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Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics

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Abstract
This paper is devoted to numerical analysis of general finite element approximations to stationary variational-hemivariational inequalities with or without constraints. The focus is on convergence under minimal solution regularity and error estimation under suitable solution regularity assumptions that cover both internal and external approximations of the stationary variational-hemivariational inequalities. A framework is developed for general variational-hemivariational inequalities, including a convergence result and a Céa type inequality. It is illustrated how to derive optimal order error estimates for linear finite element solutions of sample problems from contact mechanics.

Keywords
Hemivariational inequality, variational-hemivariational inequality, Galerkin approximation, finite element method, convergence, error estimation, contact mechanics

1. Introduction

Since the early 1980s, hemivariational inequalities have been introduced, analyzed and applied to a variety of engineering problems involving non-monotone and possibly multi-valued constitutive or interface laws for deformable bodies. Studies of hemivariational inequalities can be found in the comprehensive references [1–7]. The book by Haslinger et al. [3] is devoted to the finite element approximations of hemivariational inequalities, where convergence of the numerical methods is discussed. In recent years, efforts have been made to derive error estimates. In particular, in the work by Han et al. [8], a variational-hemivariational inequality is discussed theoretically and numerically, and an optimal order error estimate is derived for the linear finite element solution of a hemivariational or variational-hemivariational inequality under appropriate solution regularity assumptions. This is the first paper in the literature where an optimal order error estimate has been proven for a finite element method to solve a hemivariational or variational-hemivariational inequality. In the work by Barboteu et al. [9], numerical analysis is performed for solving a hyperbolic hemivariational inequality arising in dynamic frictional contact, and an optimal order error estimate with respect to both the spatial mesh-size and the temporal step-size is derived for linear finite element solutions of the problem (see also the work by Bartosz [10]). In the work by Han et al. [11], error analysis is presented for internal numerical approximations of general hemivariational inequalities, whereas in the work by Han et al. [12], error analysis is given for internal numerical approximations of more general variational-hemivariational inequalities. In a recent paper [13], convergence and error analysis

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for both internal and external approximations of elliptic hemivariational inequalities are considered. The purpose of this paper is to consider numerical approximations of stationary variational-hemivariational inequality problems by the finite element method, extending the relevant results reported [8, 11–13].

Variational-hemivariational inequalities are inequality problems where both convex and non-convex functions are involved. Interest in variational-hemivariational inequalities arises in the study of various problems in mechanics and engineering applications. They were first studied by Naniewicz and Panagiotopoulos [6,7]. Recently, a new variational-hemivariational inequality has been studied, including its numerical approximations, in the work by Han et al. [8], and a more general variational-hemivariational inequality is analyzed by Migórski et al. [14]. Solution existence and uniqueness have been proved, together with a result on the continuous dependence of the solution on the data. The variational-hemivariational inequalities studied by Han et al. [8] and Migórski et al. [14] are motivated by applications in contact mechanics. For the general variational-hemivariational inequality studied by Migórski et al. [14], we show the convergence of numerical solutions by internal or external approximation schemes under a minimal solution regularity condition, and provide Céa type inequalities which are the starting point for error estimation. Then, for some variational-hemivariational arising in contact mechanics, we derive the optimal order error estimates for the linear finite element solutions by applying Céa type inequalities.

Here is a brief description of the remaining sections. In Section 2 we review some basic notions needed in the study of variational-hemivariational inequalities. In Section 3, we introduce a general variational-hemivariational inequality. In Section 4 we describe numerical methods for solving the variational-hemivariational inequalities. The methods are allowed to be internal or external approximations. We prove convergence of the numerical solutions under the minimal solution regularity available from the existence and uniqueness result. We then present some preliminary results as the starting point for derivation of error estimates. Results on variational-hemivariational inequalities automatically reduce to corresponding ones on hemivariational inequalities, with simplified conditions. In Section 5 we introduce several contact problems and present sample optimal order error estimates for the linear finite element solutions of the contact problems by applying the results from Section 4.

2. Basic notions

Only real spaces are used in this paper. For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. When no confusion may arise, we simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{X^*\times X}$. Weak convergence is indicated by the symbol $\rightharpoonup$. The space of all linear continuous operators from one normed space X to another normed space Y is denoted by $\mathcal{L}(X,Y)$.

An operator $A\colon X\to X^*$ is said to be pseudomonotone if it is bounded and $u_n\rightharpoonup u$ in X together with $\limsup\,\langle Au_n, u_n-u\rangle_{X^*\times X}\le 0$ imply

$$\langle Au, u-v\rangle_{X^*\times X}\le \liminf\,\langle Au_n, u_n-v\rangle_{X^*\times X}\quad \forall\, v\in X.$$

The operator A is said to be radially continuous if the function $t\mapsto \langle A(u+t\,v), v\rangle$ is continuous on $[0,1]$ for any $u,v\in X$. A function $\varphi\colon K\subset X\to\mathbb{R}$ is said to be lower semicontinuous (LSC) if for any sequence $\{x_n\}\subset K$ and any $x\in K$, $x_n\to x$ in X implies $\varphi(x)\le\liminf\varphi(x_n)$. For a convex function φ, the set

$$\widetilde{\partial}\varphi(x) := \left\{x^*\in X^*\mid \varphi(v)-\varphi(x)\ge\langle x^*, v-x\rangle_{X^*\times X}\ \forall\, v\in X\right\}$$

is called the subdifferential of φ at $x\in X$. If $\widetilde{\partial}\varphi(x)$ is non-empty, any element $x^*\in\widetilde{\partial}\varphi(x)$ is called a subgradient of φ at x.

Assume $\psi : X\to\mathbb{R}$ is locally Lipschitz continuous. The generalized (16) directional derivative of ψ at $x\in X$ in the direction $v\in X$ is defined by

$$\psi^0(x;v) := \limsup_{y\to x,\,\lambda\downarrow 0}\frac{\psi(y+\lambda v)-\psi(y)}{\lambda}\,.$$

The generalized subdifferential of ψ at x is a subset of the dual space X^* given by

$$\partial\psi(x) := \{\,\zeta\in X^*\mid \psi^0(x;v)\ge\langle\zeta, v\rangle_{X^*\times X}\ \forall\, v\in X\,\}.$$

Details on properties of convex functions can be found in the work by Ekeland and Temam [15], whereas that of the subdifferential in the Clarke sense can be found in books [4, 6, 16, 17].

3. A general variational-hemivariational inequality

First, we introduce the problem. Let X, X_φ, X_j be normed spaces and $K \subset X$. Let there be given operators $A: X \to X^*$, $\gamma_\varphi : X \to X_\varphi$, $\gamma_j : X \to X_j$, and functionals $\varphi: X_\varphi \times X_\varphi \to \mathbb{R}$, $j: X_j \to \mathbb{R}$ that are assumed to be locally Lipschitz. We state the variational-hemivariational inequality as follows.

