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Numerical analysis of stationary variational-hemivariational inequalities

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Abstract Variational-hemivariational inequalities refer to the inequality problems where both convex and nonconvex functions are involved. In this paper, we consider the numerical solution of a family of stationary variational-hemivariational inequalities by the finite element method. For a variational-hemivariational inequality of a general form, we prove convergence of numerical solutions. For some particular variational-hemivariational inequalities, we provide error estimates of numerical solutions, which are of optimal order for the linear finite element method under appropriate solution regularity assumptions. Numerical results are reported on solving a variational-hemivariational inequality modeling the contact between an elastic body and a foundation with the linear finite element, illustrating the theoretically predicted optimal first order convergence and providing their mechanical interpretations.

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1 Introduction

Variational-hemivariational inequalities are inequality problems where both convex and nonconvex functions are involved. They were introduced in the pioneering

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work [19], and were further studied in [18,20]. Interest in variational-hemivariational inequalities arises in the study of various problems in mechanics and engineering applications. Their study requires arguments of Convex Analysis, including properties of the subdifferential of a convex function, and arguments of Nonsmooth Analysis, including properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions which may be nonconvex.

Recently, a new variational-hemivariational inequality is studied in [9]. The inequality involves two nonlinear operators and two nondifferentiable functionals, of which at least one is convex. Solution existence, uniqueness and data continuous dependence are shown. Moreover, the finite element method is studied for solving the inequality problem. For the first time in the literature, an optimal order error estimate is derived for the linear element solution of a hemivariational inequality under appropriate solution regularity assumptions. A more general variational-hemivariational inequality is analyzed in [17]. There, the existence and uniqueness of the solution are proved, together with a result on the continuous dependence of the solution on the data. The variational-hemivariational inequalities studied in [9,17] are motivated by applications in Contact Mechanics.

The purpose of this paper is to consider numerical approximations of stationary variational-hemivariational inequality problems by the finite element method, substantially extending the relevant result found in [9]. We note that in the comprehensive reference [12] on numerical analysis of hemivariational inequalities, only convergence of the numerical schemes is discussed and there are no error estimates. Although there have been various attempts to derive error estimates on numerical methods for solving hemivariational inequalities, optimal order error estimates were derived in [9] for the first time in the literature. For the general variational-hemivariational inequality studied in [17], we show the convergence of the numerical solution. Then, for some particular variational-hemivariational inequalities, we also derive error estimates, which are of optimal order for the linear elements. We provide numerical examples to illustrate the performance of the numerical method, including numerical convergence orders. One of the numerical examples also serves the purpose of showing the transition from variational inequalities to hemivariational inequalities.

The rest of the paper is organized as follows. In Sect. 2 we review some preliminary material needed later on in the study of variational-hemivariational inequalities, and recall an existence and uniqueness result for the general variational-hemivariational inequality from [17]. In Sect. 3 we introduce numerical methods for solving various variational-hemivariational inequalities, prove convergence and derive error estimates wherever feasible. In Sect. 4 we introduce several contact problems, in which the material behavior is modeled with a nonlinear elastic constitutive law and the contact conditions are in a subdifferential form. We list the assumptions on the data and use the abstract result in [17] to prove the unique weak solvability to the problem. In addition, we apply our results in Sect. 3 in the numerical analysis of the contact model. Finally, in Sect. 5 we present numerical simulations which represent an evidence of our error estimates and convergence results. The simulations illustrate the transition from the variational to the hemivariational case and give rise to interesting mechanical interpretations.

2 Preliminaries

All linear spaces in this paper are real. For a normed space X we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. When no confusion may arise, we simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{X^*\times X}$. Weak convergence is indicated by the symbol $\rightharpoonup$. For two normed spaces X and Y, $\mathcal{L}(X,Y)$ denotes the space of all linear continuous operators from X to Y.

We recall that an operator $A\colon X\to X^*$ is pseudomonotone if it is bounded and $u_n\rightharpoonup u$ in X together with $\limsup\,\langle Au_n,u_n-u\rangle_{X^*\times X}\le 0$ imply

$$\langle Au,u-v\rangle_{X^*\times X}\le \liminf\,\langle Au_n,u_n-v\rangle_{X^*\times X}\quad \forall\, v\in X.$$

A function $\varphi\colon K\subset X\to\mathbb{R}$ is lower semicontinuous (l.s.c.), if for any sequence $\{x_n\}\subset K$ and any $x\in K$, $x_n\to x$ in X implies $\varphi(x)\le\liminf\varphi(x_n)$. For a convex function φ, the set

$$\widetilde{\partial}\varphi(x):=\left\{x^*\in X^*\mid \varphi(v)-\varphi(x)\ge\langle x^*,v-x\rangle_{X^*\times X}\;\forall\, v\in X\right\}$$

is called the subdifferential of φ at $x\in X$. If $\widetilde{\partial}\varphi(x)$ is non-empty, an element $x^*\in\widetilde{\partial}\varphi(x)$ is called a subgradient of φ at x.

Let $\psi: X\to\mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of ψ at $x\in X$ in the direction $v\in X$ is defined by

$$\psi^0(x;v):=\limsup_{y\to x,\,\lambda\downarrow 0}\frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The Clarke subdifferential of ψ at x is a subset of the dual space X^* given by

$$\partial\psi(x):=\left\{\zeta\in X^*\mid \psi^0(x;v)\ge\langle\zeta,v\rangle_{X^*\times X}\;\forall\, v\in X\right\}.$$

We have the formula ([6]).

$$\psi^0(x;v)=\max\left\{\langle\zeta,v\rangle\mid\zeta\in\partial\psi(x)\right\}. \tag{2.1}$$

More details on the properties of the subdifferential mappings can be found in the books [8,21] in the convex case, and in the books [6,7,16,18] in the Clarke sense.

We turn now to the study of variational-hemivariational inequalities. Let X, X_φ, X_j be normed spaces, $K\subset X$ and $K_\varphi\subset X_\varphi$. Given operators $A\colon X\to X^*$, $\gamma_\varphi: X\to X_\varphi$, $\gamma_j: X\to X_j$, and functionals $\varphi\colon K_\varphi\times K_\varphi\to\mathbb{R}$, $j\colon X_j\to\mathbb{R}$, we consider the following problem.

Problem (P) *Find an element* $u\in K$ *such that*

$$\langle Au,v-u\rangle+\varphi(\gamma_\varphi u,\gamma_\varphi v)-\varphi(\gamma_\varphi u,\gamma_\varphi u)+j^0(\gamma_j u;\gamma_j v-\gamma_j u)\ge\langle f,v-u\rangle\quad\forall\, v\in K. \tag{2.2}$$

In the particular case where $\varphi \equiv 0$, Problem (P) is reduced to a stationary hemivariational inequality. Numerical analysis of the stationary hemivariational inequality is performed in [10].

For the study of Problem (P), we introduce the following assumptions on the data.

(A_1) X is a reflexive Banach space, and K is a closed and convex subset of X with $0 \in K$.

(A_2) X_φ is a Banach space, K_φ is a closed and convex subset of X_φ, $K_\varphi \supset \gamma_\varphi(K)$, and $\gamma_\varphi \in \mathcal{L}(X, X_\varphi)$: for a constant $c_\varphi > 0$,

$$\|\gamma_\varphi v\|_{X_\varphi} \le c_\varphi \|v\|_X \quad \forall\, v \in X. \tag{2.3}$$

(A_3) X_j is a Banach space and $\gamma_j \in \mathcal{L}(X, X_j)$: for a constant $c_j > 0$,

$$\|\gamma_j v\|_{X_j} \le c_j \|v\|_X \quad \forall\, v \in X. \tag{2.4}$$

(A_4) $A\colon X \to X^*$ is pseudomonotone and there exists a constant $m_A > 0$ such that

$$\langle A v_1 - A v_2, v_1 - v_2 \rangle \ge m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1 v_2 \in X. \tag{2.5}$$

(A_5) $\varphi\colon K_\varphi \times K_\varphi \to \mathbb{R}$ is such that $\varphi(z, \cdot)\colon K_\varphi \to \mathbb{R}$ is convex and l.s.c. on K_φ for all $z \in K_\varphi$, and there exists a constant $\alpha_\varphi \ge 0$ such that

$$\begin{aligned} &\varphi(z_1, z_4) - \varphi(z_1, z_3) + \varphi(z_2, z_3) - \varphi(z_2, z_4) \\ &\qquad \le \alpha_\varphi \|z_1 - z_2\|_{X_\varphi} \|z_3 - z_4\|_{X_\varphi} \quad \forall\, z_1, z_2, z_3, z_4 \in K_\varphi. \end{aligned} \tag{2.6}$$

(A_6) $j\colon X_j \to \mathbb{R}$ is locally Lipschitz, and there are constants $c_0, c_1, \alpha_j \ge 0$ such that

$$\|\xi\|_{X_j^*} \le c_0 + c_1 \|z\|_{X_j} \quad \forall\, z \in X_j,\ \forall\, \xi \in \partial j(z), \tag{2.7}$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \le \alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j. \tag{2.8}$$

(A_7)

$$\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A. \tag{2.9}$$

(A_8)

$$f \in X^*. \tag{2.10}$$

Note that in the statement of Problem (P) the function $\varphi(u, \cdot)$ is assumed to be convex whereas the function j is locally Lipschitz and is, in general, nonconvex. For this reason, we refer to the inequality (2.2) as a *variational-hemivariational inequality*. Moreover, note that Problem (P) contains as particular cases, various problems considered in the literature. We comment that the spaces X_φ and X_j are introduced to facilitate error analysis of numerical solutions of Problem (P) in later sections. For applications in Contact Mechanics, the functionals $\varphi(\cdot, \cdot)$ and $j(\cdot)$ are integrals over the contact boundary Γ_3. In such a situation, X_φ and X_j can be chosen to be $L^2(\Gamma_3; \mathbb{R}^d)$ and/or $L^2(\Gamma_3)$.

