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Minimization principles for elliptic hemivariational inequalities

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ABSTRACT

In this paper, we explore conditions under which certain elliptic hemivariational inequalities permit equivalent minimization principles. It is shown that for an elliptic variational–hemivariational inequality, under the usual assumptions that guarantee the solution existence and uniqueness, if an additional condition is satisfied, the solution of the variational–hemivariational inequality is also the minimizer of a corresponding energy functional. Then, two variants of the equivalence result are given, that are more convenient to use for applications in contact mechanics and in numerical analysis of the variational–hemivariational inequality. When the convex terms are dropped, the results on the elliptic variational–hemivariational inequalities are reduced to that on "pure" elliptic hemivariational inequalities. Finally, two representative examples from contact mechanics are discussed to illustrate application of the theoretical results.

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1. Introduction

Since the pioneering work of Panagiotopoulos in early 1980s (cf. [1]), hemivariational inequalities have been shown to be a powerful mathematical tool in modeling and studying application problems where non-smooth, non-monotone and multi-valued relations among different physical quantities are involved. Throughout the years, a rich collection of references has emerged on hemivariational inequalities. The mathematical theory of hemivariational inequalities is documented in detail in some research monographs (e.g., [2–7]) and in many journal articles. Since no closed-form solution formula can be expected for a hemivariational inequality arising in applications, numerical methods are needed to solve hemivariational inequalities. An early comprehensive reference on the numerical solution of hemivariational inequalities is [8] where convergence of finite element solutions and solution algorithms are discussed. Starting with [9], a series of recent papers are devoted to derivation of optimal order error bounds for finite element solutions of various hemivariational inequalities, e.g. [10–15]. The reference [16] provides an optimal order error bound for virtual element solutions of hemivariational inequalities in contact mechanics.

Because non-convex terms are present, in general, a hemivariational inequality represents a substationarity condition and not an optimality condition (cf. [2]). Nevertheless, non-smooth optimization methods

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have been applied to find substationary points of discrete potential functions of a hemivariational inequality (cf. [17,18]). In [19], a hemivariational inequality is studied numerically approximated through an intermediate optimization problem. The aim of this paper is to explore the possibility of existence of equivalent minimization problems for some elliptic hemivariational inequalities. Existence of equivalent minimization principles justifies the use of optimization techniques to solve hemivariational inequalities. In this paper, a hemivariational inequality refers to an inequality problem involving the Clarke generalized directional derivative or generalized subdifferential of a locally Lipschitz continuous functional that is generally not convex. When both nonsmooth convex and nonconvex functionals are present in the formulation, the hemivariational inequality is also called a variational–hemivariational inequality in the literature. In the case where no nonsmooth convex functionals are present in the formulation, we have a "pure" hemivariational inequality.

The organization of the rest of the paper is as follows. In Section 2, basic notions and results are recalled on convex subdifferentials, the generalized directional derivative and generalized subdifferential in the sense of Clarke, and on their properties. In Section 3, under certain conditions, the solution of an elliptic variational–hemivariational inequality is shown to be the minimizer of a minimization problem. In Section 4, the equivalence result in Section 3 is rephrased for elliptic variational–hemivariational inequalities in forms more convenient to use in applications or when their numerical approximation is concerned. When the nonsmooth convex functionals are dropped, the results for elliptic variational–hemivariational inequalities are reduced to those for "pure" elliptic hemivariational inequalities, and we state one such representative result in Section 4. Finally in Section 5, we discuss two representative examples from contact mechanics to illustrate application of the theoretical results.

2. Preliminaries

Study of hemivariational inequalities makes use of the generalized directional derivative and generalized subdifferential in the sense of Clarke [20,21] that we recall here. Let V be a real Banach space, and let $\Psi\colon V \to \mathbb{R}$ be locally Lipschitz continuous. Then the generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is

$$\Psi^0(u;v) := \limsup_{w\to u,\,\lambda\downarrow 0} \frac{\Psi(w+\lambda v)-\Psi(w)}{\lambda}.$$

The generalized subdifferential of Ψ at $u \in V$ is

$$\partial\Psi(u) := \left\{\eta \in V^* \mid \Psi^0(u;v) \geq \langle \eta, v\rangle \ \forall\, v \in V\right\}.$$

For variational–hemivariational inequalities, we also need the notion of convex subdifferential. Let $\Phi\colon V \to \mathbb{R}\cup\{+\infty\}$ be proper, convex and l.s.c. (lower semicontinuous). Then the (convex) subdifferential of Φ at $u \in V$ is

$$\partial\Phi(u) := \{\xi \in V^* \mid \Phi(v)-\Phi(u) \geq \langle \xi, v-u\rangle \ \forall\, v \in V\}.$$

If $\partial\Phi(u)$ is non-empty, then any element $\xi \in \partial\Phi(u)$ is called a subgradient of Φ at u.

Basic properties of the generalized directional derivative and the generalized gradient are recorded in the next two propositions (cf. [21,22]).

