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A CLASS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES WITH APPLICATIONS TO FRICTIONAL CONTACT PROBLEMS*

WEIMIN HAN†, STANISŁAW MIGÓRSKI‡, AND MIRCEA SOFONEA§

Abstract. A class of variational-hemivariational inequalities is studied in this paper. An inequality in the class involves two nonlinear operators and two nondifferentiable functionals, of which at least one is convex. An existence and uniqueness result is proved for a solution of the inequality. Continuous dependence of the solution on the data is shown. Convergence is established rigorously for finite element solutions of the inequality. An error estimate is derived which is of optimal order for the linear finite element method under appropriate solution regularity assumptions. Finally, the results are applied to a variational-hemivariational inequality arising in the study of some frictional contact problems.

Key words. variational-hemivariational inequality, Clarke subdifferential, existence and uniqueness, continuous dependence, finite element solution, convergence, error estimates, frictional contact

AMS subject classifications. 47J20, 47J22, 65K15, 65N30, 74M10, 74M15, 74S05, 74S20

1. Introduction.

The theory of variational inequalities started in early sixties and has gone through substantial development since then, see for instance [2, 3, 4, 10, 11, 12, 19, 25] and the references therein. The main ingredients in the study of variational inequalities are the arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. In contrast, the theory of hemivariational inequalities is based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions which may be nonconvex. Analysis of hemivariational inequalities, including existence and uniqueness results, can be found in [7, 16, 21, 23, 26]. Applications of the variational and hemivariational inequalities in Mechanics and Engineering Sciences, and in Contact Mechanics in particular, can be found in [9, 13, 14, 15, 17, 18, 20, 21, 25, 26, 27, 28], among others. Variational-hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are involved. Interest in their study is motivated by various problems in Mechanics (e.g., [23, 24]).

The aim of this paper is to study a new class of variational-hemivariational inequalities and to apply these results in the analysis of an elastic contact problem. New feature of the problems is reflected by the presence of two nondifferentiable functionals, one convex and the other nonconvex. In addition to the unique solvability of the inequalities, we also show the continuous dependence of the solution on the data.

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†School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China; Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA (weimin-han@uiowa.edu).

‡Faculty of Mathematics and Computer Science, Jagiellonian University, Institute of Computer Science, ul. Stanisława Łojasiewicza 6, 30348 Krakow, Poland (stanislaw.migorski@uj.edu.pl).

§Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France (sofonea@univ-perp.fr).

Moreover, we introduce and analyze numerical methods for solving the inequalities. Novel techniques are employed, leading to desired error estimates for the numerical solutions. Finally, we apply our abstract results in the study of a new model of contact, which describes the equilibrium of a nonlinear elastic body in frictional contact with a reactive foundation.

The rest of the paper is organized as follows. In Section 2 we review some preliminary material. In Section 3 we introduce the class of variational-hemivariational inequalities to be studied, state and prove an abstract existence and uniqueness result. The proof of the result is based on arguments of surjectivity for pseudomonotone operators and the Banach fixed point theorem. In Section 4 we study the continuous dependence of the solution with respect to the data. In Section 5 we study numerical methods for the variational-hemivariational inequalities, prove convergence and derive error estimates. Finally, in Section 6 we consider a contact problem in which the material behavior is modeled with a nonlinear elastic constitutive law and the frictional contact conditions are in a subdifferential form. The contact problem leads to a variational-hemivariational inequality for the displacement field, and we apply our abstract results in the analysis of the problem.

2. Preliminaries. We recall some definitions and results related to various classes of functions and nonlinear operators that are needed in the rest of the paper. More details on the material presented in this section can be found in the books [6, 7, 8, 21, 23]. All the spaces in this paper are real.

For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. The symbol w-X is used for the space X endowed with the weak topology, while 2^{X^*} represents the set of all subsets of X^*. For simplicity in exposition, in the following we always assume X is a Banach space, unless stated otherwise.

Definition 2.1. *Let $h\colon X\to\mathbb{R}$ be a locally Lipschitz function. The* generalized (Clarke) directional derivative *of h at $x\in X$ in the direction $v\in X$, denoted by $h^0(x;v)$, is defined by*

$$h^0(x;v)=\limsup_{y\to x,\ \lambda\downarrow 0}\frac{h(y+\lambda v)-h(y)}{\lambda}.$$

The generalized gradient (subdifferential) *of h at x, denoted by $\partial h(x)$, is a subset of the dual space X^* given by*

$$\partial h(x)=\{\,\zeta\in X^*\mid h^0(x;v)\ge\langle\zeta,v\rangle_{X^*\times X}\ \text{for all}\ v\in X\,\}.$$

A locally Lipschitz function h is said to be regular *(in the sense of Clarke) at $x\in X$ if for all $v\in X$ the one-sided directional derivative $h'(x;v)$ exists and $h^0(x;v)=h'(x;v)$.*

Recall that a function $\varphi\colon X\to\mathbb{R}\cup\{+\infty\}$ is proper if it is not identically equal to $+\infty$ and is lower semicontinuous if $x_n\to x$ in X implies $\varphi(x)\le\liminf\varphi(x_n)$. The effective domain of φ is denoted by $\operatorname{dom}\varphi=\{x\in X\mid\varphi(x)<+\infty\}$.

Definition 2.2. *Let $\varphi\colon X\to\mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. The mapping $\partial\varphi\colon X\to 2^{X^*}$ defined by*

$$\partial\varphi(x)=\{x^*\in X^*\mid\langle x^*,v-x\rangle_{X^*\times X}\le\varphi(v)-\varphi(x)\ \text{for all}\ v\in X\}$$

is called the subdifferential *of φ. An element $x^*\in\partial\varphi(x)$ (if any) is called a* subgradient *of φ in x.*

Let $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Denote by $\operatorname{int}\operatorname{dom}\varphi$ and $D(\partial\varphi)$ the interior of the effective domain and the domain of the subdifferential $\partial\varphi$, respectively. Then, it is known that φ is locally Lipschitz on $\operatorname{int}\operatorname{dom}\varphi$ and $\operatorname{int}\operatorname{dom}\varphi \subset D(\partial\varphi) \subset \operatorname{dom}\varphi$. In particular, if $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is convex, then it is locally Lipschitz on $\mathbb{R}^d$.

Next, we shall consider single-valued operators $A\colon X \to X^*$ as well as multivalued operators $A\colon X \to 2^{X^*}$. The following definitions hold for single-valued operators.

Definition 2.3. *An operator $A\colon X \to X^*$ is called:*
(a) bounded*, if A maps bounded sets of X into bounded sets of X^*;*
(b) monotone*, if $\langle Au - Av, u - v\rangle_{X^*\times X} \geq 0$ for all u, $v \in X$;*
(c) maximal monotone*, if it is monotone, and $\langle Au - w, u - v\rangle_{X^*\times X} \geq 0$ for any $u \in X$ implies that $w = Av$;*
(d) coercive*, if there exists a function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{t\to+\infty} \alpha(t) = +\infty$ such that $\langle Au, u\rangle_{X^*\times X} \geq \alpha(\|u\|_X)\,\|u\|_X$ for all $u \in X$;*
(e) pseudomonotone*, if it is bounded and $u_n \to u$ weakly in X together with $\limsup\langle Au_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply $\langle Au, u - v\rangle_{X^*\times X} \leq \liminf\langle Au_n, u_n - v\rangle_{X^*\times X}$ for all $v \in X$.*

It can be proved that an operator $A\colon X \to X^*$ is pseudomonotone, iff it is bounded and $u_n \to u$ weakly in X together with $\limsup\,\langle Au_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply $\lim\,\langle Au_n, u_n - u\rangle_{X^*\times X} = 0$ and $Au_n \to Au$ weakly in X^*.

For a multivalued operator $A\colon X \to 2^{X^*}$, its domain $D(A)$, range $R(A)$ and graph $Gr(A)$ are defined by

$$D(A) = \{x \in X \mid Ax \neq \emptyset\}, \quad R(A) = \bigcup\{Ax \mid x \in X\},$$
$$Gr(A) = \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}.$$

If $u_0 \in X$ we define a multivalued operator A_{u_0} by $A_{u_0}(v) = A(v + u_0)$ for all $v \in X$.

Definition 2.4. *A multivalued operator $A\colon X \to 2^{X^*}$ is called:*
(a) monotone*, if $\langle u^* - v^*, u - v\rangle_{X^*\times X} \geq 0$ for all (u, u^*), $(v, v^*) \in Gr(A)$;*
(b) maximal monotone*, if it is monotone and maximal in the sense of inclusion of graphs in the family of monotone operators from X to 2^{X^*};*
(c) coercive*, if there exists a function $c\colon \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{t\to+\infty} c(t) = +\infty$ such that $\langle u^*, u\rangle_{X^*\times X} \geq c(\|u\|_X)\,\|u\|_X$ for all $(u, u^*) \in Gr(A)$.*

The following important result is due to Rockafellar, cf. [8, Theorem 6.3.19].

Theorem 2.5. *Let φ be a proper, convex and lower semicontinuous function on X. Then $\partial\varphi\colon X \to 2^{X^*}$ is a maximal monotone operator.*

The notions of pseudomonotonicity and generalized pseudomonotonicity for multivalued operators are recalled in the following definitions.