Problem (P). *Find an element $u \in K$ such that*

$$\begin{aligned}&\langle Au, v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u)\\ &\quad + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in K.\end{aligned} \tag{1}$$

In the study of Problem (P), we will make the following assumptions.

1. (A_1). X is a reflexive Banach space, $K \subset X$ is closed and convex with $0 \in K$.
2. (A_2). X_φ is a Banach space and $\gamma_\varphi \in \mathcal{L}(X, X_\varphi)$: for a constant $c_\varphi > 0$

$$\|\gamma_\varphi v\|_{X_\varphi} \leq c_\varphi \|v\|_X \quad \forall\, v \in X. \tag{2}$$

3. (A_3). X_j is a Banach space and $\gamma_j \in \mathcal{L}(X, X_j)$: for a constant $c_j > 0$

$$\|\gamma_j v\|_{X_j} \leq c_j \|v\|_X \quad \forall\, v \in X. \tag{3}$$

4. (A_4). $A: X \to X^*$ is pseudomonotone, radially continuous, and strictly monotone, i.e. for a constant $m_A > 0$

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{4}$$

5. (A_5). With $K_\varphi := \gamma_\varphi(K)$, $\varphi: K_\varphi \times K_\varphi \to \mathbb{R}$ is such that $\varphi(z, \cdot): K_\varphi \to \mathbb{R}$ is convex and LSC for all $z \in K_\varphi$, and for a constant $\alpha_\varphi \geq 0$

$$\begin{aligned}&\varphi(z_1, z_4) - \varphi(z_1, z_3) + \varphi(z_2, z_3) - \varphi(z_2, z_4)\\ &\quad \leq \alpha_\varphi \|z_1 - z_2\|_{X_\varphi} \|z_3 - z_4\|_{X_\varphi} \quad \forall\, z_1, z_2, z_3, z_4 \in K_\varphi.\end{aligned} \tag{5}$$

6. (A_6) $j: X_j \to \mathbb{R}$ is locally Lipschitz, and for some constants $c_0, c_1, \alpha_j \geq 0$

$$\|\partial j(z)\|_{X_j^*} \leq c_0 + c_1 \|z\|_{X_j} \quad \forall\, z \in X_j, \tag{6}$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \leq \alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j. \tag{7}$$

7. (A_7)

$$\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A. \tag{8}$$

8. (A_8)

$$f \in X^*. \tag{9}$$

Since K is a non-empty, closed and convex set in X, the set K_φ introduced in (A_5) is non-empty, closed and convex in X_φ. Note that in the statement of Problem (P) the function $\varphi(z, \cdot)$ is assumed to be convex for any $z \in X_\varphi$ whereas the function j is locally Lipschitz and, in general, non-convex. Thus, equation (1) represents a variational-hemivariational inequality. The spaces X_φ and X_j are introduced to facilitate error analysis of numerical solutions of Problem (P). For applications in contact mechanics, the functionals $\varphi(\cdot, \cdot)$ and $j(\cdot)$ are integrals over the contact boundary Γ_3. In such a situation, X_φ and X_j can be chosen to be $L^2(\Gamma_3)^d$ and/or $L^2(\Gamma_3)$. For a locally Lipschitz function $j: X_j \to \mathbb{R}$, the inequality given by equation (7) is equivalent to

$$\langle \partial j(z_1) - \partial j(z_2), z_1 - z_2\rangle_{X_j^* \times X_j} \geq -\alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j, \tag{10}$$

known as a relaxed monotonicity condition. Note that if $j: X_j \to \mathbb{R}$ is convex, equation (10) is satisfied with $\alpha_j = 0$. Since $0 \in K$, we derive the following relations from equations (4), (6) and (10)

$$\langle Av, v\rangle_{X^* \times X} \geq m_A \|v\|_X^2 - c\,\|v\|_X \quad \forall\, v \in X, \tag{11}$$

$$\langle \partial j(z), z\rangle_{X_j^* \times X_j} \geq -\alpha_j \|z\|_{X_j}^2 - c_0 \|z\|_{X_j} \quad \forall\, z \in X_j, \tag{12}$$

for some constant c. The smallness assumption given by equation (8) is a variant of a similar condition used by Migórski et al. [14], so modified as to reflect the use of the spaces X_φ and X_j in the problem setting. By slightly modifying the proof in the work by Migórski et al. [14], we have the following existence and uniqueness result.

Theorem 1 *Under assumptions* (A_1) *to* (A_8)*, Problem* (P) *has a unique solution* $u \in K$.

Problem (P) contains as particular cases, various problems considered in the literature. In general, the inclusion $u \in K$ represents a constraint on the solution u. When $K = V$, we have the special case of an unconstrained problem

$$u \in V, \qquad \langle Au, v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in V. \tag{13}$$

When $\varphi(\gamma_\varphi u, \gamma_\varphi v)$ is a function of the argument $\gamma_\varphi v$ only, the two problems given by equations (1) and (13) have the forms

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(\gamma_\varphi v) - \varphi(\gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in K,$$

$$u \in V, \quad \langle Au, v-u\rangle + \varphi(\gamma_\varphi v) - \varphi(\gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in V,$$

respectively. In the case where $j \equiv 0$, the above problems become variational inequalities and they have been studied extensively in the literature. When $\varphi \equiv 0$, we have a "pure" hemivariational inequality from equation (1).

Problem (P)'. *Find an element* $u \in K$ *such that*

$$\langle Au, v-u\rangle + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{14}$$

Under the assumptions (A_1), (A_3), (A_4) and (A_6) to (A_8) (with $\alpha_\varphi = 0$ in equation (8)), Problem (P)' has a unique solution. Internal approximation methods for solving Problem (P)' are studied in the work by Han et al. [11], whereas internal approximation methods for solving Problem (P) are studied in the work by Han et al. [12]. The focus of this paper is to provide numerical analysis of Problem (P) that also covers external approximations. In the work by Han [13], numerical analysis of both internal and external approximations is conducted on the particular case of Problem (P)'. In this paper, we extend the results of Han [13] to the more general problem of variational-hemivariational inequalities.

The following result is similar to Minty's lemma for variational inequalities (cf. [18, pp. 435]), and it is useful in convergence analysis of numerical solutions.

Theorem 2 *Assume* $K \subset X$ *is convex,* $A\colon X \to X^*$ *is monotone and radially continuous, and for all* $z \in K_\varphi$, $\varphi(z,\cdot)$ *is convex on* K_φ. *Then* $u \in K$ *is a solution of Problem* (P) *if and only if it satisfies*

$$\langle Av, v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{15}$$

Proof. By the monotonicity of A, the following inequality holds

$$\langle Av, v-u\rangle \geq \langle Au, v-u\rangle \quad \forall\, u, v \in X. \tag{16}$$

It is then obvious that a solution of Problem (P) satisfies equation (15).

