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Well-posedness of a general class of elliptic mixed hemivariational–variational inequalities ☆

Weimin Han [a,*], Andaluzia Matei [b]

[a] Department of Mathematics, University of Iowa, Iowa City, IA 52242-1410, USA
[b] Department of Mathematics, University of Craiova, A.I. Cuza 13, 200585, Craiova, Romania

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ABSTRACT

In this paper, well-posedness of a general class of elliptic mixed hemivariational–variational inequalities is studied. This general class includes several classes of the previously studied elliptic mixed hemivariational–variational inequalities as special cases. Moreover, our approach of the well-posedness analysis is easily accessible, unlike those in the published papers on elliptic mixed hemivariational–variational inequalities so far. First, prior theoretical results are recalled for a class of elliptic mixed hemivariational–variational inequalities featured by the presence of a potential operator. Then the well-posedness results are extended through a Banach fixed-point argument to the same class of inequalities without the potential operator assumption. The well-posedness results are further extended to a more general class of elliptic mixed hemivariational–variational inequalities through another application of the Banach fixed-point argument. The theoretical results are illustrated in the study of a contact problem. For comparison, the contact problem is studied both as an elliptic mixed hemivariational–variational inequality and as an elliptic variational–hemivariational inequality.

1. Introduction

The main goal of this paper is to provide a well-posedness analysis of a general class of elliptic mixed hemivariational–variational inequalities of the following form:

Problem 1.1. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au, v-u\rangle + b(v-u,\lambda) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u;v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K_V, \tag{1.1}$$

$$b(u,\mu-\lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{1.2}$$

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* Corresponding author.
E-mail addresses: weimin-han@uiowa.edu (W. Han), andaluziamatei@inf.ucv.ro (A. Matei).

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In Problem 1.1, K_V and K_Λ are closed convex sets in Hilbert spaces, A is a Lipschitz continuous and strongly monotone operator, b is a continuous bilinear form, Φ is continuous and convex with respect to its second argument, Ψ is locally Lipschitz continuous and Ψ^0 denotes its generalized directional derivative in the sense of Clarke, and f is a given linear continuous functional. Precise descriptions of the assumptions on the data will be given in Section 4. Of particular relevance to applications is the case where the term $\Psi^0(u; v-u)$ in (1.1) is replaced by $I_\Delta(\psi^0(u; v-u))$, which is the integration over Δ of the generalized directional derivative of a function ψ and Δ represents a subset of the spatial domain of the application problem or a part of the boundary of the spatial domain. The corresponding elliptic mixed hemivariational–variational inequality is studied at the end of Section 4.

Elliptic mixed hemivariational–variational inequalities of special types have been studied in the literature. For example, the problem of finding $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$\begin{aligned} &\langle Au, v-u\rangle + b(v-u, \lambda) + I_\Delta(\psi^0(u; v-u)) \ge \langle f, v-u\rangle \quad \forall\, v \in K_V,\\ &b(u, \mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda \end{aligned}$$

was studied in [1], where I_Δ represents the integration over a measurable set Δ of the boundary of the spatial domain for the problem. A closely related problem of finding $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$\begin{aligned} &\langle Au, v-u\rangle + b(v-u, \lambda) + \Psi^0(u; v-u) \ge \langle f, v-u\rangle \quad \forall\, v \in K_V,\\ &b(u, \mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda \end{aligned}$$

was studied in [2,3]. The results in [2] are extended in [4] to the following problem of finding $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au, v-u\rangle + b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \ge \langle f, v-u\rangle \quad \forall\, v \in K_V, \tag{1.3}$$

$$b(u, \mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{1.4}$$

In these references, the main theoretical tools for proving solution existence are rather complicated fixed-point principles for set-valued mappings. In the more recent paper [5], under an additional assumption that A is a potential operator, the problem (1.3)–(1.4) is studied based on an equivalent saddle-point formulation. A minimax principle on the saddle-point formulation provides an elementary proof of the solution existence on the mixed hemivariational–variational inequality, thus avoiding the need of the complicated fixed-point principles for set-valued mappings.

The novelty of this paper is reflected in two aspects. First, Problem 1.1 and its analogue where Ψ^0 is replaced by $I_\Delta(\psi^0)$ are more general than the mixed hemivariational–variational inequalities studied in the references cited above. Second, we prove the well-posedness of the problems without applying the complicated fixed-point principles for set-valued mappings and our technique is easily accessible by applied mathematicians and engineers. More precisely, we start with the well-posedness results proved in [5] on the problem (1.3)–(1.4), and extend the solution existence and uniqueness without requiring the operator A to be potential. Then we further extend the well-posedness analysis to Problem 1.1 through the use of a basic fixed-point principle: the Banach fixed-point theorem.

Hemivariational inequalities are useful for applications involving non-smooth, non-monotone and set-valued relations among physical quantities. Since the pioneering work of Panagiotopoulos four decades ago [6], modelling, analysis, numerical solution and applications of hemivariational inequalities have attracted more and more attention from the research community. One has witnessed a substantial increase in the number of publications related to hemivariational inequalities in recent years. As representative recent references, one is referred to [7] for well-posedness analysis results of hemivariational inequalities, and to [8] for a survey of numerical analysis of hemivariational inequalities.

Mixed formulations are known to be useful especially in the numerical solution of problems with certain constraints. A well-known example of mixed formulations is the treatment of the incompressibility constraint for fluid flow problems. Moreover, mixed formulations are useful in developing efficient numerical methods for the computation of physical quantities other than the original unknown variable of the underlying partial differential equations. See [9] on mixed finite element methods for solving a variety of boundary value problems through their mixed formulations. Mixed hemivariational inequalities and their numerical solutions of the Stokes equations and Navier–Stokes equations can be found in [10–12].

The rest of the paper is organized as follows. In Section 2, we review preliminary materials needed later; in particular, we review results from [5] on solution existence results and minimax principles for the elliptic mixed hemivariational–variational inequality (1.3)–(1.4) when the operator A is potential. In Section 3, the elliptic mixed hemivariational–variational inequality (1.3)–(1.4) is reconsidered and solution existence is proved without assuming the operator A to be potential. In Section 4, well-posedness is proved for the more general elliptic mixed hemivariational–variational inequality (1.1)–(1.2) and its analogue where Ψ^0 is replaced by $I_\Delta(\psi^0)$. In Section 5, we apply the theoretical results in the study of a contact problem. For comparison, the contact problem is studied both as an elliptic mixed hemivariational–variational inequality and as an elliptic variational–hemivariational inequality.

2. Preliminaries

In this section, we review some results proved in [5] on the elliptic mixed hemivariational–variational inequality (1.3)–(1.4) when the operator A is assumed to be potential. In describing hemivariational inequalities, we need the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke for a locally Lipschitz continuous function [13]. Let $\Psi: V \to \mathbb{R}$ be a locally Lipschitz continuous functional defined on a real Banach space V. Then its generalized (Clarke) directional derivative at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\Psi(w+\lambda v) - \Psi(w)}{\lambda},$$

whereas the generalized subdifferential of Ψ at $u \in V$ is

$$\partial\Psi(u) := \left\{\eta \in V^* \mid \Psi^0(u;v) \ge \langle \eta, v\rangle \ \forall\, v \in V\right\}.$$

We note that if $\Psi: V \to \mathbb{R}$ is locally Lipschitz continuous and convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$. Hence, the notion of the Clarke subdifferential can be viewed as a generalization of that of the convex subdifferential. For all $\lambda \in \mathbb{R}$ and all $u \in V$, we have

$$\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u).$$

Moreover, for locally Lipschitz functions $\Psi_1, \Psi_2: V \to \mathbb{R}$, the inclusion

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{2.1}$$

holds, which is equivalent to the inequality

$$(\Psi_1 + \Psi_2)^0(u;v) \le \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{2.2}$$

Detailed discussions of the generalized directional derivative and the generalized subdifferential for locally Lipschitz continuous functionals, including their properties, can be found in several references, e.g. [13,14].