The assumption (2.9) is a smallness assumption, which is a variant of a similar condition in [17], due to the use of the spaces X_φ and X_j in the problem setting. By slightly modifying the proof in [17], we have the following existence and uniqueness result.

Theorem 2.1 *Under assumption* (A_1)–(A_8)*, Problem* (P) *has a unique solution* $u \in K$.

Note that for a locally Lipschitz function $j\colon X_j \to \mathbb{R}$, the inequality (2.8) is equivalent to

$$\langle \xi_1 - \xi_2, z_1 - z_2 \rangle_{X_j^* \times X_j} \ge -\alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_i \in X_j,\ \forall\, \xi_i \in \partial j(z_i),\ i = 1, 2, \tag{2.11}$$

known as the relaxed monotonicity condition (cf. [16] and the references therein). In addition, if $j\colon X_j \to \mathbb{R}$ is a convex function, then (2.8) or (2.11), equivalently, are satisfied with $\alpha_j = 0$, due to the monotonicity of the (convex) subdifferential. Since $0 \in K$, we derive the following relations from (2.5), (2.7) and (2.11), for some constant c:

$$\langle Av, v \rangle_{X^* \times X} \ge m_A \|v\|_X^2 - c\,\|v\|_X \quad \forall\, v \in X, \tag{2.12}$$

$$\langle \xi, z \rangle_{X_j^* \times X_j} \ge -\alpha_j \|z\|_{X_j}^2 - c_0 \|z\|_{X_j} \quad \forall\, z \in X_j,\ \forall\, \xi \in \partial j(z). \tag{2.13}$$

3 Numerical approximations

In this section, we consider numerical schemes for solving Problem (P). We keep assumptions (A_1)–(A_8) so that Problem (P) has a unique solution $u \in K$.

Let $X^h \subset X$ be a finite dimensional subspace with $h > 0$ denoting a spatial discretization parameter. Let $K^h = X^h \cap K$. Then K^h is a closed and convex subset of X^h and $0 \in K^h$. We consider the following Galerkin approximation of Problem (P).

PROBLEM (P^h) *Find an element* $u^h \in K^h$ *such that*

$$\begin{aligned} &\langle Au^h, v^h - u^h \rangle + \varphi(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u^h, \gamma_\varphi u^h) + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \\ &\quad \ge \langle f, v^h - u^h \rangle \quad \forall\, v^h \in K^h. \end{aligned} \tag{3.1}$$

The arguments of the proof of Theorem 2.1 can be applied in the setting of the finite dimensional space X^h, and we know that under assumptions (A_1)–(A_8), Problem (P^h) has a unique solution $u^h \in K^h$.

Proposition 3.1 *For some constant* $M > 0$, $\|u^h\|_X \le M\ \forall\, h > 0$.

Proof We let $v^h = 0$ in (3.1) to get

$$\langle Au^h, u^h \rangle \le \varphi(\gamma_\varphi u^h, 0) - \varphi(\gamma_\varphi u^h, \gamma_\varphi u^h) + j^0(\gamma_j u^h; -\gamma_j u^h) + \langle f, u^h \rangle. \tag{3.2}$$

For any $z \in K_\varphi$, take $z_1 = z_3 = z$ and $z_2 = z_4 = 0$ in (2.6),

$$\varphi(z, 0) - \varphi(z, z) \le \alpha_\varphi \|z\|^2_{X_\varphi} - \varphi(0, z) + \varphi(0, 0). \tag{3.3}$$

Use the lower bound ([1, p. 433])

$$\varphi(0, z) \ge c_3 + c_4 \|z\|_{X_\varphi} \quad \forall\, z \in K_\varphi$$

for some constants c_3 and c_4, not necessarily positive. Then from (3.3), we have

$$\varphi(z, 0) - \varphi(z, z) \le \alpha_\varphi \|z\|^2_{X_\varphi} + c \left(\|z\|_{X_\varphi} + 1\right). \tag{3.4}$$

Write $z_1 = z$ and take $z_2 = 0$ in (2.8),

$$j^0(z; -z) \le \alpha_j \|z\|^2_{X_j} - j^0(0; z).$$

Further, use (2.7) to get

$$j^0(z; -z) \le \alpha_j \|z\|^2_{X_j} + c_0 \|z\|_{X_j}. \tag{3.5}$$

Use (2.12), (3.4), (3.5), (2.3) and (2.4) in (3.2) to obtain

$$\left(m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2\right) \|u^h\|^2_X \le c \left(\|u^h\|_X + 1\right).$$

Since $m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2 > 0$, we deduce from the above inequality that $\|u^h\|_X$ is uniformly bounded with respect to h. □

The uniform boundedness of the numerical solutions will be useful in error analysis of the numerical method.

The focus of this section is error analysis for the numerical solution defined by Problem (P^h). We will assume, in addition, the following.

(A_9) $A : X \to X^*$ is Lipschitz continuous, i.e. for some constant $L_A > 0$,

$$\|Au - Av\|_{X^*} \le L_A \|u - v\|_X \quad \forall\, u, v \in X. \tag{3.6}$$

Note that under conditions (3.6) and (2.5), the operator A is pseudomonotone ([26, Proposition 27.6]).

(A_{10}) For fixed $z \in K_\varphi$, $\varphi(z, \cdot)$ is Lipschitz continuous: for some function $L_\varphi : K_\varphi \to \mathbb{R}_+$,

$$|\varphi(z, z_1) - \varphi(z, z_2)| \le L_\varphi(z)\, \|z_1 - z_2\|_{X_\varphi} \quad \forall\, z_1, z_2 \in K_\varphi. \tag{3.7}$$

Finally, we assume $\{K^h\}$ approximates K in the following sense:

$$\forall\, v \in K,\ \exists\, v^h \in K^h \quad \text{such that } v^h \to v \quad \text{in } X \text{ as } h \to 0. \tag{3.8}$$

All the additional assumptions are valid for a wide variety of contact conditions, cf. Sect. 4.

3.1 Convergence

We begin with an application of (2.5) with $v_1 = u$ and $v_2 = u^h$ to obtain, for any $v^h \in K^h$,

$$\begin{aligned} m_A\|u - u^h\|_X^2 \le \langle Au - Au^h, u - v^h\rangle + \langle Au, v^h - u\rangle \\ + \langle Au, u - u^h\rangle + \langle Au^h, u^h - v^h\rangle. \end{aligned} \tag{3.9}$$

From (2.2) with $v = u^h \in K$,

$$\begin{aligned} \langle Au, u - u^h\rangle \le \varphi(\gamma_\varphi u, \gamma_\varphi u^h) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ + j^0(\gamma_j u; \gamma_j u^h - \gamma_j u) - \langle f, u^h - u\rangle. \end{aligned} \tag{3.10}$$

From (3.1),

$$\begin{aligned} \langle Au^h, u^h - v^h\rangle \le \varphi(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u^h, \gamma_\varphi u^h) \\ + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) - \langle f, v^h - u^h\rangle. \end{aligned} \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), we have

$$m_A\|u - u^h\|_X^2 \le \langle Au - Au^h, u - v^h\rangle + R(v^h) + I_\varphi(v^h) + I_j(v^h), \tag{3.12}$$

where

$$\begin{aligned} R(v^h) := \langle Au, v^h - u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ + j^0(\gamma_j u; \gamma_j v^h - \gamma_j u) - \langle f, v^h - u\rangle, \end{aligned} \tag{3.13}$$

$$I_\varphi(v^h) := \varphi(\gamma_\varphi u, \gamma_\varphi u^h) + \varphi(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u^h, \gamma_\varphi u^h), \tag{3.14}$$

$$I_j(v^h) := j^0(\gamma_j u; \gamma_j u^h - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) - j^0(\gamma_j u; \gamma_j v^h - \gamma_j u). \tag{3.15}$$

Notice that for $\varepsilon > 0$ arbitrarily small, there is a constant c depending on ε such that

$$\begin{aligned} \langle Au - Au^h, u - v^h\rangle &\le L_A\|u - u^h\|_X\|u - v^h\|_X \\ &\le \varepsilon\, \|u - u^h\|_X^2 + c\, \|u - v^h\|_X^2. \end{aligned}$$

We further deduce from (3.12) that

$$(m_A - \varepsilon)\, \|u - u^h\|_X^2 \le c\, \|u - v^h\|_X^2 + R(v^h) + I_\varphi(v^h) + I_j(v^h). \tag{3.16}$$

Let us bound the terms $I_\varphi(v^h)$ and $I_j(v^h)$. By (2.6), we have

$$I_\varphi(v^h) \le \alpha_\varphi \|\gamma_\varphi u - \gamma_\varphi u^h\|_{X_\varphi} \|\gamma_\varphi u^h - \gamma_\varphi v^h\|_{X_\varphi}$$
$$\le \alpha_\varphi c_\varphi^2 \left(\|u - u^h\|_X^2 + \|u - u^h\|_X \|u - v^h\|_X \right).$$

Thus,

$$I_\varphi(v^h) \le \left(\alpha_\varphi c_\varphi^2 + \varepsilon \right) \|u - u^h\|_X^2 + c\, \|u - v^h\|_X^2 \tag{3.17}$$

for another constant c depending on $\varepsilon > 0$. We will apply the subadditivity of the generalized directional derivative:

$$j^0(z; z_1 + z_2) \le j^0(z; z_1) + j^0(z; z_2) \quad \forall\, z, z_1, z_2 \in X_j.$$