Now assume $u \in K$ satisfies equation (15). Since K is convex, for any $v \in K$ and any $t \in [0,1]$, we can replace v by $u + t\,(v-u)$ in equation (15) to obtain

$$t\,\langle A\,(u + t\,(v-u)), v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi u + t\,(\gamma_\varphi v - \gamma_\varphi u)) - \varphi(\gamma_\varphi u, \gamma_\varphi u) + t\,j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq t\langle f, v-u\rangle. \tag{17}$$

Note that

$$\varphi(\gamma_\varphi u, \gamma_\varphi u + t\,(\gamma_\varphi v - \gamma_\varphi u)) \leq t\,\varphi(\gamma_\varphi u, \gamma_\varphi v) + (1-t)\,\varphi(\gamma_\varphi u, \gamma_\varphi u).$$

We deduce from equation (17) that for $t \in (0,1)$

$$\langle A\,(u + t\,(v-u)), v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle.$$

Letting $t \to 0+$ in the above inequality, we recover the inequality given by equation (1). □

4. Numerical approximations

We then turn to numerical methods for solving Problem (P). We keep assumptions (A_1) to (A_8) so that Problem (P) has a unique solution $u \in K$.

Let $X^h \subset X$ be a finite dimensional subspace with $h > 0$ being a spatial discretization parameter. Let K^h be a closed and convex subset of X^h and $0 \in K^h$. Since K^h is not necessarily a subset of K, the function φ may not be defined on $\gamma_\varphi(K^h) \times \gamma_\varphi(K^h)$. Thus, we introduce an assumption which does not present any restriction in applications.

(A_9) $\varphi(\cdot,\cdot)$ is the restriction of a function $\varphi_0(\cdot,\cdot)\colon X_\varphi \times X_\varphi \to \mathbb{R}$ to $K_\varphi \times K_\varphi$, and $\varphi_0(z,\cdot)$ is convex and LSC for all $z \in X_\varphi$. Moreover, equation (5) extends to X_φ, with a possibly different constant α_φ

$$\begin{aligned} &\varphi_0(z_1,z_4) - \varphi_0(z_1,z_3) + \varphi_0(z_2,z_3) - \varphi_0(z_2,z_4) \\ &\quad \le \alpha_\varphi \|z_1 - z_2\|_{X_\varphi} \|z_3 - z_4\|_{X_\varphi} \quad \forall\, z_1,z_2,z_3,z_4 \in X_\varphi. \end{aligned} \tag{18}$$

Then, the Galerkin approximation of Problem (P) is the following.

Problem (P^h). *Find an element $u^h \in K^h$ such that*

$$\begin{aligned} &\langle Au^h, v^h - u^h\rangle + \varphi_0(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h) \\ &\quad + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \ge \langle f, v^h - u^h\rangle \quad \forall\, v^h \in K^h. \end{aligned} \tag{19}$$

The approximation is external if $K^h \not\subset K$, and is internal if $K^h \subset K$. The internal approximation with the choice $K^h = X^h \cap K$ is considered in the work by Han et al. [12], and that for Problem (P)' is considered in the work by Han et al. [11].

We can apply the arguments of the proof of Theorem 1 in the setting of the finite dimensional space X^h, and to conclude that under assumptions (A_1) to (A_9), Problem (P^h) has a unique solution $u^h \in K^h$. In the convergence analysis of the numerical solutions, the following uniform boundedness property will be useful.

Proposition 1 *For some constant $M > 0$, $\|u^h\|_X \le M\ \forall\, h > 0$.*

Proof. We let $v^h = 0$ in equation (19) to get

$$\langle Au^h, u^h\rangle \le \varphi_0(\gamma_\varphi u^h, 0) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h) + j^0(\gamma_j u^h; -\gamma_j u^h) + \langle f, u^h\rangle. \tag{20}$$

For any $z \in K_\varphi$, take $z_1 = z_3 = z$ and $z_2 = z_4 = 0$ in equation (5)

$$\varphi_0(z,0) - \varphi_0(z,z) \le \alpha_\varphi \|z\|^2_{X_\varphi} - \varphi_0(0,z) + \varphi_0(0,0). \tag{21}$$

Use the lower bound [18, pp. 433]

$$\varphi(0,z) \ge c_3 + c_4 \|z\|_{X_\varphi} \quad \forall\, z \in X_\varphi$$

for some constants c_3 and c_4, not necessarily positive. Then from equation (21), we have

$$\varphi_0(z,0) - \varphi_0(z,z) \le \alpha_\varphi \|z\|^2_{X_\varphi} + c\left(\|z\|_{X_\varphi} + 1\right). \tag{22}$$

Write $z_1 = z$ and take $z_2 = 0$ in equation (7)

$$j^0(z;-z) \le \alpha_j \|z\|^2_{X_j} - j^0(0;z).$$

Further, use equation (6) to get

$$j^0(z;-z) \le \alpha_j \|z\|^2_{X_j} + c_0 \|z\|_{X_j}. \tag{23}$$

Use equations (11), (22), (23), (2) and (3) in equation (20) to obtain

$$\left(m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2\right) \|u^h\|^2_X \le c\left(\|u^h\|_X + 1\right).$$

Since $m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2 > 0$, we deduce from the above inequality that $\|u^h\|_X$ is uniformly bounded with respect to h. □

The rest of this section is devoted to convergence and error analysis for the numerical solution of Problem (P^h). We list below some additional conditions to be assumed later on:

1. (A_{10}). $\gamma_\varphi : X \to X_\varphi$ and $\gamma_j : X \to X_j$ are compact operators.
2. (A_{11}). $\varphi_0(\cdot,\cdot)$ is continuous with respect to its second argument.
3. (A_{12}). $\{K^h\}_h$ approximates K in the following sense

$$v^h \in K^h \text{ and } v^h \rightharpoonup v \text{ in } X \text{ imply } v \in K; \tag{24}$$

$$\forall\, v \in K,\ \exists\, v^h \in K^h \text{ such that } v^h \to v \text{ in } X \text{ as } h \to 0. \tag{25}$$

4. (A_{13}). The operator $A : X \to X^*$ is Lipschitz continuous, i.e. for some constant $L_A > 0$

$$\|Au - Av\|_{X^*} \le L_A \|u - v\|_X \quad \forall\, u, v \in X. \tag{26}$$

We comment that the conditions given by equations (26) and (4) combined imply that the operator A is pseudomonotone [19, Proposition 27.6]. In applications to contact mechanics (cf. Section 5), γ_φ and γ_j are trace operators from an $H^1(\Omega)$-based space to $L^2(\Gamma_3)$-based spaces, the assumption (A_{10}) is automatically valid, and the assumptions (A_{11}) and (A_{13}) are usually trivially satisfied. Moreover, the assumption (A_{12}) holds true for common choices of K^h based on finite elements [20].