Following [5], from now on, we let V and Λ be two real Hilbert spaces. Their dual spaces are denoted by V^* and Λ^*. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V^* and V, or between Λ^* and Λ; it should be clear from the context which duality pairing is meant by $\langle\cdot,\cdot\rangle$. Let $K_V \subset V$ and $K_\Lambda \subset \Lambda$.

Given operators and functionals $A: V \to V^*$, $b: V \times \Lambda \to \mathbb{R}$, $\Phi: V \to \mathbb{R}$, $\Psi: V \to \mathbb{R}$, and $f \in V^*$, we consider the mixed inequality problem defined by (1.3)–(1.4).

Problem 2.1. Find $(u,\lambda)\in K_V\times K_\Lambda$ such that

$$\langle Au, v-u\rangle + b(v-u,\lambda) + \Phi(v) - \Phi(u) + \Psi^0(u;v-u) \ge \langle f, v-u\rangle \quad \forall\, v\in K_V, \tag{2.3}$$

$$b(u,\mu-\lambda)\le 0 \quad \forall\,\mu\in K_\Lambda. \tag{2.4}$$

In the study of Problem 2.1, we will use the following conditions on the problem data.

- $H(K_V)$ V is a real Hilbert space, $K_V\subset V$ is non-empty, closed and convex.
- $H(K_\Lambda)$ Λ is a real Hilbert space, $K_\Lambda\subset\Lambda$ is non-empty, closed and convex.
- $H(A)$ $A\colon V\to V^*$ is Lipschitz continuous and strongly monotone.
- $H(b)$ $b: V\times\Lambda\to\mathbb{R}$ is bilinear and bounded.
- $H(\Phi)$ $\Phi\colon V\to\mathbb{R}$ is convex and continuous.
- $H(\Psi)$ $\Psi\colon V\to\mathbb{R}$ is locally Lipschitz continuous, and there exists a constant $\alpha_\Psi\ge 0$ such that

$$\Psi^0(v_1;v_2-v_1)+\Psi^0(v_2;v_1-v_2)\le \alpha_\Psi\|v_1-v_2\|_V^2 \quad \forall\, v_1,v_2\in V. \tag{2.5}$$

- $H(f)$ $f\in V^*$.

Related to the condition $H(A)$, we will use $M_A>0$ for the Lipschitz constant:

$$\|Av_1-Av_2\|_{V^*}\le M_A\|v_1-v_2\|_V \quad \forall\, v_1,v_2\in V, \tag{2.6}$$

and use $m_A>0$ for the strong monotonicity constant:

$$\langle Av_1-Av_2, v_1-v_2\rangle \ge m_A\|v_1-v_2\|_V^2 \quad \forall\, v_1,v_2\in V. \tag{2.7}$$

Related to the condition $H(b)$, we will use $M_b>0$ for the boundedness constant:

$$|b(v,\mu)|\le M_b\|v\|_V\|\mu\|_\Lambda \quad \forall\, v\in V,\ \mu\in\Lambda. \tag{2.8}$$

Assumption $H(b)$ allows us to define an operator $B\in\mathcal{L}(V;\Lambda^*)$ by the relation

$$\langle Bv,\mu\rangle = b(v,\mu) \quad \forall\, v\in V,\ \mu\in\Lambda.$$

In $H(\Phi)$, the convex function $\Phi\colon V\to\mathbb{R}$ is assumed to be continuous, instead of l.s.c. from V to the extended real line. As is explained in [15,16], there is no loss of generality with the stronger assumption of continuity for a vast majority of applications.

When K_Λ is an unbounded set in Λ and K_V is a subspace of V, we will assume the bilinear form $b(\cdot,\cdot)$ satisfies an inf–sup condition: there exists a constant $\alpha_b>0$ such that

$$\sup_{0\ne v\in K_V}\frac{b(v,\mu)}{\|v\|_V}\ge \alpha_b\|\mu\|_\Lambda \quad \forall\,\mu\in\Lambda. \tag{2.9}$$

We recall that an operator $A\colon V\to V^*$ is potential if there exists a Gâteaux differentiable functional $F_A\colon V\to\mathbb{R}$ such that $A=F_A'$; the functional F_A is called a potential of A. Detailed discussion on potential operators can be found in [17, Section 41.3].

In the special case where A is a potential operator, Problem 2.1 admits an equivalent saddle-point reformulation. Define

$$L(v,\mu)=F_A(v)+\Phi(v)+\Psi(v)-\langle f,v\rangle + b(v,\mu), \quad v\in K_V,\ \mu\in K_\Lambda, \tag{2.10}$$

known as a Lagrangian functional. Then, we consider a saddle-point problem corresponding to Problem 2.1.

Problem 2.2. Find $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$L(u, \mu) \le L(u, \lambda) \le L(v, \lambda) \quad \forall\, v \in K_V,\, \mu \in K_\Lambda. \tag{2.11}$$

An equivalence result between Problems 2.1 and 2.2 is shown in [5].

Theorem 2.3. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, $\alpha_\Psi < m_A$, and A is a potential operator with the potential F_A. Then, Problems 2.1 and 2.2 are equivalent, i.e., $(u, \lambda) \in K_V \times K_\Lambda$ is a solution of Problem 2.1 if and only if it is a solution of Problem 2.2.*

The assumption $\alpha_\Psi < m_A$ in Theorem 2.3 is known as a smallness condition in the literature.

The next existence result on Problem 2.1 is also proved in [5]. We distinguish two cases: the case with a bounded K_Λ, and the case with an unbounded K_Λ.

Theorem 2.4. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, $\alpha_\Psi < m_A$, and A is a potential operator. Moreover, assume either*

K_Λ is bounded,

or

K_Λ is unbounded; K_V is a subspace of V; $\Phi: V \to \mathbb{R}$ is Lipschitz continuous on K_V; there exist non-negative constants c_0, c_1 and $\kappa \in [0, 2)$ such that

$$\|\partial \Psi(v)\|_{V^*} \le c_0 + c_1 \|v\|_V^\kappa \quad \forall\, v \in V; \tag{2.12}$$

and the inf–sup condition (2.9) holds.

Then Problem 2.1 has a solution $(u, \lambda) \in K_V \times K_\Lambda$, and the first component u of the solution is unique.

In the next two sections, we will consider Problem 2.1 again, without assuming A to be a potential operator, and will extend the results to the more general form, Problem 1.1.

3. A class of elliptic mixed hemivariational–variational inequality

We continue to consider Problem 2.1, but without assuming A to be a potential operator.