Proposition 2.1. *Assume that $\Psi\colon V \to \mathbb{R}$ is a locally Lipschitz function. Then the following statements are valid.*

(i) *For every $u \in V$, the function $V \ni v \mapsto \Psi^0(u;v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $\Psi^0(u;\lambda v) = \lambda\,\Psi^0(u;v)$ for all $\lambda \geq 0$, $v \in V$, and $\Psi^0(u;v_1+v_2) \leq \Psi^0(u;v_1)+\Psi^0(u;v_2)$ for all v_1, $v_2 \in V$, respectively.*

(ii) *For every $v \in V$, we have* $\Psi^0(u;v) = \max\{\langle \eta, v\rangle \mid \eta \in \partial\Psi(u)\}$.

(iii) *The function* $V \times V \ni (u,v) \mapsto \Psi^0(u;v) \in \mathbb{R}$ *is upper semi-continuous, i.e., for all* $u, v \in V$, $\{u_n\}, \{v_n\} \subset V$ *such that* $u_n \to u$ *and* $v_n \to v$ *in* V, *we have* $\limsup \Psi^0(u_n;v_n) \le \Psi^0(u;v)$.

(iv) *For every $u \in V$, the generalized gradient $\partial\Psi(u)$ is a nonempty, convex, and weakly* compact subset of V^*.*

(v) *The graph of the generalized gradient $\partial\Psi$ is closed in $V \times V^*_{w^*}$ topology, i.e., if $\{u_n\} \subset V$ and $\{\eta_n\} \subset V^*$ are sequences such that $\eta_n \in \partial\Psi(u_n)$ and $u_n \to u$ in V, $\eta_n \to \eta$ weakly* in V^*, then $\eta \in \partial\Psi(u)$.*

(vi) *If $\Psi: V \to \mathbb{R}$ is convex, then the subdifferential $\partial\Psi(u)$ in the sense of Clarke at any $u \in V$ coincides with the convex subdifferential $\partial\Psi(u)$.*

Because of Proposition 2.1(vi), we use the same symbol ∂ for the subdifferential both in the sense of Clarke and in convex analysis.

Proposition 2.2. *Let Ψ, Ψ_1, $\Psi_2 : V \to \mathbb{R}$ be locally Lipschitz functions. Then:*

(i) (scalar multiples) *The equality $\partial(\lambda\Psi)(u) = \lambda\,\partial\Psi(u)$ holds, for all $\lambda \in \mathbb{R}$ and all $u \in V$.*

(ii) (sum rules) *The inclusion*

$$\partial(\Psi_1 + \Psi_2)(u) \subseteq \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{2.1}$$

holds, or equivalently,

$$(\Psi_1 + \Psi_2)^0(u;v) \le \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{2.2}$$

Also needed is a lower bound result for a convex function (cf. [23, Lemma 11.3.5] or [24, Prop. 5.2.25]).

Lemma 2.3. *Let V be a normed space and let $\Phi: V \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and l.s.c. Then there exist a continuous linear functional $\ell_\Phi \in V^*$ and a constant $c \in \mathbb{R}$ such that*

$$\Phi(v) \ge \ell_\Phi(v) + c \quad \forall\, v \in V.$$

Consequently, there exist two constants $\overline{c}_0$ and $\overline{c}_1$ such that

$$\Phi(v) \ge \overline{c}_0 + \overline{c}_1 \|v\|_V \quad \forall\, v \in V. \tag{2.3}$$

We quote a result for a sufficient and necessary condition on strong convexity of a locally Lipschitz continuous function, characterized by the strong monotonicity of its generalized subdifferential in the sense of Clarke.

Lemma 2.4. *Let V be a real Banach space, K be a non-empty convex set in V, and $g: K \to \mathbb{R}$ be locally Lipschitz continuous. Then g is strongly convex on K with a constant $\alpha > 0$, i.e.,*

$$g(\lambda u + (1-\lambda)v) \le \lambda\, g(u) + (1-\lambda)\, g(v) - \alpha\,\lambda\,(1-\lambda)\,\|u - v\|_V^2 \quad \forall\, u, v \in K,\ \forall\, \lambda \in [0,1],$$

if and only if ∂g is strongly monotone on K with a constant 2α, i.e.,

$$\langle \xi - \eta, u - v\rangle \ge 2\,\alpha\,\|u - v\|_V^2 \quad \forall\, u, v \in K,\ \xi \in \partial g(u),\ \eta \in \partial g(v).$$

This result is stated in [25, Proposition 3.1] for K open, a condition that can be removed.

From now on, we will assume V to be a real Hilbert space. As a corollary of Lemmas 2.3 and 2.4, we have the next result.

Proposition 2.5. *Let V be a real Hilbert space, K be a non-empty closed convex set in V, and $g\colon K \to \mathbb{R}$ be a locally Lipschitz continuous and strongly convex functional on K with a constant $\alpha > 0$. Then there exist two constants $\overline{c}_0$ and $\overline{c}_1$ such that*

$$g(v) \geq \alpha \|v\|_V^2 + \overline{c}_0 + \overline{c}_1 \|v\|_V \quad \forall\, v \in K. \tag{2.4}$$

Consequently, $g(\cdot)$ is coercive on K.

