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A Pressure Projection Stabilized Mixed Finite Element Method for a Stokes Hemivariational Inequality

Min Ling[1] · **Weimin Han**[2] · **Shengda Zeng**[3]

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Abstract
This paper is devoted to the development and analysis of a pressure projection stabilized mixed finite element method, with continuous piecewise linear approximations of velocities and pressures, for solving a hemivariational inequality of the stationary Stokes equations with a nonlinear non-monotone slip boundary condition. We present an existence result for an abstract mixed hemivariational inequality and apply it for a unique solvability analysis of the numerical method for the Stokes hemivariational inequality. An optimal order error estimate is derived for the numerical solution under appropriate solution regularity assumptions. Numerical results are presented to illustrate the theoretical prediction of the convergence order.

Keywords Stokes hemivariational inequality · Stabilized mixed finite element method · Error estimates

1 Introduction

Hemivariational inequalities (HVIs) provide a useful framework to both theoretically and numerically treat many application problems in physical sciences and engineering that involve non-monotone, non-smooth and multi-valued constitutive laws among different physical quantities. Comprehensive references on modeling and mathematical analysis of HVIs include [10, 30, 31, 33, 36]. Several numerical methods have been applied to solve HVIs.

✉ Min Ling
lingmin@stu.xjtu.edu.cn

Weimin Han
weimin-han@uiowa.edu

Shengda Zeng
zengshengda@163.com

1 School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China

2 Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

3 Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China

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An early comprehensive reference on the finite element method for HVIs is [23], where convergence of the numerical solutions is shown, but no error estimate is derived. Recently, optimal order error estimates are derived for the linear finite element solutions of a variety of HVIs, e.g., [16, 18, 20, 21] for elliptic HVIs, [2, 22] for hyperbolic and parabolic HVIs, [40] for history-dependent HVIs, [13, 17] for Stokes and Navier-Stokes HVIs, cf. [19] for a summary account. In [14], the lowest order virtual element method is analyzed for an elliptic hemivariational inequality (HVI) in contact mechanics without constraint, and an optimal order error estimate is derived. The virtual element method of arbitrary order for elliptic HVIs in contact mechanics with or without constraint is studied later in [39], where an optimal order error estimate is shown with the linear virtual element solutions. The nonconforming virtual element method of an arbitrary order is constructed in [29] for solving a stationary Stokes HVI with a nonlinear slip boundary condition, and an optimal order error estimate is derived for the lowest-order element solutions. In [38], the interior penalty discontinuous Galerkin method is studied for an elliptic HVI arising in semipermeable media, and optimal convergence order is proved for the linear element solutions.

The key to the success of mixed finite element methods is the validity of the discrete inf-sup condition of the pair of finite element spaces that ensures the existence and the stability of the solution. It is not straightforward to construct such pairs of spaces. For example, for the Stokes equations and the Navier-Stokes equations, the seemingly natural choices of the $\boldsymbol{P}_1$-P_1 element pair with continuous discrete pressures and the $\boldsymbol{P}_1$-P_0 element pair with discontinuous discrete pressures are not stable. In the literature, several stabilization techniques have been introduced in order to comply with or circumvent the discrete inf-sup condition. For instance, local and global stabilized methods [24], which are based on the macroelement condition for constructing the locally stabilized formulation. Consistent stabilized methods [4, 8], which use the residual of the momentum equation in the added terms. Projection based stabilization methods including the local pressure projection stabilized method for the lowest equal order elements [7, 25–28], and local pressure gradient projection stabilized method for the quadratic equal order elements [5, 6].

Low order velocity-pressure pairs are attractive for mixed finite element simulations of incompressible flow problems [35]. This paper is on a pressure projection stabilized mixed method of the lowest equal order pair to solve a hemivariational inequality of the stationary Stokes equations with a nonlinear non-monotone slip boundary condition. In general, optimal order error estimate cannot be derived for high-order methods due to the presence of an approximation error bound term on the slip boundary. In [13], the mini element and the $\boldsymbol{P}_2$-P_1 element are both analyzed for a stationary Stokes HVI with a nonlinear slip boundary condition, and an optimal order error estimate is derived for the mini finite element solution. The nonconforming virtual element method of an arbitrary order for the same problem is analyzed later in [29], where an optimal order error estimate is derived for the lowest-order element solution. Compared with the $\boldsymbol{P}_1$-P_0 element pair with discontinuous discrete pressure, the $\boldsymbol{P}_1$-P_1 element pair with continuous discrete pressure admits fewer degrees of freedom and higher numerical convergence order for the pressure. Moreover, $\boldsymbol{P}_1$-P_1 element pair is easy to implement as the velocity and the pressure share the same function spaces. Inspired by [7], in this paper, we present the local pressure projection stabilized mixed method using continuous piecewise linear approximations of the velocity and the pressure for the stationary Stokes HVI. The method is featured by the use of an abstract operator projecting the P_1 finite element space for the pressure to the P_0 finite element space. This method is unconditionally stable and is easy to implement without extensive recoding. Moreover, we present an error estimate for the velocity and pressure, achieving the optimal order for the lowest equal order elements.

The rest of the paper is organized as follows. In Sect. 2, we review the notions of the generalized directional derivative and the subdifferential in the sense of Clarke, as well as some of their basic properties. We introduce an abstract mixed hemivariational inequality and explore its solution existence. In Sect. 3, we analyze the local pressure projection stabilized mixed finite element method for the Stokes HVI. We show the existence and uniqueness of a discrete solution and derive a priori error estimate, which is optimal with lowest equal order element solution. In Sect. 4, we report computer simulation results on numerical examples.

2 An Abstract Mixed HVI

In this section, we present a solution existence result for an abstract mixed hemivariational inequality.

We first recall the definitions of the generalized directional derivative and generalized subdifferential in the sense of Clarke for a locally Lipschitz function.

Definition 2.1 Let V be a Banach space and denote by V^* its dual. Let $\psi : V \to \mathbb{R}$ be a locally Lipschitz functional. The generalized (Clarke) directional derivative of ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\psi^0(u; v) = \limsup_{w \to u,\, \lambda \downarrow 0} \frac{\psi(w + \lambda v) - \psi(w)}{\lambda}.$$

The generalized gradient (subdifferential) of ψ at u is defined by

$$\partial\psi(u) = \{\zeta \in V^* : \psi^0(u; v) \geq \langle \zeta, v\rangle \ \forall\, v \in V\}.$$

The following result provides some basic properties of the generalized directional derivative and the generalized subdifferential ([12,Propositions 2.1.1, 2.1.2]).

Proposition 2.2 *Assume that $\psi : V \to \mathbb{R}$ is a locally Lipschitz function. Then the following statements are valid.*

(i) *For every $u \in V$, the function $\psi^0(u; \cdot) : V \to \mathbb{R}$ is positively homogeneous, i.e.*

$$\psi^0(u; \lambda v) = \lambda \psi^0(u; v) \quad \forall\, \lambda \geq 0,\ v \in V, \tag{2.1}$$

and subadditive, i.e.

$$\psi^0(u; v_1 + v_2) \leq \psi^0(u; v_1) + \psi^0(u; v_2) \quad \forall\, v_1, v_2 \in V. \tag{2.2}$$

(ii) *For any $u, v \in V$, we have*

$$\psi^0(u; v) = \max\{\langle \xi, v\rangle : \xi \in \partial\psi(u)\}. \tag{2.3}$$

(iii) *The function $\psi^0 : V \times V \to \mathbb{R}$ is upper semi-continuous, i.e., for all $u, v \in V$, $\{u_n\}, \{v_n\} \subset V$ such that $u_n \to u$ and $v_n \to v$ in V, we have*

$$\limsup_{n \to \infty} \psi^0(u_n; v_n) \leq \psi^0(u; v). \tag{2.4}$$

In the rest of this section, we consider an abstract mixed hemivariational inequality. Let V and Q be Hilbert spaces.

Problem 2.3 *Find* $(u, p) \in V \times Q$ *such that*

$$a(u, v) - d(v, p) + \Psi^0(u; v) \geq \langle f, v\rangle \quad \forall\, v \in V, \tag{2.5}$$
$$d(u, q) + S(p, q) = 0 \quad \forall\, q \in Q. \tag{2.6}$$

In the study of Problem 2.3, we impose the following assumptions on the data.

$H(a)$. $a\colon V \times V \to \mathbb{R}$ is bilinear, bounded and coercive:

$$|a(u, v)| \leq c_a \|u\|_V \|v\|_V \quad \text{and} \quad a(v, v) \geq m_a \|v\|_V^2 \quad \forall\, u, v \in V,$$

for some constants $c_a, m_a > 0$.

$H(d)$. $d\colon V \times Q \to \mathbb{R}$ is bilinear and bounded:

$$|d(v, q)| \leq c_d \|v\|_V \|q\|_Q \quad \forall\, (v, q) \in V \times Q,$$

for a constant $c_d > 0$.

$H(S)$. $S\colon Q \times Q \to \mathbb{R}$ is bilinear and bounded with $S(q, q) \geq 0$ for all $q \in Q$.

$H(B)$. There exists a constant $\beta > 0$ such that

$$\sup_{(v,q)\in V\times Q} \frac{B((u, p); (v, q))}{\|v\|_V + \|q\|_Q} \geq \beta \left(\|u\|_V + \|p\|_Q\right) \quad \forall\, (u, p) \in V \times Q, \tag{2.7}$$

where

$$B((u, p); (v, q)) = a(u, v) - d(v, p) + d(u, q) + S(p, q).$$

$H(\Psi)$. $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz, and there are constants $c_0, c_1, m_\Psi \geq 0$ such that

$$\|\partial\Psi(v)\|_{V^*} \leq c_0 + c_1 \|v\|_V \quad \forall\, v \in V, \tag{2.8}$$
$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq m_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{2.9}$$

$H(0)$. $f \in V^*$ and $m_a > m_\Psi$.