Definition 2.6. *Let X be a reflexive Banach space. A multivalued operator $A\colon X \to 2^{X^*}$ is* pseudomonotone *if:*
(a) *for every $u \in X$, the set $Au \subset X^*$ is nonempty, closed and convex;*
(b) *A is upper semicontinuous from each finite dimensional subspace of X into w-X^*;*
(c) *for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Au_n$ for all $n \geq 1$ and $\limsup\langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0$, we have that for every $v \in X$, there exists $u^*(v) \in Au$ such that*

$$\langle u^*(v), u - v\rangle_{X^*\times X} \leq \liminf\,\langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

Definition 2.7. *Let X be a reflexive Banach space. A multivalued operator $A\colon X \to 2^{X^*}$ is* generalized pseudomonotone *if for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Au_n$ for $n \geq 1$, $u_n^* \to u^*$ weakly in X^* and $\limsup \langle u_n^*, u_n - u\rangle_{X^* \times X} \leq 0$, we have $u^* \in Au$ and*

$$\lim \langle u_n^*, u_n\rangle_{X^* \times X} = \langle u^*, u\rangle_{X^* \times X}.$$

The relationship between these notions is given by the following results (cf. [8, Propositions 6.3.65 and 6.3.66]).

Proposition 2.8. *Let X be a reflexive Banach space and $A\colon X \to 2^{X^*}$ a pseudomonotone operator. Then A is generalized pseudomonotone.*

Proposition 2.9. *Let X be a reflexive Banach space and $A\colon X \to 2^{X^*}$ a bounded generalized pseudomonotone operator. If for each $u \in X$, Au is a nonempty, closed and convex subset of X^*, then A is pseudomonotone.*

Finally, we recall the following surjectivity result; see [23, Theorem 2.12].

Theorem 2.10. *Let X be a reflexive Banach space, $\overline{T}\colon X \to 2^{X^*}$ a maximal monotone operator, and $T\colon X \to 2^{X^*}$ a pseudomonotone operator. Suppose either T_{u_0} or $\overline{T}_{u_0}$ is bounded for some $u_0 \in D(\overline{T})$. Assume that there exists a function $c\colon \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in Gr(T)$, we have $\langle u^*, u - u_0\rangle_{X^* \times X} \geq c(\|u\|_X)\, \|u\|_X$. Then $T + \overline{T}$ is surjective, i.e. $R(T + \overline{T}) = X^*$.*

3. An existence and uniqueness result. Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$ and let $\Gamma \subseteq \partial\Omega$ be a measurable subset. We use the notation $\boldsymbol{x}$ for a generic point in Γ and $m(\Gamma)$ for the $(d-1)$ dimensional measure of Γ. Given an integer $s \geq 1$ we denote by V a closed subspace of $H^1(\Omega; \mathbb{R}^s)$ and let $H = L^2(\Omega; \mathbb{R}^s)$. We also use the notation $\gamma\colon V \to L^2(\Gamma; \mathbb{R}^s)$ for the trace operator, $\|\gamma\|$ for its norm in the space $\mathcal{L}(V, L^2(\Gamma; \mathbb{R}^s))$ and $\gamma^*\colon L^2(\Gamma; \mathbb{R}^s) \to V^*$ for its adjoint operator. It is known that (V, H, V^*) forms an evolution triple of spaces and the embedding $V \subset H$ is compact.

Given operators $A\colon V \to V^*$, $F\colon V \to L^2(\Gamma)$, functions φ, $j\colon \Gamma \times \mathbb{R}^s \to \mathbb{R}$ and a functional $f\colon V \to \mathbb{R}$, we consider the following problem.

Problem (P). *Find an element $u \in V$ such that*

$$\langle Au, v - u\rangle_{V^* \times V} + \int_\Gamma (Fu)\big(\varphi(\gamma v) - \varphi(\gamma u)\big)\, d\Gamma + \int_\Gamma j^0(\gamma u; \gamma v - \gamma u)\, d\Gamma \geq \langle f, v - u\rangle_{V^* \times V} \quad \text{for all } v \in V. \tag{3.1}$$

For the study of Problem (P), we introduce the following hypotheses.

$$(3.2)\quad \begin{cases} A\colon V\to V^* \text{ is} \\ \quad \text{(a) pseudomonotone and there exists } \alpha>0 \text{ such that} \\ \qquad \langle Av,v\rangle_{V^*\times V}\ge \alpha\,\|v\|_V^2 \text{ for all } v\in V; \\ \quad \text{(b) strongly monotone, i.e., there exists } m_A>0 \text{ such that} \\ \qquad \langle Av_1-Av_2,v_1-v_2\rangle_{V^*\times V}\ge m_A\|v_1-v_2\|_V^2 \text{ for all } v_1,v_2\in V. \end{cases}$$

$$(3.3)\quad \begin{cases} F\colon V\to L^2(\Gamma) \text{ and} \\ \quad \text{(a) there exists } L_F>0 \text{ such that} \\ \qquad \|Fv_1-Fv_2\|_{L^2(\Gamma)}\le L_F\|v_1-v_2\|_V \text{ for all } v_1,\,v_2\in V; \\ \quad \text{(b) } Fv\ge 0 \ \text{ a.e. on } \Gamma,\ \text{ for all } v\in V. \end{cases}$$

$$(3.4)\quad \begin{cases} \varphi\colon \Gamma\times\mathbb{R}^s\to\mathbb{R} \text{ is such that} \\ \quad \text{(a) } \varphi(\cdot,\xi) \text{ is measurable on } \Gamma \text{ for all } \xi\in\mathbb{R}^s \text{ and there} \\ \qquad \text{exists } \tilde e\in L^2(\Gamma;\mathbb{R}^s) \text{ such that } \varphi(\cdot,\tilde e(\cdot))\in L^2(\Gamma); \\ \quad \text{(b) } \varphi(\boldsymbol{x},\cdot) \text{ is convex for a.e. } \boldsymbol{x}\in\Gamma; \\ \quad \text{(c) there exists } L_\varphi>0 \text{ such that for all } \xi_1,\xi_2\in\mathbb{R}^s, \\ \qquad |\varphi(\boldsymbol{x},\xi_1)-\varphi(\boldsymbol{x},\xi_2)|\le L_\varphi\|\xi_1-\xi_2\|_{\mathbb{R}^s},\ \text{ a.e. } \boldsymbol{x}\in\Gamma. \end{cases}$$

$$(3.5)\quad \begin{cases} j\colon \Gamma\times\mathbb{R}^s\to\mathbb{R} \text{ is such that} \\ \quad \text{(a) } j(\cdot,\xi) \text{ is measurable on } \Gamma \text{ for all } \xi\in\mathbb{R}^s \text{ and there exists} \\ \qquad e\in L^2(\Gamma;\mathbb{R}^s) \text{ such that } j(\cdot,e(\cdot))\in L^1(\Gamma); \\ \quad \text{(b) } j(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R}^s \text{ for a.e. } \boldsymbol{x}\in\Gamma; \\ \quad \text{(c) } \|\partial j(\boldsymbol{x},\xi)\|_{\mathbb{R}^s}\le c_0+c_1\,\|\xi\|_{\mathbb{R}^s} \text{ for a.e. } \boldsymbol{x}\in\Gamma, \\ \qquad \text{all } \xi\in\mathbb{R}^s \text{ with } c_0,c_1\ge 0; \\ \quad \text{(d) } j^0(\boldsymbol{x},\xi_1;\xi_2-\xi_1)+j^0(\boldsymbol{x},\xi_2;\xi_1-\xi_2)\le\beta\,\|\xi_1-\xi_2\|_{\mathbb{R}^s}^2 \\ \qquad \text{for a.e. } \boldsymbol{x}\in\Gamma,\ \text{ all } \xi_1,\,\xi_2\in\mathbb{R}^s \text{ with } \beta\ge 0. \end{cases}$$

$$(3.6)\qquad f\in V^*.$$

Note that the function φ is assumed to be convex and Lipschitz continuous with respect to its second argument while the function j is locally Lipschitz with respect to the second argument and may be nonconvex. For this reason, the inequality (3.1) is a *variational-hemivariational inequality.*

Remark 3.1. The hypothesis (3.5)(d) has been introduced to guarantee the uniqueness of the solution to the variational-hemivariational inequality. It can be verified that for a locally Lipschitz function $j\colon\mathbb{R}^s\to\mathbb{R}$, the condition (3.5)(d) is equivalent to the so-called relaxed monotonicity condition

$$(3.7)\qquad (\zeta_1-\zeta_2)\cdot(\xi_1-\xi_2)\ge-\beta\,\|\xi_1-\xi_2\|_{\mathbb{R}^s}^2$$

for all ζ_i, $\xi_i\in\mathbb{R}^s$, $\zeta_i\in\partial j(\xi_i)$, $i=1,\,2$. The latter was extensively used in the literature, cf. e.g. [21] and the references therein. For particular problems, condition (3.7) is easy to verify by showing that the function

$$\mathbb{R}^s\ni\xi\mapsto j(\xi)+\frac{\beta}{2}\|\xi\|_{\mathbb{R}^s}^2\in\mathbb{R}$$

is nondecreasing. Examples of nonconvex functions which satisfy the condition (3.5) can be found in [22]. We merely remark that when $j\colon \mathbb{R}^s \to \mathbb{R}$ is convex, (3.5)(b) and (d) are satisfied with $\beta = 0$. Indeed, by convexity,

$$j^0(\xi_1; \xi_2 - \xi_1) \le j(\xi_2) - j(\xi_1) \quad \text{and} \quad j^0(\xi_2; \xi_1 - \xi_2) \le j(\xi_1) - j(\xi_2)$$

for all ξ_1, $\xi_2 \in \mathbb{R}^s$ which entails $j^0(\xi_1; \xi_2 - \xi_1) + j^0(\xi_2; \xi_1 - \xi_2) \le 0$. It means that for a convex function $j\colon \mathbb{R}^s \to \mathbb{R}$, condition (3.5)(d) or, equivalently, the condition (3.7) reduces to monotonicity of the (convex) subdifferential, i.e., $\beta = 0$.

Our existence and uniqueness result for Problem (P) is the following.