Using

$$j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \le j^0(\gamma_j u^h; \gamma_j u - \gamma_j u^h) + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u),$$

we have

$$I_j(v^h) \le \left[j^0(\gamma_j u; \gamma_j u^h - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j u - \gamma_j u^h) \right]$$
$$+ \left[j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u) - j^0(\gamma_j u; \gamma_j v^h - \gamma_j u) \right]. \tag{3.18}$$

By (2.8),

$$j^0(\gamma_j u; \gamma_j u^h - \gamma_j u) + j^0(\gamma_j u^h; \gamma_j u - \gamma_j u^h) \le \alpha_j \|\gamma_j u - \gamma_j u^h\|_{X_j}^2;$$

by (2.7),

$$\left| j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u) \right| \le \left(c_0 + c_1 \|\gamma_j u^h\|_{X_j} \right) \|\gamma_j u - \gamma_j v^h\|_{X_j},$$
$$\left| j^0(\gamma_j u; \gamma_j v^h - \gamma_j u) \right| \le \left(c_0 + c_1 \|\gamma_j u\|_{X_j} \right) \|\gamma_j u - \gamma_j v^h\|_{X_j}.$$

Thus, from (3.18), we have

$$I_j(v^h) \le \alpha_j \|\gamma_j u - \gamma_j u^h\|_{X_j}^2 + c\, \|\gamma_j u - \gamma_j v^h\|_{X_j} \tag{3.19}$$

for some constant $c > 0$ independent of h, where we used the fact that $\|\gamma_j u^h\|_{X_j}$ is uniformly bounded (cf. Proposition 3.1). Combine (3.16), (3.17), and (3.19),

$$\left(m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2 - 2\,\varepsilon \right) \|u - u^h\|_X^2$$
$$\le c\, \|u - v^h\|_X^2 + c\, \|\gamma_j u - \gamma_j v^h\|_{X_j} + R(v^h).$$

Since $\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A$, we can choose $\varepsilon = (m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2)/4 > 0$ and get

$$\|u - u^h\|_X^2 \le c\left[\|u - v^h\|_X^2 + \|\gamma_j u - \gamma_j v^h\|_{X_j} + R(v^h)\right] \quad \forall\, v^h \in K^h. \tag{3.20}$$

The residual term (3.13) can be bounded as follows:

$$\left|R(v^h)\right| \le \left[\|Au\|_{X^*} + L_\varphi(\gamma_\varphi u)\, c_\varphi + \left(c_0 + c_1\|\gamma_j u\|_{X_j}\right) c_j + \|f\|_{X^*}\right] \|u - v^h\|_X.$$

Thus, from (3.20), we have two constants $c_1, c_2 > 0$, depending on the data of the problem and the solution, such that

$$\|u - u^h\|_X^2 \le c_1\|u - v^h\|_X^2 + c_2\|u - v^h\|_X \quad \forall\, v^h \in K^h. \tag{3.21}$$

By (3.8), we can choose a sequence $\{v^h\} \subset K^h$ that converges to u. From (3.21), we then conclude the convergence

$$\|u - u^h\|_X \to 0 \quad \text{as } h \to 0.$$

Note that while (3.21) implies the convergence of the numerical method, it does not lead to optimal convergence order for error estimation.

3.2 Error estimation for the particular case $K = X$

In the special case $K = X$, we have $K^h = X^h$, and the original problem (2.2) and its approximation (3.1) become

$$\begin{aligned} &\langle Au, v - u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ &\quad + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \ge \langle f, v - u\rangle \quad \forall\, v \in X \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &\langle Au^h, v^h - u^h\rangle + \varphi(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u^h, \gamma_\varphi u^h) \\ &\quad + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \ge \langle f, v^h - u^h\rangle \quad \forall\, v^h \in X^h. \end{aligned} \tag{3.23}$$

We replace v by $2u - v$ in (3.22),

$$\begin{aligned} &\langle Au, u - v\rangle + \varphi(\gamma_\varphi u, 2\gamma_\varphi u - \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ &\quad + j^0(\gamma_j u; \gamma_j u - \gamma_j v) \ge \langle f, u - v\rangle \quad \forall\, v \in X. \end{aligned}$$

Thus,

$$\begin{aligned} \langle Au, v^h - u\rangle \le\ & \varphi(\gamma_\varphi u, 2\gamma_\varphi u - \gamma_\varphi v^h) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ & + j^0(\gamma_j u; \gamma_j u - \gamma_j v^h) - \langle f, u - v^h\rangle \quad \forall\, v^h \in X^h. \end{aligned}$$

Using this inequality in (3.12), we have

$$m_A\|u-u^h\|_X^2 \le \langle Au-Au^h, u-v^h\rangle + \tilde{I}_\varphi(v^h) + \tilde{I}_j(v^h), \tag{3.24}$$

where

$$\tilde{I}_\varphi(v^h):=I_\varphi(v^h)+\varphi(\gamma_\varphi u, 2\gamma_\varphi u-\gamma_\varphi v^h)+\varphi(\gamma_\varphi u,\gamma_\varphi v^h)-2\,\varphi(\gamma_\varphi u,\gamma_\varphi u), \tag{3.25}$$
$$\tilde{I}_j(v^h):=j^0(\gamma_j u;\gamma_j u^h-\gamma_j u)+j^0(\gamma_j u;\gamma_j u-\gamma_j v^h)+j^0(\gamma_j u^h;\gamma_j v^h-\gamma_j u^h). \tag{3.26}$$

Recall that $I_\varphi(v^h)$ is defined in (3.14) and it is bounded in (3.17). We use (3.7) to find

$$\begin{aligned}&\varphi(\gamma_\varphi u, 2\gamma_\varphi u-\gamma_\varphi v^h)+\varphi(\gamma_\varphi u,\gamma_\varphi v^h)\\ &\quad -2\,\varphi(\gamma_\varphi u,\gamma_\varphi u)\le 2\,L_\varphi(\gamma_\varphi u)\,\|\gamma_\varphi u-\gamma_\varphi v^h\|_{X_\varphi}.\end{aligned}$$

This implies, combined with (3.17), that

$$\tilde{I}_\varphi(v^h)\le \left(\alpha_\varphi c_\varphi^2+\varepsilon\right)\|u-u^h\|_X^2+c\,\|u-v^h\|_X^2+2\,L_\varphi(\gamma_\varphi u)\,\|\gamma_\varphi u-\gamma_\varphi v^h\|_{X_\varphi}. \tag{3.27}$$

Applying the inequality

$$j^0(\gamma_j u^h;\gamma_j v^h-\gamma_j u^h)\le j^0(\gamma_j u^h;\gamma_j u-\gamma_j u^h)+j^0(\gamma_j u^h;\gamma_j v^h-\gamma_j u),$$

we have

$$\begin{aligned}\tilde{I}_j(v^h)\le&\left[j^0(\gamma_j u;\gamma_j u^h-\gamma_j u)+j^0(\gamma_j u^h;\gamma_j u-\gamma_j u^h)\right]\\ &+\left[j^0(\gamma_j u;\gamma_j u-\gamma_j v^h)+j^0(\gamma_j u^h;\gamma_j v^h-\gamma_j u)\right].\end{aligned}$$

This is similar to (3.18) and we have the following analogue of (3.19):

$$\tilde{I}_j(v^h)\le \alpha_j c_j^2\|u-u^h\|_X^2+c\,\|\gamma_j u-\gamma_j v^h\|_{X_j}. \tag{3.28}$$

Combining (3.24), (3.27) and (3.28), we get

$$\begin{aligned}m_A\|u-u^h\|_X^2\le&\ \|Au-Au^h\|_X\|u-v^h\|_X+\left(\alpha_\varphi c_\varphi^2+\alpha_j c_\varphi^2+\varepsilon\right)\|u-u^h\|_X^2\\ &+c\,\|u-v^h\|_X^2+2\,L_\varphi(\gamma_\varphi u)\,\|\gamma_\varphi u-\gamma_\varphi v^h\|_{X_\varphi}\\ &+c\,\|\gamma_j u-\gamma_j v^h\|_{X_j}.\end{aligned}$$

Using the smallness assumption (2.9) and the Lipschitz condition (3.6), we deduce from the above inequality that

$$\|u - u^h\|_X^2 \\ \leq c\left(\|u - v^h\|_X^2 + \|\gamma_\varphi u - \gamma_\varphi v^h\|_{X_\varphi} + \|\gamma_j u - \gamma_j v^h\|_{X_j}\right) \quad \forall\, v^h \in X^h. \tag{3.29}$$

This is a basis for deriving error estimates.

4 Error analysis for contact problems

We illustrate applications of the framework developed in Section 3 on convergence and error estimation for numerical solutions of a number of static contact problems with elastic materials. Let Ω be the reference configuration of the elastic body, assumed to be an open, bounded, connected set in $\mathbb{R}^d$ $(d = 2, 3)$. The boundary $\Gamma = \partial\Omega$ is assumed Lipschitz continuous and is partitioned into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that meas $(\Gamma_1) > 0$. The body is in equilibrium under the action of a total body force of density $\boldsymbol{f}_0$ in Ω and a surface traction of density $\boldsymbol{f}_2$ on Γ_2, is fixed on Γ_1, and is in contact on Γ_3 with a foundation. Different contact conditions lead to different contact problems, as discussed below.