Remark 2.4 The hypothesis $H(S)$ implies that function $Q \ni q \mapsto S(q, q) \in \mathbb{R}$ is convex. Indeed, let $p, q \in Q$ and $t \in (0, 1)$ be arbitrary and denote $q_t := t\, q + (1 - t)\, p$. Then, we have

$$\begin{aligned} S(q_t, q_t) &= t^2 S(q, q) + t\,(1 - t)\, S(q, p) + (1 - t)^2 S(p, p) + t\,(1 - t)\, S(p, q) \\ &= t\, S(q, q) - t\,(1 - t)\, S(p - q, p - q) + (1 - t)\, S(p, p) \\ &\leq t\, S(q, q) + (1 - t)\, S(p, p). \end{aligned}$$

Hence, $Q \ni q \mapsto S(q, q)$ is convex.

The assumption $H(B)$ is known as the inf-sup condition for the bilinear form B. The inequality in (2.8) stands for

$$\|\eta\|_{V^*} \leq c_0 + c_1 \|v\|_V \quad \forall\, v \in V,\ \eta \in \partial\Psi(v).$$

By (2.3), (2.8) implies

$$\left|\Psi^0(u; v)\right| \leq (c_0 + c_1 \|u\|_V) \|v\|_V \quad \forall\, u, v \in V. \tag{2.10}$$

It is known that (2.9) is equivalent to ([30])

$$\langle \eta_1 - \eta_2, v_1 - v_2\rangle \geq -m_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_i \in V,\ \eta_i \in \partial\Psi(v_i),\ i = 1, 2. \tag{2.11}$$

For the solution existence of Problem 2.3, we will apply a result called the elementary Knaster-Kuratowski-Mazurkiewicz principle ([15,p. 66]). Let V be a Hilbert space and U be a nonempty subset of V. We recall that $G\colon U \to 2^V$ is a Knaster-Kuratowski-Mazurkiewicz map (or simply a KKM-map) if for any $\{v_1, v_2, \ldots, v_n\} \subset U$, its convex hull $\mathrm{co}\{v_1, v_2, \ldots, v_n\}$ is contained in $\bigcup_{i=1}^n G(v_i)$.

Lemma 2.5 *Let V be a Hilbert space, U a nonempty subset of V, and $G : U \to 2^V$ a KKM-map with closed convex values such that Gv_0 is bounded for some $v_0 \in U$. Then the intersection $\bigcap\{Gv \mid v \in U\}$ is not empty.*

For $M > 0$, define subsets $V_M \subset V$ and $Q_M \subset Q$ by

$$V_M = \{v \in V : \|v\|_V \le M\} \text{ and } Q_M = \{q \in Q : \|q\|_Q \le M\}. \tag{2.12}$$

Then we introduce an auxiliary problem:

Problem 2.6 *Find $(u, p) \in V_M \times Q_M$ such that for all $(v, q) \in V_M \times Q_M$,*

$$a(u, v-u) - d(v-u, p) + \Psi^0(u; v-u) + d(u, q-p) + S(p, q-p) \ge \langle f, v-u\rangle. \tag{2.13}$$

Theorem 2.7 *Assume that $H(a)$, $H(d)$, $H(S)$, $H(B)$, $H(\Psi)$ and $H(0)$ hold. Then Problem 2.3 has a solution.*

Proof Let $M \ge M_0$ with

$$M_0 = \max\left\{\frac{c_0 + \|f\|_{V^*}}{m_a - m_\Psi}, \frac{c_0 + \|f\|_{V^*}}{\beta}\left(\frac{c_a + c_d + c_1}{m_a - m_\Psi} + 1\right)\right\} + 1. \tag{2.14}$$

We introduce a multi-valued mapping $G\colon V_M \times Q_M \to 2^{V_M \times Q_M}$ defined by

$$\begin{aligned} G(v, q) = \{(u, p) \in V_M \times Q_M :\ & a(v, v-u) - d(v-u, p) + \inf_{\xi \in \partial\Psi(v)} \langle \xi, v-u\rangle \\ & + d(u, q-p) + S(p, q-p) \ge \langle f, v-u\rangle\} \quad \forall (v, q) \in V_M \times Q_M. \end{aligned} \tag{2.15}$$

Since $(v, q) \in G(v, q)$, $G(v, q) \ne \emptyset$ for all $(v, q) \in V_M \times Q_M$. We claim that the set $G(v, q)$ is closed in $V \times Q$. Let $\{(u_n, p_n)\} \subset G(v, q)$ be a sequence such that $(u_n, p_n) \to (u, p)$ in $V \times Q$ as $n \to \infty$, for some $(u, p) \in V_M \times Q_M$. Hence, for an arbitrary $\xi \in \partial\Psi(v)$, for each $n \in \mathbb{N}$, we have

$$a(v, v-u_n) - d(v-u_n, p_n) + \langle \xi, v-u_n\rangle + d(u_n, q-p_n) + S(p_n, q-p_n) \ge \langle f, v-u_n\rangle.$$

Passing to the upper limit $n \to \infty$ in the above inequality, we have

$$\begin{aligned} \langle f, v-u\rangle &= \lim_{n\to\infty} \langle f, v-u_n\rangle \\ &\le \lim_{n\to\infty} \Big[a(v, v-u_n) - d(v, p_n) + \langle \xi, v-u_n\rangle + d(u_n, q) + S(p_n, q)\Big] \\ &\quad - \liminf_{n\to\infty} S(p_n, p_n) \\ &\le a(v, v-u) - d(v, p) + \langle \xi, v-u\rangle + d(u, q) + S(p, q-p), \end{aligned}$$

where we have used the fact that the function $p \mapsto S(p, p)$ is continuous. Since $\xi \in \partial\Psi(v)$ is arbitrary, we conclude from the above inequality that

$$a(v, v-u) - d(v-u, p) + \inf_{\xi \in \partial\Psi(v)} \langle \xi, v-u\rangle + d(u, q-p) + S(p, q-p) \ge \langle f, v-u\rangle,$$

i.e., $(u, p) \in G(v, q)$ and $G(v, q)$ is closed. Moreover, $G(v, q)$ is convex. This is verified as follows. For $(u_0, p_0), (u_1, p_1) \in G(v, q)$ and $t \in [0, 1]$, we have, with $(u_t, p_t) := t\,(u_1, p_1) + (1 - t)\,(u_0, p_0)$,

$$\inf_{\xi \in \partial\Psi(v)} \langle \xi, v - u_t \rangle \geq t \inf_{\xi \in \partial\Psi(v)} \langle \xi, v - u_1 \rangle + (1 - t) \inf_{\xi \in \partial\Psi(v)} \langle \xi, v - u_0 \rangle,$$
$$S(p_t, q - p_t) \geq t\, S(p_1, q - p_1) + (1 - t)\, S(p_0, q - p_0),$$

where we used the convexity of $Q \ni q \mapsto S(q, q) \in \mathbb{R}$ (see Remark 2.4). Hence, $(u_t, p_t) \in G(v, q)$, i.e., $G(v, q)$ is convex. Obviously, $G(v, q) \subset V_M \times Q_M$ is bounded for any $(v, q) \in V_M \times Q_M$.

Next, we show that G is a KKM mapping. Arguing by contradiction, assume there exist $\{(v_i, q_i)\}_{i=1}^N \subset V_M \times Q_M$ such that $(v_0, q_0) \notin \bigcup_{i=1}^N G(v_i, q_i)$ where $v_0 = \sum_{i=1}^N t_i v_i$ and $q_0 = \sum_{i=1}^N t_i q_i$ for some scalars $t_i \in [0, 1]$ with $\sum_{i=1}^N t_i = 1$. For each $1 \leq i \leq N$, since $\partial\Psi(v_i)$ is weakly compact in V^*, there exists $\xi_i \in \partial\Psi(v_i)$ such that $\inf_{\xi \in \partial\Psi(v_i)} \langle \xi, v_i - v_0 \rangle = \langle \xi_i, v_i - v_0 \rangle$. Then

$$\begin{aligned} &a(v_i, v_i - v_0) - d(v_i - v_0, q_0) + \langle \xi_i, v_i - v_0 \rangle + d(v_0, q_i - q_0) \\ &\quad + S(q_0, q_i - q_0) < \langle f, v_i - v_0 \rangle. \end{aligned}$$

By $H(a)$ and (2.11),

$$a(v_i - v_0, v_i - v_0) + \langle \xi_i - \xi_0, v_i - v_0 \rangle \geq (m_a - m_\Psi)\, \|v_i - v_0\|_V^2 \geq 0 \quad \forall\, \xi_0 \in \partial\Psi(v_0).$$

Hence,

$$\begin{aligned} &a(v_0, v_i - v_0) - d(v_i - v_0, q_0) + \langle \xi_0, v_i - v_0 \rangle + d(v_0, q_i - q_0) \\ &\quad + S(q_0, q_i - q_0) < \langle f, v_i - v_0 \rangle. \end{aligned}$$

We multiply both sides of the above inequality by t_i and sum over i from 1 to N,

$$\begin{aligned} 0 &= a(v_0, v_0 - v_0) - d(v_0 - v_0, q_0) + \langle \xi_0, v_0 - v_0 \rangle + d(v_0, q_0 - q_0) + S(q_0, q_0 - q_0) \\ &\quad < \langle f, v_0 - v_0 \rangle = 0. \end{aligned}$$

This leads to a contradiction. Therefore, we conclude that G is a KKM mapping.