Theorem 3.1. *Keep the assumptions stated in Theorem 2.4, except that A is not necessarily potential. Then, Problem 2.1 has a solution $(u, \lambda) \in K_V \times K_\Lambda$, and the first component u of the solution is unique.*

Proof. For any $\theta > 0$, (2.3)–(2.4) is equivalent to

$$\begin{aligned} &(u, v-u)_V + \theta \left[b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \right] \\ &\qquad \ge (u, v-u)_V - \theta \langle Au, v-u \rangle + \theta \langle f, v-u \rangle \quad \forall\, v \in K_V, \end{aligned} \tag{3.1}$$

$$\theta\, b(u, \mu - \lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{3.2}$$

Now let $\theta \in (0, \alpha_\Psi^{-1})$. Then for any $w \in K_V$, we can apply Theorem 2.4 to conclude that there exists $(u, \lambda) \in K_V \times K_\Lambda$, u being unique, such that

$$\begin{aligned} &(u, v-u)_V + \theta \left[b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \right] \\ &\qquad \ge (w, v-u)_V - \theta \langle Aw, v-u \rangle + \theta \langle f, v-u \rangle \quad \forall\, v \in K_V, \end{aligned} \tag{3.3}$$

$$\theta\, b(u, \mu - \lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{3.4}$$

Since u is unique, we can define a mapping $P_\theta : K_V \to K_V$ by the formula

$$u = P_\theta(w), \quad w \in K_V.$$

Let us prove that the mapping $P_\theta : K_V \to K_V$ is a contraction. For any $w_1, w_2 \in K_V$, denote $u_1 = P_\theta(w_1)$ and $u_2 = P_\theta(w_2)$. Then there exist $\lambda_1, \lambda_2 \in K_\Lambda$ such that

$$\begin{aligned} &(u_1, v - u_1)_V + \theta \left[b(v - u_1, \lambda_1) + \Phi(v) - \Phi(u_1) + \Psi^0(u_1; v - u_1) \right] \\ &\qquad \ge (w_1, v - u_1)_V - \theta \langle A w_1, v - u_1 \rangle + \theta \langle f, v - u_1 \rangle \quad \forall\, v \in K_V, \end{aligned} \tag{3.5}$$

$$\theta\, b(u_1, \mu - \lambda_1) \le 0 \quad \forall\, \mu \in K_\Lambda, \tag{3.6}$$

and

$$\begin{aligned} &(u_2, v - u_2)_V + \theta \left[b(v - u_2, \lambda_2) + \Phi(v) - \Phi(u_2) + \Psi^0(u_2; v - u_2) \right] \\ &\qquad \ge (w_2, v - u_2)_V - \theta \langle A w_2, v - u_2 \rangle + \theta \langle f, v - u_2 \rangle \quad \forall\, v \in K_V, \end{aligned} \tag{3.7}$$

$$\theta\, b(u_2, \mu - \lambda_2) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{3.8}$$

We take $v = u_2$ in (3.5), $v = u_1$ in (3.7), and add the two resulting inequalities to obtain

$$\begin{aligned} \|u_1 - u_2\|_V^2 &\le \theta \left[-b(u_1 - u_2, \lambda_1 - \lambda_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) \right] \\ &\quad + (w_1 - w_2, u_1 - u_2)_V - \theta \langle A w_1 - A w_2, u_1 - u_2 \rangle. \end{aligned} \tag{3.9}$$

From (3.6) and (3.8),

$$\begin{aligned} b(u_1, \lambda_2 - \lambda_1) &\le 0, \\ b(u_2, \lambda_1 - \lambda_2) &\le 0. \end{aligned}$$

Hence,

$$-b(u_1 - u_2, \lambda_1 - \lambda_2) = b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2) \le 0.$$

Thus, we derive from (3.9) that

$$\begin{aligned} \|u_1 - u_2\|_V^2 &\le \theta \left[\Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) \right] \\ &\quad + (w_1 - w_2, u_1 - u_2)_V - \theta \langle A w_1 - A w_2, u_1 - u_2 \rangle. \end{aligned}$$

Applying the condition (2.5),

$$(1 - \theta\, \alpha_\Psi)\, \|u_1 - u_2\|_V^2 \le (w_1 - w_2, u_1 - u_2)_V - \theta \langle A w_1 - A w_2, u_1 - u_2 \rangle. \tag{3.10}$$

Let $\mathcal{J} : V^* \to V$ be the Riesz mapping. Recall that

$$\begin{aligned} \langle g, v \rangle &= (\mathcal{J} g, v)_V \quad \forall\, g \in V^*,\ v \in V, \\ \|\mathcal{J} g\|_V &= \|g\|_{V^*} \quad \forall\, g \in V^*. \end{aligned}$$

Then

$$\langle A w_1 - A w_2, u_1 - u_2 \rangle = (\mathcal{J}(A w_1 - A w_2), u_1 - u_2)_V,$$

and we can rewrite (3.10) as

$$(1 - \theta\, \alpha_\Psi)\, \|u_1 - u_2\|_V^2 \le ((w_1 - w_2) - \theta\, \mathcal{J}(A w_1 - A w_2), u_1 - u_2)_V\,.$$

Hence,

$$(1-\theta\,\alpha_\Psi)\,\|u_1-u_2\|_V \le \|(w_1-w_2)-\theta\,\mathcal{J}(Aw_1-Aw_2)\|_V. \tag{3.11}$$

Now

$$\begin{aligned}\|(w_1-w_2)-\theta\,\mathcal{J}(Aw_1-Aw_2)\|_V^2 &= \|w_1-w_2\|_V^2 - 2\,\theta\,(\mathcal{J}(Aw_1-Aw_2),w_1-w_2)_V\\ &\quad + \theta^2\|\mathcal{J}(Aw_1-Aw_2)\|_V^2.\end{aligned}$$

By assumptions $H(A)$, we have

$$\|\mathcal{J}(Aw_1-Aw_2)\|_V^2 = \|Aw_1-Aw_2\|_{V^*}^2 \le M_A^2\|w_1-w_2\|_V^2$$

and

$$(\mathcal{J}(Aw_1-Aw_2),w_1-w_2)_V = \langle Aw_1-Aw_2,w_1-w_2\rangle \ge m_A\|w_1-w_2\|_V^2.$$

So

$$\|(w_1-w_2)-\theta\,\mathcal{J}(Aw_1-Aw_2)\|_V^2 \le \left(1-2\,\theta\,m_A+\theta^2M_A^2\right)\|w_1-w_2\|_V^2.$$

Therefore, we square both sides of (3.11) and derive that

$$\|u_1-u_2\|_V^2 \le \frac{1-2\,\theta\,m_A+\theta^2M_A^2}{(1-\theta\,\alpha_\Psi)^2}\,\|w_1-w_2\|_V^2. \tag{3.12}$$

Note that

$$\frac{1-2\,\theta\,m_A+\theta^2M_A^2}{(1-\theta\,\alpha_\Psi)^2} < 1 \tag{3.13}$$

if and only if

$$\left(M_A^2-\alpha_\Psi^2\right)\theta < 2\,(m_A-\alpha_\Psi)\,. \tag{3.14}$$

Since $\alpha_\Psi < m_A \le M_A$ and $\theta \in (0,\alpha_\Psi^{-1})$, (3.14) is valid. In other words, for $\theta \in (0,\alpha_\Psi^{-1})$, (3.13) holds and the operator $P_\theta\colon K_V \to K_V$ is a contraction. By the Banach fixed-point theorem (e.g., [18, Section 5.1]), P_θ has a unique fixed-point $u \in K_V$: $P_\theta(u) = u$. Then for some $\lambda \in K_\Lambda$, the pair (u,λ) satisfies (3.1)–(3.2), or equivalently (2.3)–(2.4), i.e., it is a solution of Problem 2.1.

The uniqueness of u follows from the uniqueness of the fixed point of the operator P_θ for $\theta \in (0,\alpha_\Psi^{-1})$. It can also be proved directly. ■

4. A general class of elliptic mixed hemivariational–variational inequalities

In this section, we consider Problem 1.1, which is more general than Problem 2.1. It can also be termed as an elliptic mixed hemivariational–quasivariational inequality. For convenience, we restate the problem next.

Problem 4.1. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au,v-u\rangle + b(v-u,\lambda) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u;v-u) \ge \langle f,v-u\rangle \quad \forall\, v \in K_V, \tag{4.1}$$

$$b(u,\mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{4.2}$$

In the study of Problem 4.1, we modify $H(\Phi)$ to $H(\Phi)_2$; the subscript 2 in $H(\Phi)_2$ reminds the reader that this is a condition for the case where Φ depends on two variables.