We use $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$, and use "$\cdot$" and $\|\cdot\|$ for the canonical inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$. We denote by $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ and $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ the displacement field and the stress field, respectively. In addition, we use $\boldsymbol{\varepsilon}(\boldsymbol{u})$ to denote the linearized strain tensor. Let $\boldsymbol{\nu}$ be the unit outward normal vector, defined a.e. on Γ. For a vector field $\boldsymbol{v}$, we use $v_\nu{:=}\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau{:=}\boldsymbol{v} - v_\nu\boldsymbol{\nu}$ for the normal and tangential components of $\boldsymbol{v}$ on Γ. Similarly, for the stress field $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are defined as $\sigma_\nu{:=}(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau{:=}\boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively. Then for the contact problems under consideration, we have the elastic constitutive law

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{4.1}$$

the equilibrium equation

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{4.2}$$

the displacement boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{4.3}$$

and the traction boundary condition

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{4.4}$$

The relations (4.1)–(4.4) will be supplemented by a set of boundary conditions on Γ_3.

Note that in (4.1)–(4.4) and at various places below, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. In (4.1), $\mathcal{F}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity operator and is assumed to have the following properties:

$$\begin{cases} \text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \quad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|; \\ \text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \quad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2; \\ \text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(d) } \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{4.5}$$

To study the contact problems, we need some function spaces. For the stress and strain fields, we use the space $Q = L^2(\Omega; \mathbb{S}^d)$, which is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\, \tau_{ij}(\boldsymbol{x})\, dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q.$$

Here and below we use the summation convention upon the repeated index. The associated norm on the space Q is denoted by $\|\cdot\|_Q$. The displacement fields will be sought in the space

$$V = \left\{ \boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \right\}$$

or its subset. Since meas $(\Gamma_1) > 0$, it is known that V is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad \boldsymbol{u}, \boldsymbol{v} \in V$$

and the associated norm $\|\cdot\|_V$. For $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$ we use the same symbol $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ. By the Sobolev trace theorem we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\gamma\|\, \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V,$$

$\|\gamma\|$ being the norm of the trace operator $\gamma\colon V \to L^2(\Gamma_3; \mathbb{R}^d)$.

We consider several choices of the boundary conditions on the contact boundary Γ_3, leading to different contact problems which are examples of Problem (P) or its special cases.

4.1 Frictional contact with normal compliance and unilateral constraint

The first set of contact boundary conditions is ([17])

$$u_\nu \le g,\ \sigma_\nu + \xi_\nu \le 0,\ (u_\nu - g)\,(\sigma_\nu + \xi_\nu) = 0,\ \xi_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3, \tag{4.6}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu),\ -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\, \frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{4.7}$$

Here $g > 0$, ∂j_ν is the Clarke subdifferential of a function j_ν, and the friction bound F_b is a non-negatively valued function. The condition (4.6) models the contact with a foundation made of a rigid body covered by a layer made of elastic material, say asperities. The relation $u_\nu \le g$ restricts the allowed penetration, where g represents the thickness of the elastic layer. When there is penetration, as long as the normal displacement does not reach the bound g, the contact is described with a multivalued normal compliance condition $-\sigma_\nu = \xi_\nu \in \partial j_\nu(u_\nu)$. Thus, the unknown ξ_ν may be interpreted as the opposite of the normal stress on the contact surface. Examples of normal compliance contact conditions in the subdifferential form $-\sigma_\nu \in \partial j_\nu(u_\nu)$ can be found in [16]. To conclude, the contact condition (4.6) represents a combination of the Signorini contact condition (which models the contact with a rigid foundation) and the normal compliance condition (which models the contact with a deformable foundation). Details on the normal compliance and Signorini contact conditions can be found in [11,16,22,24]. The contact is assumed to be frictional and is described with a version of Coulomb's law of dry friction, (4.7). The friction bound F_b may depend on the normal displacement u_ν, cf. [23] for explanation. We note that when the friction bound F_b is zero, the contact boundary condition (4.7) reduces to

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \tag{4.8}$$

Thus, the frictionless contact can be viewed as a limiting special case of the general contact condition (4.7).

On the potential function $j_\nu \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, we assume

$$\left\{\begin{array}{l}\text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1 |r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \qquad \text{for all } r \in \mathbb{R} \text{ with } \overline{c}_0,\ \overline{c}_1 \ge 0; \\ \text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu} |r_1 - r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{all } r_1, r_2 \in \mathbb{R} \text{ with } \alpha_{j_\nu} \ge 0.\end{array}\right. \tag{4.9}$$

On the penetration bound $g \colon \Gamma_3 \to \mathbb{R}$ and the friction bound $F_b \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, we assume

$$g \in L^2(\Gamma_3), \quad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. on } \Gamma_3, \tag{4.10}$$

$$\left\{\begin{array}{l}\text{(a) there exists } L_{F_b} > 0 \text{ such that} \\ \qquad |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b} |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(b) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3,\ \text{for all } r \in \mathbb{R}; \\ \text{(c) } F_b(\boldsymbol{x}, r) = 0 \text{ for } r \le 0,\ F_b(\boldsymbol{x}, r) \ge 0 \text{ for } r \ge 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\end{array}\right. \tag{4.11}$$

On the densities of body forces and surface tractions we assume

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d). \tag{4.12}$$

Define $\boldsymbol{f} \in V^*$ by

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V. \tag{4.13}$$

Corresponding to the constraint $u_\nu \le g$ on Γ_3 in (4.6), we introduce a subset of the space V:

$$U := \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3\}. \tag{4.14}$$

By a standard approach (cf. [11,16]), the following weak formulation of the first contact problem can be derived.

PROBLEM (P$_1$). *Find a displacement field* $\boldsymbol{u} \in U$ *such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}))_Q + \int_{\Gamma_3} F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, d\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in U. \tag{4.15}$$

To apply the theory presented in the previous sections, we let $X = V$, $K = U$, $X_\varphi = K_\varphi = L^2(\Gamma_3; \mathbb{R}^d)$ with γ_φ the trace operator from V to X_φ, $X_j = L^2(\Gamma_3)$ with $\gamma_j \boldsymbol{v} = v_\nu$ for $\boldsymbol{v} \in V$, and

$$A\colon V \to V^*, \quad \langle A\boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in V,$$
$$\varphi\colon L^2(\Gamma_3; \mathbb{R}^d) \times L^2(\Gamma_3; \mathbb{R}^d) \to \mathbb{R},$$
$$\varphi(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_{\Gamma_3} F_b(\xi_\nu)\, \|\boldsymbol{\eta}_\tau\|\, d\Gamma \quad \text{for } \boldsymbol{\xi},\ \boldsymbol{\eta} \in L^2(\Gamma_3; \mathbb{R}^d),$$
$$j\colon L^2(\Gamma_3) \to \mathbb{R}, \quad j(\xi) = \int_{\Gamma_3} j_\nu(\xi)\, d\Gamma \quad \text{for } \xi \in L^2(\Gamma_3),$$
$$\boldsymbol{f} \in V^*, \quad \langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_3} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, dx \quad \text{for } \boldsymbol{v} \in V.$$

Observe that hypothesis (4.5) implies that the operator A satisfies condition (2.5) with $m_A = m_{\mathcal{F}}$ as well as condition (3.6). Moreover, hypothesis (4.11) implies that φ satisfies (2.6) with $\alpha_\varphi = L_{F_b}$ as well as condition (3.7). Next, hypothesis (4.9)(a) guarantees that the function j is well defined, and hypotheses (4.9)(b) and (c) imply that (2.7) holds (cf. [16, Theorem 3.47]). Condition (2.8) is a consequence of the relation

$$j^0(\xi; \eta) \le \int_{\Gamma_3} j_\nu^0(\boldsymbol{x}, \xi(\boldsymbol{x}); \eta(\boldsymbol{x}))\, d\Gamma \quad \forall\, \xi, \eta \in L^2(\Gamma_3),$$

combined with the hypothesis (4.9)(d) and is satisfied with $\alpha_j = \alpha_{j_\nu}$. Under the assumption (4.10), it is easy to see that the set (4.14) is a nonempty, closed, convex set in V. In addition, (2.10) is valid from the definition (4.13) and assumption (4.12).

Let $\lambda_{1,V} > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} \boldsymbol{u}\cdot\boldsymbol{v}\, d\Gamma \quad \forall\, \boldsymbol{v} \in V,$$

and let $\lambda_{1\nu,V} > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} u_\nu v_\nu d\Gamma \quad \forall\, \boldsymbol{v} \in V.$$

Then we may take

$$c_\varphi = \lambda_{1,V}^{-1/2}, \quad c_j = \lambda_{1\nu,V}^{-1/2}.$$

Applying Theorem 2.1, we know that under the assumptions (4.5), (4.9)–(4.12), and

$$L_{F_b}\lambda_{1,V}^{-1} + \alpha_{j_\nu}\lambda_{1\nu,V}^{-1} < m_{\mathcal{F}}, \tag{4.16}$$

Problem (P_1) has a unique solution $\boldsymbol{u} \in U$.

We now consider the finite element method of solving Problem (P_1). For simplicity, assume Ω is a polygonal/polyhedral domain and express the three parts of the boundary, Γ_k, $1 \le k \le 3$, as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \le k \le 3.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $1 \le k \le 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Construct the linear element space corresponding to $\mathcal{T}^h$:

$$V^h = \left\{\boldsymbol{v}^h \in C(\overline{\Omega}; \mathbb{R}^d) \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d,\ T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1\right\},$$

and the related finite element subset $U^h = V^h \cap U$. Assume g is a concave function. Then

$$U^h = \left\{\boldsymbol{v}^h \in V^h \mid v_\nu^h \le g \text{ at node points on } \Gamma_3\right\}.$$

Note that $\boldsymbol{0} \in U^h$. Define the following numerical method for Problem (P_1).