Applying Lemma 2.5, we know that there exists $(u, p) \in V_M \times Q_M$ such that

$$\begin{aligned} &a(v, v - u) - d(v - u, p) + \inf_{\xi \in \partial\Psi(v)} \langle \xi, v - u \rangle + d(u, q - p) + S(p, q - p) \\ &\quad \geq \langle f, v - u \rangle \quad \forall\, (v, q) \in V_M \times Q_M. \end{aligned}$$

By the definition of the generalized subdifferential, we have

$$\begin{aligned} &a(v, v - u) - d(v - u, p) + \Psi^0(v; v - u) + d(u, q - p) + S(p, q - p) \\ &\quad \geq \langle f, v - u \rangle \quad \forall\, (v, q) \in V_M \times Q_M. \end{aligned}$$

Let $(w, r) \in V_M \times Q_M$ and $t \in (0, 1)$ be arbitrary. We take $v = v_t = t\, w + (1 - t)\, u$ and $q = q_t = t\, r + (1 - t)\, p$ in the above inequality and use the positive homogeneity of $u \mapsto \Psi^0(v; u)$ to get

$$a(v_t, w - u) - d(w - u, p) + \Psi^0(v_t; w - u) + d(u, r - p) + S(p, r - p) \geq \langle f, w - u \rangle.$$

Passing to the upper limit $t \to 0$ in the above inequality, we have

$$a(u, w - u) - d(w - u, p) + \Psi^0(u; w - u) + d(u, r - p) + S(p, r - p) \geq \langle f, w - u \rangle$$

for all $(w, r) \in V_M \times Q_M$, where we have used the fact that the function Ψ^0 is upper semi-continuous. We conclude that $(u, p) \in V_M \times Q_M$ is a solution to Problem 2.6.

Let us bound $\|u\|_V$ and $\|p\|_Q$. We take $(v, q) = (0, 0) \in V_M \times Q_M$ in (2.13) to obtain

$$a(u, u) \le \Psi^0(u; -u) - S(p, p) + \langle f, u\rangle. \tag{2.16}$$

From (2.9) and (2.10),

$$\Psi^0(u; -u) \le m_\Psi \|u\|_V^2 - \Psi^0(0; u) \le m_\Psi \|u\|_V^2 + c_0\|u\|_V.$$

Since $S(q, q) \ge 0$ for all $q \in Q$, we derive from (2.16) that

$$(m_a - m_\Psi)\|u\|_V^2 \le (c_0 + \|f\|_{V^*})\|u\|_V.$$

Hence,

$$\|u\|_V \le \frac{c_0 + \|f\|_{V^*}}{m_a - m_\Psi}.$$

We then bound $\|p\|_Q$. Let $r \in Q$ be arbitrary. Then for $t \in (0, 1)$ sufficiently small, $q_t = t\,r + (1 - t)\,p \in Q_M$. Taking $q = p$ and $(v, q) = (u, q_t)$ in turn in (2.13), we obtain

$$d(v - u, p) \le a(u, v - u) + \Psi^0(u; v - u) - \langle f, v - u\rangle \quad \forall\, v \in V_M, \tag{2.17}$$

$$d(u, q) + S(p, q) = 0 \quad \forall\, q \in Q. \tag{2.18}$$

Denote by $B_V(u, 1)$ the unit closed ball in V with center u, and by $B_Q(0, 1)$ the unit closed ball in Q with center 0. Note that $B_V(u, 1) \subset V_M$. Use the inf-sup condition (2.7) with $u = 0$ to find

$$\begin{aligned}\beta\|p\|_Q &\le \sup_{(v,q)\in V\times Q} \frac{B((0, p); (v, q))}{\|v\|_V + \|q\|_Q} = \sup_{\|v\|_V+\|q\|_Q\le 1} B((0, p); (v, q))\\ &\le \sup_{\substack{v\in B_V(u,1),\\ q\in B_Q(0,1)}} B((0, p); (u - v, q)) = \sup_{\substack{v\in B_V(u,1),\\ q\in B_Q(0,1)}} \Big(d(v - u, p) + S(p, q)\Big).\end{aligned}$$

From (2.17) and (2.18),

$$\begin{aligned}\beta\|p\|_Q &\le \sup_{v\in B_V(u,1)} \Big(|a(u, v - u)| + |\Psi^0(u; v - u)| + |\langle f, v - u\rangle|\Big) + \sup_{q\in B_Q(0,1)} |d(u, q)|\\ &\le (c_a + c_d + c_1)\|u\|_V + c_0 + \|f\|_{V^*}.\end{aligned}$$

In conclusion, we see that for M_0 defined in (2.14), a solution (u^*, p^*) to Problem 2.6 with $M \ge M_0$ satisfies the inequalities

$$\|u^*\|_V < M_0 \text{ and } \|p^*\|_Q < M_0.$$

We now prove that (u^*, p^*) is also a solution to Problem 2.3. Let $(w, r) \in V \times Q$ be arbitrary. For $t \in (0, 1)$ sufficiently small, let

$$(v_t, q_t) = (t\,w + (1 - t)\,u^*, t\,r + (1 - t)\,p^*) \in V_{M_0} \times Q_{M_0}.$$

Inserting $(v, q) = (v_t, q_t)$ into (2.13) with $(u, p) = (u^*, p^*)$ gives

$$\begin{aligned}&a(u^*, w - u^*) - d(w - u^*, p^*)\\ &\quad +\Psi^0(u^*; w - u^*) + d(u^*, r - p^*) + S(p^*, r - p^*) \ge \langle f, w - u^*\rangle.\end{aligned}$$

Take $r = p^*$ and $w = u^*$ in turn in the above inequality, we obtain

$$a(u^*, w - u^*) - d(w - u^*, p^*) + \Psi^0(u^*; w - u^*) \geq \langle f, w - u^* \rangle \quad \forall w \in V,$$
$$d(u^*, r - p^*) + S(p^*, r - p^*) \geq 0 \quad \forall r \in Q.$$

Hence,

$$a(u^*, v) - d(v, p^*) + \Psi^0(u^*; v) \geq \langle f, v \rangle \quad \forall v \in V,$$
$$d(u^*, q) + S(p^*, q) = 0 \quad \forall q \in Q.$$

We see that $(u^*, p^*) \in V \times Q$ is a solution to Problem 2.3. □

3 A Stabilized Mixed FEM for Stokes HVI

We recall the Stokes HVI studied in [13]. Let $\Omega \subset \mathbb{R}^d$ ($d \leq 3$ in applications) be an open bounded connected set with a Lipschitz boundary $\Gamma = \partial\Omega$. The boundary is split into two parts: $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_S}$ with $\mathrm{meas}(\Gamma_D) > 0$, $\mathrm{meas}(\Gamma_S) > 0$, and $\Gamma_D \cap \Gamma_S = \emptyset$. We will impose a Dirichlet boundary condition on Γ_D and a slip boundary condition of friction type on Γ_S. Denote by $\boldsymbol{n}$ the unit outward normal to Γ. For a vector-valued function $\boldsymbol{u}$ on the boundary, let $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n\boldsymbol{n}$ be the normal component and the tangential component, respectively. With the flow velocity field $\boldsymbol{u}$ and the pressure p, we define the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ and the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix, $\nu > 0$ is the viscosity coefficient. Let $\sigma_n = \boldsymbol{n} \cdot \boldsymbol{\sigma}\boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n\boldsymbol{n}$ be the normal component and the tangential component of $\boldsymbol{\sigma}$.

We consider the Stokes problem

$$-\operatorname{div}(2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.1}$$
$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3.2}$$

with the following boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{3.3}$$
$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_S. \tag{3.4}$$

Here, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is a given function, and $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous with respect to its second argument. To simplify the notation, we write $\psi(\boldsymbol{u}_\tau)$ for $\psi(\boldsymbol{x}, \boldsymbol{u}_\tau)$, and $\partial\psi$ is the subdifferential of ψ in the sense of Clarke with respect to its second argument. The condition (3.4) is known as a slip boundary condition. The first part $u_n = 0$ means that the fluid can not pass through Γ_S outside the domain. The second part represents a friction condition, relating the frictional force $\boldsymbol{\sigma}_\tau$ with the tangential velocity $\boldsymbol{u}_\tau$.

Introduce function spaces

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ v_n = 0 \text{ on } \Gamma_S\},$$
$$Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q(\boldsymbol{x})\,dx = 0\right\}$$

for the velocity and pressure variables. As a consequence of Korn's inequality ([32,p. 79]),

$$\boldsymbol{V} \ni \boldsymbol{u} \mapsto \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega} := \left(\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2\,dx\right)^{\frac{1}{2}}$$

defines a norm and is equivalent to the standard $\boldsymbol{H}^1(\Omega)$-norm on $\boldsymbol{V}$. We use $\|\cdot\|_{\boldsymbol{V}} = \|\boldsymbol{\varepsilon}(\cdot)\|_{0,\Omega}$ for the norm on $\boldsymbol{V}$ and use $\|\cdot\|_{0,\Omega}$ for the norm on Q.

The mixed formulation of the Stokes HVI is as follows.