4.1. Convergence

In this subsection, we prove the convergence of the numerical solutions under the minimal solution regularity $u \in K$ available from Theorem 1.

Theorem 3 *Assume* (A_1) *to* (A_{12})*. Then*

$$u^h \to u \quad \text{in } X \text{ as } h \to 0. \tag{27}$$

Proof. The proof consists of two steps. First we show weak convergence of the numerical solutions. According to Theorem 2, the solution $u^h \in K^h$ of Problem (P^h) is characterized by the inequality

$$\begin{aligned} &\langle Av^h, v^h - u^h\rangle + \varphi_0(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h) \\ &\quad + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \ge \langle f, v^h - u^h\rangle \quad \forall\, v^h \in K^h. \end{aligned} \tag{28}$$

Note that $\{u^h\}$ is bounded in X by Proposition 1. Since X is reflexive and the operators $\gamma_\varphi : X \to X_\varphi$ and $\gamma_j : X \to X_j$ are compact, there exists a subsequence $\{u^{h'}\} \subset \{u^h\}$ and an element $w \in X$ such that

$$u^{h'} \rightharpoonup w \text{ in } X, \quad \gamma_\varphi u^{h'} \to \gamma_\varphi w \text{ in } X_\varphi, \quad \gamma_j u^{h'} \to \gamma_j w \text{ in } X_j.$$

By the assumption given by equation (24), we know that $w \in K$.

Fix an arbitrary element $v \in K$. By the assumption given by equation (25), we can find a sequence $v^{h'} \in K^{h'}$ such that $v^{h'} \to v$ in X as $h' \to 0$. Then, as $h' \to 0$

$$\begin{aligned} &Av^{h'} \to Av, \quad \langle Av^{h'}, v^{h'} - u^{h'}\rangle \to \langle Av, v - w\rangle, \\ &j^0(\gamma_j w; \gamma_j v - \gamma_j w) \ge \limsup j^0(\gamma_j u^{h'}; \gamma_j v^{h'} - \gamma_j u^{h'}), \\ &\langle f, v^{h'} - u^{h'}\rangle \to \langle f, v - w\rangle. \end{aligned}$$

From equation (18) with $z_1 = \gamma_\varphi w$, $z_2 = z_4 = \gamma_\varphi u^{h'}$ and $z_3 = \gamma_\varphi v^{h'}$, we have

$$\begin{aligned} \varphi_0(\gamma_\varphi u^{h'}, \gamma_\varphi v^{h'}) - \varphi_0(\gamma_\varphi u^{h'}, \gamma_\varphi u^{h'}) &\le \varphi_0(\gamma_\varphi w, \gamma_\varphi v^{h'}) - \varphi_0(\gamma_\varphi w, \gamma_\varphi u^{h'}) \\ &\quad + \alpha_\varphi \|\gamma_\varphi w - \gamma_\varphi u^{h'}\|_{X_\varphi} \|\gamma_\varphi v^{h'} - \gamma_\varphi u^{h'}\|_{X_\varphi}. \end{aligned} \tag{29}$$

Use this inequality in equation (28) with $h = h'$

$$\begin{aligned} &\langle Av^{h'}, v^{h'} - u^{h'} \rangle + \varphi_0(\gamma_\varphi w, \gamma_\varphi v^{h'}) - \varphi_0(\gamma_\varphi w, \gamma_\varphi u^{h'}) \\ &\quad + \alpha_\varphi \|\gamma_\varphi w - \gamma_\varphi u^{h'}\|_{X_\varphi} \|\gamma_\varphi v^{h'} - \gamma_\varphi u^{h'}\|_{X_\varphi} \\ &\quad + j^0(\gamma_j u^{h'}; \gamma_j v^{h'} - \gamma_j u^{h'}) \geq \langle f, v^{h'} - u^{h'} \rangle. \end{aligned} \tag{30}$$

Note that $\|\gamma_\varphi v^{h'} - \gamma_\varphi u^{h'}\|_{X_\varphi}$ is bounded whereas $\|\gamma_\varphi w - \gamma_\varphi u^{h'}\|_{X_\varphi} \to 0$. Thus, taking the upper limit in equation (30) as $h' \to 0$, we find

$$\langle Av, v - w \rangle + \varphi_0(\gamma_\varphi w, \gamma_\varphi v) - \varphi_0(\gamma_\varphi w, \gamma_\varphi w) + j^0(\gamma_j w; \gamma_j v - \gamma_j w) \geq \langle f, v - w \rangle.$$

This inequality holds for any $v \in K$. Applying Theorem 2, we see that w is actually the solution u of Problem (P). So $u^{h'} \rightharpoonup u$ in X. Since the limit u does not depend on the subsequence $\{u^{h'}\}$, the entire family of numerical solutions converges weakly to u, i.e. equation (27) holds.

Next, we show the strong convergence $u^h \to u$ in X as $h \to 0$. By the assumption given by equation (25), there exists a sequence $\{\overline{u}^h\}$, $\overline{u}^h \in K^h$, such that $\overline{u}^h \to u$ in X as $h \to 0$. Applying equation (4)

$$m_A \|u - u^h\|_X^2 \leq \langle Au - Au^h, u - u^h \rangle.$$

So

$$m_A \|u - u^h\|_X^2 \leq \langle Au, u - u^h \rangle - \langle Au^h, \overline{u}^h - u^h \rangle - \langle Au^h, u - \overline{u}^h \rangle.$$

From equation (19), we have

$$\begin{aligned} -\langle Au^h, \overline{u}^h - u^h \rangle &\leq \varphi_0(\gamma_\varphi u^h, \gamma_\varphi \overline{u}^h) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h) \\ &\quad + j^0(\gamma_j u^h; \gamma_j \overline{u}^h - \gamma_j u^h) - \langle f, \overline{u}^h - u^h \rangle. \end{aligned}$$

Similar to equation (29), we have

$$\begin{aligned} \varphi_0(\gamma_\varphi u^h, \gamma_\varphi \overline{u}^h) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h) &\leq \varphi_0(\gamma_\varphi u, \gamma_\varphi \overline{u}^h) - \varphi_0(\gamma_\varphi u, \gamma_\varphi u^h) \\ &\quad + \alpha_\varphi \|\gamma_\varphi u - \gamma_\varphi u^h\|_{X_\varphi} \|\gamma_\varphi \overline{u}^h - \gamma_\varphi u^h\|_{X_\varphi}. \end{aligned}$$

