Nonlinear Analysis: Real World Applications 36 (2017) 69–88

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Analysis of a general dynamic history-dependent variational–hemivariational inequality[☆]

Weimin Han[a], Stanisław Migórski[b,*], Mircea Sofonea[c]

[a] *Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA*

[b] *Faculty of Mathematics and Computer Science, Jagiellonian University in Krakow, Institute of Computer Science, ul. Lojasiewicza 6, 30348 Krakow, Poland*

[c] *Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France*

A R T I C L E I N F O

Article history:
Received 22 September 2016
Accepted 30 December 2016

Keywords:
Variational–hemivariational inequality
Clarke subdifferential
History-dependent operator
Existence and uniqueness
Frictional contact
Weak solution

A B S T R A C T

This paper is devoted to the study of a general dynamic variational–hemivariational inequality with history-dependent operators. These operators appear in a convex potential and in a locally Lipschitz superpotential. The existence and uniqueness of a solution to the inequality problem is explored through a result on a class of nonlinear evolutionary abstract inclusions involving a nonmonotone multivalued term described by the Clarke generalized gradient. The result presented in this paper is new and general. It can be applied to study various dynamic contact problems. As an illustrative example, we apply the theory on a dynamic frictional viscoelastic contact problem in which the contact is modeled by a nonmonotone Clarke subdifferential boundary condition and the friction is described by a version of the Coulomb law of dry friction with the friction bound depending on the total slip.

1. Introduction

Inequality problems in Mechanics can be divided into two main categories: that of variational inequalities, which is mainly concerned with convex functions, and that of hemivariational inequalities, which is concerned with nonconvex locally Lipschitz functions. Both variational and hemivariational inequalities are useful in a wide variety of applications, in Mechanics, Engineering, Economics, etc.

The notion of hemivariational inequality was first introduced and studied by P.D. Panagiotopoulos [1] in connection with contact problems with nonmonotone and possibly multivalued constitutive or interface

☆ This work is supported by the National Science Center of Poland under the Maestro Advanced Project no. DEC-2012/06/A/ST1/00262. The work of W.H. is also supported by NSF under grant DMS-1521684.

* Corresponding author.
E-mail address: migorski@ii.uj.edu.pl (S. Migórski).

http://dx.doi.org/10.1016/j.nonrwa.2016.12.007

laws for deformable bodies. The literature on the mathematical theory and applications of hemivariational inequalities is growing; some comprehensive references are [2–8].

Variational–hemivariational inequalities can be seen as an effective mathematical tool which combines the variational inequalities and the hemivariational inequalities and involves both convex and nonconvex energy functions. Recently, some papers are found studying various types of variational–hemivariational inequalities. For instance, an adhesive unilateral contact problem between a viscoelastic body and a deformable foundation is treated in [9], where a result is presented on unique solvability for a system consisting of a variational–hemivariational inequality and an ordinary differential equation. In [10,11], results are obtained on the existence and uniqueness of solutions for two different types of variational–hemivariational inequalities and applied these abstract results to study a static frictional contact problem and a quasistatic frictionless contact problem, respectively.

The goal of this paper is to study a general evolutionary variational–hemivariational inequality which involves history-dependent operators. The latter are operators which at any time moment depend on the history of the solution up to this time moment. Analysis of various classes of history-dependent quasistatic hemivariational inequalities and variational–hemivariational inequalities and their applications can be found in [9,12–14,6,11,15,16]. An evolutionary history-dependent variational–hemivariational inequality is studied in [17]. Different from [17], in this paper, we consider a general evolutionary variational–hemivariational inequality without a compact operator in the multivalued term and provide a new existence and uniqueness result. We allow now both convex potential and nonconvex superpotentials to depend on history-dependent operators.

Our abstract results find applications in the variational analysis of a variety of dynamic contact problems. To provide an illustrative example we consider a dynamic frictional contact problem for viscoelastic materials with long memory. In this problem the contact is described with a nonmonotone normal damped response condition with a damping coefficient depending on the normal displacement. The friction is modeled by a version of Coulomb's law of dry friction with the friction bound depending on the total slip. The weak formulation in terms of the unknown velocity of the contact problem leads to a variational–hemivariational inequality with three history-dependent operators to which our abstract results apply. Then we mention two different versions of the problem which can be treated by our abstract existence and uniqueness result.

The paper is structured as follows. In Section 2, we review some basic mathematical notation, definitions and results. In Section 3, we provide an existence and uniqueness result for an abstract subdifferential inclusion of first order considered in the framework of an evolution triple of spaces. Then, in Section 4, we show the unique solvability of a general abstract evolutionary history-dependent variational–hemivariational inequality. Finally, in Section 5, we introduce a new viscoelastic frictional contact problem, present its weak formulation, and apply the abstract result obtained in Section 4 to prove the existence of a unique weak solution to the contact problem.

2. Preliminaries

In this section we briefly review basic definitions on single-valued and multivalued operators in Banach spaces and on subdifferentials which are used later. We refer to [18–21] for more material on these topics.

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space. We denote by X^* its topological dual and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. Given a set $S \subset X$, we define $\|S\|_X = \sup\{\|s\|_X \mid s \in S\}$. A single-valued operator $A\colon X \to X^*$ is said to be hemicontinuous if the function $\lambda \mapsto \langle A(u+\lambda v), w\rangle_{X^*\times X}$ is continuous on $[0,1]$ for all $u, v, w \in X$. The operator A is demicontinuous if for all $w \in X$, the functional $u \mapsto \langle Au, w\rangle_{X^*\times X}$ is continuous, i.e., A is continuous as a mapping from X to $(w^*\text{–}X^*)$. It is monotone if $\langle Au - Av, u - v\rangle_{X^*\times X} \geq 0$ for all $u, v \in X$. It is known that for a monotone operator $A\colon X \to X^*$

with $D(A) = X$, the notions of demicontinuity and hemicontinuity coincide [20, Exercise I.9]. The operator $A\colon X \to X^*$ is said to be bounded if it maps bounded subsets of X into bounded subsets of X^*.

Let $0 < T < \infty$ and $A\colon (0,T) \times X \to X^*$. The Nemitsky (superposition) operator associated with A is the operator $\mathcal{A}\colon L^2(0,T;X) \to L^2(0,T;X^*)$ defined by $(\mathcal{A}v)(t) = A(t, v(t))$ for $v \in L^2(0,T;X)$ and a.e. $t \in (0,T)$.

A multivalued operator $A\colon X \to 2^{X^*}$ is called bounded if it maps bounded sets into bounded ones. It is called coercive if either its domain $D(A) = \{u \in X \mid Au \neq \emptyset\}$ is bounded or $D(A)$ is unbounded and

$$\lim_{\|u\|_X \to \infty,\, u \in D(A)} \frac{\inf\{\,\langle u^*, u\rangle_{X^* \times X} \mid u^* \in Au\,\}}{\|u\|_X} = +\infty.$$

We recall the notion of pseudomonotonicity of a multivalued operator.

Definition 1. Let $A\colon X \to 2^{X^*}$ be a multivalued operator and $L\colon D(L) \subset X \to X^*$ be a linear and maximal monotone operator. The operator A is pseudomonotone with respect to L or simply L-pseudomonotone if the following conditions hold:

(a) for all $u \in X$ the set Au is a nonempty, bounded, closed, and convex subset of X^*.
(b) A is upper semicontinuous (u.s.c.) from each finite dimensional subspace of X to X^* endowed with the weak topology.
(c) if $\{u_n\} \subset D(L), u_n \to u$ weakly in $X, Lu_n \to Lu$ weakly in $X^*, u_n^* \in Au_n$ is such that $u_n^* \to u^*$ weakly in X^* and $\limsup \langle u_n^*, u_n - u\rangle_{X^* \times X} \le 0$, then $u^* \in Au$ and $\langle u_n^*, u_n\rangle_{X^* \times X} \to \langle u^*, u\rangle_{X^* \times X}$.

We will need the following surjectivity result [20, Theorem 1.3.73].

Proposition 2. *Let X be a reflexive Banach space, let $L\colon D(L) \subset X \to X^*$ be a linear and maximal monotone operator. If $A\colon X \to 2^{X^*}$ is bounded, coercive, and L-pseudomonotone, then $L + A$ is surjective.*

The result is stated in [20, Theorem 1.3.73] under the hypothesis that X is strictly convex. However, by passing to an equivalent norm on X, we may always assume that X is a strictly convex Banach space.

Let E be a Banach space. A function $\varphi\colon E \to \mathbb{R} \cup \{+\infty\}$ is proper if it is not identically equal to $+\infty$, i.e., the effective domain $\operatorname{dom}\varphi = \{x \in E \mid \varphi(x) < +\infty\} \neq \emptyset$. It is lower semicontinuous (l.s.c.) if $x_n \to x$ in E implies $\varphi(x) \le \liminf \varphi(x_n)$. It is well known that a convex and l.s.c. function $\varphi\colon E \to \mathbb{R}$ is locally Lipschitz [19, Proposition 5.2.10].

The following notion of the subgradient of a convex function generalizes the classical notion of gradient. Let U be an open convex subset of E and $g\colon U \to \mathbb{R}$ be a convex function. An element $x^* \in E^*$ is called a subgradient of g at $x \in E$ if

$$g(v) \ge g(x) + \langle x^*, v - x\rangle_{E^* \times E} \quad \text{for all} \quad v \in U. \tag{2.1}$$

The set of all $x^* \in E^*$ satisfying (2.1) is called the (convex) subdifferential of g at x, and is denoted by $\partial g(x)$. Next, we recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $h\colon E \to \mathbb{R}$. The generalized directional derivative of h at $x \in E$ in the direction $v \in E$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The generalized gradient of h at $x \in E$, denoted by $\partial h(x)$, is a subset in the dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* \mid h^0(x; v) \ge \langle \zeta, v\rangle_{E^* \times E} \ \text{ for all } v \in E\}.$$

In what follows the generalized gradient of Clarke for a locally Lipschitz function and the subdifferential of a convex function will be denoted in the same way.

Finally, we recall a fixed point result (cf. [22, Lemma 7] or [23, Proposition 3.1]). For a Banach space E, the space $L^2(0,T;E)$ of vector-valued functions consists of all measurable functions $v\colon (0,T)\to E$ for which $\int_0^T \|v(t)\|_E^2 dt$ is finite.

Lemma 3. *Let E be a Banach space and $0<T<\infty$. Let $\Lambda\colon L^2(0,T;E)\to L^2(0,T;E)$ be an operator such that*

$$\|(\Lambda\eta_1)(t)-(\Lambda\eta_2)(t)\|_E^2\le c\int_0^t \|\eta_1(s)-\eta_2(s)\|_E^2\,ds$$

for all $\eta_1,\eta_2\in L^2(0,T;E)$, a.e. $t\in(0,T)$ with a constant $c>0$. Then Λ has a unique fixed point in $L^2(0,T;E)$, i.e., there exists a unique $\eta^*\in L^2(0,T;E)$ such that $\Lambda\eta^*=\eta^*$.