PROBLEM (P_1^h). *Find a displacement field* $\boldsymbol{u}^h \in U^h$ *such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}^h))_Q + \int_{\Gamma_3} F_b(u_\nu^h)\left(\|\boldsymbol{v}_\tau^h\| - \|\boldsymbol{u}_\tau^h\|\right) d\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu^h; v_\nu^h - u_\nu^h)\, d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v}^h \in U^h. \tag{4.17}$$

For an error analysis, we assume

$$\boldsymbol{u} \in H^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{\sigma}\boldsymbol{\nu} \in L^2(\Gamma_3; \mathbb{R}^d). \tag{4.18}$$

Note that for many application problems, $\boldsymbol{\sigma}\boldsymbol{\nu} \in L^2(\Gamma_3; \mathbb{R}^d)$ follows from $\boldsymbol{u} \in H^2(\Omega; \mathbb{R}^d)$; e.g., this is the case where the material is linearly elastic with suitably smooth coefficients, or where the elasticity operator $\mathcal{F}$ depends on $\boldsymbol{x}$ smoothly. We apply (3.20) to derive an error estimate. For this purpose, we need to bound the residual term defined in (3.13):

$$R(\boldsymbol{v}^h) = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}))_Q + \int_{\Gamma_3} F_b(u_\nu) \left(\|\boldsymbol{v}_\tau^h\| - \|\boldsymbol{u}_\tau\|\right) d\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu^h - u_\nu)\, d\Gamma - \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}\rangle_{V^*\times V}.$$

We follow the procedure found in [11]. Take $\boldsymbol{v} = \boldsymbol{u} \pm \boldsymbol{w}$ with $\boldsymbol{w}$ in the subset $\tilde{U}$ of U defined by

$$\tilde{U} := \left\{ \boldsymbol{w} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) \mid \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_1 \cup \Gamma_3 \right\},$$

and derive from (4.15) that

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{w} \in \tilde{U}.$$

Therefore,

$$\operatorname{Div} \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{4.19}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{4.20}$$

Then multiply (4.19) by $\boldsymbol{v} - \boldsymbol{u}$ with $\boldsymbol{v} \in U$, integrate over Ω, and integrate by parts,

$$\int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{u})\, d\Gamma - \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\, dx + \int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v} - \boldsymbol{u})\, dx = 0,$$

i.e.,

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\, dx = \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V^*\times V} + \int_{\Gamma_3} \boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{u})\, d\Gamma. \tag{4.21}$$

Thus,

$$R(\boldsymbol{v}^h) = \int_{\Gamma_3} \left[\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}^h - \boldsymbol{u}) + F_b(u_\nu)\left(\|\boldsymbol{v}_\tau^h\| - \|\boldsymbol{u}_\tau\|\right) + j_\nu^0(u_\nu; v_\nu^h - u_\nu)\right] d\Gamma,$$

and then,

$$\left|R(\boldsymbol{v}^h)\right| \le c\, \|\boldsymbol{u} - \boldsymbol{v}^h\|_{L^2(\Gamma_3)^d}. \tag{4.22}$$

Finally, from (3.20), we derive the inequality

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 \le c\left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2+\|\boldsymbol{u}-\boldsymbol{v}^h\|_{L^2(\Gamma_3)^d}\right) \quad \forall\,\boldsymbol{v}^h\in U^h. \tag{4.23}$$

Under additional solution regularity assumption

$$\boldsymbol{u}|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i};\mathbb{R}^d), \quad 1\le i\le i_3, \tag{4.24}$$

we have the optimal order error bound

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V \le c\,h. \tag{4.25}$$

We comment that similar results hold for the frictionless version of the model, i.e., where the friction condition (4.7) is replaced by (4.8). Then the problem is to solve the inequality (4.15) without the term

$$\int_{\Gamma_3} F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_\tau\|)\,d\Gamma.$$

The condition (4.16) reduces to

$$\alpha_{j_\nu}\lambda_{1\nu,V}^{-1} < m_{\mathcal{F}}.$$

The inequality (4.23) and the error bound (4.25) still hold for the linear finite element solution.

4.2 Frictional contact with normal compliance

Instead of (4.6)–(4.7), the following contact boundary conditions were considered in [9]:

$$-\sigma_\nu\in\partial j_\nu(u_\nu), \quad \|\boldsymbol{\sigma}_\tau\|\le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau=F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|}\ \text{if}\ \boldsymbol{u}_\tau\ne\boldsymbol{0} \quad \text{on}\ \Gamma_3, \tag{4.26}$$

where the potential function j_ν is assumed to satisfy (4.9), whereas the friction bound F_b is assumed to satisfy (4.11). Note that in contrast with the contact boundary condition (4.6) which involves a unilateral constraint on the displacement field, the contact condition in (4.26) does not involve such a restriction. It can be viewed as a limiting case of the (4.6) when $g\to\infty$. The weak formulation of the corresponding contact problem is the following.

PROBLEM (P$_2$). *Find a displacement field* $\boldsymbol{u}\in V$ *such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))_Q+\int_{\Gamma_3}F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_\tau\|)\,d\Gamma$$
$$+\int_{\Gamma_3} j_\nu^0(u_\nu;v_\nu-u_\nu)\,d\Gamma \ge \langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\,\boldsymbol{v}\in V. \tag{4.27}$$

This problem and its numerical approximations were studied in [9], where the functional j_ν was assumed to be Lipschitz continuous. We can apply the theory developed in Sect. 3 for error analysis of numerical solutions of Problem (P_2), with $X_\varphi = L^2(\Gamma_3; \mathbb{R}^d)$ and $X_j = L^2(\Gamma_3)$, without the need of assuming j_ν to be Lipschitz continuous.

4.3 Frictionless contact with normal compliance

The frictionless version of the contact boundary conditions (4.26) are

$$- \sigma_\nu \in \partial j_\nu(u_\nu), \quad \boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \tag{4.28}$$

These conditions can be obtained from (4.26) when the friction bound vanishes, i.e. when $F_b \equiv 0$. Such a condition represents an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motion. The weak formulation of the corresponding contact problem is the following.

PROBLEM (P_3). *Find a displacement field* $\boldsymbol{u} \in V$ *such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))_Q + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, d\Gamma \geq \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in V. \tag{4.29}$$

Note that, in contrast with the inequality (4.27) which involves both a convex and a nonconvex function, the inequality (4.29) is governed by a nonconvex function only; it is an example of a pure hemivariational inequality. In applying the theory of Sect. 3 in the study of this inequality, we choose $X_j = L^2(\Gamma_3)$ and there is no X_φ.

4.4 Frictionless contact with subdifferential boundary conditions and unilateral constraint

We turn now to a new model of frictionless contact described with two subdiferential boundary conditions, associated with a unilateral constraint for the normal displacement field. The boundary conditions are formulated as follows:

$$\begin{aligned} &\sigma_\nu = \sigma_\nu^1 + \sigma_\nu^2 + \sigma_\nu^3, \quad -\sigma_\nu^1 \in \widetilde{\partial}\varphi_\nu(u_\nu), \quad -\sigma_\nu^2 \in \partial j_\nu(u_\nu), \\ &u_\nu \leq g, \quad \sigma_\nu^3 \leq 0, \quad \sigma_\nu^3(u_\nu - g) = 0, \quad \text{on } \Gamma_3, \end{aligned} \tag{4.30}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \tag{4.31}$$

Here g is a prescribed bound, φ_ν and j_ν are given functions, and $\widetilde{\partial}\varphi_\nu$, ∂j_ν denote the convex subdifferential of φ_ν and the Clarke subdifferential of j_ν, respectively. A relevant example of the contact condition which can be cast in the general subdifferential framework (4.30) follows.

Example 4.1 We adopt assumptions (a)–(e) below in which equalities and inequalities hold on the contact surface Γ_3.

(a) The foundation is made of a rigid body covered by a layer of soft material, say asperities. Therefore, the penetration is restricted, i.e.

$$u_\nu \le g, \tag{4.32}$$

where $g > 0$ represents the thickness of the soft layer. We consider the nonhomogeneous case, i.e., g is allowed to be a function of the spatial variable $\boldsymbol{x} \in \Gamma_3$.

(b) The normal stress has an additive decomposition of the form

$$\sigma_\nu = \sigma_\nu^D + \sigma_\nu^R, \tag{4.33}$$

where the term σ_ν^D describes the reaction of the soft layer and σ_ν^R describes the reaction of the rigid body.