Combining the above three inequalities, we have

$$\begin{aligned} m_A \|u - u^h\|_X^2 &\leq \langle Au, u - u^h \rangle - \langle Au^h, u - \overline{u}^h \rangle \\ &\quad + \varphi_0(\gamma_\varphi u, \gamma_\varphi \overline{u}^h) - \varphi_0(\gamma_\varphi u, \gamma_\varphi u^h) \\ &\quad + \alpha_\varphi \|\gamma_\varphi u - \gamma_\varphi u^h\|_{X_\varphi} \|\gamma_\varphi \overline{u}^h - \gamma_\varphi u^h\|_{X_\varphi} \\ &\quad + j^0(\gamma_j u^h; \gamma_j \overline{u}^h - \gamma_j u^h) - \langle f, \overline{u}^h - u^h \rangle. \end{aligned} \tag{31}$$

Recall assumption (A_{11}). Then, as $h \to 0$

$$\begin{aligned} \varphi_0(\gamma_\varphi u, \gamma_\varphi \overline{u}^h) &\to \varphi_0(\gamma_\varphi u, \gamma_\varphi u), \\ \varphi_0(\gamma_\varphi u, \gamma_\varphi u^h) &\to \varphi_0(\gamma_\varphi u, \gamma_\varphi u). \end{aligned}$$

Also notice that $\|\overline{u}^h - u^h\|_X \to 0$ and $\|\gamma_\varphi u - \gamma_\varphi u^h\|_{X_\varphi} \to 0$. Consequently, from equation (31)

$$\limsup_{h \to 0} \|u - u^h\|_X^2 \leq 0.$$

This implies the strong convergence $u^h \to u$ in X. □

4.2. Error estimation

We now turn to the derivation of error estimates. In this subsection, we assume (A_1) to (A_9) and (A_{13}). Let $v \in K$ and $v^h \in K^h$ be arbitrary. By equation (4) with $v_1 = u$ and $v_2 = u^h$

$$m_A \|u - u^h\|_X^2 \le \langle Au - Au^h, u - u^h \rangle,$$

which is rewritten as

$$\begin{aligned} m_A \|u - u^h\|_X^2 \le\ & \langle Au - Au^h, u - v^h \rangle + \langle Au, v^h - u \rangle + \langle Au, v - u^h \rangle \\ & + \langle Au, u - v \rangle + \langle Au^h, u^h - v^h \rangle. \end{aligned} \tag{32}$$

Applying equation (1)

$$\begin{aligned} \langle Au, u - v \rangle \le\ & \varphi_0(\gamma_\varphi u, \gamma_\varphi v) - \varphi_0(\gamma_\varphi u, \gamma_\varphi u) \\ & + j^0(\gamma_j u; \gamma_j v - \gamma_j u) - \langle f, v - u \rangle. \end{aligned}$$

Applying equation (19)

$$\begin{aligned} \langle Au^h, u^h - v^h \rangle \le\ & \varphi_0(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h) \\ & + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) - \langle f, v^h - u^h \rangle. \end{aligned}$$

Using these inequalities in equation (32), after some rearrangement of the terms, we have

$$m_A \|u - u^h\|_X^2 \le \langle Au - Au^h, u - v^h \rangle + R_u(v^h, u) + R_u(v, u^h) + I_\varphi(v^h) + I_j(v, v^h), \tag{33}$$

where

$$\begin{aligned} R_u(v, w) :=\ & \langle Au, v - w \rangle + \varphi_0(\gamma_\varphi u, \gamma_\varphi v) - \varphi_0(\gamma_\varphi u, \gamma_\varphi w) \\ & + j^0(\gamma_j u; \gamma_j v - \gamma_j w) - \langle f, v - w \rangle, \end{aligned} \tag{34}$$

$$\begin{aligned} I_\varphi(v^h) :=\ & \varphi_0(\gamma_\varphi u, \gamma_\varphi u^h) + \varphi_0(\gamma_\varphi u^h, \gamma_\varphi v^h) \\ & - \varphi_0(\gamma_\varphi u, \gamma_\varphi v^h) - \varphi_0(\gamma_\varphi u^h, \gamma_\varphi u^h), \end{aligned} \tag{35}$$

$$\begin{aligned} I_j(v, v^h) :=\ & j^0(\gamma_j u; \gamma_j v - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \\ & - j^0(\gamma_j u; \gamma_j v^h - \gamma_j u) - j^0(\gamma_j u; \gamma_j v - \gamma_j u^h). \end{aligned} \tag{36}$$

Let us bound the first and the last two terms on the right-hand side of equation (33). First

$$\langle Au - Au^h, u - v^h \rangle \le L_A \|u - u^h\|_X \|u - v^h\|_X.$$

Thus, for any $\varepsilon > 0$ arbitrarily small

$$\langle Au - Au^h, u - v^h \rangle \le \varepsilon \|u - u^h\|_X^2 + c \|u - v^h\|_X^2 \tag{37}$$

for some constant c depending on ε. By equation (18), we have

$$\begin{aligned} I_\varphi(v^h) & \le \alpha_\varphi \|\gamma_\varphi u - \gamma_\varphi u^h\|_{X_\varphi} \|\gamma_\varphi u^h - \gamma_\varphi v^h\|_{X_\varphi} \\ & \le \alpha_\varphi c_\varphi^2 \left(\|u - u^h\|_X^2 + \|u - u^h\|_X \|u - v^h\|_X \right). \end{aligned}$$

Thus

$$I_\varphi(v^h) \le \left(\alpha_\varphi c_\varphi^2 + \varepsilon \right) \|u - u^h\|_X^2 + c \|u - v^h\|_X^2 \tag{38}$$

for another constant c depending on $\varepsilon > 0$. Applying the subadditivity of the generalized directional derivative

$$j^0(z; z_1 + z_2) \le j^0(z; z_1) + j^0(z; z_2) \quad \forall\, z, z_1, z_2 \in X_j,$$

we have

$$j^0(\gamma_j u; \gamma_j v - \gamma_j u) \le j^0(\gamma_j u; \gamma_j v - \gamma_j u^h) + j^0(\gamma_j u; \gamma_j u^h - \gamma_j u),$$
$$j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \le j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j u - \gamma_j u^h).$$

Thus

$$\begin{aligned} I_j(v, v^h) \le\ & j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u) - j^0(\gamma_j u; \gamma_j v^h - \gamma_j u) \\ & + j^0(\gamma_j u; \gamma_j u^h - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j u - \gamma_j u^h). \end{aligned}$$