3. Subdifferential inclusion of first order

In this section we present an existence and uniqueness result for an abstract subdifferential inclusion of first order. We treat this inclusion within the framework of an evolution triple of spaces $V\subset H\subset V^*$, where V is a reflexive and separable Banach space, H is a separable Hilbert space, the embedding $V\subset H$ is continuous, and V is dense in H. Given $0<T<+\infty$, we introduce the spaces $\mathcal{V}=L^2(0,T;V)$ and $\mathcal{W}=\{w\in\mathcal{V}\mid w'\in\mathcal{V}^*\}$, where the time derivative $w'=\partial w/\partial t$ is understood in the sense of vector-valued distributions; the dual of $\mathcal{V}$ is $\mathcal{V}^*=L^2(0,T;V^*)$. It is known that the space $\mathcal{W}$ endowed with the graph norm $\|w\|_{\mathcal{W}}=\|w\|_{\mathcal{V}}+\|w'\|_{\mathcal{V}^*}$ is a separable and reflexive Banach space. Identifying $\mathcal{H}=L^2(0,T;H)$ with its dual, we obtain the continuous embeddings $\mathcal{W}\subset\mathcal{V}\subset\mathcal{H}\subset\mathcal{V}^*$. The embedding $\mathcal{W}\subset C(0,T;H)$ is continuous, $C(0,T;H)$ being the space of continuous functions on $[0,T]$ with values in H. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is denoted by

$$\langle w,v\rangle_{\mathcal{V}^*\times\mathcal{V}}=\int_0^T\langle w(t),v(t)\rangle_{V^*\times V}\,dt\quad\text{for } w\in\mathcal{V}^*,\ v\in\mathcal{V},$$

where $\langle\cdot,\cdot\rangle_{V^*\times V}$ stands for the duality brackets of the pair (V^*,V).

Let $A\colon(0,T)\times V\to V^*$ and $\psi\colon(0,T)\times V\to\mathbb{R}$. Assume ψ is locally Lipschitz in its second argument and we denote by $\partial\psi$ the Clarke generalized gradient of ψ with respect to its second argument. Given $f:(0,T)\to V^*$ and $w_0\in H$, we consider the following evolutionary inclusion.

Problem 4. Find $w\in\mathcal{W}$ such that

$$\begin{cases} w'(t)+A(t,w(t))+\partial\psi(t,w(t))\ni f(t) & \text{a.e. } t\in(0,T),\\ w(0)=w_0.\end{cases}$$

In the study of Problem 4 we introduce the following definition.

Definition 5. A function $w\in\mathcal{W}$ is called a solution of Problem 4 if there exists $w^*\in\mathcal{V}^*$ such that

$$\begin{cases} w'(t)+A(t,w(t))+w^*(t)=f(t) & \text{a.e. } t\in(0,T),\\ w^*(t)\in\partial\psi(t,w(t)) & \text{a.e. } t\in(0,T),\\ w(0)=w_0.\end{cases}$$

We need the following hypotheses on the data.

$\underline{H(A)}$: $A\colon (0,T)\times V\to V^*$ is such that

(1) $A(\cdot,v)$ is measurable on $(0,T)$ for all $v\in V$.
(2) $A(t,\cdot)$ is demicontinuous on V for a.e. $t\in(0,T)$.
(3) $\|A(t,v)\|_{V^*}\le a_0(t)+a_1\|v\|_V$ for all $v\in V$, a.e. $t\in(0,T)$ with $a_0\in L^2(0,T), a_0\ge 0$ and $a_1\ge 0$.
(4) $A(t,\cdot)$ is strongly monotone for a.e. $t\in(0,T)$, i.e., for a constant $m_A>0$,

$$\langle A(t,v_1)-A(t,v_2),v_1-v_2\rangle_{V^*\times V}\ge m_A\|v_1-v_2\|_V^2$$

for all $v_1,v_2\in V$, a.e. $t\in(0,T)$.

$\underline{H(\psi)}$: $\psi\colon (0,T)\times V\to\mathbb{R}$ is such that

(1) $\psi(\cdot,v)$ is measurable on $(0,T)$ for all $v\in V$.
(2) $\psi(t,\cdot)$ is locally Lipschitz on V for a.e. $t\in(0,T)$.
(3) $\|\partial\psi(t,v)\|_{V^*}\le c_0(t)+c_1\|v\|_V$ for all $v\in V$, a.e. $t\in(0,T)$ with $c_0\in L^2(0,T), c_0\ge 0, c_1\ge 0$.
(4) $\partial\psi(t,\cdot)$ is relaxed monotone for a.e. $t\in(0,T)$, i.e., for a constant $m_\psi\ge 0$,

$$\langle z_1-z_2,v_1-v_2\rangle_{V^*\times V}\ge -m_\psi\|v_1-v_2\|_V^2$$

for all $z_i\in\partial\psi(t,v_i), z_i\in V^*$, $v_i\in V, i=1,\,2$, a.e. $t\in(0,T)$.

$\underline{(H_1)}$: $f\in\mathcal{V}^*, w_0\in V$.
$\underline{(H_2)}$: $m_A>\max\{m_\psi, 2\sqrt{2}\,c_1\}$.
We have the following existence and uniqueness result.

Theorem 6. *Under hypotheses $H(A), H(\psi), (H_1)$ and (H_2), Problem 4 has a unique solution.*

Proof. We prove the existence in Step 1 and the uniqueness in Step 2.

Step 1. Let $\mathcal{A}\colon\mathcal{V}\to\mathcal{V}^*$ and $\mathcal{B}\colon\mathcal{V}\to 2^{\mathcal{V}^*}$ be the Nemitsky operators corresponding to the translations of $A(t,\cdot)$ and $\partial\psi(t,\cdot)$ by the element w_0:

$$(\mathcal{A}v)(t)=A(t,v(t)+w_0),$$
$$(\mathcal{B}v)(t)=\{\,v^*\in\mathcal{V}^*\mid v^*(t)\in\partial\psi(t,v(t)+w_0)\text{ for a.e. }t\in(0,T)\,\}$$

for $v\in\mathcal{V}$ and a.e. $t\in(0,T)$. We further introduce an operator $\mathcal{F}\colon\mathcal{V}\to 2^{\mathcal{V}^*}$ defined by

$$\mathcal{F}v=\mathcal{A}v+\mathcal{B}v\quad\text{ for } v\in\mathcal{V}.$$

Define an operator $L\colon D(L)\subset\mathcal{V}\to\mathcal{V}^*$ by

$$Lv=v'\quad\text{for } v\in D(L)$$

with its domain $D(L)=\{w\in\mathcal{W}\mid w(0)=0\}$. The operator L is linear and maximal monotone [21, Proposition 32.10]. With these operators, we consider the following inclusion:

$$\begin{cases} Lw+\mathcal{F}w\ni f,\\ w(0)=0.\end{cases}\tag{3.2}$$

Then, $w\in\mathcal{W}$ is a solution of Problem 4 if and only if $w-w_0\in\mathcal{W}$ satisfies (3.2).

Let us apply Proposition 2 to prove that problem (3.2) has a solution. For this purpose, we verify that $\mathcal{F}$ has the properties required in Proposition 2.

Claim 1. *$\mathcal{F}$ is a bounded operator.*

Let $v \in \mathcal{V}$ and $v^* \in \mathcal{F}v$. Then $v^* = \mathcal{A}v + w^*$ with $w^* \in \mathcal{B}v$. From hypotheses $H(A)(3)$, $H(\psi)(3)$ and the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ for $a, b \ge 0$, we have

$$\|\mathcal{A}v\|_{\mathcal{V}^*} \le \sqrt{2}\|a_0\|_{L^2(0,T)} + 2a_1\sqrt{T}\|w_0\|_V + 2a_1\|v\|_{\mathcal{V}}, \tag{3.3}$$

$$\|w^*\|_{\mathcal{V}^*} \le \sqrt{2}\|c_0\|_{L^2(0,T)} + 2c_1\sqrt{T}\|w_0\|_V + 2c_1\|v\|_{\mathcal{V}}. \tag{3.4}$$

Combining these inequalities, we immediately deduce that $\mathcal{F}$ is a bounded operator, being the sum of two bounded operators.

Claim 2. *$\mathcal{F}$ is coercive.*

First, by $H(A)(3)$ and (4), we have

$$\begin{aligned}\langle A(t,v), v\rangle_{V^*\times V} &= \langle A(t,v) - A(t,0), v\rangle_{V^*\times V} + \langle A(t,0), v\rangle_{V^*\times V} \\ &\ge m_A\|v\|_V^2 - a_0(t)\|v\|_V \end{aligned} \tag{3.5}$$

for all $v \in V$, a.e. $t \in (0,T)$. Let $v \in \mathcal{V}$ and $v^* \in \mathcal{F}v$. Then $v^* = \mathcal{A}v + w^*$ where $w^* \in \mathcal{B}v$. Using $H(A)(3)$, (3.5) and the inequality $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$ for $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned}\langle \mathcal{A}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T [\langle A(t, v(t)+w_0), v(t)+w_0\rangle_{V^*\times V} - \langle A(t, v(t)+w_0), w_0\rangle_{V^*\times V}]\,dt \\ &\ge \int_0^T \left[m_A\|v(t)+w_0\|_V^2 - a_0(t)\|v(t)+w_0\|_V - (a_0(t) + a_1\|v(t)+w_0\|_V)\,\|w_0\|_V\right] dt \\ &\ge \frac{m_A}{2}\|v\|_{\mathcal{V}}^2 - \left(\|a_0\|_{L^2(0,T)} + a_1\sqrt{T}\|w_0\|_V\right)\|v\|_{\mathcal{V}} \\ &\quad - m_A T\|w_0\|_V^2 - 2\sqrt{T}\|w_0\|_V\|a_0\|_{L^2(0,T)} - a_1 T\|w_0\|_V^2. \end{aligned} \tag{3.6}$$

Since $w^* \in \mathcal{B}v, w^*(t) \in \partial\psi(t, v(t)+w_0)$ for a.e. $t \in (0,T)$. Using $H(\psi)(3)$, we obtain $\|w^*(t)\|_{V^*}^2 \le 2\,c_1^2\,\|v(t)\|_V^2 + 2(c_0(t) + c_1\|w_0\|_V)^2$ for a.e. $t \in (0,T)$, and then

$$\|w^*\|_{\mathcal{V}^*} \le \sqrt{2}\,c_1\|v\|_{\mathcal{V}} + c_2$$

with $c_2 = \sqrt{2}\,(\int_0^T (c_0(t) + c_1\|w_0\|_V)\,dt)^{1/2} \ge 0$. Hence

$$\langle v^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mathcal{A}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle w^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \ge \frac{m_A}{2}\|v\|_{\mathcal{V}}^2 - \sqrt{2}c_1\|v\|_{\mathcal{V}}^2 - c_3\|v\|_{\mathcal{V}} - c_4$$

with $c_3, c_4 \ge 0$. Thus, it is clear from (H_2) that $\mathcal{F}$ is coercive.

