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Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage

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Abstract

We consider a model for quasistatic frictional contact between a viscoelastic body and a foundation. The material constitutive relation is assumed to be nonlinear. The mechanical damage of the material, caused by excessive stress or strain, is described by the damage function, the evolution of which is determined by a parabolic inclusion. The contact is modeled with the normal compliance condition and the associated version of Coulomb's law of dry friction. We derive a variational formulation for the problem and prove the existence of its unique weak solution. We then study a fully discrete scheme for the numerical solutions of the problem and obtain error estimates on the approximate solutions.

Keywords: Quasistatic frictional contact; Viscoelastic material; Normal compliance; Coulomb's friction; Mechanical damage; Variational inequality; Variational analysis; Numerical analysis; Fully discrete scheme; Finite element method

1. Introduction

We model and analyze the process of quasistatic contact with friction between a viscoelastic body and a foundation, and the resulting damage caused by mechanical strain. In many engineering applications, the forces acting on the system vary periodically and so do the strains and stresses. This may cause the growth of microcracks which reduces the usefulness of the system. Therefore, accurate prediction of the damage is of considerable importance for the safe and reliable operation of mechanical equipment.

Recent models for mechanical damage derived from thermomechanical considerations can be found in [10,11]. These papers also include numerical simulations of such problems. Mathematical analysis

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of one-dimensional problems with damage appeared in [8,9], and recently, a contact problem with material damage has been investigated in [26]. The contact in that study was, however, modeled with a general damped response condition, while here, we consider a similar problem but with the normal compliance contact condition. Moreover, we also perform a numerical analysis of the model.

We consider a body made of a viscoelastic material with constitutive relation

$$\boldsymbol{\sigma} = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are the stress and linearized strain tensor fields, respectively, and β is the damage field. The latter measures the decrease in the load-bearing capacity of the material; when $\beta=1$ the material has its full capacity and when $\beta=0$ it is completely damaged. $\mathscr{A}$ and G are nonlinear constitutive functions. Finally, the dot above a variable represents the time derivative.

Following Frémond and Nedjar [10,11], the evolution of the microscopic cracks which cause the damage is governed by the differential inclusion

$$\dot{\beta} - \kappa\Delta\beta + \partial\psi_K(\beta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta), \tag{1.2}$$

where κ is a positive material constant, $K = [0, 1]$, ψ_K is the indicator function of K and $\partial\psi_K$ represents its subdifferential. ϕ is a given constitutive function describing the sources of damage in the system which results from tension or compression.

We model the contact between the viscoelastic body and the foundation with a normal compliance condition and the associated Coulomb's law of dry friction. The normal compliance contact condition was proposed in [22] and has been used extensively since then (see, e.g., [19,20,25] and the references therein).

The present paper is a continuation of [25]. The results obtained there deal with the existence and uniqueness of a weak solution for a quasistatic viscoelastic problem with normal compliance, but neither the mechanical damage to the material nor the numerical analysis were included in [25]. The novelty of this paper is the inclusion the material damage and the numerical analysis of the model.

We provide variational analysis of the mechanical problem and show the existence of a unique weak solution for the model. Then we perform numerical analysis of the problem and derive error estimates for the numerical approximations based on discrete schemes. Literature on the numerical treatment of variational inequalities is extensive (see, e.g., the monographs [12,13,18]). Of particular relevance to this paper are the results on numerical analysis of variational inequalities in plasticity and viscoelasticity (see, e.g., [14–16]).

The paper is organized as follows. In Section 2, we state the mechanical problem and discuss the contact conditions. In Section 3, we introduce the notation, list the assumptions on the data and derive the variational form of the model. Then we state the main existence and uniqueness result, Theorem 3.1, which is proven in Section 4. The proof is based on classical results of elliptic and parabolic variational inequalities and Banach's fixed point theorem. In Section 5, we analyze a fully discrete scheme for the problem. We use the finite element method to discretize the domain and a backward Euler finite difference to discretize the time derivative. We obtain error estimates for this scheme, stated in Theorem 5.2. Finally, under appropriate regularity assumptions on the exact solution, we obtain an optimal-order error estimate.

2. The model

We consider a viscoelastic body which occupies a domain Ω of $\mathbb{R}^d$ ($d = 2, 3$ in applications). The body is acted upon by time-dependent volume forces and surface tractions and may come into frictional contact with an obstacle, the so-called foundation. We assume that the boundary of Ω, denoted by Γ, is Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with $\text{meas}(\Gamma_1) > 0$. The body is held fixed on Γ_1 and therefore, the displacement vanishes there. Volume forces of density $\boldsymbol{f}_0$ act in Ω and surface tractions of density $\boldsymbol{f}_2$ are applied on Γ_2. We assume that volume forces and tractions vary slowly in time and, therefore, the accelerations in the system are negligible and the process is quasistatic. A gap g exists between the potential contact surface Γ_3 and the foundation, measured along the outward normal. We model the contact with the normal compliance condition and a version of Coulomb's law of dry friction. We use (1.1) as the constitutive law and, for the sake of simplicity, we assume a homogeneous Neumann boundary condition for the damage field. Then, given $T > 0$, a classical model for this process is the following.

Problem P**.** Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{R}_s^{d\times d}$ and a damage field $\beta : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\dot{\beta} - \kappa\Delta\beta + \partial\psi_K(\beta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0, T), \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$-\sigma_\nu = p_\nu(u_\nu - g) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$|\boldsymbol{\sigma}_\tau| \leqslant p_\tau(u_\nu - g) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.8}$$

$$|\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu - g) \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0},$$
$$|\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu - g) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda\dot{\boldsymbol{u}}_\tau, \quad \lambda \geqslant 0,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0 \quad \text{in } \Omega. \tag{2.9}$$

Here and below, $\mathbb{R}_s^{d\times d}$ denotes the space of second-order symmetric tensors on $\mathbb{R}^d$, $\boldsymbol{\nu}$ represents the unit outer normal on Γ and $\partial\beta/\partial\nu$ is the normal derivative of β on Γ. In the equilibrium equations (2.3) "Div" denotes the divergence operator. (2.4) is the boundary condition for the damage field; (2.5) and (2.6) represent the displacement and traction boundary conditions, respectively. The functions $\boldsymbol{u}_0$ and β_0 in (2.9) are prescribed and represent the initial displacements and the initial damage field, respectively.

We now comment on (2.1), (2.2) and the contact conditions (2.7), (2.8). In (2.1) the effective elastic coefficients depend on the damage field. Now, following Frémond and Nedjar, β is restricted to have values between zero and one: when $\beta = 1$ there is no damage in the material; when $\beta = 0$ the material is completely damaged; when $0 < \beta < 1$ there is a partial damage and the material has a decreased load bearing capacity. The function ϕ in (2.2) is the source of damage and depends on the damage field and on the mechanical strain.

In [10,11] the damage source function was chosen as

$$\phi_{\mathrm{Fr}}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) = \lambda_D \left(\frac{1-\beta}{\beta} \right) - \frac{1}{2}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda_w,$$

where λ_D and λ_w are two positive process parameters. We note that ϕ_{Fr} becomes unbounded when $\beta \to 0$, a condition which we do not allow here. Therefore, we may consider the global solutions which we establish below as local solutions of a problem with the damage source of the type ϕ_{Fr}, valid as long as $0 < \beta_* \leqslant \beta$, where β_* depends on the relationship between ϕ and ϕ_{Fr}. We assume that the material may recover from damage and cracks may close; thus, we do not impose the restriction $\dot{\beta} \leqslant 0$ which was used in [8–11], since there the damage was considered irreversible.

Equality (2.7) represents the normal compliance contact condition where u_ν is the normal displacement, σ_ν is the normal stress and p_ν is a prescribed function. When positive, $u_\nu - g$ represents the penetration of the surface asperities into those of the foundation. An example of a normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+, \tag{2.10}$$

where c_ν is a positive constant and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$.

Relations (2.8) are a version of Coulomb's law of dry friction. Here $\boldsymbol{\sigma}_\tau$ denotes tangential stress, $\dot{\boldsymbol{u}}_\tau$ represents the tangential velocity and p_τ is a prescribed nonnegative function, the so-called *friction bound*. According to (2.8) the tangential shear cannot exceed the maximal frictional resistance $p_\tau(u_\nu - g)$. Then, if the strict inequality holds, the surface adheres to the foundation and is in the so-called *stick* state, and when equality holds there is relative sliding, the so-called *slip* state. Therefore, at each time instant the contact surface Γ_3 is divided into three zones: the stick zone, the slip zone and the zone of separation, in which $u_\nu < g$ and there is no contact. The boundaries of these zones are unknown a priori and form free boundaries. The choice

$$p_\tau = \mu p_\nu, \tag{2.11}$$

leads to the usual Coulomb's law, and $\mu \geqslant 0$ is the coefficient of friction (see, e.g., [7] or [24]). Recently, a modified version of the Coulomb friction law was derived in [27] from thermodynamic considerations. It consists of using the friction law (2.8) with

$$p_\tau = \mu p_\nu (1 - \delta p_\nu)_+, \tag{2.12}$$

where δ is a small positive material constant related to the wear and hardness of the surface.