Proof. It is easy to see that $g\colon K \to \mathbb{R}$ is strongly convex on K with a constant $\alpha > 0$ if and only if the functional $g(v) - \alpha \|v\|_V^2$ is convex on K. Consider the extended functional

$$\varPhi(v) = \begin{cases} g(v) - \alpha \|v\|_V^2, & v \in K, \\ +\infty, & v \in V \backslash K, \end{cases}$$

which is proper, convex and l.s.c. By Lemma 2.3, we have two constants $\overline{c}_0$ and $\overline{c}_1$ such that (2.3) is valid. Therefore, (2.4) holds. Since $\alpha > 0$, the coercivity of $g(\cdot)$ on K follows from (2.4). ■

Note that strong convexity of a functional implies its strict convexity.

Finally, recall that (cf. [26, Section 41.3]) an operator $A\colon V \to V^*$ is called a potential operator if there exists a Gâteaux differentiable functional $F_A\colon V \to \mathbb{R}$ such that $A = F_A'$; the functional F_A is called a potential of A. It can be shown that if A is hemicontinuous, then A is a potential operator if and only if

$$F_A(u) - F_A(v) = \int_0^1 \langle A(v + t\,(u - v)), u - v\rangle\, dt \quad \forall\, u, v \in V,$$

where

$$F_A(v) = \int_0^1 \langle A(t\,v), v\rangle\, dt. \tag{2.5}$$

In this case, (2.5) defines a potential of A, and any other potential of A differs from F_A by a constant. In addition, if A is Gâteaux differentiable and the mapping

$$(t, s) \mapsto \langle A'(v_1 + t\, v_2 + s\, v_3)\, v_4, v_5\rangle$$

is continuous on $[0, 1] \times [0, 1]$ for all $v_i \in V$, $1 \leq i \leq 5$, then A is potential if and only if

$$\langle A'(u)\, v, w\rangle = \langle A'(u)\, w, v\rangle \quad \forall\, u, v, w \in V.$$

In particular, $A \in \mathcal{L}(V, V^*)$ is a potential operator if and only if it is symmetric:

$$\langle A v_1, v_2\rangle = \langle A v_2, v_1\rangle \quad \forall\, v_1, v_2 \in V. \tag{2.6}$$

Moreover, under the symmetry condition (2.6), a potential functional is

$$F_A(v) = \frac{1}{2}\, \langle A v, v\rangle, \quad v \in V.$$

Throughout the paper, we will use c to denote a generic positive constant whose value may change from one place to another but it is independent of other quantities of concern.

3. Minimization principle for a variational–hemivariational inequality

Consider the elliptic variational–hemivariational inequality:

Problem 3.1. Find $u \in K$ such that

$$\langle Au, v - u\rangle + \varPhi(v) - \varPhi(u) + \varPsi^0(u; v - u) \geq \langle f, v - u\rangle \quad \forall\, v \in K. \tag{3.1}$$

In the study of Problem 3.1, we introduce the following assumptions on the problem data.

$H(K)$ V is a real Hilbert space, K is a non-empty, closed and convex set in V.

$H(A)$ $A\colon V \to V^*$ is Lipschitz continuous and strongly monotone:

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.2}$$

$H(\Phi)$ $\Phi\colon V \to \mathbb{R}$ is convex and continuous.

$H(\Psi)$ $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous, and there exist non-negative constants c_0, c_1, α_Ψ such that

$$\|\partial \Psi(v)\|_{V^*} \le c_0 + c_1 \|v\|_V \quad \forall\, v \in V, \tag{3.3}$$

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \le \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.4}$$

$H(f)$ $f \in V^*$.

Here (3.3) is understood in the sense that

$$\|\eta\|_{V^*} \le c_0 + c_1 \|v\|_V \quad \forall\, v \in V,\ \forall\, \eta \in \partial \Psi(v).$$

It is known that under the assumptions $H(K)$, $H(A)$, $H(\Phi)$, $H(\Psi)$, $H(f)$ and the smallness condition

$$\alpha_\Psi < m_A, \tag{3.5}$$

Problem 3.1 has a unique solution. This result, in a somewhat different form, was first proved in [27]; see also [13].

Remark 3.2. The assumptions on the data are stated in a form more accessible and more convenient to verify in applications, although from the theoretical view-point, they are somewhat stronger than necessary. In references on solution existence and uniqueness studies, e.g. [27], the operator A is assumed to be pseudomonotone and strongly monotone, whereas $\Phi\colon V \to \mathbb{R} \cup \{+\infty\}$ is assumed to be proper, convex and l.s.c. We note that the assumption $H(A)$ is easy to verify and it implies that the operator A is pseudomonotone and strongly monotone. With application problems in mind, we consider nonsmooth convex functionals of the form $\Phi + I_K$ where $\Phi\colon V \to \mathbb{R}$ is convex and l.s.c., and I_K is the indicator function of the convex set K. Note that a l.s.c. convex function $\Phi\colon V \to \mathbb{R}$ on the Hilbert space V is continuous [28]. Hence, we introduce the assumption $H(\Phi)$.