(c) The part σ_ν^D has, in turn, an additive decomposition of the form

$$\sigma_\nu^D = \sigma_\nu^1 + \sigma_\nu^2, \tag{4.34}$$

where

$$-\sigma_\nu^1 = q_\nu(u_\nu), \quad -\sigma_\nu^2 = p_\nu(u_\nu). \tag{4.35}$$

Here q_ν and p_ν are continuous positive functions, vanishing for a negative argument. Moreover, q_ν is an increasing function and p_ν is allowed to be nonmonotone. Equalities (4.34), (4.35) show that the function σ_ν^D follows a normal compliance contact condition of the form

$$-\sigma_\nu^D = q_\nu(u_\nu) + p_\nu(u_\nu). \tag{4.36}$$

(d) The part σ_ν^R satisfies the Signorini condition in a form with a gap function, i.e.

$$\sigma_\nu^R \le 0, \quad \sigma_\nu^R(u_\nu - g) = 0. \tag{4.37}$$

The contact conditions (4.35) can be expressed in terms of the subdifferential operators. Introduce functions $\varphi_\nu : \mathbb{R} \to \mathbb{R}$ and $j_\nu : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\nu(r) = \int_0^r q_\nu(s)\, ds, \quad j_\nu(r) = \int_0^r p_\nu(s)\, ds \quad \forall r \in \mathbb{R}. \tag{4.38}$$

Then, as explained in [16,24] we have $\widetilde{\partial}\varphi(r) = \{q_\nu(r)\}$, $\partial_\nu j(r) = \{p_\nu(r)\}$ for all $r \in \mathbb{R}$. Thus, (4.35) leads to a subdifferential condition of the form

$$-\sigma_\nu^1 \in \widetilde{\partial}\varphi_\nu(u_\nu), \qquad -\sigma_\nu^2 \in \partial j(u_\nu). \tag{4.39}$$

Denote $\sigma_\nu^R = \sigma_\nu^3$. Then, gathering relations (4.33), (4.34), (4.39), (4.32) and (4.37) we obtain that u_ν and σ_ν satisfy the contact condition (4.30). We conclude that the contact model based on assumptions (4.32)–(4.37) can be cast in this abstract subdifferential setting, as claimed. □

Note that various examples of contact conditions can be considered in the form (4.30), some of them involving multivalued functions. For this reason, we proceed our analysis by considering the contact conditions in the general form (4.30). There, the function j_ν is assumed to satisfy (4.9) and $\varphi_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$\left\{\begin{array}{l}\text{(a) } \varphi_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad\ \text{exists } \widetilde{e} \in L^2(\Gamma_3) \text{ such that } \varphi_\nu(\cdot, \widetilde{e}(\cdot)) \in L^1(\Gamma_3). \\ \text{(b) } \varphi_\nu(\boldsymbol{x}, \cdot) \text{ is convex on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } |\varphi_\nu(\boldsymbol{x}, r_1) - \varphi_\nu(\boldsymbol{x}, r_2)| \le L_{\varphi_\nu}(\boldsymbol{x})\, |r_1 - r_2| \text{ for all } \boldsymbol{x} \in \Gamma_3, \\ \quad\ r_1, r_2 \in \mathbb{R}, \text{ with } L_{\varphi_\nu} \in L^2(\Gamma_3).\end{array}\right. \tag{4.40}$$

By a standard procedure, we derive the following variational formulation of the problem (4.1)–(4.4), (4.30), (4.31).

Problem (P$_4$). *Find a displacement field* $\boldsymbol{u} \in U$ *such that*

$$\begin{aligned}&\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\, dx \\ &\quad + \int_{\Gamma_3} \varphi_\nu(v_\nu)\, d\Gamma - \int_{\Gamma_3} \varphi_\nu(u_\nu)\, d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, d\Gamma \\ &\ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, d\Gamma \quad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{4.41}$$

Corresponding to this problem, the inequality (2.6) holds with $\alpha_\varphi = 0$. Assume now that the smallness condition

$$\alpha_{j_\nu} \lambda_{1\nu,V}^{-1} < m_{\mathcal{F}} \tag{4.42}$$

holds. Then, the unique solvability of Problem (P$_4$) follows from arguments similar to those used to prove the unique solvability of Problem (P$_1$).

The results in Sect. 3 apply in the study the numerical approximation of the contact problems (P$_2$)–(P$_4$) as was done for the contact problem (P$_1$) earlier in the section. In particular, under the solution regularity assumptions (4.18) and (4.24), we have the optimal order error bound (4.25).

5 Numerical simulations

In this section we report numerical simulation results. We focus on Problems (P$_1$) and (P$_2$) and, for Problem (P$_1$) we restrict to the frictionless case. The frictional case of Problem (P$_1$) does not cause additional difficulties, neither theoretically nor computationally.

The numerical algorithm used here is described in [2–5], and it is based on an iterative procedure which leads to a sequence of convex programming problems. For each "convexification" iteration, the value of the normal compliance function is fixed to a given value depending on the normal displacement solution u_ν found in the previous iteration. Then, the resulting nonsmooth convex iterative problems are solved by classical numerical methods. Furthermore, the frictional contact conditions are treated by using a numerical approach based on the combination of the penalized method and the augmented Lagrangian method. We consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact elements used for the geometrical discretization of the interface Γ_3. In the examples presented below, the discretization is based on "node-to-rigid" contact element, which is composed of one node of Γ_3 and one Lagrange multiplier node. For more details on the discretization step and Computational Contact Mechanics, we refer the reader to [13–15,25].

5.1 Physical setting and values of parameters

Let $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$ with $L > 0$ and

$$\Gamma_1 = (\{0\} \times [0, L]) \cup (\{L\} \times [0, L]), \ \ \Gamma_2 = [0, L] \times \{L\}, \ \ \Gamma_3 = [0, L] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional elastic body subjected to the action of tractions in such a way that the plane stress hypothesis is valid. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Vertical tractions act on Γ_2. No body forces are assumed to act on the body during the process. The body is in contact with an obstacle on Γ_3. The contact conditions used correspond both to Problems (P_1) and (P_2) and will be described below. The material response is governed by a linear constitutive law defined by the elasticity tensor $\mathcal{F}$ given by

$$(\mathcal{F}\boldsymbol{\tau})_{ij} = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}(\tau_{11}+\tau_{22})\delta_{ij} + \frac{E}{1+\kappa}\tau_{ij}, \quad 1 \le i, j \le 2, \ \forall \boldsymbol{\tau} \in \mathbb{S}^2.$$

Here, E and κ are Young's modulus and Poisson's ratio of the material and δ_{ij} denotes the Kronecker delta. For the computation below, we use the following data:

$$L = 1\,m, \quad E = 70\,\text{GPa}, \quad \kappa = 0.3,$$
$$\boldsymbol{f}_0 = (0, 0)\,\text{GPa}, \quad \boldsymbol{f}_2 = (0, -4)\,\text{GPa m} \quad \text{on } \Gamma_2.$$

For the numerical simulation we use linear finite elements on uniform triangulations of the domain Ω. The contact boundary Γ_3 is divided in $1/h$ parts, h being the spatial discretization parameter. The numerical solutions presented below correspond to the case $h = 1/64$ where the spatial domain is discretized into 16384 elements for a total number of degrees of freedom equal to 16770. They have been obtained by using a specific finite element library implemented in Fortran called MODULEF. Details on this library, could be found to https://www.rocq.inria.fr/modulef/english.html.

5.2 Numerical simulations for problem (P_1)

We consider a particular version of Problem (P_1), in which the contact conditions are given by

$$u_\nu \le g, \quad \sigma_\nu + k_\nu(u_\nu) \le 0, \quad (u_\nu - g)(\sigma_\nu + k_\nu(u_\nu)) = 0 \quad \text{on } \Gamma_3, \tag{5.1}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \tag{5.2}$$

Here $k_\nu : \mathbb{R} \to \mathbb{R}$ is a given function to be described below. These conditions indicate that the contact is frictionless; it follows a normal compliance condition as far as the penetration is less than the bound g and, when this bound is reached, it follows a unilateral constraint. The behavior of the foundation is an elastic-rigid one and corresponds to an obstacle made of a hard material covered by a layer composed of a soft material, say asperities, with thickness g, as depicted in Figure 1.

For the numerical simulations we choose $g = 0.02\,m$. In addition, we choose

$$k_\nu(r) = \alpha\left(\beta r^+ + p(r)\right), \quad r \in \mathbb{R}, \tag{5.3}$$

where

$$p(r) = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } r \in [0, 0.01], \\ 0.02 - r & \text{if } r \in (0.01, 0.02], \\ r - 0.02 & \text{if } r > 0.02. \end{cases} \tag{5.4}$$

Here and below r^+ represents the positive part of r, i.e. $r^+ = \max\{r, 0\}$, $\alpha > 0$ and $\beta \ge 0$ are stiffness coefficients of the foundation.

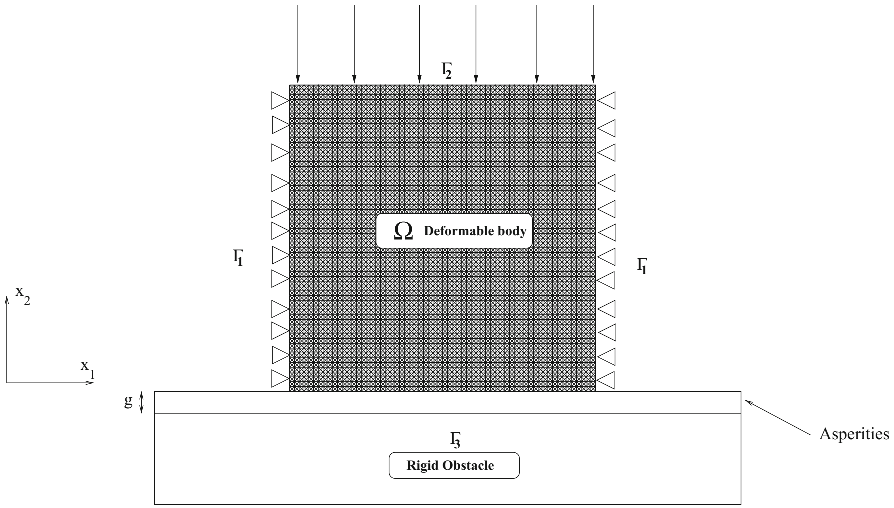

Fig. 1 Reference configuration of the two-dimensional body

Let $j_\nu : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j_\nu(r) = \int_0^r k_\nu(s)\, ds \quad \forall\, r \in \mathbb{R}. \tag{5.5}$$

Then, $\partial j_\nu(r) = \{k_\nu(r)\}$ for all $r \in \mathbb{R}$ which imply that the boundary conditions (5.1)–(5.2) are of the form (4.6)–(4.7) with $F_b = 0$.