Theorem 3.2. *Assume* (3.2)–(3.6) *and the smallness condition*

$$L_F L_\varphi \|\gamma\| + \beta\, \|\gamma\|^2 < m_A. \tag{3.8}$$

Then, if one of the following two inequalities holds,

(i) $\alpha > c_1 \sqrt{2}\, \|\gamma\|^2$,

(ii) $j^0(\boldsymbol{x}, \xi; -\xi) \le d\,(1 + \|\xi\|_{\mathbb{R}^s})$ *for all* $\xi \in \mathbb{R}^s$, *a.e.* $\boldsymbol{x} \in \Gamma$ *with* $d \ge 0$,

Problem (P) *has a unique solution* $u \in V$.

Proof of Theorem 3.2 is carried out in several steps. In the rest of the section, we assume the conditions (3.2)–(3.8). For $\eta \in V$, denote

$$z_\eta = F\eta \in L^2(\Gamma) \tag{3.9}$$

and consider the following auxiliary problem.

Problem (P$_\eta$). *Find* $u_\eta \in V$ *such that*

$$\begin{aligned} &\langle A u_\eta, v - u_\eta\rangle_{V^*\times V} + \int_\Gamma z_\eta \big(\varphi(\gamma v) - \varphi(\gamma u_\eta)\big)\, d\Gamma \\ &\quad + \int_\Gamma j^0(\gamma u_\eta; \gamma v - \gamma u_\eta)\, d\Gamma \ge \langle f, v - u_\eta\rangle_{V^*\times V} \quad \text{for all } v \in V. \end{aligned} \tag{3.10}$$

We have the following result.

Lemma 3.3. Problem (P$_\eta$) *has a unique solution* $u_\eta \in V$.

Proof. For the existence part, we apply Theorem 2.10. Define a functional $J\colon L^2(\Gamma; \mathbb{R}^s) \to \mathbb{R}$ by

$$J(v) = \int_\Gamma j(\boldsymbol{x}, v(\boldsymbol{x}))\, d\Gamma, \quad v \in L^2(\Gamma; \mathbb{R}^s).$$

Thanks to the hypotheses (3.5) (a)–(c), we have the following statements ([21, Corollary 4.15]).

$$\left\{ \begin{array}{l} \text{(i) } J \text{ is well defined and Lipschitz continuous on bounded} \\ \qquad \text{subsets of } L^2(\Gamma; \mathbb{R}^s); \\ \text{(ii) } J^0(u; v) \le \displaystyle\int_\Gamma j^0(\boldsymbol{x}, u(\boldsymbol{x}); v(\boldsymbol{x}))\, d\Gamma \quad \text{for all } u, v \in L^2(\Gamma; \mathbb{R}^s); \\ \text{(iii) } \|u^*\|_{L^2(\Gamma;\mathbb{R}^s)} \le \overline{c}_0 + \overline{c}_1\, \|u\|_{L^2(\Gamma;\mathbb{R}^s)} \quad \text{for all } u \in L^2(\Gamma; \mathbb{R}^s); \\ \qquad u^* \in \partial J(u) \text{ with } \overline{c}_0 = c_0 \sqrt{2\, m(\Gamma)} \text{ and } \overline{c}_1 = c_1 \sqrt{2}. \end{array} \right. \tag{3.11}$$

Then we define an operator $B\colon V \to 2^{V^*}$ by

$$B(v) = \gamma^* \partial J(\gamma v), \quad v \in V.$$

We claim that the operator $A+B$ is pseudomonotone and bounded from V to 2^{V^*}. To this end, note that the values of ∂J are nonempty, convex and weakly compact subsets of $L^2(\Gamma;\mathbb{R}^s)$ ([21, Proposition 3.23 (iv)]). So for every $v \in V$, the set $B(v)$ is nonempty, closed and convex in V^*. Also, by (3.11) (iii), we have

$$\|v^*\|_{V^*} \le \|\gamma^*\| \, \|\partial J(\gamma v)\|_{L^2(\Gamma;\mathbb{R}^s)} \le \|\gamma\| \, (\bar{c}_0 + \bar{c}_1 \|\gamma\| \|v\|_V) \tag{3.12}$$

for all $(v, v^*) \in Gr(B)$, implying the boundedness of the operator B. To show the claim, we apply Proposition 2.9. Then, it is sufficient to prove that B is generalized pseudomonotone.

Let v_n, $v \in V$, $v_n \to v$ weakly in V, v_n^*, $v^* \in V^*$, $v_n^* \to v^*$ weakly in V^*, $v_n^* \in B(v_n)$ and $\limsup \langle v_n^*, v_n - v\rangle_{V^*\times V} \le 0$. We prove that $v^* \in Bv$ and $\langle v_n^*, v_n\rangle_{V^*\times V} \to \langle v^*, v\rangle_{V^*\times V}$. We have $v_n^* = \gamma^*\zeta_n$ with $\zeta_n \in \partial J(\gamma v_n)$. From the bound (3.11)(iii), we know that $\{\zeta_n\}$ is bounded in $L^2(\Gamma;\mathbb{R}^s)$. Hence, by passing to a subsequence if necessary, we can assume that $\zeta_n \to \zeta$ weakly in $L^2(\Gamma;\mathbb{R}^s)$. Since the graph of $\partial J(\cdot)$ is closed in $L^2(\Gamma;\mathbb{R}^s) \times (w{-}L^2(\Gamma;\mathbb{R}^s))$-topology and $\gamma v_n \to \gamma v$ in $L^2(\Gamma;\mathbb{R}^s)$, by the compactness of the trace operator we obtain $\zeta \in \partial J(\gamma v)$. Furthermore, from $v_n^* = \gamma^*\zeta_n$ it follows that $v^* = \gamma^*\zeta$. Thus $v^* \in \gamma^*\partial J(\gamma v) = B(v)$. Clearly, we have

$$\begin{aligned}\langle v_n^*, v_n\rangle_{V^*\times V} &= \langle \gamma^*\zeta_n, v_n\rangle_{V^*\times V} = \langle \zeta_n, \gamma v_n\rangle_{L^2(\Gamma;\mathbb{R}^s)} \\ &\to \langle \zeta, \gamma v\rangle_{L^2(\Gamma;\mathbb{R}^s)} = \langle \gamma^*\zeta, v\rangle_{V^*\times V} = \langle v^*, v\rangle_{V^*\times V}.\end{aligned}$$

So the operator B is generalized pseudomonotone and, therefore, it is also pseudomonotone.

By the hypothesis (3.2)(a), it is clear (cf. Section 3.4 of [21]) that A is pseudomonotone and bounded as a multivalued operator from V to 2^{V^*}. Since the set of multivalued pseudomonotone operators is closed under addition of mappings (cf. Proposition 3.59 of [21]), we deduce that the operator $A+B$ is pseudomonotone and bounded from V to 2^{V^*}. This proves the claim.

We now establish the coercivity of the multivalued operator $A+B$ in the sense of Definition 2.4(c). First, we assume the hypothesis *(i)* of Theorem 3.2. From (3.12), we have

$$\langle v^*, v\rangle_{V^*\times V} \ge -\bar{c}_1 \|\gamma\|^2 \|v\|_V^2 - \bar{c}_0 \|\gamma\| \, \|v\|_V \quad \text{for all } v \in V,\ v^* \in Bv.$$

Hence, by (3.2) (a) we obtain

$$\begin{aligned}\langle Av + v^*, v\rangle_{V^*\times V} &\ge (\alpha - c_1\sqrt{2}\, \|\gamma\|^2)\|v\|_V^2 - \bar{c}_0 \|\gamma\| \, \|v\|_V \\ &\quad \text{for all } v \in V,\ v^* \in Bv\end{aligned} \tag{3.13}$$

which implies that the operator $A+B$ is coercive since $\alpha - c_1\sqrt{2}\,\|\gamma\|^2 > 0$.

Next, we show the coercivity under the hypothesis *(ii)* of Theorem 3.2. In this case, from the property (3.11) (ii) and (3.5) (c), we have

$$J^0(v; -v) \le d_1(1 + \|v\|_{L^2(\Gamma;\mathbb{R}^s)}) \ \text{ for all } \ v \in L^2(\Gamma;\mathbb{R}^s)$$

with $d_1 \ge 0$. Therefore, for every $v \in V$ and $\zeta \in \partial J(\gamma v)$, we deduce that

$$\langle \zeta, \gamma v\rangle_{L^2(\Gamma;\mathbb{R}^s)} \ge -J^0(\gamma v; -\gamma v) \ge -d_1 - d_2 \, \|\gamma\| \, \|v\|_V$$

with $d_1, d_2 \geq 0$. Hence, by (3.2) (a), the operator $A+B$ is coercive.

Consider a functional $\Phi_\eta \colon V \to \mathbb{R}$ defined by

$$\Phi_\eta(v) = \int_\Gamma z_\eta(\boldsymbol{x})\varphi(\boldsymbol{x}, \gamma v(\boldsymbol{x}))\, d\Gamma, \quad v \in V. \tag{3.14}$$

We observe that the hypothesis (3.4) implies that $\varphi(\cdot, \gamma v(\cdot)) \in L^2(\Gamma)$ for all $v \in V$. This property together with the fact that $z_\eta \in L^2(\Gamma)$ ensures that Φ_η is well defined. Moreover, it is clear that $\mathrm{dom}\,(\Phi_\eta) = V$. Next, since (3.3)(b) implies that $z_\eta \geq 0$ a.e. on Γ, by assumption (3.4) we infer that Φ_η is a convex continuous function. Therefore, the operator $\partial\Phi_\eta \colon V \to 2^{V^*}$ is maximal monotone with $D(\partial\Phi_\eta) = V$ (cf. Theorem 2.5).