- $H(\Phi)_2$ $\Phi\colon V\times V \to \mathbb{R}$; for any $u \in V$, $\Phi(u,\cdot)\colon V \to \mathbb{R}$ is convex and bounded above on a non-empty open set; and there exists a constant $\alpha_\Phi \ge 0$ such that

$$\Phi(u_1,v_2) - \Phi(u_1,v_1) + \Phi(u_2,v_1) - \Phi(u_2,v_2) \le \alpha_\Phi\|u_1-u_2\|\,\|v_1-v_2\| \quad \forall\, u_1,u_2,v_1,v_2 \in V. \tag{4.3}$$

Note that the assumption $\Phi(u,\cdot)\colon V \to \mathbb{R}$ being convex and bounded above on a non-empty open set implies that $\Phi(u,\cdot)$ is continuous on V (cf. [19]).

Theorem 4.2. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\Psi)$, $H(f)$, and $\alpha_\Phi+\alpha_\Psi < m_A$. Moreover, assume either*

K_Λ is bounded,

or

K_Λ is unbounded; K_V is a subspace of V; for any $u \in V$, $\Phi(u,\cdot)\colon V \to \mathbb{R}$ is Lipschitz continuous on K_V; there exist non-negative constants c_0, c_1 and $\kappa \in [0,2)$ such that

$$\|\partial\Psi(v)\|_{V^*} \le c_0 + c_1\|v\|_V^\kappa \quad \forall\, v \in V;$$

and the inf–sup condition (2.9) holds.

Then, Problem 4.1 has a solution $(u,\lambda) \in K_V \times K_\Lambda$ and the first component u of a solution is unique.

Proof. For any $w \in K_V$, we consider the auxiliary problem of finding $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au, v-u\rangle + b(v-u,\lambda) + \Phi(w,v) - \Phi(w,u) + \Psi^0(u;v-u) \ge \langle f, v-u\rangle \quad \forall\, v \in K_V, \tag{4.4}$$

$$b(u,\mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{4.5}$$

Under the stated assumptions, we apply Theorem 3.1 to conclude that there is a pair $(u,\lambda) \in K_V \times K_\Lambda$ satisfying (4.4)–(4.5) and u is unique. This allows us to define a mapping $P\colon K_V \to K_V$ by the relation

$$P(w) = u.$$

Let us show that $P\colon K_V \to K_V$ is a contraction. For any $w_1, w_2 \in K_V$, denote $u_1 = P(w_1)$, $u_2 = P(w_2)$. By the definition of the mapping P, there exist $\lambda_1, \lambda_2 \in K_\Lambda$ such that

$$\begin{aligned} &\langle Au_1, v-u_1\rangle + b(v-u_1,\lambda_1) + \Phi(w_1,v) - \Phi(w_1,u_1) + \Psi^0(u_1;v-u_1) \\ &\qquad \ge \langle f, v-u_1\rangle \quad \forall\, v \in K_V, \end{aligned} \tag{4.6}$$

$$b(u_1,\mu-\lambda_1) \le 0 \quad \forall\, \mu \in K_\Lambda \tag{4.7}$$

and

$$\begin{aligned} &\langle Au_2, v-u_2\rangle + b(v-u_2,\lambda_2) + \Phi(w_2,v) - \Phi(w_2,u_2) + \Psi^0(u_2;v-u_2) \\ &\qquad \ge \langle f, v-u_2\rangle \quad \forall\, v \in K_V, \end{aligned} \tag{4.8}$$

$$b(u_2,\mu-\lambda_2) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{4.9}$$

We take $v = u_2$ in (4.6), $v = u_1$ in (4.8), and add the two resulting inequalities to obtain

$$\begin{aligned} \langle Au_1 - Au_2, u_1-u_2\rangle \le{}& -b(u_1-u_2,\lambda_1-\lambda_2) + \Psi^0(u_1;u_2-u_1) + \Psi^0(u_2;u_1-u_2) \\ &+ \Phi(w_1,u_2) - \Phi(w_1,u_1) + \Phi(w_2,u_1) - \Phi(w_2,u_2). \end{aligned} \tag{4.10}$$

From (4.7) and (4.9),

$$b(u_1,\lambda_2-\lambda_1) \le 0,$$
$$b(u_2,\lambda_1-\lambda_2) \le 0.$$

Hence,

$$-b(u_1 - u_2, \lambda_1 - \lambda_2) = b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2) \leq 0.$$

Thus, by making use of the assumptions $H(A)$, $H(\Psi)$ and $H(\Phi)_2$, we derive from (4.10) that

$$m_A \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2 + \alpha_\Phi \|w_1 - w_2\|_V \|u_1 - u_2\|_V.$$

Then,

$$\|u_1 - u_2\|_V \leq \frac{\alpha_\Phi}{m_A - \alpha_\Psi} \|w_1 - w_2\|_V.$$

Note that the smallness assumption $\alpha_\Phi + \alpha_\Psi < m_A$ implies $\alpha_\Phi/(m_A - \alpha_\Psi) < 1$. So from the above inequality, we deduce that the mapping $P: K_V \to K_V$ is contractive. By the Banach fixed-point theorem, the mapping P has a unique fixed-point $u \in K_V$. From $u = P(u)$, it is easy to see that for some $\lambda \in K_\Lambda$, the pair $(u, \lambda) \in K_V \times K_\Lambda$ is a solution of Problem 4.1. Moreover, the first component u of the solution of Problem 4.1 is unique due to the uniqueness of the fixed-point of the mapping P. ■

Now we present a Lipschitz continuous dependence property of the solution on the right side.

Theorem 4.3. *Keep the assumptions stated in Theorem 4.2. Then, for solutions $(u_1, \lambda_1), (u_2, \lambda_2) \in K_V \times K_\Lambda$ of Problem 4.1 with $f = f_1, f_2 \in V^*$, the following inequality holds:*

$$\|u_1 - u_2\|_V \leq \frac{1}{m_A - \alpha_\Phi - \alpha_\Psi} \|f_1 - f_2\|_{V^*}. \tag{4.11}$$

Proof. By the definition of a solution of Problem 4.1, we have

$$\langle Au_1, v - u_1 \rangle + b(v - u_1, \lambda_1) + \Phi(u_1, v) - \Phi(u_1, u_1) + \Psi^0(u_1; v - u_1)$$
$$\geq \langle f_1, v - u_1 \rangle \quad \forall\, v \in K_V, \tag{4.12}$$
$$b(u_1, \mu - \lambda_1) \leq 0 \quad \forall\, \mu \in K_\Lambda \tag{4.13}$$

and

$$\langle Au_2, v - u_2 \rangle + b(v - u_2, \lambda_2) + \Phi(u_2, v) - \Phi(u_2, u_2) + \Psi^0(u_2; v - u_2)$$
$$\geq \langle f_2, v - u_2 \rangle \quad \forall\, v \in K_V, \tag{4.14}$$
$$b(u_2, \mu - \lambda_2) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{4.15}$$

We take $v = u_2$ in (4.12), $v = u_1$ in (4.14), and add the two resulting inequalities to obtain

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \leq -b(u_1 - u_2, \lambda_1 - \lambda_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2)$$
$$+ \Phi(u_1, u_2) - \Phi(u_1, u_1) + \Phi(u_2, u_1) - \Phi(u_2, u_2)$$
$$+ \langle f_1 - f_2, u_1 - u_2 \rangle. \tag{4.16}$$

As in the proof of Theorem 4.2, we deduce from (4.13) and (4.15) that

$$-b(u_1 - u_2, \lambda_1 - \lambda_2) = b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2) \leq 0.$$

Then, from (4.16) with the use of $H(A)$, $H(\Psi)$ and $H(\Phi)_2$,

$$m_A \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2 + \alpha_\Phi \|u_1 - u_2\|_V^2 + \|f_1 - f_2\|_{V^*} \|u_1 - u_2\|_V.$$

Therefore, (4.11) holds. ■

Following [4], we can introduce several Minty-type equivalent formulations of Problem 4.1. The equivalent formulations can be useful in numerical analysis of Problem 4.1, cf. [20] for such an example in the context of numerical analysis of an elliptic variational–hemivariational inequality.