Problem 3.1 *Find* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{3.5}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q, \tag{3.6}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

$$b(\boldsymbol{v}, q) = \int_\Omega q \,\mathrm{div}\boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ q \in Q,$$

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

We note that the bilinear form $a(\cdot, \cdot)$ is coercive on $\boldsymbol{V}$ due to

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2\nu \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.7}$$

Concerning the superpotential ψ, we assume the following properties:
$H(\psi)$. $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $\psi(\cdot, \boldsymbol{\xi})$ is measurable on Γ_S for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\psi(\cdot, \boldsymbol{0}) \in L^1(\Gamma_S)$;
(ii) $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_S$;
(iii) $|\boldsymbol{\eta}| \leq c_0 + c_1|\boldsymbol{\xi}| \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{\eta} \in \partial\psi(\boldsymbol{x}, \boldsymbol{\xi})$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $c_0, c_1 \geq 0$;
(iv) $\psi^0(\boldsymbol{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \psi^0(\boldsymbol{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leq m_\tau |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $m_\tau \geq 0$.

By (2.3), $H(\psi)$ (iii) implies

$$\left|\psi^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2)\right| \leq \left(c_0 + c_1|\boldsymbol{\xi}_1|\right) |\boldsymbol{\xi}_2| \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d. \tag{3.8}$$

Define a functional $\Psi : \boldsymbol{L}^2(\Gamma_S) \to \mathbb{R}$ by

$$\Psi(\boldsymbol{v}) = \int_{\Gamma_S} \psi(\boldsymbol{v})\, ds \quad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(\Gamma_S). \tag{3.9}$$

From the proof of Theorem 4.20 in [30], we have the following result.

Lemma 3.2 *Assume that* $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ *has the properties* $H(\psi)$. *Then the functional* Ψ *defined by* (3.9) *satisfies*
$H(\Psi)$. (i) $\Psi(\cdot)$ *is locally Lipschitz on* $\boldsymbol{L}^2(\Gamma_S)$*;*
(ii) $\Psi^0(\boldsymbol{u}; \boldsymbol{v}) \leq \int_{\Gamma_S} \psi^0(\boldsymbol{u}; \boldsymbol{v})\, ds \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\Gamma_S)$*;*
(iii) $\|\boldsymbol{z}\|_{0,\Gamma_S} \leq \sqrt{3|\Gamma_S|}\, c_0 + \sqrt{3}\, c_1 \|\boldsymbol{v}\|_{0,\Gamma_S} \quad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(\Gamma_S),\ \boldsymbol{z} \in \partial\Psi(\boldsymbol{v})$*;*
(iv) $\Psi^0(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \Psi^0(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \leq m_\tau \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{0,\Gamma_S}^2 \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{L}^2(\Gamma_S)$.

By the Sobolev trace theorem, we have the inequality

$$\|\boldsymbol{v}_\tau\|_{0,\Gamma_S} \leq \lambda_0^{-1/2} \|\boldsymbol{v}\|_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{3.10}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in \boldsymbol{V}, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_S} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau\, ds \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{3.11}$$

It is shown in [13] that if $\boldsymbol{f} \in \boldsymbol{V}^*$, $H(\psi)$ and $m_\tau < 2\nu\lambda_0$ hold, then Problem 3.1 has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$.

We now turn to consider a finite element method for solving Problem 3.1. For simplicity, we assume Ω is a polygonal/polyhedral domain in this section. We express $\overline{\Gamma_S}$ as the union of closed flat components with disjoint interior: $\overline{\Gamma_S} = \cup_{l=1}^{l_S} \Gamma_{S,l}$. Let $\{\mathcal{T}^h\}$ be a regular family of triangular partitions of $\overline{\Omega}$ into triangles. For a generic element $K \in \mathcal{T}^h$, denote by $h_K = \text{diam}(K)$ the diameter of K. The mesh-size of $\mathcal{T}^h$ is $h = \max\{h_K : K \in \mathcal{T}^h\}$. Let P_1 be the space of the polynomials of degree ≤ 1, and $\boldsymbol{P}_1 = (P_1)^d$ for the corresponding vector-valued polynomial space. Corresponding to the partition $\mathcal{T}^h$, we introduce finite element spaces

$$\boldsymbol{V}^h = \{\boldsymbol{v}^h \in \boldsymbol{V} : \boldsymbol{v}^h|_K \in \boldsymbol{P}_1(K)\ \forall K \in \mathcal{T}^h\}, \tag{3.12}$$

$$Q^h = \{q^h \in Q \cap C^0(\overline{\Omega}) : q^h|_K \in P_1(K)\ \forall K \in \mathcal{T}^h\}. \tag{3.13}$$

For $K \in \mathcal{T}^h$, let $\Pi^K : L^2(K) \to P_0(K)$ be the L^2-projection operator defined by

$$\Pi^K q = \frac{1}{|K|} \int_K q\, dx \quad \forall q \in L^2(K),$$

with the global operator defined as

$$\Pi^h|_K = \Pi^K \quad \forall K \in \mathcal{T}^h.$$

By [7,Lemma 2.3], it holds

$$\|q^h - \Pi^h q^h\|_{0,\Omega} \geq c\, h\, \|\nabla q^h\|_{0,\Omega} \quad \forall q^h \in Q^h. \tag{3.14}$$

From the standard finite element approximation theory ([9]), we have

$$\|p - \Pi^h p\|_{0,\Omega} \leq c\, h\, \|p\|_{1,\Omega} \quad \forall p \in H^1(\Omega). \tag{3.15}$$

Following [7, 28], we introduce a stabilized mixed finite element method for Problem 3.1.

Problem 3.3 *Find $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ such that*

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p^h) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \tag{3.16}$$

$$b(\boldsymbol{u}^h, q^h) + S^h(p^h, q^h) = 0 \quad \forall q^h \in Q^h, \tag{3.17}$$

where the stabilized term $S^h(p, q)$ is defined by

$$S^h(p, q) = \int_\Omega (p - \Pi^h p)(q - \Pi^h q)\, dx \quad \forall p, q \in Q.$$

Define a bilinear form

$$B^h((\boldsymbol{u}, p); (\boldsymbol{v}, q)) = a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q) + S^h(p, q). \tag{3.18}$$

Let us show the inf-sup condition for the bilinear form (3.18).

Lemma 3.4 *There exists a positive constant c independent of h such that*

$$\sup_{(\boldsymbol{v}^h,q^h)\in \boldsymbol{V}^h\times Q^h} \frac{B^h((\boldsymbol{u}^h,p^h);(\boldsymbol{v}^h,q^h))}{\|\boldsymbol{v}^h\|_{\boldsymbol{V}}+\|q^h\|_{0,\Omega}} \geq c\left(\|\boldsymbol{u}^h\|_{\boldsymbol{V}}+\|p^h\|_{0,\Omega}\right) \quad \forall (\boldsymbol{u}^h,p^h)\in \boldsymbol{V}^h\times Q^h. \tag{3.19}$$

Proof For any $(\boldsymbol{u}^h,p^h)\in \boldsymbol{V}^h\times Q^h$, let us construct a pair $(\boldsymbol{v}^h,q^h)\in \boldsymbol{V}^h\times Q^h$ such that

$$\frac{B^h((\boldsymbol{u}^h,p^h);(\boldsymbol{v}^h,q^h))}{\|\boldsymbol{v}^h\|_{\boldsymbol{V}}+\|q^h\|_{0,\Omega}} \geq c\left(\|\boldsymbol{u}^h\|_{\boldsymbol{V}}+\|p^h\|_{0,\Omega}\right) \tag{3.20}$$

for a constant $c>0$ independent of h. It is known that the continuous inf-sup condition is valid ([37]): there exists a positive constant β such that

$$\sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \geq \beta\,\|q\|_{0,\Omega} \quad \forall q\in Q.$$

Then, for any $p^h\in Q^h$ there exists a function $\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)$ satisfying

$$\frac{b(\boldsymbol{v},p^h)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \geq \beta\,\|p^h\|_{0,\Omega} \quad \text{and} \quad \|\boldsymbol{v}\|_{\boldsymbol{V}}=\|p^h\|_{0,\Omega}. \tag{3.21}$$

Let $\boldsymbol{v}^I$ be the first order Scott-Zhang interpolant of $\boldsymbol{v}$ ([34,Definition 2.13]). Then $\boldsymbol{v}^I\in \boldsymbol{V}^h\cap \boldsymbol{H}_0^1(\Omega)$ and for some constants $C_0, c>0$ ([34,Theorem 4.1]),

$$\|\boldsymbol{v}^I\|_{\boldsymbol{V}} \leq C_0\|\boldsymbol{v}\|_{\boldsymbol{V}} \quad \text{and} \quad \|\boldsymbol{v}-\boldsymbol{v}^I\|_{0,\Omega} \leq c\,h\,\|\boldsymbol{v}\|_{\boldsymbol{V}}. \tag{3.22}$$

Write

$$b(\boldsymbol{v}^I,p^h)=b(\boldsymbol{v},p^h)+b(\boldsymbol{v}^I-\boldsymbol{v},p^h).$$

By (3.21),

$$b(\boldsymbol{v},p^h)\geq C_1\|p^h\|_{0,\Omega}^2,$$

where $C_1=\beta>0$. Since $p^h\in Q^h$ is continuous, then it holds

$$b(\boldsymbol{v}^I-\boldsymbol{v},p^h)=-(\boldsymbol{v}^I-\boldsymbol{v},\nabla p^h)\geq -\|\boldsymbol{v}^I-\boldsymbol{v}\|_{0,\Omega}\|\nabla p^h\|_{0,\Omega}.$$