Claim 3. *$\mathcal{F}$ is L-pseudomonotone.*

First, we show the following properties of the operator $\mathcal{A}$:

$$\mathcal{A}\colon \mathcal{V} \to \mathcal{V}^* \quad \text{is demicontinuous,} \tag{3.7}$$

$$\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2\rangle_{\mathcal{V}^*\times\mathcal{V}} \ge m_A\|v_1 - v_2\|_{\mathcal{V}}^2 \quad \text{for all} \quad v_1, v_2 \in \mathcal{V}. \tag{3.8}$$

For a proof of (3.7), let $v_n \to v$ in $\mathcal{V}$. Then, by passing to a subsequence if necessary, we have $v_n(t) \to v(t)$ in V for a.e. $t \in (0,T)$ and $\|v_n(t)\|_V \le h(t)$ for a.e. $t \in (0,T)$ with $h \in L^2(0,T)$. Exploiting $H(A)(2)$, we deduce

$$A(t, v_n(t)) \to A(t, v(t)) \quad \text{weakly in } V^*, \text{ a.e. } t \in (0,T).$$

Hence, $\langle A(t, v_n(t)), \varphi(t)\rangle_{V^*\times V} \to \langle A(t, v(t)), \varphi(t)\rangle_{V^*\times V}$ for all $\varphi \in \mathcal{V}$, a.e. $t \in (0,T)$. Use $H(A)(3)$ and apply the Lebesgue dominated convergence theorem,

$$\lim \int_0^T \langle A(t, v_n(t)), \varphi(t)\rangle_{V^*\times V}\,dt = \int_0^T \langle A(t, v(t)), \varphi(t)\rangle_{V^*\times V}\,dt.$$

Thus $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$. The standard argument shows that the entire sequence $\{\mathcal{A}v_n\}$ converges weakly in $\mathcal{V}^*$ to $\mathcal{A}v$. This concludes the proof of (3.7).

The strong monotonicity property for $\mathcal{A}$ in (3.8) follows from hypothesis $H(A)(4)$.

We now prove that the operator $\mathcal{F}$ satisfies conditions (a)–(c) of Definition 1.

(a) For every $v \in \mathcal{V}$, the set $\mathcal{F}v$ is a nonempty, bounded, closed and convex in $\mathcal{V}^*$. The fact that values of the operator $\mathcal{F}$ are nonempty and convex follows from the well known property [18, Proposition 2.1.2] that values of $\partial\psi(t,\cdot)$ are nonempty and convex subsets of V^* for a.e. $t \in (0,T)$. From (3.3) and (3.4), it follows that $\|v^*\|_{\mathcal{V}^*} \le \bar{c}_0 + \bar{c}_1\|v\|_{\mathcal{V}}$ for all $v^* \in \mathcal{F}v = \mathcal{A}v + \mathcal{B}v, v \in \mathcal{V}$ with $\bar{c}_0, \bar{c}_1 \ge 0$. Hence, the set $\mathcal{F}v$ is bounded in $\mathcal{V}^*$ for all $v \in \mathcal{V}$. The set $\mathcal{B}v$ is also closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$. Indeed, let $v \in \mathcal{V}, v_n^* \in \mathcal{V}^*, v_n^* \in \mathcal{B}v, v_n^* \to v^*$ in $\mathcal{V}^*$. Passing to a subsequence if necessary, we may suppose that $v_n^*(t) \to v^*(t)$ in V^* for a.e. $t \in (0,T)$. We have

$$v_n^*(t) \in \partial\psi(t, v(t) + w_0) \quad \text{a.e. } t \in (0,T)$$

and the latter is a closed subset of V^*. Thus $v^*(t) \in \partial\psi(t, v(t) + w_0)$ for a.e. $t \in (0,T)$, i.e., $v^* \in \mathcal{B}v$, which proves the closedness of the set $\mathcal{B}v$. Hence, the set $\mathcal{F}v$ is closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$, which concludes the proof of (a).

(b) The operator $\mathcal{F}$ is u.s.c. from $\mathcal{V}$ into $2^{\mathcal{V}^*}$, where $\mathcal{V}^*$ is endowed with the weak topology. In order to show this property, we apply [19, Proposition 4.1.4]. To this end, we prove that the weak inverse image $\mathcal{F}^-(D) = \{v \in \mathcal{V} \mid \mathcal{F}v \cap D \ne \emptyset\}$ is a closed subset of $\mathcal{V}$, for every weakly closed set $D \subset \mathcal{V}^*$. Let $\{v_n\} \subset \mathcal{F}^-(D)$ be such that $v_n \to v$ in $\mathcal{V}$. We may assume, passing to a subsequence if necessary, that

$$v_n(t) \to v(t) \quad \text{in } V, \ \text{a.e. } t \in (0,T). \tag{3.9}$$

Therefore, there exists $v_n^* \in \mathcal{F}v_n \cap D$ for $n \in \mathbb{N}$, that is,

$$v_n^* = \mathcal{A}v_n + w_n^* \tag{3.10}$$

with $w_n^* \in \mathcal{B}v_n$ and $v_n^* \in D$. Since $\{v_n\}$ is bounded in $\mathcal{V}$ and the operators $\mathcal{A}$ and $\mathcal{B}$ are bounded (cf. Claim 1), we know that $\{v_n^*\}$ and $\{w_n^*\}$ are both bounded in $\mathcal{V}^*$. Thus, at least for subsequences, we may suppose that

$$v_n^* \to v^*, \qquad w_n^* \to w^* \quad \text{weakly in } \mathcal{V}^*$$

with $v^*, w^* \in \mathcal{V}^*$. Since D is weakly closed in $\mathcal{V}^*$, we have $v^* \in D$. By the definition of the operator $\mathcal{B}$, we have

$$w_n^*(t) \in \partial\psi(t, v_n(t) + w_0) \quad \text{a.e. } t \in (0,T). \tag{3.11}$$

Taking into account the convergences (3.9) and $w_n^* \to w^*$ weakly in $\mathcal{V}^*$, and the fact that $\partial\psi(t,\cdot)$ is u.s.c. with closed and convex values, we can apply a convergence theorem found in [24, p. 60] to the inclusion (3.11) and deduce $w^*(t) \in \partial\psi(t, v(t) + w_0)$ a.e. $t \in (0,T)$. Hence, $w^* \in \mathcal{B}v$.

By the demicontinuity of the operator $\mathcal{A}$ (cf. (3.7)), we have $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$. Passing to the limit in (3.10), we obtain $v^* = \mathcal{A}v + w^*$, where $w^* \in \mathcal{B}v$ and $v^* \in D$. Therefore, $v^* \in \mathcal{F}v \cap D$, implying $v^* \in \mathcal{F}^-(D)$. This proves that $\mathcal{F}^-(D)$ is closed in $\mathcal{V}$ and concludes the proof of condition (b).

(c) The condition (c) of Definition 1 holds. Let $\{v_n\} \subset D(L), v_n \to v$ weakly in $\mathcal{W}, Lv_n \to Lv$ weakly in $\mathcal{W}^*, v_n^* \in \mathcal{F}v_n, v_n^* \to v^*$ weakly in $\mathcal{V}^*$ and

$$\limsup \langle v_n^*, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} \le 0. \tag{3.12}$$

We prove that $v^* \in \mathcal{F}v$ and

$$\langle v_n^*, v_n\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle v^*, v\rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{3.13}$$

First, we observe that $\mathcal{F}\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is strongly monotone. Indeed, by $H(\psi)(4)$,

$$\begin{aligned}\langle w_1^* - w_2^*, v_1 - v_2\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \langle w_1^*(t) - w_2^*(t), v_1(t) - v_2(t)\rangle_{V^*\times V} dt \\ &\geq -m_\psi \int_0^T \|v_1(t) - v_2(t)\|_V^2 dt \\ &= -m_\psi \|v_1 - v_2\|_{\mathcal{V}}^2\end{aligned}$$

for all $w_i^* \in \mathcal{B}v_i, v_i \in \mathcal{V}, i = 1, 2$. This inequality and (3.8) together imply

$$\langle v_1^* - v_2^*, v_1 - v_2\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq (m_A - m_\psi)\, \|v_1 - v_2\|_{\mathcal{V}}^2$$

for all $v_i^* \in \mathcal{F}v_i, v_i \in \mathcal{V}, i = 1, 2$. From (H_2), it follows that the operator $\mathcal{F}$ is strongly monotone.

Next, we prove that $v_n \to v$ in $\mathcal{V}$. From the strong monotonicity of $\mathcal{F}$, we have

$$(m_A - m_\psi)\, \|v_n - v\|_{\mathcal{V}}^2 \leq \langle v_n^* - \eta, v_n - v\rangle_{\mathcal{V}^*\times\mathcal{V}}$$

for all $v_n^* \in \mathcal{F}v_n, \eta \in \mathcal{F}v$. Taking lim sup in the last inequality and using (3.12), we obtain

$$\begin{aligned}0 &\leq (m_A - m_\psi) \liminf \|v_n - v\|_{\mathcal{V}}^2 \leq (m_A - m_\psi) \limsup \|v_n - v\|_{\mathcal{V}}^2 \\ &\leq \limsup \langle v_n^*, v_n - v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \lim\langle \eta, v_n - v\rangle_{\mathcal{V}^*\times\mathcal{V}} \leq 0,\end{aligned}$$

implying $v_n \to v$ in $\mathcal{V}$.