In summary, the operator $\partial\Phi_\eta \colon V \to 2^{V^*}$ is maximal monotone with $0_V \in D(\partial\Phi_\eta)$, the operator $A+B \colon V \to 2^{V^*}$ is pseudomonotone, bounded, and satisfies the coercivity condition (3.13). We apply Theorem 2.10 to the operators $\partial\Phi_\eta$ and $A+B$, and deduce the existence of $u_\eta \in V$ such that

$$Au_\eta + Bu_\eta + \partial\Phi_\eta(u_\eta) \ni f.$$

This means that

$$Au_\eta + \gamma^*\zeta_\eta + \theta_\eta = f, \tag{3.15}$$

where $\zeta_\eta \in \partial J(\gamma u_\eta)$ and $\theta_\eta \in \partial\Phi_\eta(u_\eta)$. Using the property (3.11) (ii) of the functional J, we have

$$\langle \zeta_\eta, w\rangle_{L^2(\Gamma;\mathbb{R}^s)} \leq J^0(\gamma u_\eta; w) \leq \int_\Gamma j^0(\gamma u_\eta; w)\, d\Gamma \quad \text{for all } w \in L^2(\Gamma;\mathbb{R}^s) \tag{3.16}$$

and

$$\langle \theta_\eta, v - u_\eta\rangle_{V^*\times V} \leq \Phi_\eta(v) - \Phi_\eta(u_\eta) \quad \text{for all } v \in V. \tag{3.17}$$

For any $v \in V$, we get from (3.15) that

$$\langle Au_\eta, v - u_\eta\rangle_{V^*\times V} + \langle \zeta_\eta, \gamma v - \gamma u_\eta\rangle_{L^2(\Gamma;\mathbb{R}^s)} + \langle \theta_\eta, v - u_\eta\rangle_{V^*\times V} = \langle f, v - u_\eta\rangle_{V^*\times V}.$$

We use (3.16) and (3.17) in the equality to find that

$$\begin{aligned}&\langle Au_\eta, v - u_\eta\rangle_{V^*\times V} + \int_\Gamma j^0(\gamma u_\eta; \gamma v - \gamma u_\eta)\, d\Gamma + \int_\Gamma z_\eta\big(\varphi(\gamma v) - \varphi(\gamma u_\eta)\big)\, d\Gamma\\ &\quad \geq \langle f, v - u_\eta\rangle_{V^*\times V},\end{aligned}$$

which shows that u_η is a solution to Problem (P_η).

To show the uniqueness of a solution, let $u_1, u_2 \in V$ be two solutions to Problem (P_η). We write (3.10) for u_1 with $v = u_2$, and then for u_2 with $v = u_1$, and add the resulting inequalities. Use the strong monotonicity of A and assumption (3.5) (d) on the function j to obtain

$$m_A\|u_1 - u_2\|_V^2 \leq \beta\, \|\gamma\|^2 \|u_1 - u_2\|_V^2.$$

Applying the smallness condition (3.8), we deduce that $u_1 = u_2$. This completes the proof. □

Lemma 3.3 allows us to define an operator $\Lambda\colon V \to V$ by

$$\Lambda\eta = u_\eta, \quad \eta \in V. \tag{3.18}$$

We have the following fixed point result.

Lemma 3.4. *The operator Λ has a unique fixed point $\eta^* \in V$.*

Proof. Let η_1, $\eta_2 \in V$ and let z_i be the functions defined by $z_i = z_{\eta_i}$ for $i = 1$, 2. Denote by u_i the solution of the variational-hemivariational inequality (3.10) for $\eta = \eta_i$, i.e., $u_i = u_{\eta_i}$, $i = 1, 2$. From the definition (3.18) we have

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_V = \|u_1 - u_2\|_V. \tag{3.19}$$

We write (3.10) for $\eta = \eta_1$ with $v = u_2$, and then for $\eta = \eta_2$ with $v = u_1$. Add the resulting inequalities and use the strong monotonicity of A together with assumptions (3.4) and (3.5) on φ and j to obtain

$$m_A \|u_1 - u_2\|_V^2 \le L_\varphi \|\gamma\| \, \|z_1 - z_2\|_{L^2(\Gamma)} \|u_1 - u_2\|_V + \beta \, \|\gamma\|^2 \|u_1 - u_2\|_V^2.$$

Since $m_A - \beta\, \|\gamma\|^2 > 0$ by (3.8), we have

$$\|u_1 - u_2\|_V \le \frac{L_\varphi \|\gamma\|}{m_A - \beta\, \|\gamma\|^2} \, \|z_1 - z_2\|_{L^2(\Gamma)}. \tag{3.20}$$

Next, we use (3.9) and the property (3.3)(a) of the operator F to see that

$$\|z_1 - z_2\|_{L^2(\Gamma)} = \|F\eta_1 - F\eta_2\|_{L^2(\Gamma)} \le L_F \|\eta_1 - \eta_2\|_V.$$

Use this inequality in (3.20) to yield

$$\|u_1 - u_2\|_V \le \frac{L_F L_\varphi \|\gamma\|}{m_A - \beta\, \|\gamma\|^2} \, \|\eta_1 - \eta_2\|_V. \tag{3.21}$$

We combine (3.19) and (3.21) to deduce that

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_V \le \frac{L_F L_\varphi \|\gamma\|}{m_A - \beta\, \|\gamma\|^2} \, \|\eta_1 - \eta_2\|_V. \tag{3.22}$$

Finally, we use (3.22), the smallness assumption (3.8) and the Banach fixed point theorem to show that the operator Λ has a unique fixed point $\eta^* \in V$, which concludes the proof of the lemma. ∎

Now we complete the proof of Theorem 3.2.

Proof of Theorem 3.2. Existence. Let $\eta^* \in V$ be the fixed point of the operator Λ. It follows from (3.9) and (3.18) that the following equalities hold:

$$z_{\eta^*} = F\eta^*, \qquad u_{\eta^*} = \eta^*. \tag{3.23}$$

We write the inequality (3.10) for $\eta = \eta^*$ and then use the equalities (3.23) to conclude that the function $\eta^* \in V$ is a solution to Problem (P).

Uniqueness. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Λ and can be proved as follows. Denote by $\eta^* \in V$ the solution of the inequality (3.1) obtained above, and let $\eta \in V$ be another solution of this inequality. Also, consider the function $z_\eta \in L^2(\Gamma)$ defined by (3.9). Then, it follows that η is a solution to the variational inequality (3.10) and, since by Lemma 3.3 this inequality has a unique solution, denoted u_η, we conclude that

$$\eta = u_\eta. \tag{3.24}$$

Equality (3.24) shows that $\Lambda\eta = \eta$ where Λ is the operator defined by (3.18). Therefore, by Lemma 3.4, it follows that $\eta = \eta^*$.

Alternatively, the uniqueness of a solution can be also proved directly.

4. Continuous dependence on data. We now study the continuous dependence of the solution of Problem (P) on the data. Assume in what follows that (3.2)–(3.8) hold and denote by u the solution of Problem (P) stated in Theorem 3.2. For each $\rho > 0$ let F_ρ, j_ρ and f_ρ represent perturbed data corresponding to F, j and f, which satisfy conditions (3.3), (3.5) and (3.6), respectively. For each $\rho > 0$ we denote by $L_{F\rho}$ and β_ρ the constants involved in assumptions (3.3) and (3.5). Assume that there exists m_0 such that

$$L_{F_\rho} L_\varphi \|\gamma\| + \beta_\rho \|\gamma\|^2 \le m_0 < m_A \quad \text{for all } \rho > 0. \tag{4.1}$$

Consider the following perturbed version of Problem (P).

Problem (P$_\rho$). *Find $u_\rho \in V$ such that*

$$\langle Au_\rho, v - u_\rho\rangle_{V^*\times V} + \int_\Gamma (F_\rho u_\rho)\big(\varphi(\gamma v) - \varphi(\gamma u_\rho)\big)\, d\Gamma + \int_\Gamma j_\rho^0(\gamma u_\rho; \gamma v - \gamma u_\rho)\, d\Gamma \ge \langle f_\rho, v - u_\rho\rangle_{V^*\times V} \quad \text{for all } v \in V. \tag{4.2}$$

It follows from Theorem 3.2 that, for each $\rho > 0$, Problem (P$_\rho$) has a unique solution $u_\rho \in V$. To consider the limiting behaviour of $\{u_\rho\}$, we introduce the following assumptions:

$$\left\{\begin{array}{l} \text{There exist } G\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } g \in \mathbb{R}_+ \text{ such that} \\ \text{(a) } \|F_\rho v - Fv\|_{L^2(\Gamma)} \le G(\rho)(\|v\|_V + g) \quad \text{for all } v \in V,\ \rho > 0; \\ \text{(b) } G(\rho) \to 0 \quad \text{as } \rho \to 0. \end{array}\right. \tag{4.3}$$

$$\left\{\begin{array}{l} \text{There exist } H\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } h \in \mathbb{R}_+ \text{ such that} \\ \text{(a) } j^0(\boldsymbol{x}, \xi; \eta) - j_\rho^0(\boldsymbol{x}, \xi; \eta) \le H(\rho)(\|\xi\|_{\mathbb{R}^s} + h)\|\eta\|_{\mathbb{R}^s} \\ \qquad \text{for all } \xi, \eta \in \mathbb{R}^s, \text{ a.e. } \boldsymbol{x} \in \Gamma,\ \rho > 0; \\ \text{(b) } H(\rho) \to 0 \quad \text{as } \rho \to 0. \end{array}\right. \tag{4.4}$$

$$f_\rho \to f \quad \text{in } V^* \text{ as } \rho \to 0. \tag{4.5}$$

We have the following convergence result.