Let $\kappa_A \in [0, m_A]$ be a fixed constant.

Problem 4.4. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Av, v-u\rangle + b(v-u,\lambda) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \\ \geq \langle f, v-u\rangle + \kappa_A \|v-u\|_V^2 \quad \forall\, v \in K_V, \tag{4.17}$$

$$b(u, \mu-\lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{4.18}$$

Problem 4.5. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au, v-u\rangle + b(v,\lambda) - b(u,\mu) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \\ \geq \langle f, v-u\rangle \quad \forall\, v \in K_V,\ \mu \in K_\Lambda. \tag{4.19}$$

Problem 4.6. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Av, v-u\rangle + b(v,\lambda) - b(u,\mu) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \\ \geq \langle f, v-u\rangle + \kappa_A \|v-u\|_V^2 \quad \forall\, v \in K_V,\ \mu \in K_\Lambda. \tag{4.20}$$

Theorem 4.7. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\Psi)$, $H(f)$, and let $\kappa_A \in [0, m_A]$. Then, Problems 4.1, 4.4, 4.5, and 4.6 are all equivalent in the sense that if $(u,\lambda) \in K_V \times K_\Lambda$ is a solution of one of these problems, it is also a solution of all the other problems.*

Proof. Let $(u,\lambda) \in K_V \times K_\Lambda$ be a solution of Problem 4.1. From the monotonicity condition (2.7), we have the inequality

$$\langle Av, v-u\rangle \geq \langle Au, v-u\rangle + m_A \|v-u\|_V^2.$$

Then, it is easy to see that $(u,\lambda) \in K_V \times K_\Lambda$ is a solution of Problem 4.4.

Conversely, assume $(u,\lambda) \in K_V \times K_\Lambda$ is a solution of Problem 4.4. In (4.17), we replace the arbitrary element $v \in K_V$ by $u + t\,(v-u) \in K_V$ for any $t \in (0,1)$ and any $v \in K_V$, to obtain

$$t\,\langle A(u+t\,(v-u)), v-u\rangle + t\, b(v-u,\lambda) + \Phi(u, u+t\,(v-u)) - \Phi(u,u) + \Psi^0(u; t\,(v-u)) \\ \geq t\,\langle f, v-u\rangle + \kappa_A t^2 \|v-u\|_V^2. \tag{4.21}$$

By the convexity of Φ with respect to its second argument,

$$\Phi(u, u+t\,(v-u)) \leq t\,\Phi(u,v) + (1-t)\,\Phi(u,u).$$

Also note that $\Psi^0(u; t\,(v-u)) = t\,\Psi^0(u; v-u)$. Then from (4.21), after dividing by t,

$$\langle A(u+t\,(v-u)), v-u\rangle + b(v-u,\lambda) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \\ \geq \langle f, v-u\rangle + \kappa_A t\, \|v-u\|_V^2.$$

Take the limit $t \to 0+$ to recover (4.1). Thus, $(u,\lambda) \in K_V \times K_\Lambda$ is a solution of Problem 4.1.

Now we show the equivalence between Problems 4.1 and 4.5. Assume (4.1) and (4.2) hold. Then, for any $v \in K_V$ and any $\mu \in K_\Lambda$,

$$b(v,\lambda) - b(u,\mu) = b(v-u,\lambda) + b(u, \lambda-\mu) \geq b(v-u,\lambda).$$

Thus, (4.19) is valid. Conversely, assume (4.19) holds. We take $\mu = \lambda$ in (4.19) to recover (4.1), and take $v = u$ in (4.19) to recover (4.2).

The proof of the equivalence between Problems 4.5 and 4.6 is similar to that between Problems 4.1 and 4.4, and is hence omitted here. ■

We comment that not all the assumptions listed in Theorem 4.7 are necessary for the equivalence of the problems, e.g., (2.5) and (4.3) are not needed for the equivalence.

Finally, we focus on the case where Ψ is defined as an integral of a function ψ. In most applications, a hemivariational inequality describes a physical problem on a spatial domain in a finite-dimensional Euclidean space $\mathbb{R}^d$. Denote by Δ the domain or its sub-domain, or its boundary or part of the boundary. Let I_Δ stand for the integration over Δ. We will need a space V_ψ and a linear operator γ_ψ from V to V_ψ. Assume elements from the space V_ψ are $\mathbb{R}^m$-valued functions, for some positive integer m. In applications in contact mechanics, the operator γ_ψ is either the normal trace operator and then $m = 1$, or the tangential component trace operator and then $m = d$. We will take $V_\psi = L^2(\Delta; \mathbb{R}^m)$. Let us introduce the following assumption on the function ψ.

- $H(\psi)$ $\gamma_\psi \in \mathcal{L}(V; L^2(\Delta; \mathbb{R}^m))$; $\psi\colon \Delta \times \mathbb{R}^m \to \mathbb{R}$; $\psi(\cdot, v)$ is measurable on Δ for all $v \in \mathbb{R}^m$; $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^m$ for a.e. $\boldsymbol{x} \in \Delta$; and for some non-negative constants c and α_ψ,

$$|\partial\psi(\cdot, v)| \le c\,(1 + |v|_{\mathbb{R}^m}) \quad \forall\, v \in \mathbb{R}^m, \text{ a.e. on } \Delta, \tag{4.22}$$

$$\psi^0(v_1; v_2 - v_1) + \psi^0(v_2; v_1 - v_2) \le \alpha_\psi |v_1 - v_2|^2_{\mathbb{R}^m} \quad \forall\, v_1, v_2 \in \mathbb{R}^m, \text{ a.e. on } \Delta. \tag{4.23}$$

Consider the following analogue of Problem 4.1 in which Ψ^0 is replaced by the integral of ψ^0 over the domain Δ.

Problem 4.8. Find $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au, v - u\rangle + b(v - u, \lambda) + \Phi(u, v) - \Phi(u, u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \\ \ge \langle f, v - u\rangle \quad \forall\, v \in K_V, \tag{4.24}$$

$$b(u, \mu - \lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{4.25}$$

Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|^2_{\mathbb{R}^m}) \le c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V. \tag{4.26}$$

Define the functional

$$\Psi(v) = I_\Delta(\psi(\gamma_\psi v)), \quad v \in V. \tag{4.27}$$

Then under the assumption $H(\psi)$, similar to the results and arguments in [14, Section 3.3], it can be shown that $\Psi(\cdot)$ is well-defined and locally Lipschitz on V, and

$$\Psi^0(u; v) \le I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \quad \forall\, u, v \in V. \tag{4.28}$$

Thus, (4.23) and (4.26) imply that for any $v_1, v_2 \in V$,

$$\begin{aligned} \Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) &\le I_\Delta(\psi^0(\gamma_\psi v_1; \gamma_\psi v_2 - \gamma_\psi v_1) + \psi^0(\gamma_\psi v_2; \gamma_\psi v_1 - \gamma_\psi v_2)) \\ &\le I_\Delta(\alpha_\psi |\gamma_\psi(v_1 - v_2)|^2_{\mathbb{R}^m}) \\ &\le \alpha_\psi c_\Delta^2 \|v_1 - v_2\|_V^2, \end{aligned} \tag{4.29}$$

i.e., (2.5) is satisfied with $\alpha_\Psi = \alpha_\psi c_\Delta^2$.

Theorem 4.9. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\psi)$, $H(f)$, and $\alpha_\Phi + \alpha_\psi c_\Delta^2 < m_A$. Moreover, assume either*

K_Λ is bounded,

or

K_Λ is unbounded; K_V is a subspace of V; for any $u \in V$, $\Phi(u,\cdot): V \to \mathbb{R}$ is Lipschitz continuous on K_V; and the inf–sup condition (2.9) holds.

Then, Problem 4.8 has a solution $(u,\lambda) \in K_V \times K_\Lambda$ and the first component u of a solution is unique.