By (3.21), (3.22) and (3.14),

$$b(\boldsymbol{v}^I-\boldsymbol{v},p^h)\geq -C_2\|p^h-\Pi^h p^h\|_{0,\Omega}\|p^h\|_{0,\Omega}$$

for some constant $C_2>0$. Thus,

$$b(\boldsymbol{v}^I,p^h)\geq C_1\|p^h\|_{0,\Omega}^2-C_2\|p^h-\Pi^h p^h\|_{0,\Omega}\|p^h\|_{0,\Omega}. \tag{3.23}$$

From (3.21) and (3.22), we have

$$\|\boldsymbol{v}^I\|_{\boldsymbol{V}}\leq C_0\|p^h\|_{0,\Omega}.$$

Choosing $\boldsymbol{v}^h=\boldsymbol{u}^h-\alpha\boldsymbol{v}^I$ and $q^h=p^h$ for a positive constant α, we obtain

$$\begin{aligned} B^h((\boldsymbol{u}^h,p^h);(\boldsymbol{u}^h-\alpha\boldsymbol{v}^I,p^h)) &= a(\boldsymbol{u}^h,\boldsymbol{u}^h)-\alpha a(\boldsymbol{u}^h,\boldsymbol{v}^I)+\alpha b(\boldsymbol{v}^I,p^h)+S^h(p^h,p^h)\\ &\geq 2\nu\|\boldsymbol{u}^h\|_{\boldsymbol{V}}^2-2\nu\alpha C_0\|\boldsymbol{u}^h\|_{\boldsymbol{V}}\|p^h\|_{0,\Omega}+\alpha C_1\|p^h\|_{0,\Omega}^2\\ &\quad -\alpha C_2\|p^h-\Pi^h p^h\|_{0,\Omega}\|p^h\|_{0,\Omega}+\|p^h-\Pi^h p^h\|_{0,\Omega}^2. \end{aligned}$$

Obviously, for $\alpha > 0$ sufficiently small, we have a positive constant c that depends on α such that

$$B^h((\boldsymbol{u}^h, p^h); (\boldsymbol{u}^h - \alpha \boldsymbol{v}^I, p^h)) \geq c \left(\|\boldsymbol{u}^h\|_V + \|p^h\|_{0,\Omega} \right)^2.$$

Note that

$$\|\boldsymbol{v}^h\|_V + \|q^h\|_{0,\Omega} = \|\boldsymbol{u}^h - \alpha \boldsymbol{v}^I\|_V + \|p^h\|_{0,\Omega} \leq c(\|\boldsymbol{u}^h\|_V + \|p^h\|_{0,\Omega}).$$

Thus, we finish the proof of (3.19). □

Remark 3.5 Particularly, when $\boldsymbol{u}^h = \boldsymbol{0}$, then we have

$$\sup_{(\boldsymbol{v}^h, q^h) \in \widetilde{\boldsymbol{V}}^h \times Q^h} \frac{B^h((\boldsymbol{0}, p^h); (\boldsymbol{v}^h, q^h))}{\|\boldsymbol{v}^h\|_V + \|q^h\|_{0,\Omega}} \geq c \|p^h\|_{0,\Omega} \quad \forall p^h \in Q^h, \tag{3.24}$$

where $\widetilde{\boldsymbol{V}}^h = \boldsymbol{V}^h \cap \boldsymbol{H}_0^1(\Omega)$.

We are ready to present an existence and uniqueness result for Problem 3.3.

Theorem 3.6 *Assume $\boldsymbol{f} \in \boldsymbol{V}^*$, $H(\psi)$ and $m_\tau < 2\nu\lambda_0$. Then Problem 3.3 has a unique solution.*

Proof By Lemma 3.2, under the assumption $H(\psi)$, the functional Ψ defined by (3.9) has the properties $H(\Psi)$. Then, we can apply Theorem 2.7 to conclude that there exists $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ such that

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p^h) + \Psi^0(\boldsymbol{u}^h; \boldsymbol{v}^h) \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \tag{3.25}$$

$$b(\boldsymbol{u}^h, q^h) + S^h(p^h, q^h) = 0 \quad \forall q^h \in Q^h. \tag{3.26}$$

Since

$$\Psi^0(\boldsymbol{u}^h; \boldsymbol{v}^h) \leq \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h)\, ds,$$

we see that the solution $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ of (3.25)–(3.26) is also a solution of Problem 3.3.

For solution uniqueness, let $(\boldsymbol{u}_1^h, p_1^h), (\boldsymbol{u}_2^h, p_2^h) \in \boldsymbol{V}^h \times Q^h$ be two solutions of Problem 3.3. Then

$$a(\boldsymbol{u}_1^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_1^h) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_{1,\tau}^h; \boldsymbol{v}_\tau^h)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \tag{3.27}$$

$$b(\boldsymbol{u}_1^h, q^h) + S^h(p_1^h, q^h) = 0 \quad \forall q^h \in Q^h, \tag{3.28}$$

and

$$a(\boldsymbol{u}_2^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_2^h) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_{2,\tau}^h; \boldsymbol{v}_\tau^h)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \tag{3.29}$$

$$b(\boldsymbol{u}_2^h, q^h) + S^h(p_2^h, q^h) = 0 \quad \forall q^h \in Q^h. \tag{3.30}$$

We take $\boldsymbol{v}^h = \boldsymbol{u}_2^h - \boldsymbol{u}_1^h$ in (3.27), $\boldsymbol{v}^h = \boldsymbol{u}_1^h - \boldsymbol{u}_2^h$ in (3.29), and add the two inequalities to obtain

$$\begin{aligned} a(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{u}_1^h - \boldsymbol{u}_2^h) \leq\ & b(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, p_1^h - p_2^h) \\ & + \int_{\Gamma_S} \left[\psi^0(\boldsymbol{u}_{1,\tau}^h; \boldsymbol{u}_{2,\tau}^h - \boldsymbol{u}_{1,\tau}^h) + \psi^0(\boldsymbol{u}_{2,\tau}^h; \boldsymbol{u}_{1,\tau}^h - \boldsymbol{u}_{2,\tau}^h) \right] ds. \end{aligned} \tag{3.31}$$

Then take $q^h = p_1^h - p_2^h$ in (3.28), $q^h = p_2^h - p_1^h$ in (3.30), and add the two equalities to obtain

$$b(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, p_1^h - p_2^h) = -S^h(p_1^h - p_2^h, p_1^h - p_2^h). \tag{3.32}$$

Thus, from (3.31) and (3.32), we have

$$\begin{aligned}a(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{u}_1^h - \boldsymbol{u}_2^h) \le& -S^h(p_1^h - p_2^h, p_1^h - p_2^h)\\ &+ \int_{\Gamma_S} \left[\psi^0(\boldsymbol{u}_{1,\tau}^h; \boldsymbol{u}_{2,\tau}^h - \boldsymbol{u}_{1,\tau}^h) + \psi^0(\boldsymbol{u}_{2,\tau}^h; \boldsymbol{u}_{1,\tau}^h - \boldsymbol{u}_{2,\tau}^h)\right] ds,\end{aligned}$$

from which, by applying $H(\psi)$ (iv), (3.10) and noting that $S^h(p_1^h - p_2^h, p_1^h - p_2^h) \ge 0$,

$$2\,\nu\, \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}}^2 \le m_\tau \lambda_0^{-1} \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}}^2.$$

Hence, $\|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}} = 0$, i.e., $\boldsymbol{u}_1^h = \boldsymbol{u}_2^h$, due to the smallness condition $m_\tau < 2\nu\lambda_0$. Back to (3.32), we then have

$$S^h(p_1^h - p_2^h, p_1^h - p_2^h) = 0.$$

This equality implies that $p_1^h - p_2^h = \Pi^h(p_1^h - p_2^h)$ is a piecewise constant function. Since $p_1^h - p_2^h \in Q^h$ is continuous with a vanishing integral over Ω, we conclude that $p_1^h = p_2^h$. □

We turn to an error analysis. As a preparation, we show that the numerical solution is uniformly bounded independent of h.