Using the convergence of v_n to v in $\mathcal{V}$, and passing to a subsequence if necessary, we may assume

$$v_n(t) \to v(t) \quad \text{in } V, \text{ a.e. } t \in (0, T). \tag{3.14}$$

Consequently, from $v_n^* \in \mathcal{F}v_n$, we have

$$v_n^* = \mathcal{A}v_n + w_n^* \tag{3.15}$$

with $w_n^* \in \mathcal{B}v_n$, and thus

$$w_n^*(t) \in \partial\psi(t, v_n(t) + w_0) \quad \text{a.e. } t \in (0, T).$$

By the boundedness of the operator $\mathcal{B}$ (cf. Claim 1), we can assume that $w_n^* \to w^*$ weakly in $\mathcal{V}^*$ with $w^* \in \mathcal{V}^*$. Similarly as in the proof of condition (b), we use the convergences (3.14) and $w_n^* \to w^*$ weakly in $\mathcal{V}^*$, and apply the convergence theorem of [24, p. 60] to obtain

$$w^*(t) \in \partial\psi(t, v(t) + w_0) \quad \text{a.e. } t \in (0, T).$$

Hence, $w^* \in \mathcal{B}v$. By the demicontinuity of the operator $\mathcal{A}$ (cf. (3.7)), we obtain $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$. Passing to the limit in (3.15), we get $v^* = \mathcal{A}v + w^*$. Since $w^* \in \mathcal{B}v$, we have $v^* \in \mathcal{F}v$. From $v_n^* \to v^*$ weakly in $\mathcal{V}^*$ and $v_n \to v$ in $\mathcal{V}$, we deduce (3.13), which concludes the proof of condition (c).

Having established Claims 1–3 and noting that the operator L is linear and maximal monotone, we apply Proposition 2 to deduce that the problem (3.2) has at least one solution $w \in D(L)$. Then, $w + w_0 \in \mathcal{W}$ is a solution of Problem 4. This concludes the proof of the existence part of the theorem.

Step 2. We prove the uniqueness part of a solution to Problem 4. Assume $w_1, w_2 \in \mathcal{W}$ are two solutions. Then, there are $w_i^* \in \mathcal{V}^*$ such that

$$\begin{cases} w_i'(s) + A(t, w_i(s)) + w_i^*(s) = f(s) \quad \text{a.e. } s \in (0, T) \\ w_i^*(s) \in \partial\psi(s, w_i(s)) \quad \text{a.e. } s \in (0, T) \\ w_i(0) = w_0 \end{cases}$$

for $i = 1, 2$. We subtract the two equations for w_1 and w_2, take the result in duality with $w_1(t) - w_2(t)$, integrate from 0 to t, and note that $w_1(0) - w_2(0) = 0$ to obtain

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_H^2 + \int_0^t \langle A(s, w_1(s)) - A(s, w_2(s)), w_1(s) - w_2(s)\rangle_{V^*\times V}\, ds + \int_0^t \langle w_1^*(s) - w_2^*(s), w_1(s) - w_2(s)\rangle_{V^*\times V}\, ds = 0$$

for all $t \in [0, T]$. From hypotheses $H(A)(4)$ and $H(\psi)(4)$, we obtain

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_H^2 + (m_A - m_\psi)\int_0^t \|w_1(s) - w_2(s)\|_V^2\, ds \le 0$$

for all $t \in [0, T]$. Hence, by the smallness condition in (H_2), it follows that $w_1 = w_2$ on $[0, T]$, i.e., a solution to Problem 4 is unique. □

4. Inequality problem with history-dependent operators

In this section we study the first order variational–hemivariational inequality with history-dependent operators. The main feature of this problem is that history-dependent operators appear at several places of the inequality including a locally Lipschitz, generally nonconvex, superpotential and a convex potential. Let Y and Z be two Banach spaces. The inequality under consideration reads as follows.

Problem 7. Find $w \in \mathcal{W}$ such that

$$\begin{cases} \langle w'(t) + A(t, w(t)) + (\mathcal{R}_1 w)(t) - f(t), v - w(t)\rangle_{V^*\times V} + J^0(t, (\mathcal{S}w)(t), w(t); v - w(t)) \\ \quad + \varphi(t, (\mathcal{R}w)(t), v) - \varphi(t, (\mathcal{R}w)(t), w(t)) \ge 0 \quad \text{for all } v \in V, \text{ a.e. } t \in (0, T) \\ w(0) = w_0. \end{cases}$$

Throughout the paper, for the functional $J(t, z, v)$ defined on $(0, T) \times Z \times V$, J^0 and ∂J refer to the generalized directional derivative and the generalized gradient of Clarke with respect to the last argument v.

In the study of Problem 7 we need the following hypotheses on the data.

$\underline{H(J)}$: $J\colon (0, T) \times Z \times V \to \mathbb{R}$ is such that

(1) $J(\cdot, z, v)$ is measurable on $(0, T)$ for all $z \in Z, v \in V$.
(2) $J(t, \cdot, v)$ is continuous on Z for all $v \in V$, a.e. $t \in (0, T)$.
(3) $J(t, z, \cdot)$ is locally Lipschitz on V for all $z \in Z$, a.e. $t \in (0, T)$.
(4) $\|\partial J(t, z, v)\|_{V^*} \le c_{0J}(t) + c_{1J}\|z\|_Z + c_{2J}\|v\|_V$ for all $z \in Z, v \in V$, a.e. $t \in (0, T)$ with $c_{0J} \in L^2(0, T), c_{0J}, c_{1J}, c_{2J} \ge 0$.
(5) $J^0(t, z_1, v_1; v_2 - v_1) + J^0(t, z_2, v_2; v_1 - v_2) \le \overline{m}_{2J}\|z_1 - z_2\|_Z\|v_1 - v_2\|_V + m_{2J}\|v_1 - v_2\|_V^2$ for all $z_i \in Z, v_i \in V, i = 1, 2$, a.e. $t \in (0, T)$ with $\overline{m}_{2J} \ge 0, m_{2J} \ge 0$.

$\underline{H(\varphi)}$: $\varphi\colon (0, T) \times Y \times V \to \mathbb{R}$ is such that

(1) $\varphi(\cdot, y, v)$ is measurable on $(0, T)$ for all $y \in Y, v \in V$.
(2) $\varphi(t, \cdot, v)$ is continuous on Y for all $v \in V$, a.e. $t \in (0, T)$.
(3) $\varphi(t, y, \cdot)$ is convex and l.s.c. on V for all $y \in Y$, a.e. $t \in (0, T)$.
(4) $\|\partial\varphi(t, y, v)\|_{V^*} \le c_{0\varphi}(t) + c_{1\varphi}\|y\|_Y + c_{2\varphi}\|v\|_V$ for all $y \in Y, v \in V$, a.e. $t \in (0, T)$ with $c_{0\varphi} \in L^2(0, T), c_{0\varphi}, c_{1\varphi}, c_{2\varphi} \ge 0$.
(5) $\varphi(t, y_1, v_2) - \varphi(t, y_1, v_1) + \varphi(t, y_2, v_1) - \varphi(t, y_2, v_2) \le \beta_\varphi\|y_1 - y_2\|_Y\|v_1 - v_2\|_V$ for all $y_i \in Y, v_i \in V, i = 1, 2$, a.e. $t \in (0, T)$ with $\beta_\varphi \ge 0$.

$\underline{(H_4)}$: $\mathcal{R}_1\colon \mathcal{V} \to \mathcal{V}^*, \mathcal{R}\colon \mathcal{V} \to L^2(0,T;Y)$ and $\mathcal{S}\colon \mathcal{V} \to L^2(0,T;Z)$ are such that

(1) $\|(\mathcal{R}_1 v_1)(t) - (\mathcal{R}_1 v_2)(t)\|_{V^*} \le c_{R_1} \int_0^t \|v_1(s) - v_2(s)\|_V ds$ for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0,T)$ with $c_{R_1} > 0$.
(2) $\|(\mathcal{R} v_1)(t) - (\mathcal{R} v_2)(t)\|_Y \le c_R \int_0^t \|v_1(s) - v_2(s)\|_V ds$ for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0,T)$ with $c_R > 0$.
(3) $\|(\mathcal{S} v_1)(t) - (\mathcal{S} v_2)(t)\|_Z \le c_S \int_0^t \|v_1(s) - v_2(s)\|_V ds$ for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0,T)$ with $c_S > 0$.

$\underline{(H_5)}$: $m_A > \max\{m_{2J}, c_{2J}, c_{2\varphi}\}$.

Remark 8. It can be shown that the hypothesis $H(J)(5)$ is equivalent to the following condition

$$\langle v_1^* - v_2^*, v_1 - v_2\rangle_{V^*\times V} \ge -\overline{m}_{2J}\|z_1 - z_2\|_Z \|v_1 - v_2\|_V - m_{2J}\|v_1 - v_2\|_V^2 \tag{4.16}$$

for all $v_i^* \in \partial J(t, z_i, v_i), z_i \in Z, v_i \in V, i = 1,2$, a.e. $t \in (0,T)$. In particular, if J is independent of the variable z, the hypothesis $H(J)(5)$ (or, equivalently, (4.16)) reduces to the following relaxed monotonicity condition

$$\langle v_1^* - v_2^*, v_1 - v_2\rangle_{V^*\times V} \ge -m_{2J}\|v_1 - v_2\|_V^2 \tag{4.17}$$

for all $v_i^* \in \partial J(t, v_i), v_i \in V, i = 1,2$, a.e. $t \in (0,T)$ with $m_{2J} \ge 0$. The latter has been recently used in the literature to prove the uniqueness of the solution to the variational–hemivariational inequality. We refer to [6] for examples of nonconvex functions which satisfy the condition (4.17). Furthermore, we note that when $J(t,\cdot)$ is convex, then (4.17) holds with $m_{2J} = 0$, i.e., the condition (4.17) simplifies to the monotonicity of the (convex) subdifferential.

We have the following existence and uniqueness result.

Theorem 9. *Under hypotheses $H(A), H(J), H(\varphi), (H_1), (H_4)$ and (H_5), Problem 7 has a unique solution.*

Proof. The proof is carried out in four steps and it is based on Theorem 6 combined with a fixed-point argument.

Step 1. Let $\xi \in L^2(0,T;V^*), \eta \in L^2(0,T;Y)$ and $\zeta \in L^2(0,T;Z)$ and consider the following auxiliary problem. Find $w_{\xi\eta\zeta} \in \mathcal{W}$ such that

$$\begin{cases} \langle w'_{\xi\eta\zeta}(t) + A(t, w_{\xi\eta\zeta}(t)) - f(t) + \xi(t), v - w_{\xi\eta\zeta}(t)\rangle_{V^*\times V} + J^0(t, \zeta(t), w_{\xi\eta\zeta}(t); v - w_{\xi\eta\zeta}(t)) \\ \quad + \varphi(t, \eta(t), v) - \varphi(t, \eta(t), w_{\xi\eta\zeta}(t)) \ge 0 \quad \text{for all } v \in V, \text{ a.e. } t \in (0,T), \\ w_{\xi\eta\zeta}(0) = w_0. \end{cases} \tag{4.18}$$

To study the inequality (4.18), we define a functional $\psi_{\xi\eta\zeta}\colon (0,T) \times V \to \mathbb{R}$ by

$$\psi_{\xi\eta\zeta}(t, v) = \langle \xi(t), v\rangle_{V^*\times V} + \varphi(t, \eta(t), v) + J(t, \zeta(t), v)$$

for all $v \in V$, a.e. $t \in (0,T)$, and consider the following problem: Find $w_{\xi\eta\zeta} \in \mathcal{W}$ such that

$$\begin{cases} w'_{\xi\eta\zeta}(t) + A(t, w_{\xi\eta\zeta}(t)) + \partial\psi_{\xi\eta\zeta}(t, w_{\xi\eta\zeta}(t)) \ni f(t) \quad \text{a.e. } t \in (0,T) \\ w_{\xi\eta\zeta}(0) = w_0. \end{cases} \tag{4.19}$$

We apply Theorem 6 to prove the unique solvability of the problem (4.19). Let us check the hypotheses of Theorem 6.