Obviously, k_ν is Lipschitz continuous. Moreover, it is easy to see that the function j_ν satisfies assumptions (4.9)(a)–(c). Noting that

$$j_\nu^0(r_1; r_2) = k_\nu(r_1)\, r_2 \quad \forall\, r_1, r_2 \in \mathbb{R}, \tag{5.6}$$

the condition (4.9)(d) is equivalent to the inequality

$$(k_\nu(r_1) - k_\nu(r_2))(r_2 - r_1) \le \alpha_{j\nu}(r_1 - r_2)^2 \quad \forall\, r_1, r_2 \in \mathbb{R} \quad \text{with some } \alpha_{j\nu} \ge 0,$$

and is thus also equivalent to the statement that the function $\mathbb{R} \ni r \mapsto \alpha_{j_\nu} r + k_\nu(r) \in \mathbb{R}$ is nondecreasing for some $\alpha_{j_\nu} \ge 0$. This monotonicity property is obviously satisfied with $\alpha_{j_\nu} = \alpha$. It follows from above that condition (4.9)(d) holds, too.

We conclude from above that the choice (5.3)–(5.4) leads to a model of contact for which the weak formulation is given by the variational-hemivariational inequality (4.15) which, in our particular case $F_b = 0$, becomes

$$\begin{aligned} \boldsymbol{u} \in U, \quad & \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, d\Gamma \\ & \ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_3} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx \quad \forall\, \boldsymbol{v} \in U. \end{aligned} \tag{5.7}$$

Note also that this inequality has a unique solution, provided that α is small enough. Moreover, using (5.6) we deduce that the corresponding version of Problem (P_1) can be written, in an equivalent form as follows: find $\boldsymbol{u}$ such that

$$\begin{aligned} \boldsymbol{u} \in U, \quad & \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_3} k_\nu(u_\nu)(v_\nu - u_\nu)\, d\Gamma \\ & \ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_3} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx \quad \forall\, \boldsymbol{v} \in U. \end{aligned} \tag{5.8}$$

We examine in what follows the feature of the inequalities (5.7) and (5.8) in relation to the values of the parameter β.

If $\beta \ge 1$, then k_ν is a continuous increasing function and, therefore, j_ν is a convex function. Moreover, it can be proved that $\boldsymbol{u}$ is a solution of the inequality (5.8) if and only if

$$\boldsymbol{u} \in U, \quad \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx + \int_{\Gamma_3} [j_\nu(v_\nu) - j_\nu(v_\nu)]\, d\Gamma$$
$$\geq \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v}-\boldsymbol{u})\, dx + \int_{\Gamma_3} \boldsymbol{f}_2 \cdot (\boldsymbol{v}-\boldsymbol{u})\, dx \quad \forall\, \boldsymbol{v} \in U. \tag{5.9}$$

Since j_ν is a convex function, the inequality (5.9) is a purely variational inequality. It follows from here that inequalities (5.7) and (5.8) are, in fact, purely variational inequalities.

Next, if $0 \leq \beta < 1$, then k_ν is not a monotone function and, therefore, j_ν is not convex. This implies that in this case inequalities (5.7) and (5.8) are hemivariational inequalities.

We conclude that our contact model leads to a variational formulation (5.8) whose intrinsic nature depends on the value of the parameter β. For $\beta \geq 1$, the function k_ν is increasing and from (5.1) it follows that we have the case of a monotone normal compliance contact condition with unilateral constraint. In contrast, for $0 \leq \beta < 1$ the function k_ν is not increasing and using (5.1), we have the case of a so-called non-monotone normal compliance contact condition with unilateral constraint. Our aim in what follows is to perform simulations in the two cases above and to provide the corresponding mechanical interpretations.

5.2.1 The monotone case

In Fig. 2 we plot the deformed configuration as well as the interface forces on Γ_3, for $\alpha = 40$ while in Fig. 3 we plot that for $\alpha = 10$. In both cases we take $\beta = 2$ which guarantees that we are in the monotone case.

We observe that all nodes in Fig. 2 are in status of normal compliance, i.e. $0 \leq u_\nu < 0.02$. Moreover, the interface forces increase with respect to the penetration. This numerical result corresponds with the theoretic one. Indeed, in this case there is no contact with the rigid foundation, the normal stress reduces to its component provided by the normal compliance condition and, therefore $-\sigma_\nu = k_\nu(u_\nu)$. Since k_ν is an increasing function we deduce that the magnitude of σ_ν is increasing with the penetration.

In Fig. 3 part of the nodes are in status of normal compliance (i.e. $0 \leq u_\nu < 0.02$) and part of them are in unilateral contact (i.e. $u_\nu = 0.02$). This situation arises since the stiffness of the deformable foundation considered in Fig. 3 is $\alpha = 10$ which is lower than the one used in Fig. 2, where $\alpha = 40$. As a result, there is a complete flattening of the asperities in the center of the contact boundary. Moreover, the interface forces increase with respect to the penetration and this agrees with the theory, since we are in the monotone case. In addition, we note that the interface forces are more important on the unilateral contact zone since the component σ_ν^3 of the stress is active there.

5.2.2 The nonmonotone case

In Fig. 4 we plot the deformed configuration as well as the interface forces on Γ_3 for $\alpha = 150$ and $\beta = 0.5$ while in Fig. 5 that for $\alpha = 40$ and $\beta = 0.5$. Here we are in the nonmonotone case, since $\beta < 1$.

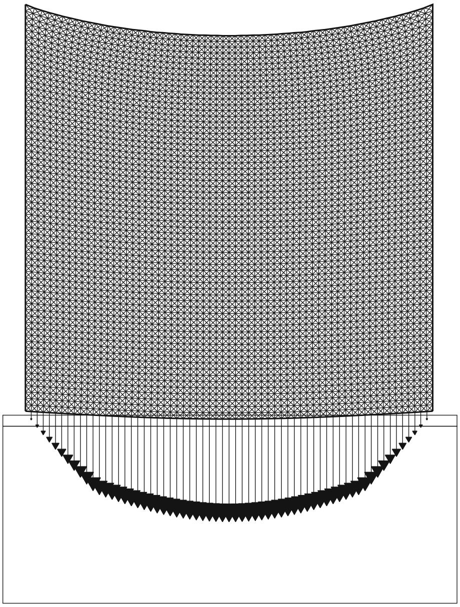

Fig. 2 Problem (P_1): deformed mesh and interface forces for a normal compliance case for $\alpha = 40$ and $\beta = 2$

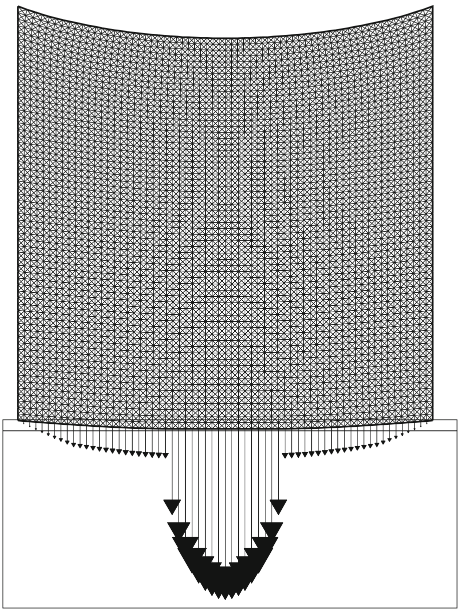

Fig. 3 Problem (P_1): deformed mesh and interface forces for a normal compliance and unilateral constraint case for $\alpha = 10$ and $\beta = 2$

Observe that in Fig. 4 all nodes are in status of normal compliance, i.e. $0 \leq u_\nu < 0.02$. Nevertheless for part of the nodes we have $0 \leq u_\nu < 0.01$ and for the other part we have $0.01 \leq u_\nu < 0.02$. We note that for $0 \leq u_\nu < 0.01$ the normal forces are increasing with respect to the penetration and for $0.01 \leq u_\nu < 0.02$ they decrease. This behaviour represent the softening property of the deformable layer. It arises since

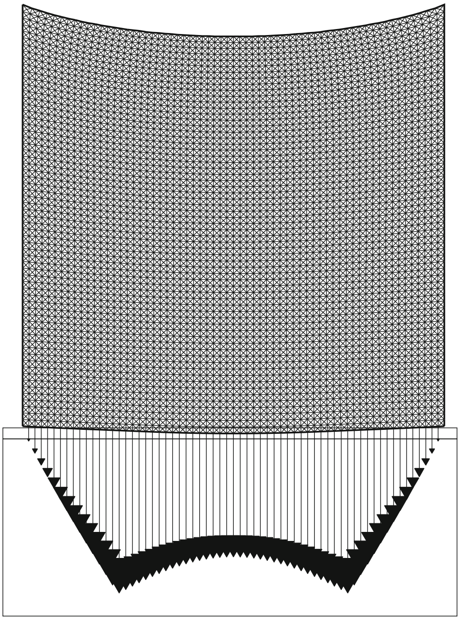

Fig. 4 Problem (P_1): deformed mesh and interface forces for a normal compliance case for $\alpha = 150$ and $\beta = 0.5$

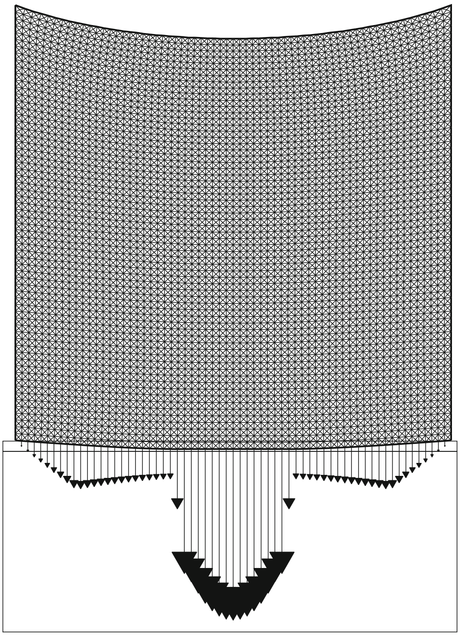

Fig. 5 Problem (P_1): deformed mesh and interface forces for a normal compliance and unilateral constraint case for $\alpha = 40$ and $\beta = 0.5$

there $-\sigma_\nu = k_\nu(u_\nu)$ and k_ν is an increasing function on $[0, 0.01]$, and it is decreasing on $[0.01, 0.02]$.