Proposition 3.7 *Assume $\boldsymbol{f} \in \boldsymbol{V}^*$, $H(\psi)$ and $m_\tau < 2\nu\lambda_0$. Then the solution component $\boldsymbol{u}^h \in \boldsymbol{V}^h$ depends Lipschitz continuously on $\boldsymbol{f}$. In particular, $\|\boldsymbol{u}^h\|_V$ is bounded independent of h.*

Proof For $i = 1, 2$, let $\boldsymbol{f}_i \in \boldsymbol{V}^*$ and $(\boldsymbol{u}_i^h, p_i^h) \in \boldsymbol{V}^h \times Q^h$ be the unique solution to Problem 3.3 corresponding to $\boldsymbol{f}_i$. Then,

$$\begin{aligned}&a(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_1^h - p_2^h) + \int_{\Gamma_S} [\psi^0(\boldsymbol{u}_{1,\tau}^h; \boldsymbol{v}_\tau^h) + \psi^0(\boldsymbol{u}_{2,\tau}^h; -\boldsymbol{v}_\tau^h)]\, ds\\ &\quad \ge \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}^h\rangle \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h,\\ &b(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, q^h) + S^h(p_1^h - p_2^h, q^h) = 0 \quad \forall\, q^h \in Q^h.\end{aligned}$$

Take $\boldsymbol{v}^h = \boldsymbol{u}_2^h - \boldsymbol{u}_1^h$ in the first inequality and $q^h = p_1^h - p_2^h$ in the second one, we obtain

$$\begin{aligned}a(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{u}_1^h - \boldsymbol{u}_2^h) \le& \int_{\Gamma_S} \left[\psi^0(\boldsymbol{u}_{1,\tau}^h; \boldsymbol{u}_{2,\tau}^h - \boldsymbol{u}_{1,\tau}^h) + \psi^0(\boldsymbol{u}_{2,\tau}^h; \boldsymbol{u}_{1,\tau}^h - \boldsymbol{u}_{2,\tau}^h)\right] ds\\ &+ b(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, p_1^h - p_2^h) + \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{u}_1^h - \boldsymbol{u}_2^h\rangle, \end{aligned}\tag{3.33}$$

and

$$b(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, p_1^h - p_2^h) = -S^h(p_1^h - p_2^h, p_1^h - p_2^h) \le 0.$$

Apply $H(\psi)$ (iv) and (3.10) in (3.33), it yileds

$$\begin{aligned}2\nu\, \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}}^2 &\le m_\tau \int_{\Gamma_S} |\boldsymbol{u}_{1,\tau}^h - \boldsymbol{u}_{2,\tau}^h|^2 ds + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{V}^*} \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}}\\ &\le m_\tau \lambda_0^{-1} \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}}^2 + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{V}^*} \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}}.\end{aligned}$$

We derive the Lipschitz continuity inequality

$$\|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_{\boldsymbol{V}} \le \frac{1}{2\nu - m_\tau \lambda_0^{-1}} \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{V}^*}.$$

In particular, this inequality implies $\|\boldsymbol{u}^h\|_{\boldsymbol{V}} \le \|\boldsymbol{f}\|_{\boldsymbol{V}^*}/(2\nu - m_\tau \lambda_0^{-1})$, i.e., $\|\boldsymbol{u}^h\|_{\boldsymbol{V}}$ is uniformly bounded independent of h. □

We now establish a Céa's inequality for the numerical solution.

Theorem 3.8 *Assume $\boldsymbol{f} \in \boldsymbol{V}^*$, $H(\psi)$ and $m_\tau < 2\nu\lambda_0$. Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ and $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ be the solutions of the Problems 3.1 and 3.3, respectively. Then there exists a positive constant c independent of h such that for all $\boldsymbol{v}^h \in \boldsymbol{V}^h$ and $q^h \in Q^h$,*

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - p^h\|_{0,\Omega} \\ &\quad \le c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega} + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S}^{1/2} \right). \end{aligned} \tag{3.34}$$

Proof Denote

$$B((\boldsymbol{u}, p); (\boldsymbol{v}, q)) = a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q).$$

Note that Problem 3.1 is equivalent to

$$\begin{aligned} &(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q, \quad B((\boldsymbol{u}, p); (\boldsymbol{v} - \boldsymbol{u}, q)) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle \\ &\qquad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q, \end{aligned} \tag{3.35}$$

whereas Problem 3.3 is equivalent to

$$\begin{aligned} &(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h, B^h((\boldsymbol{u}^h, p^h); (\boldsymbol{v}^h - \boldsymbol{u}^h, q^h)) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h \rangle \\ &\qquad \forall\, (\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h. \end{aligned} \tag{3.36}$$

Let $(\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h$ be arbitrary. Then, we have

$$\begin{aligned} 2\,\nu\, \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 &\le B^h((\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \\ &= B^h((\boldsymbol{u}^h, p^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) - S^h(q^h, p^h - q^h) \\ &\quad - B((\boldsymbol{u}, p); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \\ &\quad + B((\boldsymbol{u} - \boldsymbol{v}^h, p - q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)). \end{aligned}$$

From (3.36),

$$B^h((\boldsymbol{u}^h, p^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \le \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\, ds - \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h \rangle.$$

From (3.35) with $\boldsymbol{v} = \boldsymbol{u}^h - \boldsymbol{v}^h + \boldsymbol{u}$ and $q = p^h - q^h$,

$$-B((\boldsymbol{u}, p); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \le \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h)\, ds - \langle \boldsymbol{f}, \boldsymbol{u}^h - \boldsymbol{v}^h \rangle.$$

Hence,

$$2\,\nu\, \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 \le I_1 + I_2 + I_3, \tag{3.37}$$

where

$$I_1 = -S^h(q^h, p^h - q^h),$$
$$I_2 = B((\boldsymbol{u} - \boldsymbol{v}^h, p - q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)),$$
$$I_3 = \int_{\Gamma_S} [\psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h) + \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h)]\, ds.$$

By the definition of the stabilized term S^h, we have

$$\begin{aligned} I_1 &= S^h(p - q^h, p^h - q^h) - S^h(p, p^h - q^h) \\ &\le \|p - q^h\|_{0,\Omega}\|p^h - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\|p^h - q^h\|_{0,\Omega}. \end{aligned} \tag{3.38}$$

By the modified Cauchy inequality with an arbitrarily small $\varepsilon > 0$, one has

$$\begin{aligned} I_2 &= a(\boldsymbol{u} - \boldsymbol{v}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) - b(\boldsymbol{u}^h - \boldsymbol{v}^h, p - q^h) + b(\boldsymbol{u} - \boldsymbol{v}^h, p^h - q^h) \\ &\le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V + \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V\|p - q^h\|_{0,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}^h\|_V\|p^h - q^h\|_{0,\Omega}\right) \\ &\le \varepsilon\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V^2 + c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|p - q^h\|_{0,\Omega}^2 + \|\boldsymbol{u} - \boldsymbol{v}^h\|_V\|p^h - q^h\|_{0,\Omega}\right). \end{aligned} \tag{3.39}$$

By the subadditivity of ψ (see (2.2)), we obtain

$$\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h) \le \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{u}_\tau) + \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h),$$
$$\psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h) \le \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h) + \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau).$$

By $H(\psi)$ (iv),

$$\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{u}_\tau) + \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h) \le m_\tau|\boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h|^2.$$

By (3.8),

$$\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h) \le (c_0 + c_1|\boldsymbol{u}_\tau|)|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h|,$$
$$\psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau) \le (c_0 + c_1|\boldsymbol{u}_\tau^h|)|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h|.$$

Note that $\|\boldsymbol{u}_h\|_{0,\Gamma_S}$ is bounded by a constant independent of h. Thus,

$$\begin{aligned} I_3 &\le \int_{\Gamma_S} m_\tau|\boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h|^2\, ds + \int_{\Gamma_S} (2c_0 + c_1|\boldsymbol{u}_\tau| + c_1|\boldsymbol{u}_\tau^h|)|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h|\, ds \\ &\le m_\tau\lambda_0^{-1}\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + c\,\|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S} \\ &\le (m_\tau\lambda_0^{-1} + \varepsilon)\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V^2 + c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S}\right). \end{aligned} \tag{3.40}$$

Combine (3.37)–(3.40) to get

$$\begin{aligned} &(2\nu - m_\tau\lambda_0^{-1} - 2\varepsilon)\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V^2 \\ &\quad\le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\right)\|p^h - q^h\|_{0,\Omega} \\ &\qquad + c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|p - q^h\|_{0,\Omega}^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S}\right). \end{aligned}$$

Since $m_\tau \lambda_0^{-1} < 2\nu$, we can choose $\varepsilon = (2\nu - m_\tau \lambda_0^{-1})/4 > 0$ to get

$$\begin{aligned}\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 &\le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\right)\|p^h - q^h\|_{0,\Omega}\\ &\quad + c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + \|p - q^h\|_{0,\Omega}^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S}\right).\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}}^2 &\le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\right)\|p^h - q^h\|_{0,\Omega}\\ &\quad + c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + \|p - q^h\|_{0,\Omega}^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S}\right). \qquad (3.41)\end{aligned}$$

We next bound the error for the pressure. From the discrete inf-sup condition (3.24), we have

$$c\,\|p^h - q^h\|_{0,\Omega} \le \sup_{(\boldsymbol{v}^h, r^h)\in \widetilde{\boldsymbol{V}}^h \times Q^h} \frac{-b(\boldsymbol{v}^h, p^h - q^h) + S^h(p^h - q^h, r^h)}{\|\boldsymbol{v}^h\|_{\boldsymbol{V}} + \|r^h\|_{0,\Omega}}. \qquad (3.42)$$

Take $\boldsymbol{v}^h \in \widetilde{\boldsymbol{V}}^h$ as a test function in (3.5) and (3.16) to obtain

$$\begin{aligned}a(\boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p) &= \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle \quad \forall\, \boldsymbol{v}^h \in \widetilde{\boldsymbol{V}}^h,\\ a(\boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p^h) &= \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle \quad \forall\, \boldsymbol{v}^h \in \widetilde{\boldsymbol{V}}^h.\end{aligned}$$

Thus,

$$b(\boldsymbol{v}^h, p - p^h) = a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) \quad \forall\, \boldsymbol{v}^h \in \widetilde{\boldsymbol{V}}^h.$$

By (3.6) and (3.17),

$$S^h(p^h, r^h) = b(\boldsymbol{u} - \boldsymbol{u}^h, r^h) \quad \forall\, r^h \in Q^h.$$

Then we obtain

$$\begin{aligned}&b(\boldsymbol{v}^h, q^h - p^h) + S^h(p^h - q^h, r^h)\\ &\quad = b(\boldsymbol{v}^h, p - p^h) + b(\boldsymbol{v}^h, q^h - p) + S^h(p^h, r^h) - S^h(p, r^h) + S^h(p - q^h, r^h)\\ &\quad = a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, q^h - p) + b(\boldsymbol{u} - \boldsymbol{u}^h, r^h) - S^h(p, r^h) + S^h(p - q^h, r^h)\\ &\quad \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\right)\left(\|\boldsymbol{v}^h\|_{\boldsymbol{V}} + \|r^h\|_{0,\Omega}\right),\end{aligned}$$

which implies

$$\|p^h - q^h\|_{0,\Omega} \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\right). \qquad (3.43)$$

By the triangle inequality,

$$\|p - p^h\|_{0,\Omega} \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega}\right). \qquad (3.44)$$

Inserting (3.43) into (3.41) and using the modified Cauchy inequality, we can derive that

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_{0,\Omega} + \|p - \Pi^h p\|_{0,\Omega} + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{0,\Gamma_S}^{1/2}\right). \qquad (3.45)$$

Finally, (3.34) follows from (3.45) and (3.44). □

As a corollary of Theorem 3.8, we have an optimal order error estimate.