3. Main existence and uniqueness result

In this section, we list the assumptions imposed on the problem data, present a variational formulation of the mechanical problem and state our main existence and uniqueness result.

First, we introduce the following notation. We denote by $\mathbb{R}_s^{d\times d}$ the space of second-order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{R}_s^{d\times d}$ by

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\quad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2},\quad \forall \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij},\quad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2},\quad \forall \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{R}_s^{d\times d}.$$

Here and throughout this paper, the indices i and j run between 1 and d; the summation convention over repeated indices is used, and the index following a comma indicates a partial derivative. Next, we use the following spaces:

$$H=\{\boldsymbol{v}=(v_i)\mid v_i\in L^2(\Omega)\}=L^2(\Omega)^d,$$

$$H_1=\{\boldsymbol{v}=(v_i)\mid v_i\in H^1(\Omega)\}=H^1(\Omega)^d,$$

$$\mathscr{H}=\{\boldsymbol{\tau}=(\tau_{ij})\mid \tau_{ij}=\tau_{ji}\in L^2(\Omega)\}=L^2(\Omega)_s^{d\times d},$$

$$\mathscr{H}_1=\{\boldsymbol{\tau}\in\mathscr{H}\mid \tau_{ij,j}\in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_H=\int_\Omega u_iv_i\,\mathrm{d}x,\quad \langle\boldsymbol{\sigma},\boldsymbol{\tau}\rangle_{\mathscr{H}}=\int_\Omega \sigma_{ij}\tau_{ij}\,\mathrm{d}x,$$

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_{H_1}=\langle\boldsymbol{u},\boldsymbol{v}\rangle_H+\langle\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}},\quad \langle\boldsymbol{\sigma},\boldsymbol{\tau}\rangle_{\mathscr{H}_1}=\langle\boldsymbol{\sigma},\boldsymbol{\tau}\rangle_{\mathscr{H}}+\langle\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau}\rangle_H$$

with the associated norms $\|\cdot\|_H$, $\|\cdot\|_{\mathscr{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathscr{H}_1}$, respectively. Here $\boldsymbol{\varepsilon}:H_1\to\mathscr{H}$ and $\operatorname{Div}:\mathscr{H}_1\to H$ are the *deformation* and *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u})=\tfrac{1}{2}(u_{i,j}+u_{j,i}),\quad \operatorname{Div}\boldsymbol{\sigma}=(\sigma_{ij,j}).$$

For an element $\boldsymbol{v}\in H_1$ we denote by $\boldsymbol{v}$ its trace on Γ and by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$ its *normal* and *tangential* components on the boundary. We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* traces of $\boldsymbol{\sigma}\in\mathscr{H}_1$. If $\boldsymbol{\sigma}$ is a regular function (e.g., C^1) then $\sigma_\nu=(\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma\nu}-\sigma_\nu\boldsymbol{\nu}$. The following Green formula holds:

$$\langle\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}}+\langle\operatorname{Div}\boldsymbol{\sigma},\boldsymbol{v}\rangle_H=\int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,\mathrm{d}\Gamma\quad \forall\boldsymbol{v}\in H_1. \tag{3.1}$$

Let V be the closed subspace of H_1 given by

$$V=\{\boldsymbol{v}\in H_1\mid \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\}.$$

Since $\operatorname{meas}(\Gamma_1)>0$ and Γ is Lipschitz, Korn's inequality (see, e.g., [23]) holds,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathscr{H}}\geqslant c\|\boldsymbol{v}\|_{H_1}\quad \forall\boldsymbol{v}\in V, \tag{3.2}$$

where here and below c represents a positive constant which may change its value from place to place and may depend on the input data. We define the inner product $\langle\cdot,\cdot\rangle_V$ on V by

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_V=\langle\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}}. \tag{3.3}$$

It follows from (3.2) and (3.3) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V, thus, $(V, \|\cdot\|_V)$ is a real Hilbert space.

We note that the assumption that Γ is Lipschitz continuous is sufficient for our purposes. First, it ensures that the outer normal $\boldsymbol{v}$ is defined a.e. on Γ, and then the normal and tangential components of various functions make sense. Second, it is sufficient for Korn's inequality (3.2) to hold true, see, e.g., [5,23,28] and also [21], where a detailed proof can be found.

Finally, if $(X, \|\cdot\|_X)$ is a real Hilbert space, we use the standard notation for $L^p(0,T;X)$ and Sobolev spaces $H^k(0,T;X)$, $K \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$. We also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, with norms

$$\|x\|_{C(0,T;X)} = \max_{t\in[0,T]} \|x(t)\|_X, \quad \|x\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X,$$

respectively. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product space endowed with the canonical inner product $\langle \cdot,\cdot \rangle_{X_1\times X_2}$ and norm $\|\cdot\|_{X_1\times X_2}$. For further details we refer the reader to [1,7,17,24].

To study the mechanical problem (2.1)–(2.9) we make the following assumptions on the data.

The *viscosity operator* $\mathscr{A} : \Omega \times \mathbb{R}_s^{d\times d} \to \mathbb{R}_s^{d\times d}$ satisfies:

(a) there exists $L_{\mathscr{A}} > 0$ such that

$$\|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathscr{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{R}_s^{d\times d}, \text{ a.e. } \boldsymbol{x} \in \Omega,$$

(b) there exists $m > 0$ such that

$$(\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{R}_s^{d\times d}, \text{ a.e. } \boldsymbol{x} \in \Omega, \tag{3.4}$$

(c) $\boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is Lebesgue measurable on $\Omega \quad \forall \boldsymbol{\varepsilon} \in \mathbb{R}_s^{d\times d}$,
(d) $\boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \mathbf{0}) \in \mathscr{H}$.

The *elasticity operator* $G : \Omega \times \mathbb{R}_s^{d\times d} \times \mathbb{R} \to \mathbb{R}_s^{d\times d}$ satisfies:

(a) there exists $L_G > 0$ such that

$$\begin{aligned} &\|G(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \beta_1) - G(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2)\| \leqslant L_G(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\beta_1 - \beta_2|) \\ &\quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{R}_s^{d\times d}, \ \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega, \end{aligned} \tag{3.5}$$

(b) $\boldsymbol{x} \mapsto G(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta)$ is Lebesgue measurable on $\Omega \quad \forall \boldsymbol{\varepsilon} \in \mathbb{R}_s^{d\times d}$, $\beta \in \mathbb{R}$,
(c) $\boldsymbol{x} \mapsto G(\boldsymbol{x}, \mathbf{0}, 0) \in \mathscr{H}$.

The *damage source function* $\phi : \Omega \times \mathbb{R}_s^{d\times d} \times \mathbb{R} \to \mathbb{R}$ satisfies:

(a) there exists $L_\phi > 0$ such that

$$\begin{aligned} &|\phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \beta_1) - \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2)| \leqslant L_\phi(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\beta_1 - \beta_2|) \\ &\quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{R}_s^{d\times d}, \ \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega, \end{aligned} \tag{3.6}$$

(b) $\boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta)$ is Lebesgue measurable on $\Omega \quad \forall \boldsymbol{\varepsilon} \in \mathbb{R}_s^{d\times d}$, $\beta \in \mathbb{R}$,
(c) $\boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \mathbf{0}, 0) \in L^2(\Omega)$.

The *normal compliance function* p_ν and the *friction bound* p_τ, where $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ $(r = \nu, \tau)$ satisfy:

(a) there exists $L_r > 0$ such that

$$|p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \leqslant L_r |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \tag{3.7}$$

(b) $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u)$ is Lebesgue measurable on Γ_3 $\quad \forall u \in \mathbb{R}$,

(c) $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u) = 0$ for $u \leqslant 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.

Assumptions (3.4) on the viscosity operator are rather routine, and effectively follow from the linearized case. And so are the assumptions (3.5) on G. However, assumptions (3.6) on ϕ are more delicate. Indeed, the function ϕ_{Fr}, which was derived from first principles, doesn't satisfy them. The issue is that once the damage is complete, $\beta = 0$, the mechanical system breaks down, and we have "quenching" of the solution. Therefore, one can consider ϕ as a truncated version of ϕ_{Fr}, valid as long as $0 < \beta_* \leqslant \beta$. Thus, we may consider the problem below as an approximation of the "real" problem.

Assumptions (3.7) on p_ν and p_τ are fairly general. The main restriction is the requirement that asymptotically the functions grow at most linearly. Clearly, the function defined in (2.10) satisfies this condition. We also observe that if the functions p_ν and p_τ are related by (2.11) or (2.12) and p_ν satisfies condition (3.7)(a), then p_τ does too with $L_\tau = \mu L_\nu$.