We also comment that in the well-posedness theory for variational–hemivariational inequalities such as Problem 3.1, it is common to assume that V is a reflexive Banach space. In this paper, however, we explore existence of minimization principles for variational–hemivariational inequalities and assume V is a Hilbert space.

We note that (3.4) is equivalent to the following (cf. [7, p. 124]):

$$\langle \eta_1 - \eta_2, v_1 - v_2\rangle \ge -\alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V,\ \eta_1 \in \partial \Psi(v_1),\, \eta_2 \in \partial \Psi(v_2). \tag{3.6}$$

Let us explore conditions under which Problem 3.1 has an equivalent minimization principle. For this purpose, we further assume $A\colon V \to V^*$ to be a potential operator with a potential F_A.

Problem 3.3. Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} E(v) \tag{3.7}$$

where

$$E(v) = F_A(v) + \Phi(v) + \Psi(v) - \langle f, v\rangle, \quad v \in V. \tag{3.8}$$

We first present some properties on the energy functional $E(\cdot)$.

Proposition 3.4. *Assume $H(K)$, $H(A)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, (3.5), and assume $A\colon V \to V^*$ is a potential operator with the potential $F_A(\cdot)$ defined in (2.5). Then $E(\cdot)$ is locally Lipschitz and strongly convex on V. Moreover, $E(\cdot)$ is coercive on K.*

Proof. Since each term on the right side of (3.8) in defining $E(\cdot)$ is locally Lipschitz on V, so is $E(\cdot)$. For the generalized subdifferential of $E(\cdot)$, we repeatedly apply the summation rule (2.1), and note that for a locally Lipschitz continuous convex functional, the generalized subdifferential and the convex subdifferential coincide. As a result, we have

$$\partial E(v) \subset Av + \partial \Phi(v) + \partial \Psi(v) - f \quad \forall\, v \in V. \tag{3.9}$$

For the strong convexity of $E(\cdot)$, by Lemma 2.4, it is sufficient to prove the strong monotonicity of $\partial E(\cdot)$. For any $v_1, v_2 \in V$, let $\zeta_i \in \partial E(v_i)$, $i = 1, 2$. Write

$$\zeta_i = Av_i + \xi_i + \eta_i - f, \quad \xi_i \in \partial \Phi(v_i),\ \eta_i \in \partial \Psi(v_i).$$

Then we have

$$\begin{aligned}\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle &= \langle Av_1 - Av_2, v_1 - v_2\rangle + \langle \xi_1 - \xi_2, v_1 - v_2\rangle + \langle \eta_1 - \eta_2, v_1 - v_2\rangle \\ &\ge m_A \|v_1 - v_2\|_V^2 + 0 - \alpha_\Psi \|v_1 - v_2\|_V^2 \\ &= (m_A - \alpha_\Psi)\, \|v_1 - v_2\|_V^2.\end{aligned}$$

By (3.5), $m_A - \alpha_\Psi > 0$. Hence, $\partial E(\cdot)$ is strongly monotone and thus $E(\cdot)$ is strongly convex on V.

Applying Proposition 2.5, we know that $E(\cdot)$ is coercive on K. ■

Now we are ready to present the main result of the section, on the equivalence between Problems 3.1 and 3.3.

Theorem 3.5. *Assume $H(K)$, $H(A)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, (3.5), and assume that $A\colon V \to V^*$ is a potential operator with the potential $F_A(\cdot)$. Then Problem 3.3 has a unique solution which is also the unique solution of Problem 3.1.*

Proof. By Proposition 3.4, $E(\cdot)$ is continuous, strictly convex and coercive on K. Therefore, Problem 3.3 has a unique solution $u \in K$ (cf. [23, §3.3.2]). This solution is characterized by the condition

$$0 \in \partial\,(E(u) + I_K(u)) \subset \partial E(u) + \partial I_K(u),$$

where $I_K(\cdot)$ is the indicator function of the set K. By (3.9), this condition is equivalent to (3.1), i.e., u is a solution of Problem 3.1. Since Problem 3.1 admits a unique solution, we conclude that Problem 3.3 has a unique solution which is also the unique solution of Problem 3.1. ■

4. Some variants

We provide some variants of the theoretical results shown in the previous section.

First, instead of Problem 3.1, we consider its following variant (cf. [11]).