From hypotheses $H(J)(1)$, (2) and $H(\varphi)(1)$, (2), we know that $J(\cdot,\cdot,v)$ and $\varphi(\cdot,\cdot,v)$ are Carathéodory functions for all $v \in V$. Thus, since $t \mapsto \xi(t), t \mapsto \zeta(t)$ and $t \mapsto \eta(t)$ are measurable on $(0,T)$, we infer that the functional $\psi_{\xi\eta\zeta}(\cdot, v)$ is also measurable on $(0,T)$ for all $v \in V$, i.e., $H(\psi)(1)$ holds. Since $\varphi(t, y, \cdot)$ is convex and l.s.c. for all $y \in Y$, a.e. $t \in (0,T)$, we find that $\varphi(t, y, \cdot)$ is locally Lipschitz [19, Proposition 5.2.10]. Thanks to condition $H(J)(3)$, we conclude that the functional $\psi_{\xi\eta\zeta}(t,\cdot)$ is locally Lipschitz on V for a.e. $t \in (0,T)$, i.e., $H(\psi)(2)$ is satisfied.

From the fact that $J(t,z,\cdot)$ and $\varphi(t,y,\cdot)$ are locally Lipschitz for all $z \in Z$ and $y \in Y$, a.e. $t \in (0,T)$, we have [19, Proposition 5.6.23]

$$\partial\psi_{\xi\eta\zeta}(t,v) \subset \xi(t) + \partial\varphi(t,\eta(t),v) + \partial J(t,\zeta(t),v) \tag{4.20}$$

for all $v \in V$ and a.e. $t \in (0,T)$. Hence

$$\begin{aligned}\|\partial\psi_{\xi\eta\zeta}(t,v)\|_{V^*} &\le \|\xi(t)\|_{V^*} + \|\partial J(t,\zeta(t),v)\|_{V^*} + \|\partial\varphi(t,\eta(t),v)\|_{V^*}\\ &\le \|\xi(t)\|_{V^*} + c_{0J}(t) + c_{1J}\|\zeta(t)\|_Z + c_{2J}\|v\|_V + c_{0\varphi}(t) + c_{1\varphi}\|\eta(t)\|_Y + c_{2\varphi}\|v\|_V\\ &= c_0(t) + c_1\|v\|_V\end{aligned}$$

for all $v \in V$, a.e. $t \in (0,T)$, where $c_0 \in L^2(0,T), c_0 \ge 0$ and $c_1 = \max\{c_{2J}, c_{2\varphi}\} \ge 0$. So, condition $H(\psi)(3)$ is satisfied.

Taking into account hypothesis $H(\varphi)(3)$, we know that $\partial\varphi(t,y,\cdot)$ is maximal monotone for all $y \in Y$, a.e. $t \in (0,T)$ [19, Theorem 6.3.19]. Using the monotonicity of $\partial\varphi(t,y,\cdot)$ and condition $H(J)(5)$, we obtain

$$\begin{aligned}&\langle\partial\psi_{\xi\eta\zeta}(t,v_1) - \partial\psi_{\xi\eta\zeta}(t,v_2), v_1 - v_2\rangle_{V^*\times V}\\ &\quad= \langle\partial J(t,\zeta(t),v_1) - \partial J(t,\zeta(t),v_2), v_1 - v_2\rangle_{V^*\times V} + \langle\partial\varphi(t,\eta(t),v_1) - \partial\varphi(t,\eta(t),v_2), v_1 - v_2\rangle_{V^*\times V}\\ &\quad\ge -m_{2J}\,\|v_1 - v_2\|_V^2\end{aligned}$$

for all $v_1, v_2 \in V$, a.e. $t \in (0,T)$. Therefore, condition $H(\psi)(4)$ holds with $m_\psi = m_{2J}$, which completes the proof of $H(\psi)$. In addition, since $m_\psi = m_{2J}$, it is clear that (H_5) implies (H_2).

Applying Theorem 6, we deduce that the problem (4.19) has a unique solution $w_{\xi\eta\zeta} \in \mathcal{W}$. By the relation (4.20), it is obvious that $w_{\xi\eta\zeta} \in \mathcal{W}$ is also a solution to the following inclusion: Find $w_{\xi\eta\zeta} \in \mathcal{W}$ such that

$$\begin{cases} w'_{\xi\eta\zeta}(t) + A(t, w_{\xi\eta\zeta}(t)) + \partial J(t,\zeta(t), w_{\xi\eta\zeta}(t))\\ \quad +\partial\varphi(t,\eta(t), w_{\xi\eta\zeta}(t)) + \xi(t) \ni f(t) \quad \text{a.e. } t \in (0,T),\\ w_{\xi\eta\zeta}(0) = w_0. \end{cases} \tag{4.21}$$

Furthermore, it is clear from the definitions of the Clarke and the convex subdifferentials that every solution to problem (4.21) is also a solution to the problem (4.18). This completes the proof that the problem (4.18) has a solution $w_{\xi\eta\zeta} \in \mathcal{W}$.

Step 2. We show that a solution to the problem (4.18) is unique. Let $w_1, w_2 \in \mathcal{W}$ be solutions to the problem (4.18) and, for simplicity, we skip the subscripts ξ, η and ζ for this part of the proof. Take $w_2(t)$ as the test function in the inequality for w_1, take $w_1(t)$ as the test function in the inequality for w_2, and add the two resulting inequalities to get

$$\begin{aligned}&\langle w_1'(t) - w_2'(t), w_1(t) - w_2(t)\rangle_{V^*\times V} + \langle A(t,w_1(t)) - A(t,w_2(t)), w_1(t) - w_2(t)\rangle_{V^*\times V}\\ &\quad\le J^0(t,\zeta(t),w_1(t); w_2(t) - w_1(t)) + J^0(t,\zeta(t),w_2(t); w_1(t) - w_2(t))\end{aligned}$$

for a.e. $t \in (0,T)$. Integrating the above inequality over the time interval $(0,t)$, noting $w_1(0) - w_2(0) = 0$, and using conditions $H(A)(4)$ and $H(J)(5)$, we have

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_H^2 + m_A\int_0^t \|w_1(s) - w_2(s)\|_V^2 ds \le m_{2J}\int_0^t \|w_1(s) - w_2(s)\|_V^2 ds$$

for all $t \in [0,T]$. Hence, due to assumption (H_5), we obtain

$$\|w_1(t) - w_2(t)\|_H^2 = 0 \quad \text{for all } t \in [0,T].$$

Thus, $w_1 = w_2$, i.e., a solution to the problem (4.18) is unique.

Step 3. Define an operator $\Lambda\colon L^2(0,T;V^*\times Y\times Z)\to L^2(0,T;V^*\times Y\times Z)$ by

$$\Lambda(\xi,\eta,\zeta)=(\mathcal{R}_1 w_{\xi\eta\zeta},\mathcal{R}w_{\xi\eta\zeta},\mathcal{S}w_{\xi\eta\zeta})\quad \text{for all}\ \ (\xi,\eta,\zeta)\in L^2(0,T;V^*\times Y\times Z),$$

where $w_{\xi\eta\zeta}\in\mathcal{W}$ denotes the unique solution to the problem (4.18) corresponding to (ξ,η,ζ).

We apply Lemma 3 to show that the operator Λ has a unique fixed point. Let $(\xi_i,\eta_i,\zeta_i)\in L^2(0,T;V^*\times Y\times Z), i=1,2$ and $w_1=w_{\xi_1\eta_1\zeta_1}, w_2=w_{\xi_2\eta_2\zeta_2}$ be the unique solutions to the problem (4.18) corresponding to (ξ_1,η_1,ζ_1) and (ξ_2,η_2,ζ_2), respectively. We have

$$\begin{aligned}&\langle w_1'(t)+A(t,w_1(t))-f(t)+\xi_1(t),w_2(t)-w_1(t)\rangle_{V^*\times V}\\ &\quad+J^0(t,\zeta_1(t),w_1(t);w_2(t)-w_1(t))+\varphi(t,\eta_1(t),w_2(t))-\varphi(t,\eta_1(t),w_1(t))\ge 0\end{aligned}$$

for a.e. $t\in(0,T)$ and

$$\begin{aligned}&\langle w_2'(t)+A(t,w_2(t))-f(t)+\xi_2(t),w_1(t)-w_2(t)\rangle_{V^*\times V}\\ &\quad+J^0(t,\zeta_2(t),w_2(t);w_1(t)-w_2(t))+\varphi(t,\eta_2(t),w_1(t))-\varphi(t,\eta_2(t),w_2(t))\ge 0\end{aligned}$$

for a.e. $t\in(0,T)$, and $w_1(0)=w_2(0)=w_0$. Adding these two inequalities, we obtain

$$\begin{aligned}&\langle w_1'(t)-w_2'(t),w_2(t)-w_1(t)\rangle_{V^*\times V}+\langle A(t,w_1(t))-A(t,w_2(t)),w_2(t)-w_1(t)\rangle_{V^*\times V}\\ &\quad+J^0(t,\zeta_1(t),w_1(t);w_2(t)-w_1(t))+J^0(t,\zeta_2(t),w_2(t);w_1(t)-w_2(t))\\ &\quad+\varphi(t,\eta_1(t),w_2(t))-\varphi(t,\eta_1(t),w_1(t))+\varphi(t,\eta_2(t),w_1(t))-\varphi(t,\eta_2(t),w_2(t))\\ &\ge\langle\xi_1(t),w_2(t)-w_1(t)\rangle_{V^*\times V}-\langle\xi_2(t),w_1(t)-w_2(t)\rangle_{V^*\times V}\end{aligned}$$

for a.e. $t\in(0,T)$. We integrate the above inequality on $(0,t)$, and use conditions $H(A)(4)$, $H(J)(5)$ and $H(\varphi)(5)$ to get