By equation (7)

$$j^0(\gamma_j u; \gamma_j u^h - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j u - \gamma_j u^h) \le \alpha_j \|\gamma_j u - \gamma_j u^h\|_{X_j}^2.$$

Moreover

$$\left| j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u)\right| \le \left(c_0 + c_1 \|\gamma_j u^h\|_{X_j}\right) \|\gamma_j v^h - \gamma_j u\|_{X_j},$$
$$\left| j^0(\gamma_j u; \gamma_j v^h - \gamma_j u)\right| \le \left(c_0 + c_1 \|\gamma_j u\|_{X_j}\right) \|\gamma_j v^h - \gamma_j u\|_{X_j}.$$

Combining the above four inequalities and using the fact that $\|\gamma_j u^h\|_{X_j}$ is uniformly bounded (cf. Proposition 1), we find that

$$I_j(v, v^h) \le \alpha_j \|\gamma_j u - \gamma_j u^h\|_{X_j}^2 + c\, \|\gamma_j u - \gamma_j v^h\|_{X_j} \tag{39}$$

for some constant $c > 0$ independent of h. Using equations (37), (38) and (39) in equation (33), we have

$$\begin{aligned} \left(m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2 - 2\,\varepsilon\right) \|u - u^h\|_X^2 \le\ & c\, \|u - v^h\|_X^2 + c\, \|\gamma_j u - \gamma_j v^h\|_{X_j} \\ & + R_u(v^h, u) + R_u(v, u^h). \end{aligned}$$

Recall the smallness assumption, $\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A$. We then choose $\varepsilon = (m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2)/4 > 0$ and get the inequality

$$\|u - u^h\|_X^2 \le c \inf_{v^h \in K^h} \left[\|u - v^h\|_X^2 + \|\gamma_j u - \gamma_j v^h\|_{X_j} + R_u(v^h, u)\right] + c \inf_{v \in K} R_u(v, u^h).$$

We summarize the result in the form of a theorem.

Theorem 4 *Assume* (A_1) *to* (A_9) *and* (A_{13}). *Then for the solution u of Problem* (P) *and the solution* u^h *of Problem* (P^h)*, we have the Céa type inequality*

$$\begin{aligned} \|u - u^h\|_X \le\ & c \inf_{v^h \in K^h} \left[\|u - v^h\|_X + \|\gamma_j u - \gamma_j v^h\|_{X_j}^{1/2} + R_u(v^h, u)^{1/2}\right] \\ & + c \inf_{v \in K} R_u(v, u^h)^{1/2}. \end{aligned} \tag{40}$$

We remark that in the literature on error analysis of numerical solutions of variational inequalities, it is standard that the Céa type inequalities involve square root of approximation error of the solution in certain norms due to the inequality form of the problems; cf. [21–23].

To proceed further, we need to bound the residual term given by equation (34) and this depends on the problem to be solved. We illustrate this point in Section 5 in the context of a contact problem.

In the special case where $K = X$, we have $K^h = X^h$. Then the approximation is always internal. The error analysis in the work by Han et al. [12] applies, and we have the following Céa type inequality

$$\|u - u^h\|_X^2 \le c \inf_{v^h \in X^h} \left(\|u - v^h\|_X^2 + \|\gamma_\varphi u - \gamma_\varphi v^h\|_{X_\varphi} + \|\gamma_j u - \gamma_j v^h\|_{X_j}\right). \tag{41}$$

5. Error analysis for sample contact problems

In this section, we show how to derive error estimates for numerical solutions of several static contact problems by applying the theoretical results developed in Section 4. The physical setting of a contact problem is as follows: the reference configuration of an elastic body is an open, bounded, connected set $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3 in applications) with a Lipschitz boundary $\Gamma = \partial\Omega$ partitioned into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}\,(\Gamma_1) > 0$. The body is in equilibrium under the action of a volume force of density $\boldsymbol{f}_0$ in Ω and a surface traction of density $\boldsymbol{f}_2$ on Γ_2; it is fixed on Γ_1 and is in contact on Γ_3 with a foundation.

Let us introduce some notation useful in the description of the contact problems. We use the symbol $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$, and "$\cdot$" and "$\|\cdot\|$" denote the canonical inner product and norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$. We use $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ for the displacement, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)/2$ for the linearized strain tensor, and $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ for the stress field. Let $\boldsymbol{\nu}$ be the unit outward normal vector, which is defined a.e. on Γ. For a vector field $\boldsymbol{v}$, $v_\nu := \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu\boldsymbol{\nu}$ are the normal and tangential components of $\boldsymbol{v}$ on Γ. For the stress field $\boldsymbol{\sigma}$, $\sigma_\nu := (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ are its normal and tangential components on the boundary. For the stress and strain fields, we will use the Hilbert space $Q = L^2(\Omega;\mathbb{S}^d)$ with the canonical inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx, \quad \boldsymbol{\sigma},\boldsymbol{\tau} \in Q.$$

The displacement will be sought in the space

$$V = \left\{\boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\right\}$$

or its subset. Since $\text{meas}\,(\Gamma_1) > 0$, by Korn's inequality, V is a Hilbert space with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \quad \boldsymbol{u},\boldsymbol{v} \in V.$$

We will use the same symbol $\boldsymbol{v}$ for the trace of a function $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ on Γ.

5.1. A normal compliance frictional contact problem with a unilateral constraint

The equations and conditions for this contact problem are

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{42}$$

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{43}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{44}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{45}$$

supplemented by the following contact conditions [14]

$$u_\nu \le g,\ \sigma_\nu + \xi_\nu \le 0,\ (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0,\ \xi_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3, \tag{46}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu),\ -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{47}$$

In these equations and conditions, equation (42) is the elastic constitutive law, equation (43) represents the equilibrium equation, equation (44) is the displacement boundary condition and equation (45) describes the traction boundary condition. In equation (42), $\mathcal{F}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity operator and is assumed to have the following properties:

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega,\\
\qquad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|;\\
\text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega,\\
\qquad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2;\\
\text{(c) } \mathcal{F}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d;\\
\text{(d) } \mathcal{F}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{48}$$

On the densities of the body force and the surface traction, we assume

$$\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2;\mathbb{R}^d). \tag{49}$$

Define $\boldsymbol{f} \in V^*$ by

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V. \tag{50}$$

In the normal contact condition given by equation (46), the relation $u_\nu \le g$ restricts the allowed penetration, where g represents the thickness of the elastic layer. We assume $g\colon \Gamma_3 \to \mathbb{R}$ satisfies

$$g \in L^2(\Gamma_3), \quad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{51}$$