Theorem 3.9 *Assume $\boldsymbol{f} \in \boldsymbol{V}^*$, $H(\psi)$ and $m_\tau < 2\nu\lambda_0$. Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ and $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ be the solutions of the Problems 3.1 and 3.3, respectively. Assume $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $\boldsymbol{u}|_{\Gamma_{S,l}} \in \boldsymbol{H}^2(\Gamma_{S,l})$, $1 \le l \le l_S$, and $p \in H^1(\Omega)$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|p - p^h\|_{0,\Omega} \le c\,h. \tag{3.46}$$

Proof Let $\boldsymbol{u}^I \in \boldsymbol{V}^h$ be the finite element interpolant of $\boldsymbol{u}$, and let p^I be the L^2 projection of p to Q^h. Then from the finite element approximation theory ([1, 9, 11]), we have

$$\|\boldsymbol{u} - \boldsymbol{u}^I\|_V \le c\,h\|\boldsymbol{u}\|_{2,\Omega},$$

$$\|\boldsymbol{u} - \boldsymbol{u}^I\|_{0,\Gamma_S} \le c\,h^2 \left(\sum_{l=1}^{l_S} \|\boldsymbol{u}\|^2_{2,\Gamma_{S,l}} \right)^{\frac{1}{2}},$$

and

$$\|p - p^I\|_{0,\Omega} \le c\,h\|p\|_{1,\Omega}.$$

Take $\boldsymbol{v}^h = \boldsymbol{u}^I$ and $q^h = p^I$ in (3.34), we deduce the optimal order error estimate (3.46). □

4 Numerical Examples

In this section, we report numerical results obtained using the local pressure projection stabilized mixed method. The main goal of the experiment is to verify the numerical convergence orders of (3.46) for the lowest equal order elements in two dimensions.

For the numerical examples, we let $\Omega = (0, 1)\times(0, 1)$, $\Gamma_S = (0, 1)\times\{0\}$ and $\Gamma_D = \Gamma\backslash\Gamma_S$, and use uniform triangulations. Here, the unit interval $[0, 1]$ is divided into $1/h$ equal parts, and we use h as the mesh-size. We choose $\nu = 1$. For positive parameters $a > b$ and α, we let

$$\mu(t) = (a - b)e^{-\alpha t} + b, \quad \psi(\boldsymbol{u}_\tau) = \int_0^{|\boldsymbol{u}_\tau|} \mu(t)\,dt.$$

Then the slip boundary condition $-\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau)$ from (3.4) is equivalent to

$$|\boldsymbol{\sigma}_\tau| \le \mu(0) \quad \text{if } \boldsymbol{u}_\tau = \boldsymbol{0}, \quad \boldsymbol{\sigma}_\tau = -\mu(|\boldsymbol{u}_\tau|)\,\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \quad \text{if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_S. \tag{4.1}$$

It can be verified that for this choice of ψ, $H(\psi)$ (iv) is satisfied with $m_\tau = \alpha(a - b)$.

Example 1 The source function is chosen as

$$\boldsymbol{f}(x, y) = \begin{bmatrix} 4\pi^2\,(\sin(2\pi x) + \sin(2\pi y) - 2\sin(2\pi y)\cos(2\pi x)) \\ -4\pi^2\,(\sin(2\pi x) + \sin(2\pi y) - 2\sin(2\pi x)\cos(2\pi y)) \end{bmatrix}.$$

We take $a = 9.01$, $b = 9.0$, and $\alpha = 10$ for the function ψ in the numerical tests.

Example 2 In this example, let

$$\boldsymbol{f}(x, y) = \begin{bmatrix} -20(6x^2 - 6x + 1)(2y^3 - 3y^2 + y) - 60x^2(x - 1)^2(2y - 1) + 2(2y - 1) \\ 20(2x^3 - 3x^2 + x)(6y^2 - 6y + 1) + 60(2x - 1)y^2(y - 1)^2 + 2(2x - 1) \end{bmatrix}.$$

We take $a = 0.255$, $b = 0.25$, and $\alpha = 10$ for the function ψ in the numerical tests.

Table 1 Numerical errors of the lowest equal order elements for Example 1

h	$\|\boldsymbol{u}^* - \boldsymbol{u}^h\|_{0,\Omega}$	Order	$\|\boldsymbol{u}^* - \boldsymbol{u}^h\|_{\boldsymbol{V}}$	Order	$\|p^* - p^h\|_{0,\Omega}$	Order
2^{-2}	4.8714e-01	–	5.2789	–	3.6049	–
2^{-3}	1.4299e-01	1.7684	2.7362	0.9481	1.1434	1.6566
2^{-4}	4.0183e-02	1.8313	1.3899	0.9771	4.3835e-01	1.3832
2^{-5}	1.0407e-02	1.9490	6.8917e-01	1.0121	1.5918e-01	1.4615

Table 2 Numerical errors of the lowest equal order elements for Example 2

h	$\|\boldsymbol{u}^* - \boldsymbol{u}^h\|_{0,\Omega}$	Order	$\|\boldsymbol{u}^* - \boldsymbol{u}^h\|_{\boldsymbol{V}}$	Order	$\|p^* - p^h\|_{0,\Omega}$	Order
2^{-2}	1.9256e-02	–	1.8702e-01	–	1.6517e-01	–
2^{-3}	7.0943e-03	1.4406	1.0589e-01	0.8206	8.0673e-02	1.0338
2^{-4}	1.7384e-03	2.0289	5.1189e-02	1.0486	2.4786e-02	1.7026
2^{-5}	4.6116e-04	1.9144	2.5812e-02	0.9878	8.7783e-03	1.4975

The discrete problem 3.3 is solved by an algorithm based on a sequence of convex programming problems, cf. [3]. The main idea of the iterative algorithm is to update the value of the function μ at each iteration with the numerical solution found in the previous iteration.

For initialization, choose a small value ε used in a stopping criterion of the iteration and choose an initial guess $(\boldsymbol{u}_0^h, p_0^h) \in \boldsymbol{V}^h \times Q^h$. Then for $\ell = 0, 1, \ldots$, find $(\boldsymbol{u}_{\ell+1}^h, p_{\ell+1}^h) \in \boldsymbol{V}^h \times Q^h$ as the solution of the following problem

$$
\begin{aligned}
& a(\boldsymbol{u}_{\ell+1}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_{\ell+1}^h) + \int_{\Gamma_S} \boldsymbol{\lambda}_{\ell+1}^h \cdot \boldsymbol{v}_\tau^h \, ds = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \\
& b(\boldsymbol{u}_{\ell+1}^h, q^h) + S^h(p_{\ell+1}^h, q^h) = 0 \quad \forall q^h \in Q^h,
\end{aligned}
$$

with

$$
\boldsymbol{\lambda}_{\ell+1}^h \in \mu(|\boldsymbol{u}_{\ell,\tau}^h|)\partial|\boldsymbol{u}_{\ell+1,\tau}^h| \quad \text{on } \Gamma_S
$$

until

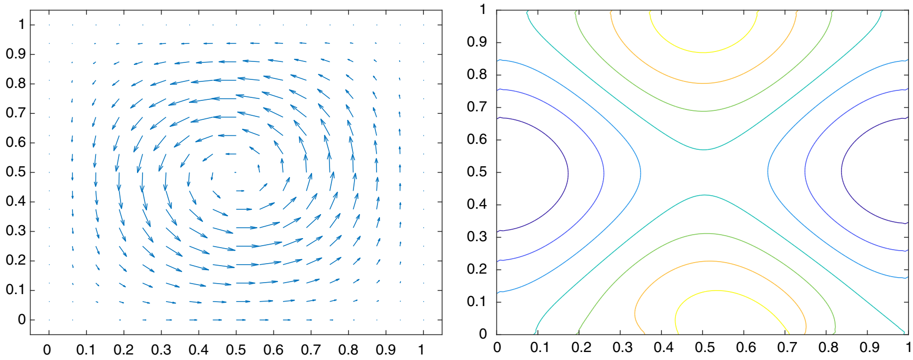
$$
\begin{aligned}
& \|\boldsymbol{u}_{\ell+1}^h - \boldsymbol{u}_\ell^h\|_{0,\Omega} \le \varepsilon \|\boldsymbol{u}_\ell^h\|_{0,\Omega}, \quad \|p_{\ell+1}^h - p_\ell^h\|_{0,\Omega} \le \varepsilon \|p_\ell^h\|_{0,\Omega}, \quad \text{and} \\
& \|\boldsymbol{\lambda}_{\ell+1}^h - \boldsymbol{\lambda}_\ell^h\|_{0,\Gamma_S} \le \varepsilon \|\boldsymbol{\lambda}_\ell^h\|_{0,\Gamma_S}.
\end{aligned}
$$

The sequence $\{\boldsymbol{\lambda}_\ell^h\}$ can be viewed as a sequence approximating a Lagrange multiplier $\boldsymbol{\lambda}^h$.