Problem 4.1. Find $u \in K$ such that

$$\langle Au, v - u\rangle + \Phi(\gamma_\Phi v) - \Phi(\gamma_\Phi u) + \Psi^0(\gamma_\Psi u; \gamma_\Psi v - \gamma_\Psi u) \ge \langle f, v - u\rangle \quad \forall\, v \in K. \tag{4.1}$$

The corresponding minimization problem is

Problem 4.2. Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} E(v) \tag{4.2}$$

where

$$E(v) = F_A(v) + \Phi(\gamma_\Phi v) + \Psi(\gamma_\Psi v) - \langle f, v\rangle. \tag{4.3}$$

We keep the assumptions $H(K)$, $H(A)$, and modify $H(\Phi)$, $H(\Psi)$ as follows.
$H(\Phi)'$ V_Φ is a real Banach space, $\gamma_\Phi \in \mathcal{L}(V; V_\Phi)$, $\Phi\colon V_\Phi \to \mathbb{R}$ is convex and continuous.
$H(\Psi)'$ V_Ψ is a real Banach space, $\gamma_\Psi \in \mathcal{L}(V; V_\Psi)$, $\Psi\colon V_\Psi \to \mathbb{R}$ is locally Lipschitz continuous, and there exist constants c_0, c_1, α_Ψ such that

$$\|\partial\Psi(v)\|_{V_\Psi^*} \le c_0 + c_1\|v\|_{V_\Psi} \quad \forall\, v \in V_\Psi, \tag{4.4}$$

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \le \alpha_\Psi \|v_1 - v_2\|_{V_\Psi}^2 \quad \forall\, v_1, v_2 \in V_\Psi. \tag{4.5}$$

An analogue of Theorem 3.5 is the following.

Theorem 4.3. *Assume $H(K)$, $H(A)$, $H(\Phi)'$, $H(\Psi)'$, and $H(f)$. In addition assume that $A\colon V \to V^*$ is a potential operator with the potential $F_A(\cdot)$ and*

$$\alpha_\Psi \|\gamma_\Psi\|^2 < m_A. \tag{4.6}$$

Then Problem 4.2 has a unique solution which is also the unique solution of Problem 4.1.

Now consider another variant of Problem 3.1. In most applications, a hemivariational inequality describes a physical problem on a spatial domain in a finite-dimensional Euclidean space. Denote by Δ the domain or its sub-domain, or its boundary or part of the boundary. Let I_Δ stand for the integration over Δ. Instead of the space V_Ψ, we will need a space V_ψ and a linear operator γ_ψ from V to V_ψ. Assume elements from the space V_ψ are $\mathbb{R}^m$-valued functions, for some positive integer m.

Problem 4.4. Find $u \in K$ such that

$$\langle Au, v - u\rangle + \Phi(v) - \Phi(u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \ge \langle f, v - u\rangle \quad \forall\, v \in K. \tag{4.7}$$

The corresponding minimization problem is

Problem 4.5. Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} E(v) \tag{4.8}$$

where

$$E(v) = F_A(v) + \Phi(v) + I_\Delta(\psi(\gamma_\psi v)) - \langle f, v\rangle. \tag{4.9}$$

In the study of Problems 4.4 and 4.5, we will replace the assumption $H(\Psi)$ by $H(\psi)$:
$H(\psi)$ V_ψ is a real Banach space; $\gamma_\psi \in \mathcal{L}(V; V_\psi)$; $\psi\colon \Delta \times \mathbb{R}^m \to \mathbb{R}$; $\psi(\cdot, v)$ is measurable on Δ for all $v \in \mathbb{R}^m$; $\psi(\cdot, v(\cdot)) \in L^1(\Delta)$ for all $v \in V_\psi$; $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^m$ for a.e. $\boldsymbol{x} \in \Delta$, and there exist non-negative constants $\tilde{c}_0, \tilde{c}_1, \alpha_\psi$ such that

$$|\partial\psi(v)|_{\mathbb{R}^m} \le \tilde{c}_0 + \tilde{c}_1 |v|_{\mathbb{R}^m} \quad \forall\, v \in \mathbb{R}^m, \tag{4.10}$$

$$\psi^0(v_1; v_2 - v_1) + \psi^0(v_2; v_1 - v_2) \le \alpha_\psi |v_1 - v_2|_{\mathbb{R}^m}^2 \quad \forall\, v_1, v_2 \in \mathbb{R}^m. \tag{4.11}$$

Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|_{\mathbb{R}^m}^2) \le c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V. \tag{4.12}$$

Define the functional

$$\Psi(v) = I_\Delta(\psi(\gamma_\psi v)), \quad v \in V.$$

Then ([6, Section 3.3])

$$\Psi^0(u; v) \le I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \quad \forall\, u, v \in V. \tag{4.13}$$

Thus, for $u, v \in V$, by using (4.10) and (4.12), we have

$$\begin{aligned} \Psi^0(u; v) &\le I_\Delta((\tilde{c}_0 + \tilde{c}_1 |\gamma_\psi u|_{\mathbb{R}^m})\, |\gamma_\psi v|_{\mathbb{R}^m}) \\ &\le \left(\tilde{c}_0 \sqrt{|\Delta|} + \tilde{c}_1 \|\gamma_\psi u\|_{L^2(\Delta)}\right) \|\gamma_\psi v\|_{L^2(\Delta)} \\ &\le \left(\tilde{c}_0 c_\Delta \sqrt{|\Delta|} + \tilde{c}_1 c_\Delta^2 \|u\|_V\right) \|v\|_V, \end{aligned}$$

i.e., (3.3) is satisfied with $c_0 = \tilde{c}_0 c_\Delta \sqrt{|\Delta|}$, $c_1 = \tilde{c}_1 c_\Delta^2$. Similarly, for $v_1, v_2 \in V$, by using (4.11) and (4.12),