Theorem 4.1. *Assume* (4.3)–(4.5). *Then the solution u_ρ of* Problem (P$_\rho$) *converges in V to the solution u of* Problem (P), *i.e.,*

$$u_\rho \to u \quad \text{in } V \text{ as } \rho \to 0. \tag{4.6}$$

Proof. Let $\rho > 0$. We take $v = u$ in (4.2) and $v = u_\rho$ in (3.1) and add the resulting inequalities to obtain

$$\begin{aligned}(4.7)\quad \langle Au_\rho - Au, u_\rho - u\rangle_{V^*\times V} \le& \int_\Gamma (F_\rho u_\rho - Fu)\big(\varphi(\gamma u) - \varphi(\gamma u_\rho)\big)\, d\Gamma \\ &+ \int_\Gamma \Big(j_\rho^0(\gamma u_\rho; \gamma u - \gamma u_\rho) + j^0(\gamma u; \gamma u_\rho - \gamma u)\Big)\, d\Gamma \\ &+ \langle f_\rho - f, u_\rho - u\rangle_{V^*\times V}.\end{aligned}$$

Let us bound each term in the previous inequality. First, it follows from assumption (3.2)(b) that

$$(4.8)\qquad \langle Au_\rho - Au, u_\rho - u\rangle_{V^*\times V} \ge m_A\|u_\rho - u\|_V^2.$$

Next, we use the properties (3.3), (3.4) of the operator F_ρ and the function φ, respectively, combined with assumption (4.3)(a) to see that

$$\begin{aligned}&\int_\Gamma (F_\rho u_\rho - Fu)\big(\varphi(\gamma u) - \varphi(\gamma u_\rho)\big)\, d\Gamma \\ &\quad\le L_\varphi \|F_\rho u_\rho - Fu\|_{L^2(\Gamma)}\, \|\gamma u_\rho - \gamma u\|_{L^2(\Gamma;\mathbb{R}^s)} \\ &\quad\le L_\varphi \|\gamma\| \Big(\|F_\rho u_\rho - F_\rho u\|_{L^2(\Gamma)} + \|F_\rho u - Fu\|_{L^2(\Gamma)}\Big)\, \|u_\rho - u\|_V \\ &\quad\le L_\varphi \|\gamma\| \Big(L_{F_\rho}\|u_\rho - u\|_V + G(\rho)(\|u\|_V + g)\Big)\, \|u_\rho - u\|_V.\end{aligned}$$

Therefore,

$$\begin{aligned}(4.9)\quad \int_\Gamma (F_\rho u_\rho - Fu)\big(\varphi(\gamma u) - \varphi(\gamma u_\rho)\big)\, d\Gamma \le& L_\varphi L_{F_\rho}\|\gamma\|\|u_\rho - u\|_V^2 \\ &+ L_\varphi G(\rho)\, \|\gamma\|\, (\|u\|_V + g)\, \|u_\rho - u\|_V.\end{aligned}$$

We use the property (3.5)(d) of the function j_ρ combined with assumption (4.4) to get

$$\begin{aligned}&\int_\Gamma \Big(j_\rho^0(\gamma u_\rho; \gamma u - \gamma u_\rho) + j^0(\gamma u; \gamma u_\rho - \gamma u)\Big)\, d\Gamma \\ &\quad= \int_\Gamma \Big(j_\rho^0(\gamma u_\rho; \gamma u - \gamma u_\rho) + j_\rho^0(\gamma u; \gamma u_\rho - \gamma u)\Big)\, d\Gamma \\ &\qquad+ \int_\Gamma \Big(j^0(\gamma u; \gamma u_\rho - \gamma u) - j_\rho^0(\gamma u; \gamma u_\rho - \gamma u)\Big)\, d\Gamma \\ &\quad\le \beta_\rho\|\gamma\|^2\|u_\rho - u\|_V^2 + H(\rho)\int_\Gamma (\|\gamma u\|_{\mathbb{R}^s} + h)\|\gamma u_\rho - \gamma u\|_{\mathbb{R}^s}\, d\Gamma.\end{aligned}$$

Therefore,

$$\begin{aligned}(4.10)\quad &\int_\Gamma \Big(j_\rho^0(\gamma u_\rho; \gamma u - \gamma u_\rho) + j^0(\gamma u; \gamma u_\rho - \gamma u)\Big)\, d\Gamma \\ &\quad\le \beta_\rho\|\gamma\|^2\|u_\rho - u\|_V^2 + H(\rho)\, \|\gamma\| \big(\|\gamma\|\, \|u\|_V + h\sqrt{m(\Gamma)}\big)\, \|u_\rho - u\|_V.\end{aligned}$$

Finally, note that

$$\langle f_\rho - f, u_\rho - u\rangle_{V^*\times V} \le \|f_\rho - f\|_{V^*}\|u_\rho - u\|_V. \tag{4.11}$$

We combine now inequalities (4.7)–(4.11) to deduce that

$$\begin{aligned} m_A\|u_\rho - u\|_V^2 &\le L_\varphi L_{F_\rho}\|\gamma\|\|u_\rho - u\|_V^2 + L_\varphi G(\rho)\,\|\gamma\|\,(\|u\|_V + g)\,\|u_\rho - u\|_V \\ &\quad + \beta_\rho\|\gamma\|^2\|u_\rho - u\|_V^2 + H(\rho)\,\|\gamma\|(\|\gamma\|\,\|u\|_V + h\sqrt{m(\Gamma)})\,\|u_\rho - u\|_V \\ &\quad + \|f_\rho - f\|_{V^*}\|u_\rho - u\|_V \end{aligned}$$

which yields

$$\begin{aligned} &\big(m_A - \beta_\rho\|\gamma\|^2 - L_\varphi L_{F_\rho}\|\gamma\|\big)\|u_\rho - u\|_V \\ &\quad \le L_\varphi G(\rho)\,\|\gamma\|\,(\|u\|_V + g) + H(\rho)\,\|\gamma\|(\|\gamma\|\,\|u\|_V + h\sqrt{m(\Gamma)}) + \|f_\rho - f\|_{V^*}. \end{aligned}$$

We apply assumption (4.1) to see that

$$\begin{aligned} (m_A - m_0)\|u_\rho - u\|_V &\le L_\varphi G(\rho)\,\|\gamma\|\,(\|u\|_V + g) \\ &\quad + H(\rho)\,\|\gamma\|(\|\gamma\|\,\|u\|_V + h\sqrt{m(\Gamma)}) + \|f_\rho - f\|_{V^*}. \end{aligned} \tag{4.12}$$

Theorem 4.1 is now a consequence of inequality (4.12) combined with assumptions (4.3)(b), (4.4)(b) and (4.5). ∎

5. Numerical approximations. In this section, we consider numerical schemes for solving Problem (P). We make the assumptions stated in Theorem 3.2 so that a unique solution $u \in V$ is guaranteed for Problem (P). Let $V^h \subset V$ be a finite dimensional subspace with $h > 0$ denoting a spatial discretization parameter. We consider the following approximation of Problem (P).

Problem (P^h). *Find an element $u^h \in V^h$ such that*

$$\begin{aligned} &\langle Au^h, v^h - u^h\rangle_{V^*\times V} + \int_\Gamma (Fu^h)\big(\varphi(\gamma v^h) - \varphi(\gamma u^h)\big)\,d\Gamma \\ &\quad + \int_\Gamma j^0(\gamma u^h; \gamma v^h - \gamma u^h)\,d\Gamma \ge \langle f, v^h - u^h\rangle_{V^*\times V} \quad \text{for all } v^h \in V^h. \end{aligned} \tag{5.1}$$

The arguments of the proof of Theorem 3.2 can be applied in the setting of the finite dimensional space V^h, and we know that under the assumptions given in Theorem 3.2, Problem (P^h) has a unique solution $u^h \in V^h$. The focus of this section is error analysis for the numerical solution defined by Problem (P^h).

Theorem 5.1. *Assume the conditions stated in Theorem* 3.2. *Moreover, assume $A\colon V \to V^*$ is Lipschitz continuous, i.e., there exists $L_A > 0$ such that*

$$\|Au - Av\|_{V^*} \le L_A\|u - v\|_V \quad \text{for all } u, v \in V, \tag{5.2}$$

and, in addition, $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^s$ for a.e. $\boldsymbol{x} \in \Gamma$, with a Lipschitz constant $L_j > 0$ which does not depend on $\boldsymbol{x}$. Then, there exists a constant c independent of h such that

$$\|u - u^h\|_V \le c \inf_{v^h \in V^h}\left(\|u - v^h\|_V + \|\gamma u - \gamma v^h\|^{1/2}_{L^2(\Gamma;\mathbb{R}^s)}\right). \tag{5.3}$$

Proof. By the strong monotonicity of A, (3.2) (b),

$$m_A\|u-u^h\|_V^2 \le \langle Au-Au^h, u-u^h\rangle_{V^*\times V}. \tag{5.4}$$

Let $v^h \in V^h$ be an arbitrary function from V^h. We write

$$\begin{aligned}\langle Au-Au^h, u-u^h\rangle_{V^*\times V} &= \langle Au-Au^h, u-v^h\rangle_{V^*\times V} + \langle Au, v^h-u\rangle_{V^*\times V}\\ &\quad + \langle Au, u-u^h\rangle_{V^*\times V} + \langle Au^h, u^h-v^h\rangle_{V^*\times V}.\end{aligned} \tag{5.5}$$

We take $v=u^h$ in (3.1) to obtain

$$\begin{aligned}\langle Au, u-u^h\rangle_{V^*\times V} &\le \int_\Gamma (Fu)\big(\varphi(\gamma u^h)-\varphi(\gamma u)\big)\,d\Gamma\\ &\quad + \int_\Gamma j^0(\gamma u;\gamma u^h-\gamma u)\,d\Gamma - \langle f, u^h-u\rangle_{V^*\times V}.\end{aligned} \tag{5.6}$$

By (5.1),

$$\begin{aligned}\langle Au^h, u^h-v^h\rangle_{V^*\times V} &\le \int_\Gamma (Fu^h)\big(\varphi(\gamma v^h)-\varphi(\gamma u^h)\big)\,d\Gamma\\ &\quad + \int_\Gamma j^0(\gamma u^h;\gamma v^h-\gamma u^h)\,d\Gamma - \langle f, v^h-u^h\rangle_{V^*\times V}.\end{aligned} \tag{5.7}$$