The forces and tractions are assumed to satisfy

$$\boldsymbol{f}_0 \in C([0,T]; H), \quad \boldsymbol{f}_2 \in C([0,T]; L^2(\Gamma_2)^d) \tag{3.8}$$

and the gap function satisfies

$$g \in L^2(\Gamma_3), \quad g \geqslant 0 \text{ a.e. on } \Gamma_3. \tag{3.9}$$

The initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \beta_0 \in \mathscr{K}, \tag{3.10}$$

where $\mathscr{K}$ represents the set of admissible damage functions defined by

$$\mathscr{K} = \{\xi \in H^1(\Omega) \mid \xi \in K \text{ a.e. in } \Omega\}.$$

Next, we define $\boldsymbol{f} : [0,T] \to V$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, \mathrm{d}\Gamma \tag{3.11}$$

for all $\boldsymbol{v} \in V$, $t \in [0,T]$. We note that conditions (3.8) imply

$$\boldsymbol{f} \in C([0,T]; V). \tag{3.12}$$

Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form

$$a(\xi, \eta) = \kappa \int_\Omega \nabla \xi \cdot \nabla \eta \, \mathrm{d}x \tag{3.13}$$

and $j : V \times V \to \mathbb{R}$ be the functional

$$j(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_3} (p_\nu(v_\nu - g) w_\nu + p_\tau(v_\nu - g)|\boldsymbol{w}_\tau|) \, \mathrm{d}\Gamma. \tag{3.14}$$

By assumptions on p_ν and p_τ, we see that for $\boldsymbol{v} \in V$, $p_\nu(v_\nu - g), p_\tau(v_\nu - g) \in L^2(\Gamma_3)$. Thus, the functional $j(\cdot, \cdot)$ is well defined on $V \times V$.

The variational formulation of the quasistatic problem with normal compliance, friction and damage is as follows.

Problem P_V**.** Find a displacement field $\boldsymbol{u} : [0,T] \to V$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathscr{H}$ and a damage field $\beta : [0,T] \to H^1(\Omega)$ such that

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)), \tag{3.15}$$

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\rangle_{\mathscr{H}} + j(\boldsymbol{u}(t), \boldsymbol{w}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geqslant \langle \boldsymbol{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_V \quad \forall \boldsymbol{w} \in V \tag{3.16}$$

for all $t \in [0,T]$,

$$\beta(t) \in \mathscr{K},$$

$$\langle \dot{\beta}(t), \xi - \beta(t)\rangle_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t)) \geqslant \langle \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)), \xi - \beta(t)\rangle_{L^2(\Omega)} \quad \forall \xi \in \mathscr{K}, \tag{3.17}$$

for a.e. $t \in (0,T)$, and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0. \tag{3.18}$$

This formulation is obtained by using arguments similar to those in [25,26].

Our main existence and uniqueness result, which we establish in the next section, is:

Theorem 3.1. *Assume that* (3.4)–(3.10) *hold. Then problem* P_V *has a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$. *Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1([0,T]; V), \quad \boldsymbol{\sigma} \in C([0,T]; \mathscr{H}_1), \quad \beta \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

We conclude that under assumptions (3.4)–(3.10), the mechanical problem (2.1)–(2.9) has a unique weak solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$. Furthermore, it follows that the problem with an unbounded damage source function ϕ_{Fr} has a unique local weak solution.

Next, for numerical purposes, we formulate the problem in terms of the velocities instead of displacements. Let

$$\boldsymbol{v} = \dot{\boldsymbol{u}}, \tag{3.19}$$

denote the velocity field. Then

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)\,\mathrm{d}s \quad \forall t \in [0,T], \tag{3.20}$$

since $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Relations (3.15) and (3.16) can now be combined, thus,

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)) + G(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\boldsymbol{v}(t))\rangle_{\mathscr{H}} + j(\boldsymbol{u}(t), \boldsymbol{w}) - j(\boldsymbol{u}(t), \boldsymbol{v}(t))$$
$$\geqslant \langle \boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}(t)\rangle_V \quad \forall \boldsymbol{w} \in V,\ t \in [0,T]. \tag{3.21}$$

This inequality is used in Section 5 in the numerical analysis of problem P_V.

We now list the properties of the solution of P_V which are needed in Section 5. Let $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$ be the solution then, for all $t \in [0,T]$, the following equalities hold:

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{a.e. in } \Omega, \tag{3.22}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{v} = \boldsymbol{f}_2(t) \quad \text{a.e. on } \Gamma_2, \tag{3.23}$$

$$-\sigma_\nu(t) = p_\nu(u_\nu(t) - g) \quad \text{a.e. on } \Gamma_3. \tag{3.24}$$

Using assumptions (3.7)–(3.9), the proof of (3.22)–(3.24) follow from standard arguments (see, e.g., [18]).

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on classical results for elliptic and parabolic variational inequalities and fixed point arguments, and is similar to those used in [25,26]. It is carried out in several steps. We assume that (3.4)–(3.10) hold and, to simplify the notation, we do not indicate explicitly the dependence on t.

Let $\boldsymbol{\eta} \in C([0,T];\mathscr{H})$ and $\theta \in C([0,T];L^2(\Omega))$ be given. In the first step we consider the following auxiliary problems.

Problem P_η^1. Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_\eta$: $[0,T] \to \mathscr{H}$ such that

$$\boldsymbol{\sigma}_\eta = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta) + \boldsymbol{\eta}, \tag{4.1}$$

$$\langle \boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta)\rangle_{\mathscr{H}} + j(\boldsymbol{u}_\eta, \boldsymbol{w}) - j(\boldsymbol{u}_\eta, \dot{\boldsymbol{u}}_\eta) \geqslant \langle \boldsymbol{f}, \boldsymbol{w} - \dot{\boldsymbol{u}}_\eta\rangle_V \quad \forall \boldsymbol{w} \in V \tag{4.2}$$

for all $t \in [0,T]$, and

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.3}$$

Problem P_θ^2. Find a damage field $\beta_\theta : [0,T] \to H^1(\Omega)$ such that

$$\beta_\theta \in \mathscr{K}, \quad \langle \dot{\beta}_\theta, \xi - \beta_\theta\rangle_{L^2(\Omega)} + a(\beta_\theta, \xi - \beta_\theta) \geqslant \langle \theta, \xi - \beta_\theta\rangle_{L^2(\Omega)} \quad \forall \xi \in \mathscr{K} \tag{4.4}$$

for a.e. $t \in (0,T)$, and

$$\beta_\theta(0) = \beta_0. \tag{4.5}$$

To solve problem P_η^1 we need the following result.

Proposition 4.1. *Let $\boldsymbol{g} \in C([0,T];V)$. Then there exists a function $\boldsymbol{v}_{\eta g} \in C([0,T];V)$ such that for all $t \in [0,T]$,*

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta g}), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta g})\rangle_{\mathscr{H}} + \langle \boldsymbol{\eta}, \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta g})\rangle_{\mathscr{H}} + j(\boldsymbol{g}, \boldsymbol{w}) - j(\boldsymbol{g}, \boldsymbol{v}_{\eta g}) \geqslant \langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{v}_{\eta g}\rangle_V \quad \forall \boldsymbol{w} \in V. \tag{4.6}$$

Proof. It follows from classical results for elliptic variational inequalities (see, e.g., [3]) that there exists a unique function $\boldsymbol{v}_{\eta g} : [0,T] \to V$ which solves (4.6). To establish the regularity claim $\boldsymbol{v}_{\eta g} \in C([0,T];V)$, let $t_1, t_2 \in [0,T]$ and denote by $\boldsymbol{\eta}_i = \boldsymbol{\eta}(t_i)$, $\boldsymbol{g}_i = \boldsymbol{g}(t_i)$, $\boldsymbol{f}_i = \boldsymbol{f}(t_i)$ and $\boldsymbol{v}_i = \boldsymbol{v}_{\eta g}(t_i)$, $i = 1,2$. Using algebraic manipulations we obtain from (4.6) that

$$\begin{aligned}\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \boldsymbol{\varepsilon}(\boldsymbol{v}_2)\rangle_{\mathscr{H}} &\leqslant \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_V \\ &+ \langle \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{\varepsilon}(\boldsymbol{v}_2) - \boldsymbol{\varepsilon}(\boldsymbol{v}_1)\rangle_{\mathscr{H}} + j(\boldsymbol{g}_1, \boldsymbol{v}_2) - j(\boldsymbol{g}_1, \boldsymbol{v}_1) + j(\boldsymbol{g}_2, \boldsymbol{v}_1) - j(\boldsymbol{g}_2, \boldsymbol{v}_2).\end{aligned} \tag{4.7}$$

Moreover, it follows from (3.4) and (3.3) that

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \boldsymbol{\varepsilon}(\boldsymbol{v}_2)\rangle_{\mathscr{H}} \geqslant m\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2 \tag{4.8}$$

and (3.7) implies

$$j(\boldsymbol{g}_1,\boldsymbol{v}_2)-j(\boldsymbol{g}_1,\boldsymbol{v}_1)+j(\boldsymbol{g}_2,\boldsymbol{v}_1)-j(\boldsymbol{g}_2,\boldsymbol{v}_2)\leqslant c\|\boldsymbol{g}_1-\boldsymbol{g}_2\|_V\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V. \tag{4.9}$$

Using now (4.7)–(4.9) we obtain

$$\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V\leqslant c(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_V+\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_{\mathscr{H}}+\|\boldsymbol{g}_1-\boldsymbol{g}_2\|_V). \tag{4.10}$$

The fact that $\boldsymbol{v}_{\eta g}\in C([0,T];V)$ follows from the inequality (4.10) and the regularity assumptions on $\boldsymbol{f}$, $\boldsymbol{\eta}$ and $\boldsymbol{g}$. □

Now, we prove the following existence and uniqueness result for problem P_η^1.