$$\begin{aligned} \Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) &\le I_\Delta(\psi^0(\gamma_\psi v_1; \gamma_\psi v_2 - \gamma_\psi v_1) + \psi^0(\gamma_\psi v_2; \gamma_\psi v_1 - \gamma_\psi v_2)) \\ &\le I_\Delta(\alpha_\psi |\gamma_\psi(v_1 - v_2)|^2_{\mathbb{R}^m}) \\ &\le \alpha_\psi c_\Delta^2 \|v_1 - v_2\|_V^2, \end{aligned}$$

i.e., (3.4) is satisfied with $\alpha_\Psi = \alpha_\psi c_\Delta^2$.

With the above preparations, we derive from Theorem 3.5 the next result.

Theorem 4.6. *Assume $H(K)$, $H(A)$, $H(\Phi)$, $H(\psi)$, $H(f)$, $A\colon V \to V^*$ is a potential operator with the potential $F_A(\cdot)$, and $\alpha_\psi c_\Delta^2 < m_A$. Then Problems 4.4 and 4.5 are equivalent in the sense that they have the same unique solution.*

We now consider the particular case of hemivariational inequalities, i.e., the case where $\Phi \equiv 0$. Corresponding to Problems 3.1 and 3.3, we consider the problems:

Problem 4.7. Find $u \in K$ such that

$$\langle Au, v - u\rangle + \Psi^0(u; v - u) \ge \langle f, v - u\rangle \quad \forall\, v \in K. \tag{4.14}$$

Problem 4.8. Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} E(v) \tag{4.15}$$

where

$$E(v) = F_A(v) + \Psi(v) - \langle f, v\rangle. \tag{4.16}$$

The counterpart of Theorem 3.5 is the next result.

Theorem 4.9. *Assume $H(K)$, $H(A)$, $H(\Psi)$, $H(f)$, (3.5), and that $A\colon V \to V^*$ is a potential operator with the potential $F_A(\cdot)$. Then Problem 4.8 has a unique solution which is also the unique solution of Problem 4.7.*

As another particular case, consider Problems 4.4 and 4.5 with $\Phi = 0$.

Problem 4.10. Find $u \in K$ such that

$$\langle Au, v - u\rangle + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \ge \langle f, v - u\rangle \quad \forall\, v \in K. \tag{4.17}$$

Problem 4.11. Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} E(v) \tag{4.18}$$

where

$$E(v) = F_A(v) + I_\Delta(\psi(\gamma_\psi v)) - \langle f, v \rangle. \tag{4.19}$$

Then from Theorem 4.6, we have the following equivalence result.

Theorem 4.12. *Assume $H(K)$, $H(A)$, $H(\psi)$, $H(f)$, $\alpha_\psi c_\Delta^2 < m_A$, and that $A\colon V \to V^*$ is a potential operator with the potential $F_A(\cdot)$. Then Problems 4.10 and 4.11 are equivalent in the sense that they have the same unique solution.*

Note that we can get similar results for further special cases where $K = V$. Since it is trivial to deduce results for such further special cases, we omit the detail.

5. Applications in contact mechanics

We illustrate application of the theoretical results developed in previous sections on two representative contact problems between a linear elastic body and a rigid foundation in this section. The contact problems are studied in e.g. [29]. Let Ω represent the reference configuration of the elastic body. We assume Ω is open, bounded and connected in $\mathbb{R}^d$ $(d \le 3)$ with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. Then the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on Γ.

The displacement is a vector in $\mathbb{R}^d$, whereas the stress and strain tensors belong to the space $\mathbb{S}^d$ of second order symmetric tensors on $\mathbb{R}^d$. Over the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, we use "$\cdot$" and "$|\cdot|$" for the canonical inner products and the induced norms. The linearized strain tensor associated with a displacement field $\boldsymbol{u}$ is denoted by $\boldsymbol{\varepsilon}(\boldsymbol{u})$. For a vector field $\boldsymbol{v}$, we use $v_\nu := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ for the normal and tangential components of $\boldsymbol{v}$ on Γ. Similarly, for the stress field $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are defined as $\sigma_\nu := (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively.

Let $\mathbb{C}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ be the elasticity operator of the elastic body. As usual in the literature, we assume $\mathbb{C} = (C_{ijkl})_{1 \le i,j,k,l \le d}$ is symmetric, bounded, and pointwise stable:

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad 1 \le i, j, k, l \le d, \tag{5.1}$$

$$C_{ijkl} \in L^\infty(\Omega), \quad 1 \le i, j, k, l \le d, \tag{5.2}$$

$$(\mathbb{C}\boldsymbol{\epsilon}) \cdot \boldsymbol{\epsilon} \ge m_{\mathbb{C}} |\boldsymbol{\epsilon}|^2, \quad m_{\mathbb{C}} > 0, \ \forall\, \boldsymbol{\epsilon} \in \mathbb{S}^d. \tag{5.3}$$