Combining (5.4)–(5.7), we have

$$m_A\|u-u^h\|_V^2 \le E_1+E_2+E_3+E_4, \tag{5.8}$$

where

$$E_1 = \langle Au-Au^h, u-v^h\rangle_{V^*\times V},$$

$$\begin{aligned}E_2 &= \langle Au, v^h-u\rangle_{V^*\times V} + \int_\Gamma (Fu)\big(\varphi(\gamma u)-\varphi(2\gamma u-\gamma v^h)\big)\,d\Gamma\\ &\quad - \int_\Gamma j^0(\gamma u;\gamma u-\gamma v^h)\,d\Gamma - \langle f, v^h-u\rangle_{V^*\times V},\end{aligned}$$

$$\begin{aligned}E_3 &= \int_\Gamma \big(Fu-Fu^h\big)\,\big(\varphi(\gamma u^h)-\varphi(\gamma v^h)\big)\,d\Gamma\\ &\quad + \int_\Gamma (Fu)\,\big(\varphi(\gamma v^h)+\varphi(2\gamma u-\gamma v^h)-2\,\varphi(\gamma u)\big)\,d\Gamma,\end{aligned}$$

$$E_4 = \int_\Gamma \big[j^0(\gamma u;\gamma u^h-\gamma u)+j^0(\gamma u^h;\gamma v^h-\gamma u^h)+j^0(\gamma u;\gamma u-\gamma v^h)\big]\,d\Gamma.$$

Let us bound each of the terms E_j, $1\le j\le 4$. First, by the Lipschitz continuity of A,

$$E_1 \le \|Au-Au^h\|_{V^*}\|u-v^h\|_V \le L_A\|u-u^h\|_V\|u-v^h\|_V. \tag{5.9}$$

Next, we replace v by $2u - v$ in (3.1) to get

$$\langle Au, u-v\rangle_{V^*\times V} + \int_\Gamma (Fu)\big(\varphi(2\gamma u - \gamma v) - \varphi(\gamma u)\big)\, d\Gamma$$
$$+ \int_\Gamma j^0(\gamma u; \gamma u - \gamma v)\, d\Gamma \ge \langle f, u-v\rangle_{V^*\times V} \quad \text{for all } v \in V.$$

Then,

$$E_2 \le 0. \tag{5.10}$$

Applying (3.3) (a) and (3.4) (c), we have

$$\begin{aligned}
&\int_\Gamma \left(Fu - Fu^h\right) \left(\varphi(\gamma u^h) - \varphi(\gamma v^h)\right) d\Gamma \\
&\quad \le \|Fu - Fu^h\|_{L^2(\Gamma)} \|\varphi(\gamma u^h) - \varphi(\gamma v^h)\|_{L^2(\Gamma;\mathbb{R}^s)} \\
&\quad \le L_F L_\varphi \|\gamma\|\, \|u - u^h\|_V \|u^h - v^h\|_V \\
&\quad \le L_F L_\varphi \|\gamma\|\, \|u - u^h\|_V^2 + L_F L_\varphi \|\gamma\|\, \|u - u^h\|_V \|u - v^h\|_V,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Gamma (Fu) \left(\varphi(\gamma v^h) + \varphi(2\gamma u - \gamma v^h) - 2\,\varphi(\gamma u)\right) d\Gamma \\
&\quad \le \|Fu\|_{L^2(\Gamma;\mathbb{R}^s)} \|\varphi(\gamma v^h) + \varphi(2\gamma u - \gamma v^h) - 2\,\varphi(\gamma u)\|_{L^2(\Gamma;\mathbb{R}^s)} \\
&\quad \le 2\, L_\varphi \|Fu\|_{L^2(\Gamma;\mathbb{R}^s)} \|\gamma u - \gamma v^h\|_{L^2(\Gamma;\mathbb{R}^s)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
E_3 \le{}& L_F L_\varphi \|\gamma\|\, \|u - u^h\|_V^2 + L_F L_\varphi \|\gamma\|\, \|u - u^h\|_V \|u - v^h\|_V \\
&+ 2\, L_\varphi \|Fu\|_{L^2(\Gamma;\mathbb{R}^s)} \|\gamma u - \gamma v^h\|_{L^2(\Gamma;\mathbb{R}^s)}.
\end{aligned} \tag{5.11}$$

Finally, to bound E_4, we note that

$$j^0(\gamma u^h; \gamma v^h - \gamma u^h) \le j^0(\gamma u^h; \gamma v^h - \gamma u) + j^0(\gamma u^h; \gamma u - \gamma u^h) \quad \text{a.e. on } \Gamma_3.$$

Use the condition (3.5) (d),

$$j^0(\gamma u; \gamma u^h - \gamma u) + j^0(\gamma u^h; \gamma u - \gamma u^h) \le \beta\, \|\gamma u - \gamma u^h\|_{\mathbb{R}^s}^2 \quad \text{a.e. on } \Gamma_3.$$

Also,

$$j^0(\gamma u^h; \gamma v^h - \gamma u) \le L_j \|\gamma v^h - \gamma u\|_{\mathbb{R}^s},$$
$$j^0(\gamma u; \gamma u - \gamma v^h) \le L_j \|\gamma u - \gamma v^h\|_{\mathbb{R}^s},$$

a.e. on Γ_3. Hence,

$$E_4 \le \beta\, \|\gamma\|^2 \|u - u^h\|_V^2 + 2\, L_j \|\gamma u - \gamma v^h\|_{L^2(\Gamma;\mathbb{R}^s)}. \tag{5.12}$$

Using (5.9)–(5.12) in (5.8), we have

$$m_A\|u-u^h\|_V^2 \le (L_A + L_F L_\varphi \|\gamma\|)\,\|u-u^h\|_V \|u-v^h\|_V$$
$$+ \left(L_F L_\varphi\|\gamma\| + \beta\,\|\gamma\|^2\right)\|u-u^h\|_V^2$$
$$+ 2\left(L_\varphi \|Fu\|_{L^2(\Gamma;\mathbb{R}^s)} + L_j\right)\|\gamma u - \gamma v^h\|_{L^2(\Gamma;\mathbb{R}^s)}.$$

Recall the condition (3.8) and bound the first term on the right as follows:

$$(L_A + L_F L_\varphi\|\gamma\|)\,\|u-u^h\|_V\|u-v^h\|_V \le \delta\,\|u-u^h\|_V^2 + C(\delta)\,\|u-v^h\|_V^2$$

with a sufficiently small $\delta > 0$. Then we derive from the above relation the following inequality

$$\|u-u^h\|_V^2 \le c\left(\|u-v^h\|_V^2 + \|\gamma u - \gamma v^h\|_{L^2(\Gamma;\mathbb{R}^s)}\right)$$

where c represents a positive constant which does not depend on h. Since $v^h \in V^h$ is arbitrary, we the conclude the error bound (5.3). □

The inequality (5.3) is the basis for convergence analysis and error estimation. Indeed, let $\{V^h\}_{h>0}$ be a family of finite dimensional subspaces of V such that

(5.13) for any $v \in V$, there exists $v^h \in V^h$ such that $v^h \to v$ in V as $h \to 0^+$.

In other words, any function from the space V can be approximated by functions from V^h when $h \to 0^+$. We have the following convergence result.

Corollary 5.2. *Assume* (5.13) *and the conditions stated in Theorem* 5.1. *Then we have convergence of the numerical solutions:*

(5.14) $$\|u-u^h\|_V \to 0 \quad \text{as } h \to 0.$$

To present a result of concrete error estimate, we consider the finite element method. Let us assume Ω is a polygonal/polyhedral domain and express the parts of the boundary, Γ and $\partial\Omega\backslash\Gamma$ as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma} = \cup_{i=1}^{i_0}\Gamma_{(i)}, \quad \overline{\partial\Omega\backslash\Gamma} = \cup_{i=i_0+1}^{i_1}\Gamma_{(i)}.$$

We introduce a regular family of partitions $\{\mathcal{T}^h\}$ of $\overline{\Omega}$ into triangles/tetrahedrons. The triangulations are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{(i)}$, $1 \le i \le i_1$, in the sense that if the intersection of one side/face of an element with one of the sets $\Gamma_{(i)}$, $1 \le i \le i_1$, has a positive measure relative to $\Gamma_{(i)}$, then the side/face lies entirely in $\Gamma_{(i)}$. For the finite dimensional subspace V^h, we use the linear element corresponding to $\mathcal{T}^h$:

$$V^h = \left\{v^h \in V \mid v^h|_T \in \mathbb{P}_1(T)^d \text{ for all } T \in \mathcal{T}^h\right\},$$

where $\mathbb{P}_1(T)$ is the space of polynomials of degree less than or equal to one over T. Then the property (5.13) is valid. Using the standard finite element interpolation error estimates ([1, 5]), we can derive the following error estimate from Theorem 5.1.

Corollary 5.3. *Under the assumptions stated in Theorem* 5.1 *as well as the solution regularities $u \in H^2(\Omega;\mathbb{R}^d)$ and $\gamma u|_{\Gamma_{(i)}} \in H^2(\Gamma_{(i)};\mathbb{R}^d)$, $1 \le i \le i_0$, we have the optimal order error bound*

(5.15) $$\|u-u^h\|_V \le c\,h.$$

6. A frictional contact problem. Several static contact problems with elastic materials lead to a variational-hemivariational inequality of the form (3.1) in which the unknown is the displacement field. For a variety of such inequalities, the results in Section 3–5 can be applied. We illustrate this point here on a representative contact problem.

The physical setting is the following. An elastic body occupies a regular domain Ω of $\mathbb{R}^d$ ($d = 2, 3$) with its boundary $\Gamma = \partial\Omega$ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that the measure of Γ_1, denoted $m(\Gamma_1)$, is positive. The body is clamped on Γ_1 and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and volume forces of density $\boldsymbol{f}_0$ act in Ω. The body is allowed to arrive in contact with an obstacle on Γ_3, the so-called foundation. The contact is modeled with a nonmonotone normal compliance condition associated with one version of Coulomb's law of dry friction. We are interested in the equilibrium process of the mechanical state of the body.