Proposition 4.2. *There exists a unique solution of problem P_η^1 such that $\boldsymbol{u}_\eta\in C^1([0,T];V)$, $\boldsymbol{\sigma}_\eta\in C([0,T];\mathscr{H}_1)$.*

Proof. We consider the operator $\Lambda_\eta: C([0,T];V)\to C([0,T];V)$ defined by

$$\Lambda_\eta\boldsymbol{g}(t)=\boldsymbol{u}_0+\int_0^t\boldsymbol{v}_{\eta g}(s)\,\mathrm{d}s\quad\forall\boldsymbol{g}\in C([0,T];V),\quad t\in[0,T], \tag{4.11}$$

where $\boldsymbol{v}_{\eta g}$ is the solution of (4.6). We show that this operator has a unique fixed point $\boldsymbol{g}_\eta\in C([0,T];V)$. To this end, let $\boldsymbol{g}_1,\boldsymbol{g}_2\in C([0,T];V)$ and denote by $\boldsymbol{v}_i=\boldsymbol{v}_{\eta g_i}$, $i=1,2$, the corresponding solutions of (4.6). Using (4.11) we obtain

$$\|\Lambda_\eta\boldsymbol{g}_1(t)-\Lambda_\eta\boldsymbol{g}_2(t)\|_V\leqslant c\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V\,\mathrm{d}s\quad\forall t\in[0,T]. \tag{4.12}$$

Using estimates similar to those in the proof of Proposition 4.1 (see (4.10)) we have

$$\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V\leqslant c\|\boldsymbol{g}_1(s)-\boldsymbol{g}_2(s)\|_V\quad\forall s\in[0,T]$$

and, taking into account (4.12), it follows that

$$\|\Lambda_\eta\boldsymbol{g}_1(t)-\Lambda_\eta\boldsymbol{g}_2(t)\|_V\leqslant c\int_0^t\|\boldsymbol{g}_1(s)-\boldsymbol{g}_2(s)\|_V\,\mathrm{d}s\quad\forall t\in[0,T]. \tag{4.13}$$

Reiterating this inequality n times we obtain

$$\|\Lambda_\eta^n\boldsymbol{g}_1-\Lambda_\eta^n\boldsymbol{g}_2\|_{C([0,T];V)}\leqslant\frac{c^n}{n!}\|\boldsymbol{g}_1-\boldsymbol{g}_2\|_{C([0,T];V)}.$$

This shows that for n large enough the operator Λ_η^n is a contraction in $C([0,T],V)$. Thus, there exists a unique element $\boldsymbol{g}_\eta\in C([0,T],V)$ such that $\Lambda_\eta^n\boldsymbol{g}_\eta=\boldsymbol{g}_\eta$ and $\boldsymbol{g}_\eta$ is also the unique fixed point of Λ_η.

Next, let $\boldsymbol{v}_\eta\in C([0,T];V)$, $\boldsymbol{u}_\eta\in C^1([0,T];V)$ and $\boldsymbol{\sigma}_\eta\in C([0,T];\mathscr{H})$ be given by

$$\boldsymbol{v}_\eta=\boldsymbol{v}_{\eta g_\eta}, \tag{4.14}$$

$$\boldsymbol{u}_\eta(t)=\boldsymbol{u}_0+\int_0^t\boldsymbol{v}_\eta(s)\,\mathrm{d}s\quad\forall t\in[0,T], \tag{4.15}$$

$$\boldsymbol{\sigma}_\eta=\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta)+\boldsymbol{\eta}. \tag{4.16}$$

Clearly, $\{\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta\}$ satisfies (4.1) and (4.3). Moreover, by (4.15), (4.14) and (4.11) it follows that $\boldsymbol{u}_\eta = \boldsymbol{g}_\eta$ and $\dot{\boldsymbol{u}}_\eta = \boldsymbol{v}_\eta$. Therefore, if we let $\boldsymbol{g} = \boldsymbol{g}_\eta$ in (4.6) we obtain (4.2). Choosing $\boldsymbol{w} = \dot{\boldsymbol{u}}_\eta \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^d$ in (4.2) we get

$$\langle \boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\rangle_{\mathscr{H}} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_V \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega)^d \quad \text{on } [0,T]$$

and using (3.11) we find

$$\operatorname{Div} \boldsymbol{\sigma}_\eta + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{on } [0,T]. \tag{4.17}$$

Now, assumption (3.8) and Eq. (4.17) imply that $\boldsymbol{\sigma} \in C([0,T]; \mathscr{H}_1)$.

This establishes the existence part in Proposition 4.2. The uniqueness part follows directly from (4.1)–(4.3), using (3.4), (3.7) and a Gronwall-type inequality. □

We prove next the unique solvability of the auxiliary problem P_θ^2.

Proposition 4.3. *There exists a unique solution for problem P_θ^2, and*

$$\beta_\theta \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

Proof. It follows from the coercivity of the form a defined by (3.13) and standard results for parabolic variational inequalities (see, e.g., [2, p. 124]). □

As a consequence of Propositions 4.2 and 4.3, and assumptions (3.5) and (3.6), we may define an operator $\Lambda : C([0,T]; \mathscr{H} \times L^2(\Omega)) \to C([0,T]; \mathscr{H} \times L^2(\Omega))$ by

$$\Lambda(\boldsymbol{\eta}, \theta) = (G(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta), \beta_\theta), \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta), \beta_\theta)) \tag{4.18}$$

for all $(\boldsymbol{\eta}, \theta) \in C([0,T]; \mathscr{H} \times L^2(\Omega))$.

Proposition 4.4. *The operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in C([0,T]; \mathscr{H} \times L^2(\Omega))$.*

Proof. Let $(\boldsymbol{\eta}_1, \theta_1)$, $(\boldsymbol{\eta}_2, \theta_2) \in C([0,T]; \mathscr{H} \times L^2(\Omega))$. Using (4.18), (3.5) and (3.6) we deduce that

$$\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathscr{H} \times L^2(\Omega)} \leqslant c(\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V + \|\beta_{\theta_1}(t) - \beta_{\theta_2}(t)\|_{L^2(\Omega)}) \tag{4.19}$$

for all $t \in [0,T]$. It follows from (4.15) that

$$\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V \leqslant c \int_0^t \|\boldsymbol{v}_{\eta_1}(s) - \boldsymbol{v}_{\eta_2}(s)\|_V \,\mathrm{d}s \quad \forall t \in [0,T]. \tag{4.20}$$

Using (4.1), (4.2) and estimates similar to those in the proof of Proposition 4.1 (see (4.10)) we find

$$\|\boldsymbol{v}_{\eta_1}(s) - \boldsymbol{v}_{\eta_2}(s)\|_V \leqslant c(\|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathscr{H}} + \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|_V) \quad \forall s \in [0,T]. \tag{4.21}$$

Combining (4.20) and (4.21), and using a Gronwall-type inequality we have

$$\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V \leqslant c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathscr{H}} \,\mathrm{d}s \quad \forall t \in [0,T]. \tag{4.22}$$

On the other hand, (4.4), (4.5) imply that

$$\|\beta_{\theta_1}(t) - \beta_{\theta_2}(t)\|_{L^2(\Omega)} \leqslant c \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} \,\mathrm{d}s \quad \forall t \in [0,T]. \tag{4.23}$$

Using now (4.19), (4.22) and (4.23) we find

$$\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathscr{H}\times L^2(\Omega)}\leqslant c\int_0^t\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|_{\mathscr{H}\times L^2(\Omega)}\,\mathrm{d}s\quad\forall t\in[0,T]. \tag{4.24}$$

Proposition 4.4 follows now from (4.24) and the Banach fixed-point theorem. □

We have now all the ingredients to prove Theorem 3.1.