The contact condition given by equation (46) represents a combination of the Signorini contact condition for contact with a rigid foundation and the normal compliance condition for contact with a deformable foundation. Details on the normal compliance and Signorini contact conditions can be found in previous works [4, 22, 24]. The tangential contact condition given by equation (47) describes a version of Coulomb's law of dry friction. The friction bound F_b may depend on the normal displacement u_ν [25]. We assume the potential function $j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ and the friction bound $F_b\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ have the properties:

$$\left\{\begin{array}{l}
\text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\
\qquad \text{exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_3); \\
\text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\
\text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1 |r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3\ \forall\, r \in \mathbb{R} \text{ with } \overline{c}_0,\ \overline{c}_1 \ge 0; \\
\text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu} |r_1 - r_2|^2 \\
\qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } r_1, r_2 \in \mathbb{R} \text{ with } \alpha_{j_\nu} \ge 0.
\end{array}\right. \tag{52}$$

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{F_b} > 0 \text{ such that} \\
\qquad |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b} |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3; \\
\text{(b) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3, \text{ for all } r \in \mathbb{R}; \\
\text{(c) } F_b(\boldsymbol{x}, r) = 0 \text{ for } r \le 0,\ F_b(\boldsymbol{x}, r) \ge 0 \text{ for } r \ge 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{53}$$

The displacement will be sought from the following subset of the space V

$$U := \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3\}.$$

By a standard approach (cf. [4, 22]), the following weak formulation of the first contact problem can be derived.

Problem (P$_1$). *Find a displacement field $\boldsymbol{u} \in U$ such that*

$$\begin{aligned}
(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}))_Q &+ \int_{\Gamma_3} F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, d\Gamma \\
&+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in U.
\end{aligned} \tag{54}$$

We can apply the results of the previous section in the numerical analysis of Problem (P$_1$). Let $X = V$, $K = U$, $X_\varphi = L^2(\Gamma_3)^d$ with γ_φ the trace operator from V to X_φ, $X_j = L^2(\Gamma_3)$ with $\gamma_j \boldsymbol{v} = v_\nu$ for $\boldsymbol{v} \in V$. Then, $\alpha_\varphi = L_{F_b}$ and $\alpha_j = \alpha_{j_\nu}$. Applying Theorem 1, we know that Problem (P$_1$) has a unique solution $\boldsymbol{u} \in U$ under the stated assumptions, and equation (8) takes the form

$$L_{F_b} \lambda_{1,V}^{-1} + \alpha_{j_\nu} \lambda_{1\nu,V}^{-1} < m_{\mathcal{F}}, \tag{55}$$

where $\lambda_{1,V} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} \boldsymbol{u} \cdot \boldsymbol{v}\, d\Gamma \quad \forall\, \boldsymbol{v} \in V,$$

and $\lambda_{1\nu,V} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} u_\nu v_\nu d\Gamma \quad \forall\, \boldsymbol{v} \in V.$$

Let us use the finite element method to solve Problem (P$_1$). For brevity, assume Ω is a polygonal/polyhedral domain and express the three parts of the boundary, Γ_k, $1 \le k \le 3$, as unions of closed flat components with disjoint interiors

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \le k \le 3.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $1 \le k \le 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Construct the linear element space corresponding to $\mathcal{T}^h$

$$V^h = \left\{ \boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d,\ T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1 \right\},$$

and the related finite element subset

$$U^h = \left\{ \boldsymbol{v}^h \in V^h \mid v_\nu^h \le g \text{ at node points on } \Gamma_3 \right\}.$$

Note that $\boldsymbol{0} \in U^h$ and in general $U^h \not\subset U$ unless g is concave. Then the finite element method for Problem (P$_1$) is the following.

Problem (P$_1^h$). *Find a displacement field $\boldsymbol{u}^h \in U^h$ such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}^h))_Q + \int_{\Gamma_3} F_b(u_\nu^h)\left(\|\boldsymbol{v}_\tau^h\| - \|\boldsymbol{u}_\tau^h\|\right) d\Gamma \qquad (56)$$

$$+ \int_{\Gamma_3} j_\nu^0(u_\nu^h; v_\nu^h - u_\nu^h)\, d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h \rangle_{V^*\times V} \quad \forall\, \boldsymbol{v}^h \in U^h.$$

For error analysis, we begin with a simplified version of the Céa type inequality derived from equation (40) in the abstract framework.

Theorem 5 *Assume the solution regularity*

$$\boldsymbol{u} \in H^2(\Omega)^d, \quad \boldsymbol{\sigma}\boldsymbol{\nu} \in L^2(\Gamma_3)^d. \qquad (57)$$

Then for the solution $\boldsymbol{u} \in U$ of Problem (P$_1$) *and the solution $\boldsymbol{u}^h \in U^h$ of Problem* (P$_1^h$)*, we have the Céa type inequality*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V \le c \inf_{\boldsymbol{v}^h \in U^h} \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V + \|\boldsymbol{u} - \boldsymbol{v}^h\|_{L^2(\Gamma_3)^d}^{1/2} \right) + c \inf_{\boldsymbol{v} \in U} \|\boldsymbol{v} - \boldsymbol{u}^h\|_{L^2(\Gamma_3)^d}^{1/2}. \qquad (58)$$

Proof. For the contact problem (P), the residual term defined in equation (34) is

$$R_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{w}) = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{w}))_Q + \int_{\Gamma_3} F_b(u_\nu)\left(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{w}_\tau\|\right) d\Gamma$$

$$+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - w_\nu)\, d\Gamma - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{w} \rangle_{V^*\times V}.$$

We follow a procedure described in detail in the work by Han and Sofonea [22], so as to bound the residual. We let $\boldsymbol{v} = \boldsymbol{u} \pm \boldsymbol{w}$ in equation (54), where the arbitrary function $\boldsymbol{w} \in U$ is such that $\boldsymbol{w} \in C^\infty(\overline{\Omega})^d$ and $\boldsymbol{w} = \boldsymbol{0}$ on $\Gamma_1 \cup \Gamma_3$. Then we have the identity

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{V^*\times V}.$$

From this identity we can deduce that

$$\mathrm{Div}\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}))+\boldsymbol{f}_0=\boldsymbol{0} \quad \text{in } \Omega, \tag{59}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu}=\boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{60}$$

We multiply the equation (59) by $\boldsymbol{v}-\boldsymbol{w}$ with $\boldsymbol{v},\boldsymbol{w}\in V$, integrate over Ω, and perform an integration by parts to obtain

$$\int_{\partial\Omega}\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{w})\,d\Gamma-\int_{\Omega}\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w})\,dx+\int_{\Omega}\boldsymbol{f}_0\cdot(\boldsymbol{v}-\boldsymbol{w})\,dx=0.$$

Using the homogeneous Dirichlet boundary condition of $\boldsymbol{v}-\boldsymbol{w}$ on Γ_1 and the traction boundary condition given by equation (60) on Γ_2, we further have

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}))_Q=\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{w}\rangle_{V^*\times V}+\int_{\Gamma_3}\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{w})\,d\Gamma. \tag{61}$$

Therefore

$$R_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{w})=\int_{\Gamma_3}\left[\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{w})+F_b(u_\nu)\left(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{w}_\tau\|\right)+j_\nu^0(u_\nu;v_\nu-w_\nu)\right]d\Gamma.$$

Then

$$|R_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{w})|\le c\,\|\boldsymbol{v}-\boldsymbol{w}\|_{L^2(\Gamma_3)^d}. \tag{62}$$

Now the inequality given by equation (58) follows from equation (40). □

Regarding the solution regularity equation (57), we comment that for many application problems, $\boldsymbol{\sigma}\boldsymbol{\nu}\in L^2(\Gamma_3)^d$ follows from $\boldsymbol{u}\in H^2(\Omega)^d$. Next, we present one sample error estimate.