We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below, the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. We denote by $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the displacement vector, the stress tensor, and the linearized strain tensor, respectively. Recall that $\varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$, where $u_{i,j} = \partial u_i/\partial x_j$. We use $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \ \boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for all } \ \boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. For a vector field, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively.

The classical formulation of the contact problem is stated as follows.

PROBLEM (P_C). *Find a displacement field* $\boldsymbol{u} \colon \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} \colon \Omega \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{6.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{6.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{6.3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{6.4}$$

$$-\sigma_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3, \tag{6.5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \ \text{ if } \ \boldsymbol{u}_\tau \neq \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{6.6}$$

We now present a short description of the equations and conditions in PROBLEM (P_C) and we refer the reader to [14, 21, 28] for more details and mechanical interpretation. Equation (6.1) is the constitutive law for elastic materials in which $\mathcal{F}$ represents the elasticity operator. Equation (6.2) is the equilibrium equation for the static contact process. On Γ_1, we have the clamped boundary condition (6.3), and on Γ_2, the surface traction boundary condition (6.4). Relation (6.5) is the contact condition in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν, and (6.6) represents a version of static Coulomb's law of dry friction. Here F_b is a given positive function, the friction bound.

We note that (6.5) represents the nonmonotone contact condition with normal compliance, which describes the contact with a reactive foundation. In this condition the scalar u_ν, when positive, may be interpreted as the interpenetration between the asperities of the body's contact surface and the foundation. The condition can model the contact with an elastic, rigid-perfect plastic and rigid-plastic foundation. Due to the nonmonotonicity of ∂j_ν, the condition allows to describe the hardening of the softening phenomena of the foundation. Various examples and mechanical interpretation associated with this contact condition can be found in [21].

The friction bound F_b in (6.6) is assumed to depend on the normal displacement u_ν. This assumption is reasonable for any version of Coulomb's friction law when associated with the normal compliance contact condition. Indeed, in the Coulomb's law the friction bound is assumed to depend on the normal stress which, in turn, in the normal compliance condition, is assumed to depend on the normal displacement. Consequently, dependence of the friction bound on the normal displacement results, as is shown in (6.6). This dependence makes sense from the physical point of view, since it takes into account the influence of the asperities of the contact surface on the friction law. Assume, for instance, that F_b is an increasing function which vanishes for a negative argument. We deduce that when there is no penetration (i.e. when $u_\nu < 0$) then the friction bound vanishes, when the penetration is small then the friction bound is small and, finally, when the penetration is large then the friction bound is large. Physically, when there is no penetration, there is separation between the body and the foundation and so there is no friction, justifying a vanishing friction bound; when the penetration is small, contact takes place at the tips of the asperities, justifying the small value of the friction bound; when the penetration becomes larger, the actual contact area increases, justifying a larger value of the friction bound.

In the study of Problem (P_C) we use standard notation for Lebesgue and Sobolev spaces. For all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ we still denote by $\boldsymbol{v}$ the trace of $\boldsymbol{v}$ on Γ, and we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on Γ. We introduce spaces V and $\mathcal{H}$ defined by

$$V = \{\ \boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \Gamma_1\ \},$$
$$\mathcal{H} = \{\ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{S}^d) \mid \tau_{ij} = \tau_{ji},\ 1 \le i,j \le d\ \}.$$

The space $\mathcal{H}$ is a real Hilbert space with the canonical inner product given by

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx,$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. Since $m(\Gamma_1) > 0$, it is well known that V is a real Hilbert space with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \boldsymbol{u},\boldsymbol{v} \in V \tag{6.7}$$

and the associated norm $\|\cdot\|_V$. By the Sobolev trace theorem,

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\gamma\|\,\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V, \tag{6.8}$$

$\|\gamma\|$ being the norm of the trace operator $\gamma\colon V \to L^2(\Gamma_3;\mathbb{R}^d)$.

We make the following assumptions on the problem data. For the elasticity operator $\mathcal{F}$, assume

$$(6.9)\qquad \begin{cases} \mathcal{F}\colon \Omega\times\mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \text{(a) there exists } L_{\mathcal{F}}>0 \text{ such that} \\ \quad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\| \\ \qquad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \ \text{a.e. } \boldsymbol{x}\in\Omega; \\ \text{(b) there exists } m_{\mathcal{F}}>0 \text{ such that} \\ \quad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2 \\ \qquad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \ \text{a.e. } \boldsymbol{x}\in\Omega; \\ \text{(c) the mapping } \ \boldsymbol{x}\mapsto\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d; \\ \text{(d) } \mathcal{F}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{0} \ \text{ a.e. } \boldsymbol{x}\in\Omega. \end{cases}$$

For the friction bound F_b and the potential function j_ν, assume

$$(6.10)\qquad \begin{cases} F_b\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that} \\ \text{(a) there exists } L_{F_b}>0 \text{ such that} \\ \quad |F_b(\boldsymbol{x},r_1)-F_b(\boldsymbol{x},r_2)| \le L_{F_b}|r_1-r_2| \\ \qquad \text{for all } \ r_1,r_2\in\mathbb{R}, \ \text{a.e. } \boldsymbol{x}\in\Gamma_3; \\ \text{(b) the mapping } \boldsymbol{x}\mapsto F_b(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3, \\ \quad \text{for all } r\in\mathbb{R}; \\ \text{(c) } F_b(\boldsymbol{x},r)\ge 0 \ \text{ for all } r\in\mathbb{R}, \ \text{a.e. } \boldsymbol{x}\in\Gamma_3; \\ \text{(d) } F_b(\boldsymbol{x},0)=0 \ \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases}$$

$$(6.11)\qquad \begin{cases} j_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R} \text{ is such that} \\ \text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R} \text{ and there} \\ \quad \text{exists } \overline{e}\in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,\overline{e}(\cdot))\in L^1(\Gamma_3); \\ \text{(b) } j_\nu(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x}\in\Gamma_3; \\ \text{(c) } |\partial j_\nu(\boldsymbol{x},r)| \le \overline{c}_0+\overline{c}_1\,|r| \text{ for a.e. } \boldsymbol{x}\in\Gamma_3, \\ \quad \text{for all } r\in\mathbb{R} \text{ with } \overline{c}_0,\ \overline{c}_1\ge 0; \\ \text{(d) } j_\nu^0(\boldsymbol{x},r_1;r_2-r_1)+j_\nu^0(\boldsymbol{x},r_2;r_1-r_2)\le\overline{\beta}\,|r_1-r_2|^2 \\ \quad \text{for a.e. } \boldsymbol{x}\in\Gamma_3, \ \text{all } r_1,\ r_2\in\mathbb{R} \text{ with } \overline{\beta}\ge 0. \end{cases}$$

For the densities of body forces and surface tractions, assume

$$(6.12)\qquad \boldsymbol{f}_0\in L^2(\Omega;\mathbb{R}^d), \quad \boldsymbol{f}_2\in L^2(\Gamma_2;\mathbb{R}^d).$$

Following a standard approach (cf. [14, 21]), we can derive the next variational formulation for Problem (P_C).

Problem (P_V). *Find a displacement field* $\boldsymbol{u}\in V$ *such that*

$$(6.13)\qquad (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}+\int_{\Gamma_3}F_b(u_\nu)(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_\tau\|)\,d\Gamma$$
$$+\int_{\Gamma_3}j_\nu^0(u_\nu;v_\nu-u_\nu)\,d\Gamma \ge \langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v}\in V,$$

where $\boldsymbol{f}\in V^*$ is given by

$$(6.14)\qquad \langle\boldsymbol{f},\boldsymbol{v}\rangle_{V^*\times V}=(\boldsymbol{f}_0,\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)}+(\boldsymbol{f}_2,\boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \text{for all } \boldsymbol{v}\in V.$$

We have the following existence and uniqueness result for PROBLEM (P_V).

THEOREM 6.1. *Assume the hypotheses* (6.9)–(6.12) *and the smallness condition*

$$L_{F_b}\|\gamma\| + \overline{\beta}\,\|\gamma\|^2 \le m_{\mathcal{F}}. \tag{6.15}$$

Assume moreover that one of the following inequalities hold:

(i) $m_{\mathcal{F}} > \overline{c}_1\sqrt{2}\,\|\gamma\|^2$,

(ii) $j_\nu^0(\boldsymbol{x}, r; -r) \le \overline{d}\,(1+|r|)$ *for all* $r \in \mathbb{R}$, *a.e.* $\boldsymbol{x} \in \Gamma_3$ *with* $\overline{d} \ge 0$.

Then PROBLEM (P_V) *has a unique solution* $\boldsymbol{u} \in V$.