Proof. The assumption $H(\psi)$ implies that the functional Ψ defined by (4.27) satisfies $H(\Psi)$. Therefore, by Theorem 4.2 with $\kappa = 1$, Problem 4.1 has at least one solution $(u,\lambda) \in K_V \times K_\Lambda$. By (4.28), we recognize that $(u,\lambda) \in K_V \times K_\Lambda$ satisfies (4.24)–(4.25).

To prove the uniqueness of the first component u of the solution, let $(u_1,\lambda_1), (u_2,\lambda_2) \in K_V \times K_\Lambda$ be two solutions of Problem 4.8. Similar to the proof of Theorem 4.2, we have the following analogue of (4.10):

$$\begin{aligned}\langle Au_1 - Au_2, u_1 - u_2\rangle \le &-b(u_1 - u_2, \lambda_1 - \lambda_2)\\ &+ I_\Delta(\psi^0(\gamma_\psi u_1; \gamma_\psi u_2 - \gamma_\psi u_1) + \psi^0(\gamma_\psi u_2; \gamma_\psi u_1 - \gamma_\psi u_2))\\ &+ \Phi(u_1,u_2) - \Phi(u_1,u_1) + \Phi(u_2,u_1) - \Phi(u_2,u_2),\end{aligned}$$

where

$$-b(u_1 - u_2, \lambda_1 - \lambda_2) = b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2) \le 0.$$

Hence, by applying $H(A)$, (4.29) and $H(\Phi)_2$,

$$m_A\|u_1 - u_2\|_V^2 \le \alpha_\psi c_\Delta^2 \|u_1 - u_2\|_V^2 + \alpha_\Phi \|u_1 - u_2\|_V^2.$$

Then,

$$\left(m_A - \alpha_\psi c_\Delta^2 - \alpha_\Phi\right) \|u_1 - u_2\|_V^2 \le 0.$$

Since $m_A - \alpha_\psi c_\Delta^2 - \alpha_\Phi > 0$, we conclude that $u_1 - u_2 = 0$. In other words, the first component u of the solution is unique. ■

5. An example in contact mechanics

In this section we discuss an example from contact mechanics to illustrate the application of the theoretical results shown in previous sections.

Consider the deformation of a deformable body whose initial configuration is a Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d \le 3)$. Since $\partial\Omega$ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)^T$ is defined a.e. on $\partial\Omega$. For an $\mathbb{R}^d$-valued function $\boldsymbol{u}$ on the boundary, its normal and tangential components are $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. For an $\mathbb{S}^d$-valued function $\boldsymbol{\sigma}$ on the boundary, we call $\sigma_\nu = \boldsymbol{\nu}\cdot\boldsymbol{\sigma}\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary.

We adopt the summation convention over a repeated index. The indices i and j range between 1 and d. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\begin{aligned}&\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| \equiv |\boldsymbol{v}|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,\\ &\boldsymbol{\sigma}:\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\sigma}| \equiv |\boldsymbol{\sigma}|_{\mathbb{S}^d} = (\boldsymbol{\sigma}:\boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.\end{aligned}$$

To describe the contact problem, we split the boundary $\Gamma = \partial\Omega$ into four mutually disjoint parts: $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3} \cup \overline{\Gamma_4}$ with $|\Gamma_1| > 0$, $|\Gamma_3| + |\Gamma_4| > 0$. The classical formulation of the contact problem is to find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.1}$$

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{5.4}$$

$$-\sigma_\nu \in \partial\psi_\nu(u_\nu), \quad |\boldsymbol{\sigma}_\tau| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \quad \text{on } \Gamma_3, \tag{5.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0},\ \sigma_\nu \le 0,\ u_\nu \le 0,\ \sigma_\nu u_\nu = 0 \quad \text{on } \Gamma_4. \tag{5.6}$$

Let us briefly comment on the equations and conditions in this problem. The equilibrium equation is represented by (5.1) where $\boldsymbol{f}_0$ is the density of the body force. The material is assumed to be elastic and (5.2) is the constitutive law in which $\mathcal{E}$ represents the elasticity operator. The condition (5.3) means that the body is fixed on Γ_1. The condition (5.4) gives a surface traction condition on Γ_2, where $\boldsymbol{f}_2$ is a given density of the applied surface traction. Two types of contact conditions are considered on two different parts, Γ_3 and Γ_4, of the contact boundary. On Γ_3, we have a possibly non-monotone contact condition with normal compliance on contact of the deformable body with a reactive foundation, described by the first part of (5.5). The function ψ_ν is locally Lipschitz continuous and it may be non-convex. The second and third relations of (5.5) provide a version of static Coulomb's law of dry friction, associated with the normal contact condition; here F_b is a given friction bound and it is allowed to depend on the normal displacement. The two relations are equivalent to

$$|\boldsymbol{\sigma}_\tau| \le F_b(u_\nu), \quad \boldsymbol{\sigma}_\tau \cdot \boldsymbol{u}_\tau + F_b(u_\nu)\,|\boldsymbol{u}_\tau| = 0 \quad \text{on } \Gamma_3. \tag{5.7}$$

Mechanical interpretation of (5.5) can be found in [21]. On Γ_4, we specify a frictionless unilateral contact condition, cf. (5.6). For more details on (5.6) the reader may consult, for instance, [22,23]. Note that the assumption "$|\Gamma_3| + |\Gamma_4| > 0$" allows the possibility that $|\Gamma_3| > 0$ and $\Gamma_4 = \emptyset$, or $|\Gamma_4| > 0$ and $\Gamma_3 = \emptyset$; in the former case, the contact problem contains the contact condition (5.5) only, whereas in the latter case, the contact problem contains the contact condition (5.6) only.

The elasticity operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is assumed to have the following properties:

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{\mathcal{E}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\
\qquad |\mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \le L_{\mathcal{E}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|; \\
\text{(b) there exists } m_{\mathcal{E}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\
\qquad (\mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{E}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2; \\
\text{(c) } \mathcal{E}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\
\text{(d) } \mathcal{E}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{5.8}$$

For the potential function $\psi_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, assume

$$\left\{\begin{array}{l}
\text{(a) } \psi_\nu(\cdot, z) \text{ is measurable on } \Gamma_3 \text{ for all } z \in \mathbb{R} \text{ and } \psi_\nu(\cdot, 0) \in L^1(\Gamma_3); \\
\text{(b) } \psi_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\
\text{(c) } |\partial\psi_\nu(\boldsymbol{x}, z)| \le c\,(1 + |z|) \text{ for all } z \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3; \\
\text{(d) } \psi_\nu^0(\boldsymbol{x}, z_1; z_2 - z_1) + \psi_\nu^0(\boldsymbol{x}, z_2; z_1 - z_2) \le \alpha_{\psi_\nu} |z_1 - z_2|^2 \\
\qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } z_1, z_2 \in \mathbb{R} \text{ with } \alpha_{\psi_\nu} \ge 0.
\end{array}\right. \tag{5.9}$$

For the friction bound $F_b : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, assume

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{F_b} > 0 \text{ such that for all } z_1, z_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\
\qquad |F_b(\boldsymbol{x}, z_1) - F_b(\boldsymbol{x}, z_2)| \le L_{F_b} |z_1 - z_2|; \\
\text{(b) } F_b(\cdot, z) \text{ is measurable on } \Gamma_3, \text{ for all } z \in \mathbb{R}; \\
\text{(c) } F_b(\boldsymbol{x}, z) \ge 0 \text{ for } z \in \mathbb{R} \text{ and } F_b(\boldsymbol{x}, 0) = 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{5.10}$$

The contact problem will be studied as an elliptic mixed hemivariational–variational inequality, and for comparison, it will be also studied as an elliptic variational–hemivariational inequality. For this purpose, we will apply a result from [16, Theorem 4.10].