In Fig. 5 for part of the nodes we have $0 \leq u_\nu \leq 0.01$, for other part we have $0.01 \leq u_\nu \leq 0.02$ and, finally, for the remainder part we have $u_\nu = 0.02$. We note

Fig. 6 Problem (P_1): deformed mesh and interface forces for a normal compliance and unilateral constraint case for $\alpha = 40$ and $\beta = 0$

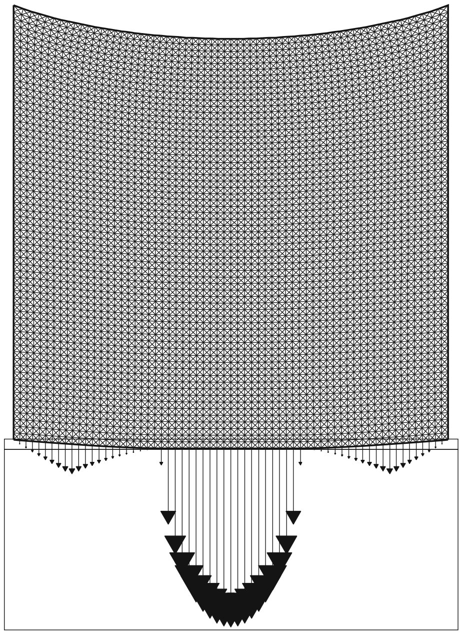

Table 1 Problem (P_1): relative errors in energy norm for contact with normal compliance and unilateral constraint

h	1/4	1/8	1/16	1/32	1/64
Relative error	0.187	0.0891	0.0433	0.0213	0.0105

that for $0 \leq u_\nu < 0.01$ the normal forces are increasing with respect to the penetration and for $0.01 \leq u_\nu < 0.02$ they decrease. Moreover, the interface forces increase when $u_\nu = 0.02$ since, there, the component σ_3 of the stress is active.

Finally, in Fig. 6 we present our numerical results in the case $\alpha = 40$ and $\beta = 0$. In this case, the non-monotonicity reaches its peak and, therefore, the average iterations number of the "convexification" procedure to solve the problem is larger than that needed in the previous cases. Note that the value of σ_ν on the contact boundary decreases when $0.01 \leq u_\nu \leq 0.02$ and converges to zero for a node located in the transition area between the normal compliance zone and the unilateral constraint zone.

5.2.3 Numerical solution errors

We report relative numerical solution errors in the energy norm $\|\boldsymbol{u}_{\text{ref}} - \boldsymbol{u}^h\|_E / \|\boldsymbol{u}_{\text{ref}}\|_E$ in Table 1 and Fig. 7, where the energy norm is defined by

$$\|\boldsymbol{v}\|_E := (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{v})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q^{1/2}$$

which is equivalent to the norm $\|\boldsymbol{v}\|_V$. Since the true solution $\boldsymbol{u}$ is not available, we use instead the numerical solution corresponding to a fine discretization of Ω

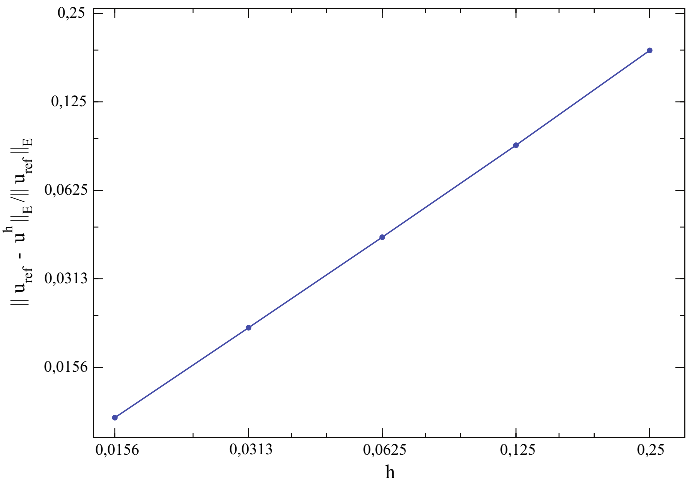

Fig. 7 Problem (P_1): relative errors in energy norm for contact with normal compliance and unilateral constraint

with $h = 1/256$ as the "reference" solution $\boldsymbol{u}_{\text{ref}}$ in computing the solution errors. This fine discretization corresponds to a problem with 132,098 degrees of freedom, 131,072 elements and is computed in 2462 CPU time (expressed in seconds) on an IBM computer equipped with Intel Dual core processors (Model 5148, 2.33 GHz). The curve of the relative numerical solution errors is asymptotically linear, which is consistent with the theoretically predicted optimal linear convergence of the numerical solution established in Sect. 4.

5.3 Numerical simulations for problem (P_2)

In this problem, the contact is frictional with normal compliance. The foundation reacts elastically. A representative numerical simulation result is shown in Fig. 8 for the problem with the choice

$$j_\nu(r) = \int_0^r k_\nu(s)\,ds, \quad k_\nu(r) = \alpha(\beta r^+ + p(r)), \quad F_b(r) = \mu\, k_\nu(r)$$

and $\alpha = 150$, $\beta = 0, 5$, $\mu = 1$. Note that this corresponds to a nonmonotone normal compliance associated with the classical Coulomb's law of dry friction with the friction coefficient μ. The normal stress is not aligned along the normal direction due to the frictional contact.

Numerical solution errors We report relative numerical solution errors in the energy norm in Table 2 and Fig. 9. Again, we use the numerical solution with $h = 1/256$ as the "reference" solution $\boldsymbol{u}_{\text{ref}}$ in computing the solution errors. This fine discretization for the "reference" solution $\boldsymbol{u}_{\text{ref}}$ corresponds to a problem with 132,098 degrees of freedom, 131,072 elements and is computed in 5700 CPU time (expressed in seconds) on an IBM computer equipped with Intel Dual core processors (Model 5148, 2.33

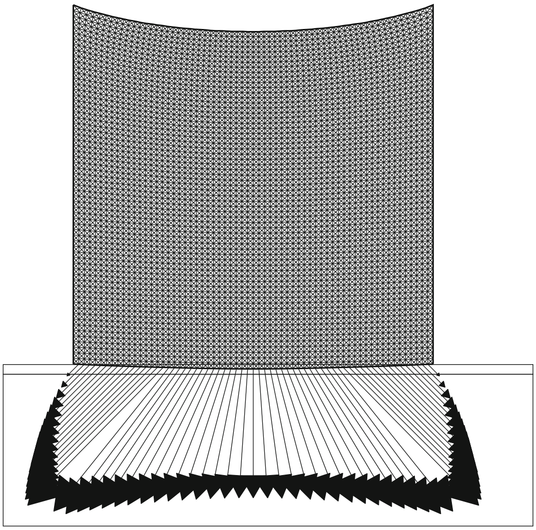

Fig. 8 Problem (P_1): deformed mesh and interface forces for a normal compliance case with friction for $\alpha = 150$ and $\beta = 0.5$

Table 2 Problem (P_2): relative errors in energy norm for a normal compliance case with friction

h	1/4	1/8	1/16	1/32	1/64
Relative error	0.212	0.104	0.0519	0.0261	0.0128

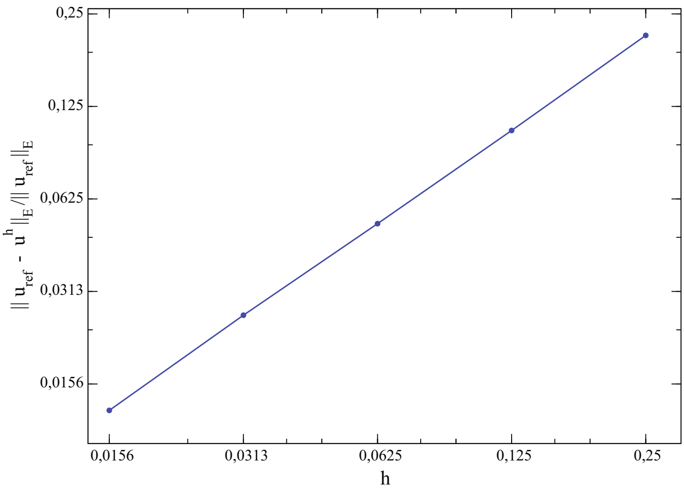

Fig. 9 Problem (P_2): relative errors in energy norm for frictional contact with normal compliance

GHz). The curve of the relative numerical solution errors is asymptotically linear, which is consistent with the theoretically predicted optimal linear convergence of the numerical solution established in Sect. 4.

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