$$\begin{aligned}&\frac12\|w_1(t)-w_2(t)\|_H^2-\frac12\|w_1(0)-w_2(0)\|_H^2+m_A\int_0^t\|w_1(s)-w_2(s)\|_V^2ds\\ &\le\overline{m}_{2J}\int_0^t\|\zeta_1(s)-\zeta_2(s)\|_Z\|w_1(s)-w_2(s)\|_V^2ds+m_{2J}\int_0^t\|w_1(s)-w_2(s)\|_V^2ds\\ &\quad+\beta_\varphi\int_0^t\|\eta_1(s)-\eta_2(s)\|_Y\,\|w_1(s)-w_2(s)\|_Vds+\int_0^t\|\xi_1(s)-\xi_2(s)\|_{V^*}\|w_1(s)-w_2(s)\|_Vds\end{aligned}$$

for all $t\in[0,T]$. Next, using hypothesis (H_5) and the Hölder inequality, we have

$$\begin{aligned}(m_A-m_{2J})\,\|w_1-w_2\|^2_{L^2(0,t;V)}&\le\overline{m}_{2J}\|\zeta_1-\zeta_2\|_{L^2(0,t;Z)}\|w_1-w_2\|_{L^2(0,t;V)}\\ &\quad+\beta_\varphi\|\eta_1-\eta_2\|_{L^2(0,t;Y)}\|w_1-w_2\|_{L^2(0,t;V)}\\ &\quad+\|\xi_1-\xi_2\|_{L^2(0,t;V^*)}\|w_1-w_2\|_{L^2(0,t;V)}\end{aligned}$$

for all $t\in[0,T]$. Hence

$$\|w_1-w_2\|_{L^2(0,t;V)}\le c\left(\|\zeta_1-\zeta_2\|_{L^2(0,t;Z)}+\|\eta_1-\eta_2\|_{L^2(0,t;Y)}+\|\xi_1-\xi_2\|_{L^2(0,t;V^*)}\right)\tag{4.22}$$

for all $t\in[0,T]$, where c is a positive constant. By the definition of operator Λ, hypothesis (H_4), condition (4.22) and the Jensen inequality, we can verify that

$$\begin{aligned}&\|\Lambda(\xi_1,\eta_1,\zeta_1)(t)-\Lambda(\xi_2,\eta_2,\zeta_2)(t)\|^2_{V^*\times Y\times Z}\\ &\quad=\|(\mathcal{R}_1w_1)(t)-(\mathcal{R}_1w_2)(t)\|^2_{V^*}+\|(\mathcal{R}w_1)(t)-(\mathcal{R}w_2)(t)\|^2_Y+\|(\mathcal{S}w_1)(t)-(\mathcal{S}w_2)(t)\|^2_Z\\ &\quad\le\left(c_{R_1}\int_0^t\|w_1(s)-w_2(s)\|_V\,ds\right)^2+\left(c_R\int_0^t\|w_1(s)-w_2(s)\|_V\,ds\right)^2+\left(c_S\int_0^t\|w_1(s)-w_2(s)\|_V\,ds\right)^2\\ &\quad\le c\,\|w_1-w_2\|^2_{L^2(0,t;V)}\\ &\quad\le c\left(\|\zeta_1-\zeta_2\|^2_{L^2(0,t;Z)}+\|\eta_1-\eta_2\|^2_{L^2(0,t;Y)}+\|\xi_1-\xi_2\|^2_{L^2(0,t;V^*)}\right),\end{aligned}$$

i.e.,

$$\|\varLambda(\xi_1,\eta_1,\zeta_1)(t)-\varLambda(\xi_2,\eta_2,\zeta_2)(t)\|^2_{V^*\times Y\times Z}\le c\int_0^t\|(\xi_1,\eta_1,\zeta_1)(s)-(\xi_2,\eta_2,\zeta_2)(s)\|^2_{V^*\times Y\times Z}\,ds \tag{4.23}$$

for a.e. $t\in(0,T)$. By Lemma 3, we deduce that there exists a unique fixed point (ξ^*,η^*,ζ^*) of $\varLambda$, i.e.,

$$(\xi^*,\eta^*,\zeta^*)\in L^2(0,T;V^*\times Y\times Z)\quad\text{and}\quad \varLambda(\xi^*,\eta^*,\zeta^*)=(\xi^*,\eta^*,\zeta^*).$$

Step 4. Let $(\xi^*,\eta^*,\zeta^*)\in L^2(0,T;V^*\times Y\times Z)$ be the unique fixed point of the operator $\varLambda$. Let $w_{\xi^*\eta^*\zeta^*}\in\mathcal{W}$ be the unique solution to the problem (4.18) corresponding to (ξ^*,η^*,ζ^*). From the definition of the operator $\varLambda$, we have

$$\xi^*=\mathcal{R}_1(w_{\xi^*\eta^*\zeta^*}),\qquad \eta^*=\mathcal{R}(w_{\xi^*\eta^*\zeta^*})\quad\text{and}\quad \zeta^*=\mathcal{S}(w_{\xi^*\eta^*\zeta^*}).$$

Using these relations in (4.18), we know that $w_{\xi^*\eta^*\zeta^*}$ is the unique solution of Problem 7. This completes the proof of the theorem. □

5. A frictional contact problem

Various dynamic contact problems with elastic, viscoelastic or viscoplastic materials lead to evolutionary problems of the form given in Problem 7 in which the unknown is the velocity field. For such inequalities, Theorem 9 may be applied. We illustrate this point here on a dynamic viscoelastic contact problem. For this purpose, we need some additional notation.

Given $d\in\mathbb{N}$, we use the symbol $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{aligned}&\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i, &&\|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}\quad\text{for all } \boldsymbol{u}=(u_i),\ \boldsymbol{v}=(v_i)\in\mathbb{R}^d,\\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}, &&\|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}\quad\text{for all } \boldsymbol{\sigma}=(\sigma_{ij}),\ \boldsymbol{\tau}=(\tau_{ij})\in\mathbb{S}^d,\end{aligned}$$

respectively. Everywhere below $\varOmega$ will represent a regular domain of $\mathbb{R}^d$ $(d=2,\ 3)$ with a boundary $\partial\varOmega$ partitioned into three disjoint measurable parts $\varGamma_1,\varGamma_2$ and $\varGamma_3$, such that the measure of $\varGamma_1$, denoted $m(\varGamma_1)$, is positive. We use the notation $\boldsymbol{x}=(x_i)$ for a typical point in $\varOmega\cup\partial\varOmega$ and we denote by $\boldsymbol{\nu}=(\nu_i)$ the outward unit normal on $\partial\varOmega$. Here and below, the indices i,j,k,l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. We denote by $\boldsymbol{u}=(u_i),\boldsymbol{\sigma}=(\sigma_{ij})$, and $\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u}))$ the displacement vector, the stress tensor, and linearized strain tensor, respectively. Sometimes we do not indicate explicitly the dependence of the variables on the spatial variable $\boldsymbol{x}$. Recall that the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are

$$\varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}\,(u_{i,j}+u_{j,i})$$

where $u_{i,j}=\partial u_i/\partial x_j$. For a vector field, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on $\partial\varOmega$ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$, respectively.

The physical setting is the following. A viscoelastic body occupies, in its reference configuration, a regular domain $\varOmega$. The body is clamped on $\varGamma_1$ and so the displacement field vanishes there. Time-dependent surface tractions of density $\boldsymbol{f}_2$ act on $\varGamma_2$ and time-dependent volume forces of density $\boldsymbol{f}_0$ act in $\varOmega$. The body is in permanent contact on $\varGamma_3$ with a device, say a piston. The contact is modeled with a nonmonotone normal damped response condition associated with a total slip-dependent version of Coulomb's law of dry friction. We are interested in the evolutionary process of the mechanical state of the body, in the time interval of interest $(0,T)$ with $T>0$. The mathematical model of the contact problem is stated as follows.

Problem 10. Find a displacement field $\boldsymbol{u}\colon \Omega \times (0,T) \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \Omega \times (0,T) \to \mathbb{S}^d$ such that for all $t \in (0,T)$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}'(s))\,ds \quad \text{in } \Omega, \tag{5.24}$$

$$\rho\,\boldsymbol{u}''(t) = \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \quad \text{in } \Omega, \tag{5.25}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.26}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{5.27}$$

$$-\sigma_\nu(t) \in k(u_\nu(t))\partial j_\nu(u_\nu'(t)) \quad \text{on } \Gamma_3, \tag{5.28}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(t)\| \le F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big),\\ &-\boldsymbol{\sigma}_\tau = F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\frac{\boldsymbol{u}_\tau'(t)}{\|\boldsymbol{u}_\tau'(t)\|} \quad \text{if } \boldsymbol{u}_\tau'(t) \ne \boldsymbol{0} \end{aligned}\right. \quad \text{on } \Gamma_3, \tag{5.29}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(0) = \boldsymbol{w}_0 \quad \text{in } \Omega. \tag{5.30}$$

We briefly comment on the equations and conditions in Problem 10 and refer the reader to [25,6,26] for more details and mechanical interpretation. Eq. (5.24) is the constitutive law for viscoelastic materials in which $\mathcal{A}$ represents the viscosity operator, $\mathcal{B}$ represents the elasticity operator and $\mathcal{C}$ is the relaxation tensor. Such kind of equations have been used in [27–29], for instance. Eq. (5.25) is the equation of motion in which ρ denotes the density of mass. For simplicity, in what follows we set $\rho \equiv 1$. We have the clamped boundary condition (5.26) on Γ_1 and the surface traction boundary condition (5.27) on Γ_2.

Relation (5.28) is the multivalued contact condition with normal damped response in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν and k is a damper coefficient. Condition (5.29) represents a version of Coulomb's law of dry friction in which F_b is a given positive function, the friction bound. Details on such a frictional contact condition is found in [6] and some references therein. However, note that in contrast to the conditions used in the literature, in (5.28) the damper coefficient is allowed to depend on the normal displacement $u_\nu(t)$. In addition, in (5.28) the friction bound may depend on the quantity

$$S(\boldsymbol{x},t) = \int_0^t \|\boldsymbol{u}_\tau(\boldsymbol{x},s)\|\,ds$$

which represents the total slip (or, alternatively, the accumulated slip) at the point $\boldsymbol{x} \in \Gamma_3$ over the time period $[0,t]$. Considering such a dependence is reasonable from the physical point of view, since it incorporates the changes on the contact surface resulting from sliding.

Finally, conditions (5.30) are the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ represent the initial displacement and the initial velocity, respectively.