Proof of Theorem 3.1

Existence: Let $\{\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*}\}$ be the solution of (4.1)–(4.3) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and let β_{θ^*} be the solution of (4.4), (4.5) for $\theta=\theta^*$. Since $\boldsymbol{\eta}^*=G(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}),\beta_{\theta^*})$ and $\theta^*=\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}),\beta_{\theta^*})$, it is straightforward to see that $\{\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\beta_{\theta^*}\}$ is a solution of problem (3.15)–(3.18) such that $\boldsymbol{u}_{\eta^*}\in C^1([0,T];V)$, $\boldsymbol{\sigma}_{\eta^*}\in C([0,T];\mathscr{H}_1)$ and $\beta_{\theta^*}\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$.

Uniqueness: Let $\{\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\beta_{\eta^*}\}$ be the solution of (3.15)–(3.18) obtained above and let $\{\boldsymbol{u},\boldsymbol{\sigma},\beta\}$ be another solution of the problem such that $\boldsymbol{u}\in C^1([0,T];V)$, $\boldsymbol{\sigma}\in C([0,T];\mathscr{H}_1)$ and $\beta\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$. We denote by $\boldsymbol{\eta}\in C([0,T];\mathscr{H})$ and $\theta\in C([0,T];L^2(\Omega))$ the functions

$$\boldsymbol{\eta}=G(\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta),\quad \theta=\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta). \tag{4.25}$$

Now, (3.15), (3.16) and (3.18) imply that $\{\boldsymbol{u},\boldsymbol{\sigma}\}$ is a solution of the variational problem P_η^1. Using Proposition 4.2 it follows that this problem has a unique solution $\boldsymbol{u}_\eta\in C^1([0,T];V)$, $\boldsymbol{\sigma}\in C([0,T];\mathscr{H}_1)$ and so we conclude that

$$\boldsymbol{u}=\boldsymbol{u}_\eta,\quad \boldsymbol{\sigma}=\boldsymbol{\sigma}_\eta. \tag{4.26}$$

Next, (3.17), (3.18) and a similar argument yield

$$\beta=\beta_\theta. \tag{4.27}$$

Using now (4.18), (4.26), (4.27) and (4.25) we obtain $\Lambda(\boldsymbol{\eta},\theta)=(\boldsymbol{\eta},\theta)$ and by the uniqueness of the fixed point of the operator Λ we deduce

$$\boldsymbol{\eta}=\boldsymbol{\eta}^*,\quad \theta=\theta^*. \tag{4.28}$$

The uniqueness of the solution is now a consequence of (4.26)–(4.28). □

5. Discrete approximation

We introduce and analyze a fully discrete approximation scheme for the problem. We discretize both the space and time variables. Let $V^h\subset V$ be a finite-dimensional space and $\mathscr{K}^h\subset\mathscr{K}$ be a nonempty, finite-dimensional closed convex set. We discuss below how to construct them. In addition, we introduce a uniform partition of the time interval $[0,T]$ with the step-size $k=T/N$ and the nodes $t_n=nk,\ n=0,1,\dots,N$. We use the notation $z_n=z(t_n)$ for a continuous function $z(t)$. For a sequence $\{z_n\}_{n=0}^N$ we denote by $\delta z_n=(z_n-z_{n-1})/k$ the divided difference. In this section, no summation is implied over the repeated index n, and the generic constant c does not depend on k,h or n.

Keeping in mind (3.21), (3.17) and (3.18), we introduce the following fully discretized approximation of problem P_V.

Problem P_V^{hk}. Find $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ and $\beta^{hk} = \{\beta_n^{hk}\}_{n=0}^N \subset \mathscr{K}^h$ such that

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\rangle_{\mathscr{H}} + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}^h) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk})$$
$$\geqslant \langle \boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk}\rangle_V - \langle G(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \beta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\rangle_{\mathscr{H}} \quad \forall \boldsymbol{w}^h \in V^h, \tag{5.1}$$

$$\langle \delta\beta_n^{hk}, \xi^h - \beta_n^{hk}\rangle_{L^2(\Omega)} + a(\beta_n^{hk}, \xi^h - \beta_n^{hk}) \geqslant \langle \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \beta_{n-1}^{hk}), \xi^h - \beta_n^{hk}\rangle_{L^2(\Omega)} \quad \forall \xi^h \in \mathscr{K}^h \tag{5.2}$$

for $n = 1, \ldots, N$, and

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \quad \beta_0^{hk} = \beta_0^h. \tag{5.3}$$

Here $\boldsymbol{u}_0^h \in V^h$ and $\beta_0^h \in \mathscr{K}^h$ are appropriate approximations of $\boldsymbol{u}_0$ and β_0, and $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N$ and $\{\boldsymbol{v}_n^{hk}\}_{n=0}^N$ are related by

$$\boldsymbol{v}_n^{hk} = \delta \boldsymbol{u}_n^{hk} \tag{5.4}$$

and

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + k\sum_{j=1}^{n} \boldsymbol{v}_j^{hk}. \tag{5.5}$$

A mathematical induction argument shows that the fully discrete approximation problem admits a unique solution.

We turn now to obtain a bound on the errors $\{\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\}$ and $\{\beta_n - \beta_n^{hk}\}$. To this end, we make the following assumptions on the regularity of the solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$, for an integer $l \geqslant 1$:

$$\boldsymbol{u}_0 \in H^{l+1}(\Omega)^d, \quad \boldsymbol{\sigma}\boldsymbol{v} \in C([0,T]; L^2(\Gamma)^d), \tag{5.6}$$

$$\boldsymbol{v} \in W^{1,1}(0,T;V) \cap C([0,T]; H^{l+1}(\Omega)^d), \quad \boldsymbol{v}_\tau \in C([0,T]; H^{l+1}(\Gamma_3)^d), \tag{5.7}$$

$$\beta \in C([0,T]; H^{l+1}(\Omega)) \cap H^2(0,T; L^2(\Omega)), \quad \dot\beta \in L^2(0,T; H^l(\Omega)). \tag{5.8}$$

It is not difficult to see that $\dot\beta, \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta), \Delta\beta \in C([0,T]; L^2(\Omega))$. We remark that under these assumptions inequality (3.17) holds for all $t \in [0,T]$. Next, we let $\boldsymbol{w} = \boldsymbol{v}_n^{hk}$ in (3.21) at $t = t_n$ to obtain

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) + G(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \beta_n), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_n)\rangle_{\mathscr{H}} + j(\boldsymbol{u}_n, \boldsymbol{v}_n^{hk}) - j(\boldsymbol{u}_n, \boldsymbol{v}_n) \geqslant \langle \boldsymbol{f}_n, \boldsymbol{v}_n^{hk} - \boldsymbol{v}_n\rangle_V.$$

We add this inequality to (5.1) with $\boldsymbol{w}^h = \boldsymbol{w}_n^h \in V^h$. After a rearrangement, we obtain

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\rangle_{\mathscr{H}}$$
$$\leqslant \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h)\rangle_{\mathscr{H}}$$
$$- \langle G(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \beta_n) - G(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \beta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h)\rangle_{\mathscr{H}}$$
$$+ R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n) + J(\boldsymbol{u}_n, \boldsymbol{u}_{n-1}^{hk}; \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h). \tag{5.9}$$

Here, the term $R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n)$ is defined as

$$R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n) = \langle \boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h) - \boldsymbol{\varepsilon}(\boldsymbol{v}_n)\rangle_{\mathscr{H}} + j(\boldsymbol{u}_n, \boldsymbol{w}_n^h) - j(\boldsymbol{u}_n, \boldsymbol{v}_n) - \langle \boldsymbol{f}_n, \boldsymbol{w}_n^h - \boldsymbol{v}_n\rangle_V.$$

Using relations (3.22)–(3.24) and the boundary condition $\boldsymbol{w}_n^h - \boldsymbol{v}_n = \boldsymbol{0}$ on Γ_1, we have

$$R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n) = \int_{\Gamma_3} [(\boldsymbol{\sigma}_n)_\tau \cdot ((\boldsymbol{w}_n^h)_\tau - (\boldsymbol{v}_n)_\tau) + p_\tau((u_n)_\nu - g)(|(\boldsymbol{w}_n^h)_\tau| - |(\boldsymbol{v}_n)_\tau|)]\,\mathrm{d}\Gamma.$$

Thus, we obtain the estimate

$$|R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n)| \leqslant c\|(\boldsymbol{w}_n^h)_\tau - (\boldsymbol{v}_n)_\tau\|_{L^2(\Gamma_3)^d}, \tag{5.10}$$

where the constant c depends on the solution. The term $J(\boldsymbol{u}_n, \boldsymbol{u}_{n-1}^{hk}; \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h)$ is defined as

$$J(\boldsymbol{u}_n, \boldsymbol{u}_{n-1}^{hk}; \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h) = j(\boldsymbol{u}_n, \boldsymbol{v}_n^{hk}) - j(\boldsymbol{u}_n, \boldsymbol{w}_n^h) + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}_n^h) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}).$$