Then the elastic constitutive law is

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega. \tag{5.4}$$

To describe boundary conditions for the contact problem, we partition the boundary Γ into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that meas $(\Gamma_1) > 0$ and meas $(\Gamma_3) > 0$. We assume the body is fixed on Γ_1, is in equilibrium under the action of a total body force of density $\boldsymbol{f}_0$ in Ω and a surface traction of density $\boldsymbol{f}_2$ on Γ_2, and is in potential contact on Γ_3 with the rigid foundation. Then the equilibrium equation of the body is

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega. \tag{5.5}$$

The homogeneous displacement boundary condition is specified on Γ_1:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.6}$$

and the traction boundary condition is specified on Γ_2:

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{5.7}$$

Different contact conditions on Γ_3 will lead to different hemivariational inequalities.

To study the contact problems, we will use the space

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \right\} \tag{5.8}$$

and its subspace/subset for the displacement fields. Since meas $(\Gamma_1) > 0$, Korn's inequality (cf. [30]) implies that V is a Hilbert space with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \quad \boldsymbol{u},\boldsymbol{v} \in V,$$

and the associated norm $\|\cdot\|_V$ is equivalent to the standard $H^1(\Omega;\mathbb{R}^d)$-norm over V. For $\boldsymbol{v} \in V$, we use the same symbol $\boldsymbol{v}$ for its trace on Γ. We use $Q = L^2(\Omega;\mathbb{S}^d)$ as the space for the stress and strain fields; this is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx$$

and the associated norm $\|\cdot\|_Q$.

We assume on the densities of body forces and surface tractions that

$$\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2;\mathbb{R}^d) \tag{5.9}$$

and define $\boldsymbol{f} \in V^*$ by the formula

$$\langle \boldsymbol{f},\boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0,\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2,\boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\,\boldsymbol{v} \in V. \tag{5.10}$$

Example 5.1. In the first example, we consider a bilateral contact problem with friction. The contact boundary conditions are

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_3. \tag{5.11}$$

The feature of bilateral contact is reflected by the condition $u_\nu = 0$. We assume the potential function $\psi_\tau\colon \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ has the following properties.

$$\left\{\begin{array}{l}
\text{(a) } \psi_\tau(\cdot,\boldsymbol{z}) \text{ is measurable on } \Gamma_3 \text{ for all } \boldsymbol{z} \in \mathbb{R}^d \text{ and } \psi_\tau(\cdot,\boldsymbol{0}) \in L^1(\Gamma_3);\\
\text{(b) } \psi_\tau(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Gamma_3;\\
\text{(c) } |\partial\psi_\tau(\boldsymbol{x},\boldsymbol{z})| \le \bar{c}_0 + \bar{c}_1\|\boldsymbol{z}\| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3,\\
\qquad \text{for all } \boldsymbol{z} \in \mathbb{R}^d \text{ with } \bar{c}_0,\,\bar{c}_1 \ge 0;\\
\text{(d) } \psi_\tau^0(\boldsymbol{x},\boldsymbol{z}_1;\boldsymbol{z}_2-\boldsymbol{z}_1) + \psi_\tau^0(\boldsymbol{x},\boldsymbol{z}_2;\boldsymbol{z}_1-\boldsymbol{z}_2) \le \alpha_{\psi_\tau}\|\boldsymbol{z}_1-\boldsymbol{z}_2\|^2\\
\qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } \boldsymbol{z}_1,\boldsymbol{z}_2 \in \mathbb{R}^d \text{ with } \alpha_{\psi_\tau} \ge 0.
\end{array}\right. \tag{5.12}$$

With V from (5.8), we define the subspace

$$V_1 = \{\boldsymbol{v} \in V \mid v_\nu = 0 \text{ on } \Gamma_3\}. \tag{5.13}$$

We take $V_\psi = L^2(\Gamma_3;\mathbb{R}^d)$, $m = d$, and let $\gamma_\psi\colon V \to V_\psi$ be the tangential component trace operator: $\gamma_\psi\boldsymbol{v} = \boldsymbol{v}_\tau$ for $\boldsymbol{v} \in V$. Then $\gamma_\psi \in \mathcal{L}(V;V_\psi)$. Let $c_\psi = \lambda_1^{-1/2}$ where $\lambda_1 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V_1, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda\int_{\Gamma_3} \boldsymbol{u}_\tau\cdot\boldsymbol{v}_\tau da \quad \forall\,\boldsymbol{v} \in V_1.$$

Then

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_\psi \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V_1. \tag{5.14}$$

The weak formulation of the contact problem defined by (5.5)–(5.7) and (5.11) is the following (note that V_1 is a subspace).

