+; V)$ if and only if u_λ is a solution to Problem 20.*

Proof. The function q defined by (5.5) is a regular function in the sense of Clarke; moreover, the generalized directional derivative of q is

$$q^0(s; r) = k(s)\,r, \quad r, s \in \mathbb{R}.$$

A simple computation based on (5.6) shows that

$$j^0(u; v) = q^0(u(L); v(L)), \quad u, v \in V.$$

It follows from above that

$$j^0(u; v) = k(u(L))v(L) \quad \forall\, u, v \in V. \tag{5.9}$$

Lemma 21 is now is a direct consequence of notation (5.1)–(5.4) combined with equality (5.9). □

The next result concerns the operator P.

Lemma 22. *The operator (5.2) is a penalty operator of the set U, i.e., it satisfies condition (2.9) with $X = V$ and $K = U$.*

Proof. Let $u, w \in V$ and $v \in U$. From (5.2), (4.10) and (4.6) it is easy to see that

$$|\langle Pu - Pv, w\rangle| \le k_0^2 L_p \|u - v\|_V \|w\|_V,$$
$$\langle Pu - Pv, u - v\rangle \ge 0.$$

These inequalities show that P is a Lipschitz continuous and monotone operator; therefore, (2.9)(a) holds.

Assume $Pu = 0$. Then, $\langle Pu, u\rangle_V = 0$ which implies that

$$p(u(L) - g)u(L) = 0. \tag{5.10}$$

This equality combined with assumption (4.10) implies that $u(L) \le g$, i.e., $u \in U$. Conversely, if $u \in U$ it follows that $u(L) \le g$. Using assumption (4.10)(d), we deduce that $p(u(L) - g) = 0$. From the definition (5.2) of the operator P, we deduce that $\langle Pu, v\rangle = 0$ for all $v \in V$, which implies that $Pu = 0$. It follows from above that P satisfies the condition (2.9)(b) which concludes the proof. □

Let us now complete the proof of Theorem 18.

Proof. We apply Theorems 2 and 5 with $X = V$, $K = U$ and $Y = L^2(0, L)$. Observe that the set (4.13) is nonempty, closed and convex in V. We use properties (4.7) of the constitutive function $\mathcal{F}$ to see that the operator A given by (5.1) satisfies the inequalities

$$\langle Au - Av, w\rangle \le L_{\mathcal{F}} \|u - v\|_V \|w\|_V,$$
$$\langle Au - Av, u - v\rangle \ge m_{\mathcal{F}} \|u - v\|_V^2$$

for any $u, v, w \in V$. Hence, A satisfies condition (2.3) with constant $m_A = m_{\mathcal{F}}$. From the definition (5.3) and assumption (4.8), we have

$$\|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_{L^2(0,L)} \le \max_{r \in [0,n]} |\mathcal{R}(r)| \int_0^t \|u_1(s) - u_2(s)\|_V \, ds$$

for all $n \in \mathbb{N}$, $t \in [0, n]$ and $u_1, u_2 \in C(\mathbb{R}_+; V)$. This inequality shows that the operator $\mathcal{S}$ satisfies condition (2.4) with $s_n = \max_{r \in [0,n]} |\mathcal{R}(r)|$.

Define the functional $\varphi : L^2(0, L) \times V \times V \to \mathbb{R}$ by

$$\varphi(y, u, v) = (y, v')_{L^2(0,L)} \quad \forall\, y \in L^2(0, L),\ u, v \in V.$$

Then, it is easy to see that φ satisfies condition (2.5) and (2.12) with $\alpha_\varphi = 0$, $\beta_\varphi = 1$ and $c_\varphi(r_1, r_2) = r_1$, for all $r_1, r_2 \in \mathbb{R}_+$. Moreover, note that

$$\varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t)) = ((\mathcal{S}u)(t), v' - u')_{L^2(0,L)} \quad \forall\, u \in C(\mathbb{R}_+, V),\ v \in V,\ t \in \mathbb{R}_+. \tag{5.11}$$

By standard arguments on subdifferential calculus (cf. [11, Theorem 3.47]) and (4.9)(c) it follows that the function j defined by (5.6) satisfies conditions (2.6)(a) and (b). In addition, using (4.9)(d) we have

$$\begin{aligned} j^0(u; v - u) + j^0(v; u - v) &= \big(k(u(L)) - k(v(L))\big)(u(L) - v(L)) \\ &\le \alpha_k |u(L) - v(L)|^2 \end{aligned}$$

for all $u, v \in V$. Therefore, using the trace inequality (4.6) we see that j satisfies condition (2.6)(c) with $\alpha_j = \alpha_k k_0^2$. Since $\alpha_\varphi = 0$, we conclude from (4.12) that the smallness condition (2.7) holds, too.