In the numerical experiment, we let $\varepsilon = 10^{-7}$ and determine the initial guess $(\boldsymbol{u}_0^h, p_0^h) \in \boldsymbol{V}^h \times Q^h$ as the solution of the problem

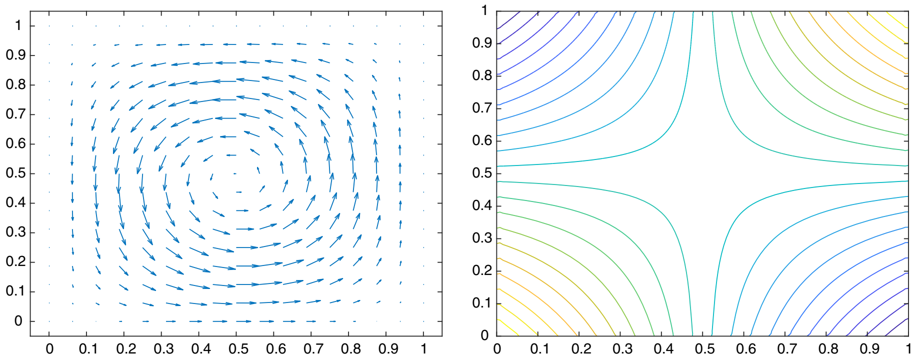
$$
\begin{aligned}
& a(\boldsymbol{u}_0^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_0^h) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \\
& b(\boldsymbol{u}_0^h, q^h) + S^h(p_0^h, q^h) = 0 \quad \forall q^h \in Q^h.
\end{aligned}
$$

Since the true solution is unknown, we use the numerical solution with $h = 2^{-8}$ as the reference solution $(\boldsymbol{u}^*, p^*)$ to compute the numerical solution errors. In Table 1-2, we report the errors $\|\boldsymbol{u}^* - \boldsymbol{u}^h\|_{0,\Omega}$, $\|\boldsymbol{u}^* - \boldsymbol{u}^h\|_{\boldsymbol{V}}$, and $\|p^* - p^h\|_{0,\Omega}$ for $h = 2^{-n}$, $2 \le n \le 5$. We observe that the numerical convergence orders of the velocity in the $\boldsymbol{V}$-norm are around 1 with



(a) Plot of velocity **(b)** Plot of pressure

Fig. 1 Numerical solution $(\boldsymbol{u}^h, p^h)$ in Example 1



(a) Plot of velocity **(b)** Plot of pressure

Fig. 2 Numerical solution $(\boldsymbol{u}^h, p^h)$ in Example 2

respect to the mesh-size h, which matches the theoretical result in Theorem 3.9. However, the numerical convergence orders for the pressure in the L^2-norm appear to be higher than the predicted order of one. The numerical solutions $(\boldsymbol{u}^h, p^h)$ are also shown in Figs. 1, 2.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

References

1. Atkinson, K., Han, W.: Theoretical Numerical Analysis: A Functional Analysis Framework, 3rd edn. Springer, New York (2009)
2. Barboteu, M., Bartosz, K., Han, W., Janiczko, T.: Numerical analysis of a hyperbolic hemivariational inequality arising in dynamic contact. SIAM J. Numer. Anal. **53**, 527–550 (2015)
3. Barboteu, M., Bartosz, K., Kalita, P.: An analytical and numerical approach to a bilateral contact problem with nonmonotone friction. Int. J. Appl. Math. Comput. Sci. **23**, 263–276 (2013)
4. Barth, T., Bochev, P.B., Gunzburger, M., Shadid, J.: A taxonomy of consistently stabilized finite element methods for the Stokes problem. SIAM J. Sci. Comput. **25**, 1585–1607 (2004)
5. Becker, R., Braack, M.: A finite elements pressure gradient stabilization for the Stokes equations based on local projections. Calc. **38**, 173–199 (2000)
6. Blasco, J., Codina, R.: Stabilization finite elements method for the transient Navier-Stokes equations based on a pressure gradient projection. Comput. Methods Appl. Mech. Engrg. **182**, 277–300 (2000)
7. Bochev, P.B., Dohrmann, C.R., Gunzburger, M.D.: Stabilization of low-order mixed finite elements for the Stokes equations. SIAM J. Numer. Anal. **44**, 82–101 (2006)
8. Bochev, P.B., Gunzburger, M.D.: An absolutely stable pressure-Poisson stabilized finite element method for the Stokes equations. SIAM J. Numer. Anal. **42**, 1189–1207 (2004)
9. Brenner, S.C., Scott, R.L.: The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics, Vol. 15, Springer-Verlag (2008)
10. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications. Springer, New York (2007)
11. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North Holland, Amsterdam (1978)
12. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, Interscience, New York (1983)
13. Fang, C., Czuprynski, K., Han, W., Cheng, X.L., Dai, X.: Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition. IMA J. Numer. Anal. **40**, 2696–2716 (2020)
14. Feng, F., Han, W., Huang, J.: Virtual element method for an elliptic hemivariational inequality with applications to contact mechanics. J. Sci. Comput. **81**, 2388–2412 (2019)
15. Granas, A., Dugundji, J.: Fixed Point Theory. Springer-Verlag, New York Inc (2003)
16. Han, W.: Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics. Math. Mech. Solids **23**, 279–293 (2018)
17. Han, W., Czuprynski, K., Jing, F.: Mixed finite element method for a hemivariational inequality of stationary Navier-Stokes equations. J. Sci. Comput. **89**, (2021), article number 8
18. Han, W., Migórski, S., Sofonea, M.: A class of variational-hemivariational inequalities with applications to frictional contact problems. SIAM J. Math. Anal. **46**, 3891–3912 (2014)
19. Han, W., Sofonea, M.: Numerical analysis of hemivariational inequalities in contact mechanics. Acta Numer. **28**, 175–286 (2019)
20. Han, W., Sofonea, M., Barboteu, M.: Numerical analysis of elliptic hemivariational inequalities. SIAM J. Numer. Anal. **55**, 640–663 (2017)
21. Han, W., Sofonea, M., Danan, D.: Numerical analysis of stationary variational-hemivariational inequalities. Numer. Math. **139**, 563–592 (2018)
22. Han, W., Wang, C.: Numerical analysis of a parabolic hemivariational inequality for semipermeable media. J. Comput. Appl. Math. **389**, (2021), article number 113326
23. Haslinger, J., Miettinen, M., Panagiotopoulos, P.D.: Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications. Kluwer Academic, Dordrecht (1999)
24. He, Y., Wang, A., Mei, L.: Stabilized finite element method for the stationary Navier-Stokes equations. J. Eng. Math. **51**, 367–380 (2005)
25. Jing, F., Han, W., Zhang, Y., Yan, W.: Analysis of an a posteriori error estimator for a variational inequality governed by the Stokes equations. J. Comput. Appl. Math. **372**, (2020), article number 112721
26. Li, J., Jing, F., Chen, Z., Liu, X.: A priori and a posteriori estimates of stabilized mixed finite volume methods for the incompressible flow arising in arteriosclerosis. J. Comput. Appl. Math. **363**, 35–52 (2020)
27. Li, Y., Li, K.: Pressure projection stabilized finite element method for Navier-Stokes equations with nonlinear slip boundary conditions. Comput. **87**, 113–133 (2010)
28. Li, Y., Li, K.: Pressure projection stabilized finite element method for Stokes problem with nonlinear slip boundary conditions. J. Comput. Appl. Math. **235**, 3673–3682 (2011)
29. Ling, M., Wang, F., Han, W.: The nonconforming virtual element method for a stationary Stokes hemivariational inequality with slip boundary condition. J. Sci. Comput. **85**, (2020), article number 56
30. Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems. Springer-Verlag, New York (2013)

31. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker, New York (1995)
32. Nečas, J., Hlaváček, I.: Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction. Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York (1981)
33. Panagiotopoulos, P.D.: Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer-Verlag, Berlin (1993)
34. Scott, R.L., Zhang, S.: Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comput. **54**, 483–493 (1990)
35. Silvester, D.J.: Optimal low order finite element methods for incompressible flow. Comput. Methods Appl. Mech. Engrg. **111**, 357–368 (1994)
36. Sofonea, M., Migórski, S.: Variational-Hemivariational Inequalities with Applications. Chapman & Hall/CRC Press, Boca Raton-London (2018)
37. Temam, R.: Navier-Stokes Equations: Theory and Numerical Analysis. North-Holland, Amsterdam (1979)
38. Wang, F., Qi, H.: A discontinuous Galerkin method for an elliptic hemivariational inequality for semipermeable media. Appl. Math. Lett. **109**, (2020), article number 106572
39. Wang, F., Wu, B., Han, W.: The virtual element method for general elliptic hemivariational inequalities. J. Comput. Appl. Math. **389**, (2021), article number 113330
40. Xu, W., Huang, Z., Han, W., Chen, W., Wang, C.: Numerical analysis of history-dependent hemivariational inequalities and applications to viscoelastic contact problems with normal penetration. Comput. Math. Appl. **77**, 2596–2607 (2019)