Proof. We apply Theorem 3.2 with $\Gamma = \Gamma_3 \subset \partial\Omega$, $s = d$,

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{6.16}$$

$$(F\boldsymbol{v})(\boldsymbol{x}) = F_b(\boldsymbol{x}, v_\nu(\boldsymbol{x})) \quad \text{for } \boldsymbol{v} \in V,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{6.17}$$

$$j(\boldsymbol{x}, \boldsymbol{\xi}) = j_\nu(\boldsymbol{x}, \xi_\nu) \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{6.18}$$

$$\varphi(\boldsymbol{x}, \boldsymbol{\xi}) = \|\boldsymbol{\xi}_\tau\| \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{6.19}$$

and $f \equiv \boldsymbol{f} \in V^*$ defined by (6.14). Then it can be verified that for A defined by (6.16), (6.9) implies (3.2) with $\alpha = m_A = m_{\mathcal{F}}$; for F defined by (6.17), (6.10) implies (3.3) with $L_F = L_{F_b}\|\gamma\|$; the function $\varphi\colon \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ defined by (6.19) satisfies (3.4) with $L_\varphi = 1$; and (6.12) implies (3.6). For the function j defined by (6.18), the conditions (3.5)(a) and (b) follow from (6.11)(a) and (b), respectively. The properties (3.5)(c) and (d) are consequences of the relations

$$\partial j(\boldsymbol{x}, \boldsymbol{\xi}) \subset \partial j_\nu(\boldsymbol{x}, \xi_\nu)\boldsymbol{\nu},\ \ j^0(\boldsymbol{x}, \boldsymbol{\xi}; \boldsymbol{\eta}) \le j_\nu^0(\boldsymbol{x}, \xi_\nu; \eta_\nu) \ \text{ for all } \ \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Gamma_3$$

combined with the hypothesis (6.11)(c) and (d). Thus, (3.5) holds with $\beta = \overline{\beta}$ and $c_1 = \overline{c}_1$. Therefore, Theorem 6.1 holds as a corollary of Theorem 3.2. ∎

In addition, Theorem 4.1 can be used to study the dependence of the weak solution of PROBLEM (P_V) with respect to perturbations of the data and to prove its continuous dependence on the friction bound, the normal compliance function, and the densities of body forces and surface tractions. We omit the detail here. Instead, we provide an example of functions F_b, $F_{b\rho}$ and j_ν, $j_{\nu\rho}$ which satisfies conditions (6.10) and (6.11), such that the corresponding operators F, F_ρ defined by (6.17) satisfy the condition (4.3) and the coresponding functions j, j_ρ defined by (6.18) satisfy the condition (4.4). In the following, ρ is a positive parameter.

Consider the functions μ and μ_ρ which satisfy

$$\left\{\begin{array}{ll} \text{(a) } \mu \in L^\infty(\Gamma_3), & \mu(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(b) } \mu_\rho \in L^\infty(\Gamma_3), & \mu_\rho(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } \mu_\rho \to \mu \quad \text{in } L^\infty(\Gamma_3) \quad \text{as } \rho \to 0. & \end{array}\right. \tag{6.20}$$

We define the functions F_b and $F_{b\rho}$ by equalities

$$F_b(\boldsymbol{x}, r) = \mu(\boldsymbol{x})\, r^+, \quad F_{b\rho}(\boldsymbol{x}, r) = \mu_\rho(\boldsymbol{x})\, r^+ \quad \text{for all } r \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,$$

where r^+ represents the positive part of r. Let $F\colon V \to L^2(\Gamma_3)$ and $F_\rho\colon V \to L^2(\Gamma_3)$ be the operators defined by

$$(F\boldsymbol{v})(\boldsymbol{x}) = F_b(\boldsymbol{x}, v_\nu(\boldsymbol{x})), \quad (F_\rho\boldsymbol{v})(\boldsymbol{x}) = F_{b\rho}(\boldsymbol{x}, v_\nu(\boldsymbol{x})) \quad \text{for all } \boldsymbol{v} \in V,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.$$

Then, the functions F_b and $F_{b\rho}$ satisfy conditions (6.10), and it follows from the trace inequality (6.8) that

$$\|F_\rho \boldsymbol{v} - F\boldsymbol{v}\|_{L^2(\Gamma_3)} \le \|\gamma\|\,\|\mu_\rho - \mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V. \tag{6.21}$$

We use inequality (6.21) and assumption (6.20)(c) to see that (4.3) holds.

Define $p_\nu\colon \mathbb{R} \to \mathbb{R}$ by

$$p_\nu(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ r & \text{if} \quad 0 \le r < 1, \\ 2 - r & \text{if} \quad 1 \le r < 2, \\ \sqrt{r-2} + r - 2 & \text{if} \quad 2 \le r < 6, \\ r & \text{if} \quad r \ge 6. \end{cases} \tag{6.22}$$

This function is continuous, yet neither monotone nor Lipschitz continuous. Then define the function $j_\nu\colon \mathbb{R} \to \mathbb{R}$ by

$$j_\nu(r) = \int_0^r p_\nu(s)\,ds \quad \text{for all } r \in \mathbb{R}. \tag{6.23}$$

Note that j_ν is not convex. Since $j_\nu'(r) = p_\nu(r)$ for all $r \in \mathbb{R}$, j_ν is a C^1 function, and thus is a locally Lipschitz function. Since $|p_\nu(r)| \le |r|$ for $r \in \mathbb{R}$, we know that j_ν satisfies (6.11)(c). The function $r \mapsto r + p_\nu(r) \in \mathbb{R}$ is nondecreasing and therefore,

$$(p_\nu(r_1) - p_\nu(r_2))(r_2 - r_1) \le (r_1 - r_2)^2 \quad \text{for all } r_1, r_2 \in \mathbb{R}.$$

We combine this inequality with equality $j_\nu^0(r_1; r_2) = p_\nu(r_1) r_2$, valid for all r_1, $r_2 \in \mathbb{R}$, to see that condition (6.11)(d) is satisfied with $\overline{\beta} = 1$. Hence, j_ν satisfies the hypothesis (6.11).

Consider now the function $p_{\nu\rho}\colon \mathbb{R} \to \mathbb{R}$ defined by

$$p_{\nu\rho}(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ r & \text{if} \quad 0 \le r < 1 - \rho, \\ \frac{\rho-1}{\rho+1} r - \frac{2(\rho-1)}{\rho+1} & \text{if} \quad 1 - \rho \le r < 2, \\ \sqrt{r-2} + r - 2 & \text{if} \quad 2 \le r < 6, \\ r & \text{if} \quad r \ge 6 \end{cases} \tag{6.24}$$

and define the function $j_{\nu\rho}\colon \mathbb{R} \to \mathbb{R}$ by

$$j_{\nu\rho}(r) = \int_0^r p_{\nu\rho}(s)\,ds \quad \text{for all } r \in \mathbb{R}.$$

A similar argument shows that $j_{\nu\rho}$ satisfies the hypothesis (6.11).

Let $j\colon \mathbb{R}^d \to \mathbb{R}$ and $j_\rho\colon \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ be the functions given by

$$j(\boldsymbol{\xi}) = j_\nu(\xi_\nu), \quad j_\rho(\boldsymbol{\xi}) = j_{\nu\rho}(\xi_\nu) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d.$$

Then

$$j^0(\boldsymbol{\xi}; \boldsymbol{\eta}) - j_\rho^0(\boldsymbol{\xi}; \boldsymbol{\eta}) = \big(p_\nu(\xi_\nu) - p_{\nu\rho}(\xi_\nu)\big)\eta_\nu \quad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d, \tag{6.25}$$

and

$$
(6.26) \qquad |p_\nu(r) - p_{\nu\rho}(r)| \le \frac{2\rho}{\rho+1}\,|r| \quad \text{for all } r\in\mathbb{R}.
$$

We now combine inequalities (6.25) and (6.26) to see that (4.4) holds. We conclude from here that the convergence result in Theorem 4.1 can be used in the study of the corresponding frictional contact problem.

Finally, we consider numerical approximations of Problem (P_V). Here, the results in Section 5 apply. Let $\{V^h\}_{h>0}$ be a family of finite dimensional subspaces of V. Then we consider the following approximation of Problem (P_V).

Problem (P_V^h). *Find a displacement field $\boldsymbol{u}^h \in V^h$ such that*

$$
(6.27) \quad (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^h))_{\mathcal{H}} + \int_{\Gamma_3} F_b(u_\nu^h)(\|\boldsymbol{v}_\tau^h\| - \|\boldsymbol{u}_\tau^h\|)\, d\Gamma
$$
$$
+ \int_{\Gamma_3} j_\nu^0(u_\nu^h; v_\nu^h - u_\nu^h)\, d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v}^h \in V^h.
$$

To apply Theorem 5.1, we verify the assumptions of the theorem. Using (6.9)(a), we easily establish the Lipschitz continuity of A,

$$
\|A\boldsymbol{u} - A\boldsymbol{v}\|_{V^*} \le L_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V.
$$

So (5.2) holds. In applications, for j_ν defined by (6.23), without loss of generality, we may assume $p_\nu(r)$ to be bounded for $r > 0$ since if the normal component of the displacement u_ν becomes too large, then part of the contact surface breaks down and the formulation of Problem (P_V) is no longer suitable for the contact process. Hence, it can be assumed that the Lipschitz constant for j_ν is independent of the arguments of j_ν. Thus, Theorem 5.1 can be applied, and we have from (5.3) that

$$
(6.28) \qquad \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \le c \inf_{\boldsymbol{v}^h \in V^h} \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V + \|u_\nu - v_\nu^h\|_{L^2(\Gamma_3)}^{1/2} \right).
$$

Hence, if the finite dimensional subspaces $\{V^h\}_{h>0}$ are chosen so that any function in V can be approximated by a sequence of functions from $\{V^h\}_{h>0}$ as $h \to 0$, then we have the convergence of the numerical solutions:

$$
\|\boldsymbol{u} - \boldsymbol{u}^h\|_V \to 0 \quad \text{as } h \to 0.
$$

This is the case when we use the finite element method to construct V^h. As a sample, assume Ω is a polygonal/polyhedral domain and express the three parts of the boundary, Γ_k, $1 \le k \le 3$, as unions of closed flat components with disjoint interiors:

$$
\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \le k \le 3.
$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $1 \le k \le 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Then use the linear element corresponding to $\mathcal{T}^h$:

$$
V^h = \left\{ \boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d,\ T \in \mathcal{T}^h, \text{ and } \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1 \right\}.
$$

Then under additional solution regularity assumptions $\boldsymbol{u} \in H^2(\Omega;\mathbb{R}^d)$ and $u_\nu|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i};\mathbb{R}^d)$, $1 \le i \le i_3$, we have the optimal order error bound

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V \le c\,h.$$

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