Theorem 5.1. *Assume $H(K_V)$, $H(A)$, $H(\Phi)_2$, $H(\psi)$, $H(f)$, and $\alpha_\Phi+\alpha_\psi c_\Delta^2 < m_A$. Then there is a unique solution to the problem*

$$u \in K_V, \quad \langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \geq \langle f, v-u\rangle \quad \forall\, v \in K_V.$$

We will use the space

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \right\} \tag{5.11}$$

for the displacement variable. The trace of $\boldsymbol{v} \in V$ on the boundary is denoted by the same symbol $\boldsymbol{v}$. Since $|\Gamma_1| > 0$, Korn's inequality holds (cf. [24, p. 79]): for some constant $c > 0$,

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \leq c\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \forall\, \boldsymbol{v} \in V.$$

Consequently, we can define the norm $\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)}$, which is equivalent to the standard norm $\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}$ on V. Define a subset of V:

$$K = \{\boldsymbol{v} \in V \mid v_\nu \leq 0 \text{ on } \Gamma_4\}. \tag{5.12}$$

Let $\lambda_\nu > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} u_\nu v_\nu\, da \quad \forall\, \boldsymbol{v} \in V,$$

and let $\lambda_\tau > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau\, da \quad \forall\, \boldsymbol{v} \in V.$$

Then we have the trace inequalities

$$\|v_\nu\|_{L^2(\Gamma_3)} \leq \lambda_\nu^{-1/2} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{5.13}$$

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq \lambda_\tau^{-1/2} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.14}$$

Study of the contact problem as an elliptic variational–hemivariational inequality. In the derivation of the elliptic variational–hemivariational inequality for the contact problem defined by (5.1)–(5.6), we assume that there is a smooth function $\boldsymbol{u} \in V$ such that (5.1)–(5.6) hold pointwise. For any smooth function $\boldsymbol{v} \in V$, we multiply (5.1) by $(\boldsymbol{v}-\boldsymbol{u})$, integrate over Ω, and perform an integration by parts to obtain

$$-\int_\Gamma \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v}-\boldsymbol{u})\, da + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx = \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v}-\boldsymbol{u})\, dx. \tag{5.15}$$

By applying the boundary conditions (5.3) and (5.4), we have

$$-\int_\Gamma \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v}-\boldsymbol{u})\, da = -\int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v}-\boldsymbol{u})\, da - \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v}-\boldsymbol{u})\, da - \int_{\Gamma_4} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v}-\boldsymbol{u})\, da.$$

To apply the boundary conditions (5.5) and (5.6), we write

$$\boldsymbol{\sigma\nu} \cdot (\boldsymbol{v}-\boldsymbol{u}) = \sigma_\nu (v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau).$$

On Γ_3, by (5.5), we have

$$-\sigma_\nu(v_\nu - u_\nu) \le \psi_\nu^0(u_\nu; v_\nu - u_\nu),$$
$$-\boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \le F_b(u_\nu)\left(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\right),$$

where the second inequality can be easily derived by making use of (5.7). Hence,

$$-\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, da \le \int_{\Gamma_3} \left[\psi_\nu^0(u_\nu; v_\nu - u_\nu) + F_b(u_\nu)\left(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\right)\right] da.$$

By applying the boundary condition (5.6),

$$-\int_{\Gamma_4} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, da = -\int_{\Gamma_4} \sigma_\nu(v_\nu - u_\nu)\, da \le 0, \tag{5.16}$$

where we use the relations $\sigma_\nu u_\nu = 0$ and $\sigma_\nu v_\nu \ge 0$ on Γ_4. Summarizing, we arrive at the following variational–hemivariational inequality: find a displacement field $\boldsymbol{u} \in K$ such that

$$\begin{aligned} \langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle &+ \int_{\Gamma_3} \left[\psi_\nu^0(u_\nu; v_\nu - u_\nu) + F_b(u_\nu)\left(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\right)\right] da \\ &\ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in K, \end{aligned} \tag{5.17}$$

where the operator $A\colon V \to V^*$ and the functional $\boldsymbol{f} \in V^*$ are defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{5.18}$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V. \tag{5.19}$$

Let us apply Theorem 5.1 to the inequality (5.17). Take $K_V = K$, defined in (5.12). Obviously, $H(K_V)$ is valid. Define the functional Φ by

$$\Phi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} F_b(u_\nu)\, |\boldsymbol{v}_\tau|\, da \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.20}$$

By (5.8), $A\colon V \to V^*$ is Lipschitz continuous and is strongly monotone with

$$\langle A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle \ge m_{\mathcal{E}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2 \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V.$$

We take $\Delta = \Gamma_3$, $\psi = \psi_\nu$, γ_ψ as the normal trace operator:

$$\gamma_\psi \boldsymbol{v} = v_\nu \quad \forall\, \boldsymbol{v} \in V,$$

$m = 1$ and $V_\psi = L^2(\Gamma_3)$. We have $c_\Delta^2 = \lambda_\nu^{-1}$ from (5.13). For any $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$,

$$\begin{aligned} \Phi(\boldsymbol{u}_1, \boldsymbol{v}_2) &- \Phi(\boldsymbol{u}_1, \boldsymbol{v}_1) + \Phi(\boldsymbol{u}_2, \boldsymbol{v}_1) - \Phi(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ &= \int_{\Gamma_3} \left(F_b(u_{1,\nu}) - F_b(u_{2,\nu})\right)\left(|\boldsymbol{v}_{2,\tau}| - |\boldsymbol{v}_{1,\tau}|\right) da \\ &\le L_{F_b} \int_{\Gamma_3} |u_{1,\nu} - u_{2,\nu}|\, |\boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau}|\, da \\ &\le L_{F_b} \|u_{1,\nu} - u_{2,\nu}\|_{L^2(\Gamma_3)} \|\boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau}\|_{L^2(\Gamma_3)^d} \\ &\le L_{F_b} \lambda_\nu^{-1/2} \lambda_\tau^{-1/2} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V. \end{aligned}$$

So (4.3) holds with $\alpha_\Phi = L_{F_b} \lambda_\nu^{-1/2} \lambda_\tau^{-1/2}$.

Applying Theorem 5.1, we see that under the stated assumptions on the data and the smallness condition

$$L_{F_b}\lambda_\nu^{-1/2}\lambda_\tau^{-1/2} + \alpha_{\psi_\nu}\lambda_\nu^{-1} < m_{\mathcal{E}}, \tag{5.21}$$

the variational–hemivariational inequality (5.17) has a unique solution $\boldsymbol{u} \in K_V$.

Study of the contact problem as an elliptic mixed variational–hemivariational inequality. We now turn to a mixed hemivariational–variational inequality for the contact problem. For simplicity, in this part of the paper, we assume Ω is a $C^{1,1}$ domain so that the unit outward normal vector $\boldsymbol{\nu}$ is Lipschitz continuous on the boundary Γ. We choose $\lambda = -\sigma_\nu$ on Γ_4 as the Lagrangian multiplier for the mixed hemivariational–variational inequality of the contact problem. For this purpose, we first introduce the space

$$Z = \left\{ z \in L^2(\Gamma_4) \mid z = v_\nu \text{ a.e. on } \Gamma_4 \text{ for some } \boldsymbol{v} \in V \right\}. \tag{5.22}$$

This is a Hilbert space with the norm

$$\|z\|_Z = \inf\left\{ \|\boldsymbol{v}\|_V \mid z = v_\nu \text{ a.e. on } \Gamma_4 \right\}. \tag{5.23}$$

Since Ω is a $C^{1,1}$ domain, the normal trace operator $\boldsymbol{v} \mapsto v_\nu$ is surjective from $H^1(\Omega;\mathbb{R}^d)$ to $H^{1/2}(\Gamma)$ (cf. [25, Theorem 5.5]). So in the case $\overline{\Gamma_1} \cap \overline{\Gamma_4} = \emptyset$, we may simply view the space Z as the restriction of $H^{1/2}(\Gamma)$ over Γ_4.