In the study of Problem 10 we use standard notation for Lebesgue and Sobolev spaces. For $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$, we use the same symbol $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on $\partial\Omega$ and we use the notation v_ν and $\boldsymbol{v}_\tau$ for its normal and tangential traces. In addition, we introduce spaces V and Q as follows:

$$V = \left\{ \boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \Gamma_1 \right\}, \qquad Q = L^2(\Omega;\mathbb{S}^d).$$

Both are real Hilbert spaces with inner products

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for } \boldsymbol{u},\boldsymbol{v} \in V,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx \quad \text{for } \boldsymbol{\sigma},\boldsymbol{\tau} \in Q.$$

The associated norms are $\|\cdot\|_V$ and $\|\cdot\|_Q$. By the Sobolev trace theorem,

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\gamma\|\,\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v}\in V. \tag{5.31}$$

Here and below $\|\gamma\|$ represents the norm of the trace operator $\gamma\colon V\to L^2(\Gamma_3;\mathbb{R}^d)$.

With the space V defined above and the space $H=L^2(\Omega;\mathbb{R}^d)$, we introduce the spaces $V^*,\mathcal{V},\mathcal{V}^*$ and $\mathcal{W}$ as in Section 3. Define a space of fourth order tensor fields

$$\mathbf{Q}_\infty = \{\ \mathcal{E}=(\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\ \ 1\le i,j,k,l\le d\ \}.$$

This is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \sum_{0\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le \|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q \quad \text{for all } \mathcal{E}\in\mathbf{Q}_\infty,\ \boldsymbol{\tau}\in Q. \tag{5.32}$$

We now list assumptions on the problem data. For the viscosity operator $\mathcal{A}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$, assume

$$\begin{cases} \text{(a) there exists } L_{\mathcal{A}}>0 \text{ such that } \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\| \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(b) there exists } m_{\mathcal{A}}>0 \text{ such that } (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2 \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c) the mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \quad \text{for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d. \\ \text{(d) } \mathcal{A}(\boldsymbol{x},\mathbf{0})=\mathbf{0} \quad \text{a.e. } \boldsymbol{x}\in\Omega. \end{cases} \tag{5.33}$$

For the elasticity operator $\mathcal{B}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$, assume

$$\begin{cases} \text{(a) there exists } L_{\mathcal{B}}>0 \text{ such that } \|\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\| \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(b) the mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \quad \text{for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d. \\ \text{(c) } \mathcal{B}(\boldsymbol{x},\mathbf{0})=\mathbf{0} \quad \text{a.e. } \boldsymbol{x}\in\Omega. \end{cases} \tag{5.34}$$

For the relaxation tensor $\mathcal{C}$, assume

$$\mathcal{C}\in C(0,T;\mathbf{Q}_\infty). \tag{5.35}$$

For the potential function $j_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}$, assume

$$\begin{cases} \text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R} \text{ and there} \\ \quad \text{exists } \overline{e}\in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,\overline{e}(\cdot))\in L^1(\Gamma_3). \\ \text{(b) } j_\nu(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \quad \text{for a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(c) } |\partial j_\nu(\boldsymbol{x},r)|\le\overline{c}_0 \quad \text{for a.e. } \boldsymbol{x}\in\Gamma_3, \quad \text{for all } r\in\mathbb{R} \text{ with } \overline{c}_0\ge 0. \\ \text{(d) } j_\nu^0(\boldsymbol{x},r_1;r_2-r_1)+j_\nu^0(\boldsymbol{x},r_2;r_1-r_2)\le\overline{\beta}\,|r_1-r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x}\in\Gamma_3,\ \text{all } r_1,\ r_2\in\mathbb{R} \text{ with } \overline{\beta}\ge 0. \end{cases} \tag{5.36}$$

For the damper coefficient $k\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$, assume

$$\begin{cases} \text{(a) the mapping } \boldsymbol{x}\mapsto k(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3, \quad \text{for any } r\in\mathbb{R}. \\ \text{(b) there are constants } k_1,\ k_2 \text{ such that } 0<k_1\le k(\boldsymbol{x},r)\le k_2 \\ \qquad \text{for all } r\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(c) there exists } L_k>0 \text{ such that } |k(\boldsymbol{x},r_1)-k(\boldsymbol{x},r_2)|\le L_k|r_1-r_2| \\ \qquad \text{for all } r_1,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases} \tag{5.37}$$

For the friction bound $F_b\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, assume

$$\begin{cases} \text{(a) the mapping } \boldsymbol{x} \mapsto F_b(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \quad \text{for any } r \in \mathbb{R}. \\ \text{(b) there exists } L_{F_b} > 0 \text{ such that } \ |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b}|r_1 - r_2| \\ \qquad \text{for all } \ r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) the mapping } \boldsymbol{x} \mapsto F_b(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{5.38}$$

Finally, for the densities of body forces, surface tractions and the initial data, assume

$$\boldsymbol{f}_0 \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \qquad \boldsymbol{f}_2 \in L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)), \quad \boldsymbol{u}_0,\ \boldsymbol{w}_0 \in V. \tag{5.39}$$

Remark 11. We provide an example of the function which satisfies hypothesis (5.36). For simplicity, we skip its dependence on the spatial variable. Let $j_\nu\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$j_\nu(r) = \begin{cases} 0, & \text{if } r < 0, \\ \dfrac{a-b}{2a}\, r^2 + b\, r, & \text{if } 0 \le r \le a, \\ a\, r + \dfrac{a(b-a)}{2}, & \text{if } r > a, \end{cases}$$

with $0 < a < b$. It is clear that j_ν is a locally Lipschitz and nonconvex function, and its Clarke subgradient is given by

$$\partial j_\nu(r) = \begin{cases} 0, & \text{if } r < 0, \\ [0,b], & \text{if } r = 0, \\ \dfrac{a-b}{a}\, r + b, & \text{if } 0 < r \le a, \\ a, & \text{if } r > a. \end{cases}$$

Hence, $|\partial j_\nu(r)| \le b$ for all $r \in \mathbb{R}$ and so j_ν satisfies (5.36)(c) with $\overline{c}_0 = b$. Next, we check that the function $\mathbb{R} \ni r \mapsto j_\nu(r) + \frac{b-a}{a} \in \mathbb{R}$ is nondecreasing, and therefore, by [6, Corollary 3.53], condition (5.36)(d) holds with $\overline{\beta} = \frac{b-a}{a}$. More examples can be found in [30] and [6, Section 7.1].

We now turn to the weak formulation of Problem 10. Let $\boldsymbol{v} \in V$ and $t \in (0,T)$. We multiply Eq. (5.25) by $\boldsymbol{v} - \boldsymbol{u}'(t)$, integrate over Ω, perform an integration by parts, and apply the boundary conditions (5.26) and (5.27) to obtain

$$\begin{aligned} &\int_\Omega \boldsymbol{u}''(t)\cdot(\boldsymbol{v} - \boldsymbol{u}'(t))\,dx + \int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}'(t)))\,dx \\ &\quad = \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v} - \boldsymbol{u}'(t))\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{v} - \boldsymbol{u}'(t))\,d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{u}'(t))\,d\Gamma. \end{aligned} \tag{5.40}$$

From (5.28), (5.29) and the definition of the subdifferential, we have

$$F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\,\big(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau'(t)\|\big) + k(u_\nu(t))\,j_\nu^0(u_\nu'(t); v_\nu - u_\nu'(t)) + \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{u}'(t)) \ge 0 \quad \text{a.e. on } \Gamma_3,$$

which implies that

$$\begin{aligned} &\int_{\Gamma_3} F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\,\big(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau'(t)\|\big)\,d\Gamma \\ &\quad + \int_{\Gamma_3} k(u_\nu(t))\,j_\nu^0(u_\nu'(t); v_\nu - u_\nu'(t))\,d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{u}'(t))\,d\Gamma \ge 0. \end{aligned} \tag{5.41}$$

Define $\boldsymbol{f}\colon (0,T) \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2(t), \gamma\boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \tag{5.42}$$

for all $\boldsymbol{v} \in V$ and all $t \in (0,T)$. Then, combining (5.40)–(5.42), we find that

$$\int_\Omega \boldsymbol{u}''(t)\cdot(\boldsymbol{v}-\boldsymbol{u}'(t))\,dx + (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}'(t)))_Q + \int_{\Gamma_3} F\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}'_\tau(t)\|)\,d\Gamma$$
$$+ \int_{\Gamma_3} k(u_\nu(t))\, j_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t))\,d\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{u}'(t)\rangle_{V^*\times V}. \tag{5.43}$$

We now use the constitutive law (5.24) and inequality (5.43) to obtain the following weak formulation of Problem 10 in terms of the displacement.

Problem 12. Find a displacement field $\boldsymbol{u}\colon (0,T)\to V$ such that for a.e. $t\in(0,T)$,

$$\int_\Omega \boldsymbol{u}''(t)\cdot(\boldsymbol{v}-\boldsymbol{u}'(t))\,dx + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}'(t)))_Q$$
$$+ (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}'(t)))_Q + \Big(\int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}'(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}'(t))\Big)_Q$$
$$+ \int_{\Gamma_3} F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}'_\tau(t)\|)\,d\Gamma$$
$$+ \int_{\Gamma_3} k(u_\nu(t))\, j_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t))\,d\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{u}'(t)\rangle_{V^*\times V}, \tag{5.44}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(0) = \boldsymbol{w}_0. \tag{5.45}$$

The unique solvability of Problem 12 is provided in the following result.

Theorem 13. *Assume the hypotheses (5.33)–(5.39). If*

$$m_{\mathcal{A}} > \overline{\beta}\, k_2 \|\gamma\|^2, \tag{5.46}$$

then Problem 12 has a unique solution with regularity $\boldsymbol{u}\in\mathcal{V}$, $\boldsymbol{u}'\in\mathcal{W}$.

Proof. Let $Y = Z = L^2(\Gamma_3)$. We introduce operators $A\colon (0,T)\times V\to V^*$, $\mathcal{R}_1\colon \mathcal{V}\to\mathcal{V}^*$, $\mathcal{R}\colon \mathcal{V}\to L^2(0,T;Y)$ and $\mathcal{S}\colon \mathcal{V}\to L^2(0,T;Z)$ as follows:

$$\langle A\boldsymbol{w},\boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for all } \boldsymbol{w},\boldsymbol{v}\in V,\ t\in(0,T), \tag{5.47}$$

$$\langle(\mathcal{R}_1\boldsymbol{w})(t),\boldsymbol{v}\rangle_{V^*\times V} = \Big(\mathcal{B}\Big(\int_0^t \boldsymbol{\varepsilon}(\boldsymbol{w}(s))\,ds + \boldsymbol{u}_0\Big),\boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_Q$$
$$+ \Big(\int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{w}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_Q \quad \text{for all } \boldsymbol{w}\in\mathcal{V},\ \boldsymbol{v}\in V,\ t\in(0,T), \tag{5.48}$$

$$(\mathcal{R}\boldsymbol{w})(t) = \int_0^t \Big\|\int_0^s \boldsymbol{w}_\tau(r)\,dr + \boldsymbol{u}_{0\tau}\Big\|\,ds \quad \text{for all } \boldsymbol{w}\in\mathcal{V},\ t\in(0,T), \tag{5.49}$$

$$(\mathcal{S}\boldsymbol{w})(t) = \int_0^t w_\nu(s)\,ds + u_{0\nu} \quad \text{for all } \boldsymbol{w}\in\mathcal{V},\ t\in(0,T). \tag{5.50}$$

Also, define functions $J\colon (0,T)\times Z\times V\to\mathbb{R}$ and $\varphi\colon (0,T)\times Y\times V\to\mathbb{R}$ by

$$J(t,z,\boldsymbol{v}) = \int_{\Gamma_3} k(z)\, j_\nu(v_\nu)\,d\Gamma \quad \text{for all } z\in Z,\ \boldsymbol{v}\in V,\ t\in(0,T), \tag{5.51}$$

$$\varphi(t,y,\boldsymbol{v}) = \int_{\Gamma_3} F_b(y)\,\|\boldsymbol{v}_\tau\|\,d\Gamma \quad \text{for all } y\in Y,\ \boldsymbol{v}\in V,\ t\in(0,T). \tag{5.52}$$

With the notation above we consider the following problem in terms of the velocity.