Using (3.14) and (3.7) we get

$$J(\boldsymbol{u}_n, \boldsymbol{u}_{n-1}^{hk}; \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h) \leqslant c\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V). \tag{5.11}$$

By using assumptions (3.4) and (3.5), we find from (5.9) and (5.11) that

$$\begin{aligned}\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leqslant c\{&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V \\ &+(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\beta_n - \beta_{n-1}^{hk}\|_{L^2(\Omega)})(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V) + |R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n)|\}.\end{aligned} \tag{5.12}$$

We then obtain

$$\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leqslant c\{\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \|\beta_n - \beta_{n-1}^{hk}\|_{L^2(\Omega)}^2 + |R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n)|\}. \tag{5.13}$$

We estimate the term $\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V$. Recall that $W^{1,1}(0,T;V) \subset C([0,T];V)$ and

$$\|\boldsymbol{v}\|_{C([0,T];V)} \leqslant c\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)} \quad \forall \boldsymbol{v} \in W^{1,1}(0,T;V).$$

Since

$$\begin{aligned}\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk} &= \boldsymbol{u}_0 + \int_0^{t_n} \boldsymbol{v}(s)\,\mathrm{d}s - \boldsymbol{u}_0^h - k\sum_{j=1}^{n-1} \boldsymbol{v}_j^{hk} \\ &= k\sum_{j=1}^{n-1}(\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}) + \boldsymbol{u}_0 - \boldsymbol{u}_0^h + \sum_{j=1}^{n-1}\left(\int_{t_{j-1}}^{t_j} \boldsymbol{v}(s)\,\mathrm{d}s - \boldsymbol{v}_j k\right) + \int_{t_{n-1}}^{t_n} \boldsymbol{v}(s)\,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\left\|\sum_{j=1}^{n-1}\left(\boldsymbol{v}_j k - \int_{t_{j-1}}^{t_j} \boldsymbol{v}(s)\,\mathrm{d}s\right)\right\|_V &= \left\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\boldsymbol{v}_j - \boldsymbol{v}(s))\,\mathrm{d}s\right\|_V = \left\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\int_s^{t_j}\dot{\boldsymbol{v}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right\|_V \\ &\leqslant \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\int_s^{t_j}\|\dot{\boldsymbol{v}}(\tau)\|_V\,\mathrm{d}\tau\,\mathrm{d}s \leqslant ck\|\dot{\boldsymbol{v}}\|_{L^1(0,T;V)}\end{aligned}$$

and

$$\left\|\int_{t_{n-1}}^{t_n} \boldsymbol{v}(s)\,\mathrm{d}s\right\|_V^2 \leqslant \left(\int_{t_{n-1}}^{t_n}\|\boldsymbol{v}(s)\|_V\,\mathrm{d}s\right)^2 \leqslant k^2\|\boldsymbol{v}\|_{C([0,T];V)}^2,$$

we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V \leqslant k \sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + ck\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}.$$

Now,

$$\left(k \sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V\right)^2 \leqslant ck \sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2,$$

thus,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 \leqslant c\left\{k \sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + k^2\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}^2\right\}. \tag{5.14}$$

Similarly, we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \leqslant c\left\{k \sum_{j=1}^{n} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2\right\}. \tag{5.15}$$

It follows from (5.8) that $\dot\beta \in C([0,T];L^1(\Omega))$, therefore,

$$\|\beta_n - \beta_{n-1}^{hk}\|_{L^2(\Omega)} \leqslant \|\beta_{n-1} - \beta_{n-1}^{hk}\|_{L^2(\Omega)} + k\|\dot\beta\|_{C([0,T];L^1(\Omega))}. \tag{5.16}$$

We combine (5.13)–(5.16) and obtain

$$\begin{aligned}\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leqslant c\Bigg\{&\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + k^2\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}^2\\ &+ k^2\|\dot\beta\|_{C([0,T];L^1(\Omega))}^2 + |R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n)|\\ &+ k\sum_{j=1}^{n} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\beta_{n-1} - \beta_{n-1}^{hk}\|_{L^2(\Omega)}^2\Bigg\}.\end{aligned} \tag{5.17}$$

We turn to estimate $\|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}$. We choose $\xi = \beta_n^{hk}$ in (3.17) at $t = t_n$:

$$\langle \dot\beta_n, \beta_n^{hk} - \beta_n\rangle_{L^2(\Omega)} + a(\beta_n, \beta_n^{hk} - \beta_n) \geqslant \langle \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \beta_n), \beta_n^{hk} - \beta_n\rangle_{L^2(\Omega)}.$$

Adding this inequality to (5.2), with $\xi^h = \xi_n^h \in \mathscr{K}^h$, yields

$$\begin{aligned}&\langle \delta(\beta_n - \beta_n^{hk}), \beta_n - \beta_n^{hk}\rangle_{L^2(\Omega)} + a(\beta_n - \beta_n^{hk}, \beta_n - \beta_n^{hk})\\ &\quad\leqslant \langle \delta\beta_n - \dot\beta_n, \beta_n - \beta_n^{hk}\rangle_{L^2(\Omega)} + \langle \delta(\beta_n - \beta_n^{hk}), \beta_n - \xi_n^h\rangle_{L^2(\Omega)}\\ &\qquad + a(\beta_n - \beta_n^{hk}, \beta_n - \xi_n^h) - \langle \delta\beta_n, \beta_n - \xi_n^h\rangle_{L^2(\Omega)} - a(\beta_n, \beta_n - \xi_n^h)\\ &\qquad + \langle \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \beta_n), \beta_n - \xi_n^h\rangle_{L^2(\Omega)} + \langle \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \beta_n) - \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \beta_{n-1}^{hk}), \xi_n^h - \beta_n^{hk}\rangle_{L^2(\Omega)}.\end{aligned} \tag{5.18}$$

We estimate each term on the right-hand side. For the term

$$\langle \delta(\beta_n - \beta_n^{hk}), \beta_n - \beta_n^{hk}\rangle_{L^2(\Omega)} = \frac{1}{k}\,(\|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}^2 - \langle \beta_n - \beta_n^{hk}, \beta_{n-1} - \beta_{n-1}^{hk}\rangle_{L^2(\Omega)}),$$

we have

$$\langle\delta(\beta_n-\beta_n^{hk}),\beta_n-\beta_n^{hk}\rangle_{L^2(\Omega)}\geqslant\frac{1}{2k}\,(\|\beta_n-\beta_n^{hk}\|^2_{L^2(\Omega)}-\|\beta_{n-1}-\beta_{n-1}^{hk}\|^2_{L^2(\Omega)}). \tag{5.19}$$

We use (5.19), replace n by j in (5.18) and then sum over $j=1,\dots,n$ to obtain

$$\begin{aligned}
&\|\beta_n-\beta_n^{hk}\|^2_{L^2(\Omega)}-\|\beta_0-\beta_0^{hk}\|^2_{L^2(\Omega)}+k\sum_{j=1}^{n}a(\beta_j-\beta_j^{hk},\beta_j-\beta_j^{hk})\\
&\quad\leqslant k\sum_{j=1}^{n}\langle\delta\beta_j-\dot\beta_j,\beta_j-\beta_j^{hk}\rangle_{L^2(\Omega)}+\langle\beta_n-\beta_n^{hk},\beta_n-\xi_n^h\rangle_{L^2(\Omega)}\\
&\qquad-\sum_{j=1}^{n-1}\langle\beta_j-\beta_j^{hk},(\beta_{j+1}-\xi_{j+1}^h)-(\beta_j-\xi_j^h)\rangle_{L^2(\Omega)}-\langle\beta_0-\beta_0^{hk},\beta_1-\xi_1^h\rangle_{L^2(\Omega)}\\
&\qquad+k\sum_{j=1}^{n}a(\beta_j-\beta_j^{hk},\beta_j-\xi_j^h)\\
&\qquad+k\sum_{j=1}^{n}[\langle\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\beta_j),\beta_j-\xi_j^h\rangle_{L^2(\Omega)}-\langle\delta\beta_j,\beta_j-\xi_j^h\rangle_{L^2(\Omega)}-a(\beta_j,\beta_j-\xi_j^h)]\\
&\qquad+k\sum_{j=1}^{n}\langle\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\beta_j)-\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}),\beta_{j-1}^{hk}),\xi_j^h-\beta_j^{hk}\rangle_{L^2(\Omega)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\|\beta_n-\beta_n^{hk}\|^2_{L^2(\Omega)}+k\sum_{j=1}^{n}\|\nabla(\beta_j-\beta_j^{hk})\|^2_{L^2(\Omega)}\\
&\quad\leqslant c\Bigg\{\|\beta_0-\beta_0^{h}\|^2_{L^2(\Omega)}+k\sum_{j=1}^{n}\|\delta\beta_j-\dot\beta_j\|_{L^2(\Omega)}\|\beta_j-\beta_j^{hk}\|_{L^2(\Omega)}\\
&\qquad+\|\beta_n-\beta_n^{hk}\|_{L^2(\Omega)}\|\beta_n-\xi_n^h\|_{L^2(\Omega)}+\|\beta_0-\beta_0^{hk}\|_{L^2(\Omega)}\|\beta_1-\xi_1^h\|_{L^2(\Omega)}\\
&\qquad+k\sum_{j=1}^{n-1}\|\beta_j-\beta_j^{hk}\|_{L^2(\Omega)}\|(\beta_{j+1}-\xi_{j+1}^h)-(\beta_j-\xi_j^h)\|_{L^2(\Omega)}\\
&\qquad+k\sum_{j=1}^{n}\|\nabla(\beta_j-\beta_j^{hk})\|_{L^2(\Omega)}\|\nabla(\beta_j-\xi_j^h)\|_{L^2(\Omega)}\\
&\qquad+k\sum_{j=1}^{n}\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\beta_j)-\delta\beta_j+\kappa\Delta\beta_j\|_{L^2(\Omega)}\|\beta_j-\xi_j^h\|_{L^2(\Omega)}\\
&\qquad+k\sum_{j=1}^{n}(\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_{L^2(\Omega)}+\|\beta_j-\beta_j^{hk}\|_{L^2(\Omega)})(\|\beta_j-\beta_j^{hk}\|_{L^2(\Omega)}+\|\beta_j-\xi_j^h\|_{L^2(\Omega)})\Bigg\}.
\end{aligned}$$