We can use a compactness argument to see that if $u_n \to u$ weakly in V, then $u_n(L) \to u(L)$; therefore, equality (5.9) implies that

$$j^0(u_n; v - u_n) \to j^0(u; v - u) \quad \forall\, v \in V.$$

It follows that condition (2.11) is satisfied. Note also that assumption (4.11) and definition (5.4) imply that $f \in C(\mathbb{R}_+; V^*)$. Finally, we recall Lemma 22 which shows that the operator P satisfies condition (2.9) with $X = V$ and $K = U$.

We conclude from the above that the assumptions of Theorems 2 and 5 are satisfied. Therefore, with the equality (5.11), it follows from Theorem 2 that there exists a unique solution $u \in C(\mathbb{R}_+; U)$ to Problem 19. Moreover, it follows from Theorem 2 that, for each $\lambda > 0$, there exists a unique solution $u_\lambda \in C(\mathbb{R}_+; U)$ to Problem 20 and, in addition, the convergence (4.14) holds. Theorem 18 is now a consequence of Lemma 21. □

References

[1] C. Baiocchi, A. Capelo, Variational and Quasivariational Inequalities: Applications To Free-Boundary Problems, John Wiley, Chichester, 1984.
[2] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, New York, 1984.
[3] R. Glowinski, J.-L. Lions, R. Trémolières, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.
[4] W. Han, B.D. Reddy, Plasticity: Mathematical Theory and Numerical Analysis, second ed., Springer-Verlag, New York, 2013.
[5] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, in: Studies in Advanced Mathematics, vol. 30, Americal Mathematical Society, RI–International Press, Providence, Somerville, MA, 2002.
[6] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, 1988.
[7] D. Kinderlehrer, G. Stampacchia, An Introduction To Variational Inequalities and their Applications, in: Classics in Applied Mathematics, vol. 31, SIAM, Philadelphia, 2000.
[8] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications, Birkhäuser, Boston, 1985.
[9] M. Sofonea, A. Matei, Mathematical Models in Contact Mechanics, in: London Mathematical Society Lecture Note Series, vol. 398, Cambridge University Press, Cambridge, 2012.
[10] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities, in: Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.
[11] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, in: Advances in Mechanics and Mathematics, vol. 26, Springer, New York, 2013.
[12] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.
[13] P.D. Panagiotopoulos, Hemivariational Inequalities, in: Applications in Mechanics and Engineering, Springer-Verlag, Berlin, 1993.
[14] M. Sofonea, A. Matei, History-dependent quasivariational inequalities arising in Contact Mechanics, European J. Appl. Math. 22 (2011) 471–491.
[15] M. Sofonea, Y. Xiao, Fully history-dependent quasivariational inequalities in contact mechanics, Appl. Anal. 95 (2016) 2464–2484.
[16] M. Sofonea, F. Pătrulescu, Penalization of history-dependent variational inequalities, European J. Appl. Math. 25 (2014) 155–176.
[17] K. Kamran, M. Barboteu, W. Han, M. Sofonea, Numerical analysis of history-dependent quasivariational inequalities with applications in contact mechanics, Math. Model. Numer. Anal. M2AN 48 (2014) 919–942.
[18] S. Migórski, A. Ochal, M. Sofonea, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, Nonlinear Anal. RWA 12 (2011) 3384–3396.
[19] S. Migórski, A. Ochal, M. Sofonea, History-dependent variational-hemivariational inequalities in contact mechanics, Nonlinear Anal. RWA 22 (2015) 604–618.
[20] M. Sofonea, W. Han, S. Migórski, Numerical analysis of history-dependent variational-hemivariational inequalities with applications to contact problems, European J. Appl. Math. 26 (2015) 427–452.
[21] M. Sofonea, S. Migórski, A class of history-dependent variational-hemivariational inequalities, Nonlinear Differential Equations Appl. 23 (2016) 38. http://dx.doi.org/10.1007/s00030-016-0391-0.
[22] S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, J. Elasticity 127 (2017) 151–178.
[23] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction To Nonlinear Analysis: Applications, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[24] E. Zeidler, Nonlinear Functional Analysis and Applications, Vol. II/B, Springer, New York, 1990.
[25] M. Shillor, M. Sofonea, J.J. Telega, Models and Analysis of Quasistatic Contact, in: Lect. Notes Phys., vol. 655, Springer, Berlin, Heidelberg, 2004.