We then introduce the dual space of Z:

$$\Lambda = Z^* \tag{5.24}$$

endowed with the ordinary dual norm

$$\|\mu\|_\Lambda = \sup_{z\in Z} \frac{\langle \mu, z\rangle_{\Lambda\times Z}}{\|z\|_Z}. \tag{5.25}$$

As usual, the duality $\langle \mu, z\rangle_{\Lambda\times Z}$ extends the $L^2(\Gamma_4)$ inner product, i.e.,

$$\langle \mu, z\rangle_{\Lambda\times Z} = \int_{\Gamma_4} \mu\, z\, da \quad \text{for } \mu \in L^2(\Gamma_4),\ z \in Z.$$

Now we define the bilinear form

$$b(\boldsymbol{v}, \mu) = \langle \mu, v_\nu\rangle_{\Lambda\times Z}, \quad \boldsymbol{v} \in V,\ \mu \in \Lambda. \tag{5.26}$$

Note that if $\lambda = -\sigma_\nu \in L^2(\Gamma_4)$, then

$$b(\boldsymbol{v}, \lambda) = -\int_{\Gamma_4} \sigma_\nu v_\nu\, da \text{ for } \boldsymbol{v} \in V.$$

The continuity of the bilinear form b is readily seen:

$$|b(\boldsymbol{v}, \mu)| \le \|\mu\|_\Lambda \|v_\nu\|_Z \le \|\mu\|_\Lambda \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V,\ \mu \in \Lambda.$$

To establish the inf–sup condition for the bilinear form b, let us first prove that the infimum in the definition of the norm $\|z\|_Z$ in (5.23) can be achieved.

Lemma 5.2. *For any $z \in Z$,*

$$\inf\left\{ \|\boldsymbol{v}\|_V \mid z = v_\nu \text{ a.e. on } \Gamma_4 \right\} = \min\left\{ \|\boldsymbol{v}\|_V \mid z = v_\nu \text{ a.e. on } \Gamma_4 \right\}. \tag{5.27}$$

Proof. For any given $z \in Z$, denote

$$\alpha = \inf\left\{\|\boldsymbol{v}\|_V \mid z = v_\nu \text{ a.e. on } \Gamma_4\right\}.$$

Then there is a sequence $\{\boldsymbol{v}_n\} \subset V$ such that $v_{n,\nu} = z$ on Γ_4 and $\|\boldsymbol{v}_n\|_V \to \alpha$ as $n \to \infty$.

Since $\{\boldsymbol{v}_n\}$ is a bounded sequence in the Hilbert space V, we have a subsequence, still denoted as $\{\boldsymbol{v}_n\}$, and an element $\boldsymbol{v}_0 \in V$ such that

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}_0 \quad \text{in } V.$$

Due to the l.s.c. of the norm (cf. [18, Exercise 3.3.6]),

$$\|\boldsymbol{v}_0\|_V \le \liminf \|\boldsymbol{v}_n\|_V = \alpha.$$

By the definition of α, we must have

$$\|\boldsymbol{v}_0\|_V = \alpha.$$

The two properties $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in V and $\|\boldsymbol{v}_n\|_V \to \|\boldsymbol{v}_0\|_V$ imply (cf. [18, Exercise 2.7.3])

$$\boldsymbol{v}_n \to \boldsymbol{v}_0 \quad \text{in } V.$$

In particular, since $v_{n,\nu} = z$ on Γ_4, we also have $v_{0,\nu} = z$ on Γ_4. In other words, the infimum on the left side of (5.27) is achieved at the element $\boldsymbol{v}_0$. ■

Return to the definition (5.25) for the norm $\|\mu\|_\Lambda$. For any $z \in Z$, by Lemma 5.2, we have the existence of $\boldsymbol{v}_z \in V$ with $v_{z,\nu} = z$ on Γ_4 such that $\|z\|_Z = \|\boldsymbol{v}_z\|_V$. Then,

$$\|\mu\|_\Lambda = \sup_{z\in Z} \frac{\langle \mu, z\rangle_{\Lambda\times Z}}{\|z\|_Z} = \sup_{z\in Z} \frac{\langle \mu, v_{z,\nu}\rangle_{\Lambda\times Z}}{\|\boldsymbol{v}_z\|_V}.$$

Thus,

$$\|\mu\|_\Lambda \le \sup_{\boldsymbol{v}\in V} \frac{\langle \mu, v_\nu\rangle_{\Lambda\times Z}}{\|\boldsymbol{v}\|_V} = \sup_{\boldsymbol{v}\in V} \frac{b(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_V},$$

i.e., the inf–sup condition for the bilinear form b holds, with the constant $\alpha_b = 1$ in (2.9).

Further, introduce a subset of Λ:

$$K_\Lambda = \left\{\mu \in \Lambda \mid \langle \mu, v_\nu\rangle \le 0\ \forall\, \boldsymbol{v} \in V,\ v_\nu \le 0 \text{ on } \Gamma_4\right\}. \tag{5.28}$$

In the derivation of the mixed hemivariational–variational inequality, instead of (5.16), we write

$$-\int_{\Gamma_4} \boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,da = -\int_{\Gamma_4} \sigma_\nu (v_\nu - u_\nu)\,da = b(\boldsymbol{v}-\boldsymbol{u},\lambda).$$

Moreover, for any $\mu \in K_\Lambda$, we note that

$$b(\boldsymbol{u},\mu-\lambda) = \int_{\Gamma_4} u_\nu(\mu-\lambda)\,da = \int_{\Gamma_4} u_\nu \mu\,da \le 0.$$

Hence, the mixed formulation for the contact problem (5.1)–(5.6) is to find $(\boldsymbol{u},\lambda) \in V \times K_\Lambda$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle + b(\boldsymbol{v}-\boldsymbol{u},\lambda) + \int_{\Gamma_3} \left[\psi_\nu^0(u_\nu; v_\nu - u_\nu) + F_b(u_\nu)\left(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\right)\right] da$$
$$\ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in V, \tag{5.29}$$
$$b(\boldsymbol{u},\mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda, \tag{5.30}$$

where $A{:}\, V \to V^*$ is defined by (5.18) and $\boldsymbol{f} \in V^*$ is defined by (5.19).

Let us apply Theorem 4.9 with $K_V = V$, K_Λ defined by (5.28), Φ defined by (5.20), $b: V \times \Lambda \to \mathbb{R}$ defined by (5.26), and the functional f defined by $\boldsymbol{f}$. Obviously, $H(K_V)$ and $H(K_\Lambda)$ hold. We already know that $A: V \to V^*$ is Lipschitz continuous and is strongly monotone with the monotonicity constant $m_{\mathcal{E}}$. Moreover, we have verified the continuity and the inf–sup condition for the bilinear form $b: V \times \Lambda \to \mathbb{R}$. We take $\Delta = \Gamma_3$, $\psi = \psi_\nu$, γ_ψ as the normal trace operator:

$$\gamma_\psi \boldsymbol{v} = v_\nu \quad \forall\, \boldsymbol{v} \in V,$$

$m = 1$ and $V_\psi = L^2(\Gamma_3)$. As previously, $H(\Phi)_2$ and $H(\psi)$ are valid, $c_\Delta^2 = \lambda_\nu^{-1}$, and $\alpha_\Phi = L_{F_b} \lambda_\nu^{-1/2} \lambda_\tau^{-1/2}$.

By Theorem 4.9, under the stated assumptions on the data and the smallness condition (5.21), the mixed hemivariational–variational inequality (5.29)–(5.30) has a solution $(\boldsymbol{u}, \lambda) \in V \times K_\Lambda$, and the first component $\boldsymbol{u}$ of the solution is unique.

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