Using (5.14) and (5.16) we obtain

$$\|\beta_n-\beta_n^{hk}\|^2_{L^2(\Omega)}+k\sum_{j=1}^n\|\nabla(\beta_j-\beta_j^{hk})\|^2_{L^2(\Omega)}$$
$$\leqslant c\left\{\|\beta_0-\beta_0^h\|^2_{L^2(\Omega)}+\|\beta_1-\xi_1^h\|^2_{L^2(\Omega)}+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|^2_V\right.$$
$$+k^2(\|\boldsymbol{v}\|^2_{W^{1,1}(0,T;V)}+\|\dot\beta\|^2_{C([0,T];L^1(\Omega))})+k\sum_{j=1}^n\|\delta\beta_j-\dot\beta_j\|^2_{L^2(\Omega)}$$
$$+\|\beta_n-\xi_n^h\|^2_{L^2(\Omega)}+k\sum_{j=1}^n\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\beta_j)-\delta\beta_j+\kappa\Delta\beta_j\|_{L^2(\Omega)}\|\beta_j-\xi_j^h\|_{L^2(\Omega)}$$
$$+k\sum_{j=1}^n\|\nabla(\beta_j-\xi_j^h)\|^2_{L^2(\Omega)}+k^{-1}\sum_{j=1}^n\|(\beta_{j+1}-\xi_{j+1}^h)-(\beta_j-\xi_j^h)\|^2_{L^2(\Omega)}$$
$$\left.+k\sum_{j=1}^n\|\beta_j-\beta_j^{hk}\|^2_{L^2(\Omega)}+k\sum_{j=1}^{n-1}\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|^2_V\right\}. \tag{5.20}$$

Now, (5.17) and (5.20) imply

$$\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|^2_V+\|\beta_n-\beta_n^{hk}\|^2_{L^2(\Omega)}+k\sum_{j=1}^n\|\nabla(\beta_j-\beta_j^{hk})\|^2_{L^2(\Omega)}$$
$$\leqslant c\left\{\|\beta_0-\beta_0^h\|^2_{L^2(\Omega)}+\|\beta_1-\xi_1^h\|^2_{L^2(\Omega)}+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|^2_V\right.$$
$$+k^2(\|\boldsymbol{v}\|^2_{W^{1,1}(0,T;V)}+\|\dot\beta\|^2_{C([0,T];L^1(\Omega))})+k\sum_{j=1}^n\|\delta\beta_j-\dot\beta_j\|^2_{L^2(\Omega)}$$
$$+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|^2_V+\|\beta_n-\xi_n^h\|^2_{L^2(\Omega)}$$
$$+k\sum_{j=1}^n\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\beta_j)-\delta\beta_j+\kappa\Delta\beta_j\|_{L^2(\Omega)}\|\beta_j-\xi_j^h\|_{L^2(\Omega)}$$
$$+k\sum_{j=1}^n\|\nabla(\beta_j-\xi_j^h)\|^2_{L^2(\Omega)}+k^{-1}\sum_{j=1}^{n-1}\|(\beta_{j+1}-\xi_{j+1}^h)-(\beta_j-\xi_j^h)\|^2_{L^2(\Omega)}$$
$$\left.+|R_n(\boldsymbol{w}_n^h,\boldsymbol{v}_n)|+k\sum_{j=1}^n\|\beta_j-\beta_j^{hk}\|^2_{L^2(\Omega)}+k\sum_{j=1}^n\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|^2_V\right\}. \tag{5.21}$$

To proceed, we need the following discrete version of the Gronwall inequality.

Proposition 5.1. *Assume that* $\{g_n\}_{n=1}^N$ *and* $\{e_n\}_{n=1}^N$ *are two sequences of nonnegative numbers, satisfying*

$$e_n \leqslant c g_n + ck \sum_{j=1}^{n} e_j.$$

Then,

$$e_n \leqslant c \left(g_n + k \sum_{j=1}^{n} g_j \right), \quad n = 1, \ldots, N.$$

Moreover,

$$\max_{1 \leqslant n \leqslant N} e_n \leqslant c \max_{1 \leqslant n \leqslant N} g_n.$$

Proof. Let $E_n = k \sum_{j=1}^{n} e_j$. Then,

$$E_n - E_{n-1} = k e_n \leqslant ckg_n + ckE_n$$

and hence $(1-ck)E_n - E_{n-1} \leqslant ckg_n$, which we rewrite as

$$(1-ck)^n E_n - (1-ck)^{n-1} E_{n-1} \leqslant ck(1-ck)^{n-1} g_n.$$

By an induction argument we find

$$(1-ck)^n E_n \leqslant ck \sum_{j=1}^{n} (1-ck)^{j-1} g_j.$$

Therefore, $E_n \leqslant ck \sum_{j=1}^{n} (1-ck)^{j-n-1} g_j$. Now, for k sufficiently small, $1 - ck \geqslant e^{-2ck}$ and thus, for $j \leqslant n$ and $n \leqslant N$,

$$(1-ck)^{j-n-1} \leqslant e^{2ck(n+1-j)} \leqslant e^{2ckN} \leqslant c.$$

Hence, $E_n \leqslant ck \sum_{j=1}^{n} g_j$. Then, the inequality $e_n \leqslant cg_n + cE_n$ completes the proof. □

Applying Proposition 5.1 to (5.21) yields

$$\max_n (\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}^2) + k \sum_{j=1}^{N} \|\nabla(\beta_j - \beta_j^{hk})\|_{L^2(\Omega)}^2$$

$$\leqslant c \Bigg\{ \|\beta_0 - \beta_0^h\|_{L^2(\Omega)}^2 + \|\beta_1 - \xi_1^h\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + k^2 (\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}^2 + \|\dot{\beta}\|_{C([0,T];L^1(\Omega))}^2)$$

$$+ k \sum_{j=1}^{N} \|\delta\beta_j - \dot{\beta}_j\|_{L^2(\Omega)}^2 + \max_n [\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + |R_n(\boldsymbol{w}_n^h, \boldsymbol{v}_n)|]$$

$$+ \max_n \Bigg[\|\beta_n - \xi_n^h\|_{L^2(\Omega)}^2 + k \sum_{j=1}^{n} \|\nabla(\beta_j - \xi_j^h)\|_{L^2(\Omega)}^2$$

$$+k^{-1}\sum_{j=1}^{n-1}\|(\beta_{j+1}-\xi_{j+1}^h)-(\beta_j-\xi_j^h)\|_{L^2(\Omega)}^2$$

$$\left.\left.+k\sum_{j=1}^{n}\|\phi(\varepsilon(\boldsymbol{u}_j),\beta_j)-\delta\beta_j+\kappa\Delta\beta_j\|_{L^2(\Omega)}\|\beta_j-\xi_j^h\|_{L^2(\Omega)}\right]\right\} \tag{5.22}$$

for each $\boldsymbol{w}_n^h\in V^h,\ \xi_n^h\in\mathscr{K}^h,\ n=0,1,\ldots,N$.