Problem 5.2. Find a displacement field $\boldsymbol{u} \in V_1$ such that

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in V_1. \tag{5.15}$$

Problem 5.2 has a unique solution under the stated assumptions on the problem data and

$$\alpha_{\psi_\tau} < \lambda_1 m_{\mathbb{C}}.$$

Let us apply Theorem 4.12 and first we examine the assumptions of the theorem. The set V_1 defined by (5.13) obviously satisfies $H(K)$. The operator $A\colon V \to V^*$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$$

satisfies $H(A)$. Moreover, A is symmetric, thanks to (5.1)–(5.3). So $A\colon V \to V^*$ is a potential operator with the potential

$$F_A(\boldsymbol{v}) = \frac{1}{2}\,(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \quad \boldsymbol{v} \in V.$$

The assumption $H(\psi)$ follows from (5.12), and $H(f)$ is valid due to (5.9). Note that $m_A = m_{\mathbb{C}}$, and α_{ψ_τ} is given in (5.12)(d). Moreover, $c_\Delta = c_\psi = \lambda_1^{-1/2}$. Therefore, under the stated conditions, Problem 5.2 is equivalent to the minimization problem:

Problem 5.3. Find a displacement field $u \in V_1$ such that

$$E(u) \le E(v) \quad \forall\, v \in V_1,$$

where the energy functional is

$$E(v) = \frac{1}{2}\,(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} \psi(\boldsymbol{v}_\tau)\, da - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V}.$$

This completes the first example. □

Example 5.4. Consider a static frictionless unilateral contact problem between a linear elastic body and a rigid foundation. On the contact boundary Γ_3, we assume the contact is frictionless:

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3, \tag{5.16}$$

and satisfies the unilateral relations

$$u_\nu \le g,\ \sigma_\nu + \xi_\nu \le 0,\ (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0,\ \xi_\nu \in \partial\psi(u_\nu) \quad \text{on } \Gamma_3. \tag{5.17}$$

The relations (5.17) model a frictionless contact with a foundation made of a rigid body covered by a layer made of elastic material. Penetration is restricted by the relation $u_\nu \le g$, where g is non-negative valued and it represents the thickness of the elastic layer. When there is penetration and the normal displacement does not reach the bound g, the contact is described by a multi-valued normal compliance condition: $-\sigma_\nu = \xi_\nu \in \partial\psi(u_\nu)$. We assume the potential function $\psi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ has the following properties:

$$\begin{cases} \text{(a) } \psi(\cdot, z) \text{ is measurable on } \Gamma_3 \text{ for all } z \in \mathbb{R} \text{ and } \psi(\cdot, 0) \in L^1(\Gamma_3); \\ \text{(b) } \psi(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } |\partial\psi(\boldsymbol{x}, z)| \le \bar{c}_0 + \bar{c}_1 |z| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \qquad \text{for all } z \in \mathbb{R} \text{ with } \bar{c}_0,\ \bar{c}_1 \ge 0; \\ \text{(d) } \psi^0(\boldsymbol{x}, z_1; z_2 - z_1) + \psi^0(\boldsymbol{x}, z_2; z_1 - z_2) \le \alpha_\psi |z_1 - z_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{all } z_1, z_2 \in \mathbb{R} \text{ with } \alpha_\psi \ge 0. \end{cases} \tag{5.18}$$

With V from (5.8), we define the set

$$K = \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3\}. \tag{5.19}$$

We take $V_\psi = L^2(\Gamma_3)$, $m = 1$, and let $\gamma_\psi\colon V \to V_\psi$ be the normal component trace operator: $\gamma_\psi \boldsymbol{v} = v_\nu$ for $\boldsymbol{v} \in V$. Then $\gamma_\psi \in \mathcal{L}(V; V_\psi)$. Let $c_\psi = \lambda_2^{-1/2}$ where $\lambda_2 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} u_\nu v_\nu da \quad \forall\, \boldsymbol{v} \in V.$$

Then

$$\|v_\nu\|_{L^2(\Gamma_3)} \le c_\psi \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.20}$$

The weak formulation of the contact problem defined by (5.5)–(5.7), (5.16) and (5.17) is the following (cf. [29]).

Problem 5.5. Find a displacement field $\boldsymbol{u} \in K$ such that

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}))_Q + \int_{\Gamma_3} \psi^0(u_\nu; v_\nu - u_\nu)\, da \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in K. \tag{5.21}$$

Problem 5.5 has a unique solution under the stated assumptions on the problem data and

$$\alpha_\psi < \lambda_2 m_{\mathbb{C}}.$$

Let us apply Theorem 4.12 and first we examine the assumptions of the theorem. The set K defined by (5.19) obviously satisfies $H(K)$. The operator $A\colon V \to V^*$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$$

satisfies $H(A)$ and is symmetric, thanks to (5.1)–(5.3). The assumption $H(\psi)$ follows from (5.18), and $H(f)$ is valid due to (5.9). Note that $m_A = m_{\mathbb{C}}$, and α_ψ is given in (5.18)(d). Moreover, $c_\Delta = c_\psi = \lambda_2^{-1/2}$. Therefore, under the stated conditions, Problem 5.5 is equivalent to the minimization problem:

Problem 5.6. Find a displacement field $u \in K$ such that

$$E(u) \le E(v) \quad \forall\, v \in K,$$

where the energy functional is

$$E(v) = \frac{1}{2}\, (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} \psi(v_\nu)\, da - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V}.$$

This concludes the second example. □

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