Theorem 6 *Assume the solution regularity equation* (57)*. Additionally, assume* Γ_3 *is a flat component of the boundary* $\partial\Omega$ *and*

$$\boldsymbol{u}|_{\Gamma_3}\in H^2(\Gamma_3;\mathbb{R}^d),\quad g|_{\Gamma_3}\in H^2(\Gamma_3). \tag{63}$$

Then for the solution $\boldsymbol{u}\in U$ *of Problem* (P_1) *and the solution* $\boldsymbol{u}^h\in U^h$ *of Problem* (P_1^h)*, we have the optimal order error bound*

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V\le c\,h. \tag{64}$$

Proof. We apply equation (58). The first part on the right side of equation (58) is bounded by

$$c\left(\|\boldsymbol{u}-\Pi^h\boldsymbol{u}\|_V+\|\boldsymbol{u}-\Pi^h\boldsymbol{u}\|_{L^2(\Gamma_3)^d}^{1/2}\right),$$

where $\Pi^h\boldsymbol{u}\in U^h$ is the finite element interpolant of $\boldsymbol{u}$. By the standard finite element interpolation error bounds [18, 26, 27], we have

$$\|\boldsymbol{u}-\Pi^h\boldsymbol{u}\|_V\le c\,h\,\|\boldsymbol{u}\|_{H^2(\Omega)^d},$$
$$\|\boldsymbol{u}-\Pi^h\boldsymbol{u}\|_{L^2(\Gamma_3)^d}\le c\,h^2\|\boldsymbol{u}\|_{H^2(\Gamma_3)^d}.$$

The second part on the right side of equation (58) is treated as in the work by Han [13] and is omitted here. Putting these bounds together, we get the error estimate given by equation (64). □

5.2. A frictional contact problem with normal compliance

For this contact problem, the equations and conditions given by equations (42) to (45) are supplemented by the following contact conditions [8]

$$-\sigma_\nu\in\partial j_\nu(u_\nu),\quad \|\boldsymbol{\sigma}_\tau\|\le F_b(u_\nu),\quad -\boldsymbol{\sigma}_\tau=F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|}\ \text{if } \boldsymbol{u}_\tau\ne\boldsymbol{0}\quad \text{on } \Gamma_3. \tag{65}$$

The potential function j_ν is assumed to satisfy equation (52), whereas the friction bound F_b is assumed to satisfy equation (53).

The contact condition in equation (65) does not involve the unilateral constraint and it can be viewed as a limiting case of the condition given by equation (46) as $g \to \infty$. The weak formulation of the corresponding contact problem is the following.

Problem (P_2). *Find a displacement field* $\boldsymbol{u} \in V$ *such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))_Q + \int_{\Gamma_3} F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\,d\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\,d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in V.$$

Since Problem (P_2) is posed over a space, error analysis of numerical methods for Problem (P_2) is simpler than that for Problem (P_1). In particular, the counterpart of equation (58) in Theorem 5 reduces to

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V \le c \inf_{\boldsymbol{v}^h \in V^h} \left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V + \|\boldsymbol{u}-\boldsymbol{v}^h\|^{1/2}_{L^2(\Gamma_3)^d} \right). \tag{66}$$

Then, under the solution regularity assumptions, equation (57) and

$$\boldsymbol{u}|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i}; \mathbb{R}^d), \quad g|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i}), \quad 1 \le i \le i_3,$$

the optimal order error bound equation (64) holds for the linear finite element solutions of Problem (P_2) and the proof is simpler in the sense that it follows directly from equation (66) and the standard finite element interpolation error estimates.

5.3. A unilateral contact problem with a frictionless subdifferential boundary condition

Here, the equations and conditions given by equations (42) to (45) are supplemented by the following contact conditions

$$\sigma_\nu = \sigma_\nu^1 + \sigma_\nu^2 + \sigma_\nu^3, \quad -\sigma_\nu^1 \in \widetilde{\partial}\varphi_\nu(u_\nu), \quad -\sigma_\nu^2 \in \partial j_\nu(u_\nu), \tag{67}$$
$$u_\nu \le g, \quad \sigma_\nu^3 \le 0, \quad \sigma_\nu^3(u_\nu - g) = 0, \qquad \text{on } \Gamma_3,$$
$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on } \Gamma_3. \tag{68}$$

We refer the reader to the work by Han et al. [12] for a mechanical interpretation of the condition given by equation (67). Assume $j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies equation (52), and assume $\varphi_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$\left\{ \begin{array}{l} \text{(a) } \varphi_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad \text{exists } \widetilde{e} \in L^2(\Gamma_3) \text{ such that } \varphi_\nu(\cdot, \widetilde{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(b) } \varphi_\nu(\boldsymbol{x}, \cdot) \text{ is convex on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \end{array} \right. \tag{69}$$

The weak formulation of the corresponding contact problem is the following.

Problem (P_3). *Find a displacement field* $\boldsymbol{u} \in U$ *such that*

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_{\Gamma_3} [\varphi_\nu(v_\nu) - \varphi_\nu(u_\nu)]\,d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\,d\Gamma$$
$$\ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in U.$$

Error analysis of numerical methods for Problem (P_3) can be done similar to that of Problem (P_1). The counterpart of equation (58) in Theorem 5 is

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V \le c \inf_{\boldsymbol{v}^h \in U^h} \left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V + \|u_\nu - v_\nu^h\|^{1/2}_{L^2(\Gamma_3)} \right) + c \inf_{\boldsymbol{v} \in U} \|\boldsymbol{v}-\boldsymbol{u}^h\|^{1/2}_{L^2(\Gamma_3)^d}.$$

Then, under the assumptions stated in Theorem 6, the optimal order error bound equation (64) holds also for the linear finite element solutions of Problem (P_3).

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