We use the finite element method to construct the sets V^h and $\mathscr{K}^h$. We first introduce a finite element space to approximate $H^1(\Omega)$ (see, e.g., [4] for detail). For the sake of simplicity, we assume that Ω is a polygon. Let $\mathscr{T}^h$ be a regular finite element partition of Ω in such a way that if a side of an element lies on the boundary, the side belongs entirely to one of the subsets $\Gamma_1,\ \Gamma_2$ and Γ_3. As is customary, the symbol h denotes the maximal diameter of the elements. We define $X^h\subset H^1(\Omega)$ to be the finite element space consisting of piecewise polynomials of degree less than or equal to l, corresponding to the partition $\mathscr{T}^h$. Then, we use

$$V^h=(X^h)^d\cap V \quad\text{and}\quad \mathscr{K}^h=X^h\cap\mathscr{K},$$

to approximate V and $\mathscr{K}$. We note that the polynomial degrees for the functions in V^h and $\mathscr{K}^h$ do not have to be equal. The argument here can be easily extended to the case where different polynomial degrees are used for V^h and $\mathscr{K}^h$. Next, we introduce the finite element interpolation operators Π^h. When $\boldsymbol{v}(t)\in C(\bar{\Omega})$, we use $\Pi^h\boldsymbol{v}(t)$ to denote the standard finite element interpolant of $\boldsymbol{v}(t)$ (cf. [4]); while if $\boldsymbol{v}(t)\notin C(\bar{\Omega})$, we use $\Pi^h\boldsymbol{v}(t)$ to denote Clément's interpolant introduced in [6]. Also, we use the same symbol Π^h for the interpolation of $\boldsymbol{v}(t)$ on Γ_3 and for the interpolation of $\beta(t)$ onto $\mathscr{K}^h$. It can be verified that $\beta(t)\in\mathscr{K}$ implies $\Pi^h\beta(t)\in\mathscr{K}^h$ and $(\Pi^h\beta)_t=\Pi^h\dot{\beta}$. Under the regularity conditions (5.6)–(5.8), we have the following interpolation error estimates:

$$\|\boldsymbol{u}_0-\Pi^h\boldsymbol{u}_0\|_V\leqslant ch^l|\boldsymbol{u}_0|_{H^{l+1}(\Omega)^d}, \tag{5.23}$$

$$\|\beta_0-\Pi^h\beta_0\|_{L^2(\Omega)}\leqslant ch^l|\beta_0|_{H^l(\Omega)} \tag{5.24}$$

and for all $t\in[0,T]$,

$$\|\boldsymbol{v}(t)-\Pi^h\boldsymbol{v}(t)\|_V\leqslant ch^l|\boldsymbol{v}(t)|_{H^{l+1}(\Omega)^d}, \tag{5.25}$$

$$\|\boldsymbol{v}_\tau(t)-\Pi^h\boldsymbol{v}_\tau(t)\|_{L^2(\Gamma_3)^d}\leqslant ch^{l+1}|\boldsymbol{v}_\tau(t)|_{H^{l+1}(\Gamma_3)^d}, \tag{5.26}$$

$$\|\dot{\beta}(t)-\Pi^h\dot{\beta}(t)\|_{L^2(\Omega)}\leqslant ch^l|\dot{\beta}(t)|_{H^l(\Omega)}, \tag{5.27}$$

$$\|\beta(t)-\Pi^h\beta(t)\|_{L^2(\Omega)}\leqslant ch^{l+1}|\beta(t)|_{H^{l+1}(\Omega)}, \tag{5.28}$$

$$\|\beta(t)-\Pi^h\beta(t)\|_{H^1(\Omega)}\leqslant ch^l|\beta(t)|_{H^{l+1}(\Omega)}. \tag{5.29}$$

Now, we choose the initial values to be

$$\boldsymbol{u}_0^h=\Pi^h\boldsymbol{u}_0,\quad \beta_0^h=\Pi^h\beta_0. \tag{5.30}$$

Next, we choose $\boldsymbol{w}_n^h = \Pi^h \boldsymbol{v}_n,\ \xi_n^h = \Pi^h \beta_n,\ n = 1,\dots,N$ in (5.22). We use the regularity of solutions (5.6)–(5.8) to estimate terms in inequality (5.22). We write

$$\delta\beta_j - \dot{\beta}_j = \frac{1}{k}\int_{t_{j-1}}^{t_j} (\dot{\beta}(t) - \dot{\beta}(t_j))\,\mathrm{d}t = \frac{1}{k}\int_{t_{j-1}}^{t_j}\int_{t_j}^{t} \ddot{\beta}(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Then, it is easy to see that

$$k\sum_{j=1}^{N} \|\delta\beta_j - \dot{\beta}_j\|_{L^2(\Omega)}^2 \leqslant k^2 \|\ddot{\beta}\|_{L^2(0,T;L^2(\Omega))}^2.$$

With $\xi_j^h = \Pi^h\beta(t_j)$, we have

$$(\beta_{j+1} - \xi_{j+1}^h) - (\beta_j - \xi_j^h) = (\beta_{j+1} - \beta_j) - \Pi^h(\beta_{j+1} - \beta_j) = k(\delta\beta_{j+1} - \Pi^h\delta\beta_{j+1}).$$

Now, $\delta\beta_{j+1} = \frac{1}{k}\int_{t_j}^{t_{j+1}} \dot{\beta}(t)\,\mathrm{d}t$, and so

$$\|\delta\beta_{j+1}\|_{H^l(\Omega)}^2 \leqslant \frac{1}{k}\int_{t_j}^{t_{j+1}} \|\dot{\beta}(t)\|_{H^l(\Omega)}^2\,\mathrm{d}t.$$

From the error estimate (5.27) we infer that

$$\|(\beta_{j+1} - \xi_{j+1}^h) - (\beta_j - \xi_j^h)\|_{L^2(\Omega)}^2 \leqslant ckh^{2l}\int_{t_j}^{t_{j+1}} \|\dot{\beta}(t)\|_{H^l(\Omega)}^2\,\mathrm{d}t.$$

Therefore,

$$k^{-1}\sum_{j=1}^{n-1} \|(\beta_{j+1} - \xi_{j+1}^h) - (\beta_j - \xi_j^h)\|_{L^2(\Omega)}^2 \leqslant ch^{2l}\|\dot{\beta}\|_{L^2(0,T;H^l(\Omega))}^2.$$

The other terms on the right-hand side of (5.22) can be estimated directly using the interpolation error estimates (5.23)–(5.29).

Finally, collecting all the estimates above, we obtain the following error estimate from (5.22):

$$\max_n(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}^2) + k\sum_{j=1}^{N} \|\nabla(\beta_j - \beta_j^{hk})\|_{L^2(\Omega)}^2 \leqslant c(k^2 + h^{\min\{2l,l+1\}}),$$

which, using (5.15), can be replaced by

$$\max_n(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\delta\boldsymbol{u}_n - \delta\boldsymbol{u}_n^{hk}\|_V^2 + \|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}^2) + k\sum_{j=1}^{N} \|\nabla(\beta_j - \beta_j^{hk})\|_{L^2(\Omega)}^2$$
$$\leqslant c(k^2 + h^{\min\{2l,l+1\}}).$$

This yields the following estimate. But first, we note that the finite-dimensional sets V^h and $\mathscr{K}^h$ are constructed from piecewise polynomials of degree less than or equal to l, corresponding to a regular finite element triangulation of the domain. Also, we use (5.30) for the discrete initial values.

Then, we have the following result for the discrete scheme P_V^{hk}.

Theorem 5.2. *The approximate problem P_V^{hk} has a unique solution. Under assumptions* (5.6)–(5.8) *the following error estimate holds*:

$$\max_n(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\delta\boldsymbol{u}_n - \delta\boldsymbol{u}_n^{hk}\|_V + \|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}) + k\left[\sum_{j=1}^{N}\|\nabla(\beta_j - \beta_j^{hk})\|^2_{L^2(\Omega)}\right]^{1/2}$$
$$\leqslant c(k + h^{\min\{l,(l+1)/2\}}). \tag{5.31}$$

The constant c depends on the norms $\|\boldsymbol{u}_0\|_{H^{l+1}(\Omega)^d}$, $\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta) - \dot{\beta} + \kappa\Delta\beta\|_{C([0,T];L^2(\Omega))}$, $\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}$, $\|\boldsymbol{v}\|_{C([0,T];H^{l+1}(\Omega)^d)}$, $\|\boldsymbol{v}\|_{C([0,T];H^{l+1}(\Gamma_3)^d)}$, $\|\beta\|_{C([0,T];H^{l+1}(\Omega))}$, $\|\beta\|_{H^2(0,T;L^2(\Omega))}$ *and* $\|\dot{\beta}\|_{L^2(0,T;H^l(\Omega))}$.

In particular, when $l = 1$ and (5.6)–(5.8) hold we have the optimal-order error estimate

$$\max_n(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\delta\boldsymbol{u}_n - \delta\boldsymbol{u}_n^{hk}\|_V + \|\beta_n - \beta_n^{hk}\|_{L^2(\Omega)}) + k\left[\sum_{j=1}^{N}\|\nabla(\beta_j - \beta_j^{hk})\|^2_{L^2(\Omega)}\right]^{1/2}$$
$$\leqslant c(k + h).$$

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