Problem 14. Find $\boldsymbol{w} \in \mathcal{W}$ such that

$$\begin{cases} \langle \boldsymbol{w}'(t) + A(t, \boldsymbol{w}(t)) + (\mathcal{R}_1 \boldsymbol{w})(t) - \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle_{V^* \times V} + J^0(t, (\mathcal{S}\boldsymbol{w})(t), \boldsymbol{w}(t); \boldsymbol{v} - \boldsymbol{w}(t)) \\ \quad + \varphi(t, (\mathcal{R}\boldsymbol{w})(t), \boldsymbol{v}) - \varphi(t, (\mathcal{R}\boldsymbol{w})(t), \boldsymbol{w}(t)) \geq 0 \quad \text{for all } \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T), \\ \boldsymbol{w}(0) = \boldsymbol{w}_0. \end{cases}$$

We apply Theorem 9 in studying Problem 14. For this purpose, we check that the hypotheses $H(A), H(J), H(\varphi), (H_1), (H_4)$ and (H_5) are satisfied. Note that A, J and φ do not depend explicitly on the temporal variable. First, it follows from (5.33) and [6, Theorem 7.3] that the operator A defined by (5.47) satisfies hypothesis $H(A)$ with $m_A = m_{\mathcal{A}}$.

By hypotheses (5.36) and (5.37), it is clear that for the function J defined by (5.51), conditions $H(J)(2)$ and (3) hold. From [6, Proposition 3.37(ii) & Theorem 3.47] and the relation $\partial(j_\nu(\boldsymbol{x}, \xi_\nu)) \subset \partial j_\nu(\boldsymbol{x}, \xi_\nu)\boldsymbol{\nu}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Gamma_3$, we deduce that the hypothesis $H(J)(4)$ is satisfied with $c_{0J}(t) = k_2 \overline{c}_0 \|\gamma\| \sqrt{m(\Gamma_3)}$ and $c_{1J} = c_{2J} = 0$. Next, from (5.36) and (5.37), [6, Theorem 3.47], and the relation $|\xi_\nu| \leq \|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, we have

$$\begin{aligned} & J^0(t, z_1, \boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + J^0(t, z_2, \boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \\ & \quad \leq \int_{\Gamma_3} \Big[k(z_1) j_\nu^0(v_{1\nu}; v_{2\nu} - v_{1\nu}) + k(z_2) j_\nu^0(v_{2\nu}; v_{1\nu} - v_{2\nu}) \Big] \, d\Gamma \\ & \quad \leq \int_{\Gamma_3} \Big[\big(k(z_1) - k(z_2)\big) j_\nu^0(v_{1\nu}; v_{2\nu} - v_{1\nu}) + k(z_2) \Big(j_\nu^0(v_{1\nu}; v_{2\nu} - v_{1\nu}) + j_\nu^0(v_{2\nu}; v_{1\nu} - v_{2\nu}) \Big) \Big] \, d\Gamma \\ & \quad \leq \overline{c}_0 L_k \int_{\Gamma_3} |z_1(\boldsymbol{x}) - z_2(\boldsymbol{x})| \, \|\boldsymbol{v}_1(\boldsymbol{x}) - \boldsymbol{v}_2(\boldsymbol{x})\| \, d\Gamma + k_2 \overline{\beta} \int_{\Gamma_3} \|\boldsymbol{v}_1(\boldsymbol{x}) - \boldsymbol{v}_2(\boldsymbol{x})\|^2 \, d\Gamma \\ & \quad \leq \overline{m}_{2J} \|z_1 - z_2\|_Z \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V + m_{2J} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2, \end{aligned}$$

where $\overline{m}_{2J} = \overline{c}_0 L_k \|\gamma\|$ and $m_{2J} = \overline{\beta} k_2 \|\gamma\|^2$. Hence $H(J)(5)$ follows.

From (5.38) and the convexity of the norm function, it follows that the function φ defined by (5.52) satisfies $H(\varphi)(2)$ and (3). Exploiting again [6, Proposition 3.37(ii), Theorem 3.47] and the relation $|F_b(\boldsymbol{x}, r)| \leq L_{F_b}|r| + |F_b(\boldsymbol{x}, 0)|$ for all $r \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_3$ (a consequence of (5.38)), we see that the hypothesis $H(\varphi)(4)$ holds with $c_{0\varphi}(t) = \sqrt{2}\|\gamma\|\|F_b(0)\|_Y, c_{1\varphi} = \sqrt{2} L_{F_b} \|\gamma\|$ and $c_{2\varphi} = 0$. By (5.38) and the Hölder inequality, we have

$$\begin{aligned} & \varphi(t, y_1, \boldsymbol{v}_2) - \varphi(t, y_1, \boldsymbol{v}_1) + \varphi(t, y_2, \boldsymbol{v}_1) - \varphi(t, y_2, \boldsymbol{v}_2) \\ & \quad = \int_{\Gamma_3} \big(F_b(y_1) - F(y_2)\big)\big(\|\boldsymbol{v}_{2\tau}\| - \|\boldsymbol{v}_{1\tau}\|\big) \, d\Gamma \\ & \quad \leq L_{F_b} \int_{\Gamma_3} |y_1(\boldsymbol{x}) - y_2(\boldsymbol{x})| \, \|\boldsymbol{v}_1(\boldsymbol{x}) - \boldsymbol{v}_2(\boldsymbol{x})\| \, d\Gamma \\ & \quad \leq L_{F_b} \|y_1 - y_2\|_Y \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ & \quad \leq L_{F_b} \|\gamma\| \, \|y_1 - y_2\|_Y \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \end{aligned}$$

for all $y_i \in Y, \boldsymbol{v}_i \in V, i = 1, 2$, a.e. $t \in (0,T)$, i.e., $H(\varphi)(5)$ holds with $\beta_\varphi = L_{F_b}\|\gamma\|$.

Moreover, the operators $\mathcal{R}_1, \mathcal{R}$ and $\mathcal{S}$ defined by (5.48)–(5.50), respectively, under the hypotheses (5.34) and (5.35), satisfy (H_4). For details, we refer to [31]. Condition (H_1) is a consequence of the assumption (5.39) and the definition (5.42). Finally, the condition (H_5) follows from (5.46). We have verified all hypotheses of Theorem 9. Then, we can deduce that Problem 14 has a unique solution $\boldsymbol{w} \in \mathcal{W}$. Define a function

$\boldsymbol{u}\colon (0,T) \to V$ by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0 \quad \text{for all } t \in (0,T).$$

It follows from the above that $\boldsymbol{u} \in \mathcal{V}$ with $\boldsymbol{u}' \in \mathcal{W}$ is the unique solution to Problem 12. This completes the proof of Theorem 13. □

A couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (5.24) and (5.44) is called a *weak solution* to Problem 10. We conclude that, under the assumptions of Theorem 13, Problem 10 has a unique weak solution. Moreover the solution has the regularity

$$\boldsymbol{u} \in \mathcal{V}, \qquad \boldsymbol{u}' \in \mathcal{W}, \qquad \boldsymbol{\sigma} \in L^2(0,T;Q), \qquad \mathrm{Div}\,\boldsymbol{\sigma} \in \mathcal{V}^*.$$

We end this section with a remark that a similar unique weak solvability result can be obtained if we replace the boundary conditions (5.28), (5.29) in Problem 10 with the following frictional contact conditions:

$$\left.\begin{aligned} &|\sigma_\nu(t)| \le F\Big(\int_0^t u_\nu^+(s)\,ds\Big),\\ &\sigma_\nu(t) = \begin{cases} 0 & \text{if } u_\nu < 0\\ F\Big(\int_0^t u_\nu^+(s)\,ds\Big) & \text{if } u_\nu > 0, \end{cases} \end{aligned}\right\} \quad \text{on } \Gamma_3, \tag{5.53}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \mu(u_\nu(t))\partial j_\tau(\dot{\boldsymbol{u}}_\tau(t)) \quad \text{on } \Gamma_3. \tag{5.54}$$

Indeed, it is easy to see that Problem (5.24)–(5.27), (5.30), (5.53), (5.54) leads to a variational formulation in velocities given by Problem 10, with an appropriate choice of operators $\mathcal{R}, \mathcal{S}$ and functionals J, φ. For the convenience of the reader we precise that relation (5.53) represents a multivalued contact condition with normal compliance and memory effects, in which F is a given function and r^+ denotes the positive part of r. Condition (5.54) represents the friction law in which ∂j_τ denotes the Clarke subdifferential of the nonconvex function j_τ and μ is a given coefficient of friction, which vanishes for a negative argument. Its dependence of the normal displacement u_ν guarantees the fact that when there is separation then the contact is frictionless.

The same remark can be formulated in the case when the boundary conditions (5.28), (5.29) are replaced by the frictional contact conditions

$$-\sigma_\nu(t) = p_\nu(u_\nu(t)) \quad \text{on } \Gamma_3, \tag{5.55}$$

$$\|\boldsymbol{\sigma}_\tau(t)\| \le \mu|\sigma_\nu|, \qquad -\boldsymbol{\sigma}_\tau = \mu|\sigma_\nu|\frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|} \quad \text{if } \dot{\boldsymbol{u}}_\tau(t) \ne \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{5.56}$$

Here, condition (5.55) represents the normal compliance condition in which p_ν is a given function and (5.56) in the classical Columb's law of dry friction in which μ denotes the coefficient of friction.

We conclude from the above examples that our abstract result find applications in various dynamic frictional contact problems, as claimed in Section 1 of this paper.

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