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THE KAČANOV METHOD FOR A NONLINEAR VARIATIONAL INEQUALITY OF THE SECOND KIND ARISING IN ELASTOPLASTICITY***

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Abstract

The authors first prove a convergence result on the Kačanov method for solving general nonlinear variational inequalities of the second kind and then apply the Kačanov method to solve a nonlinear variational inequality of the second kind arising in elastoplasticity. In addition to the convergence result, an a posteriori error estimate is shown for the Kačanov iterates. In each step of the Kačanov iteration, one has a (linear) variational inequality of the second kind, which can be solved by using a regularization technique. The Kačanov iteration and the regularization technique together provide approximations which can be readily computed numerically. An a posteriori error estimate is derived for the combined effect of the Kačanov iteration and the regularization.

Keywords Kačanov method, Nonlinear variational inequality of the second kind, Convergence, Regularization, A posteriori error estimate

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§1. Introduction

The Kačanov method is an iteration method for solving nonlinear problems, via linearization. An early reference on the method is [8], where the method is applied to compute a stationary magnetic field in nonlinear media. Convergence of the method is proved in the context of the particular application there, though the technique of the proof is rather general. The method is applied to a nonlinear elasticity problem in [9]. For applications of the method in solving variational inequalities for transonic flows in gas dynamics see [2,4]. A general convergence result of the method is presented in [10] (see also [13]) for solving a nonlinear variational inequality of the first kind (i.e., it is an inequality because the problem is posed over a non-empty closed convex subset rather than over a whole space). In [6], the method is applied to solve some nonlinear problems. Convergence of the method applied to the problems follows from the convergence result in [10] and [13]. A posteriori error estimates are derived which can be used to bound the true error of an iterant by two consecutive iterates. Some numerical examples are presented in [6], showing the efficiency of the Kačanov method and the a posteriori error estimates.

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In this paper, we analyze the Kačanov method in solving general nonlinear variational inequalities of the second kind (i.e., these are inequalities because of the presence of non-differentiable terms in problem formulations). We will first prove a convergence result for the Kačanov method in solving nonlinear variational inequalities of the mixed kind (i.e., they are inequalities both because they involve non-differentiable terms and are posed over non-empty closed convex subsets). The convergence result is a generalization of that in [10] and [13]. This is done in Section 2. In Section 3, we apply the Kačanov method to solve a nonlinear variational inequality of the second kind arising in elastoplasticity, and show the convergence of the method. In Section 4, we present an a posteriori error estimate for the Kačanov iterates. In each step of the Kačanov iteration, one has a (linear) variational inequality of the second kind. Because of the difficulty caused by the non-differentiable term in the variational inequality, we use a regularization technique to approximate the problem in the last section. The Kačanov iteration and the regularization technique together provide approximations which can be readily computed numerically. An a posteriori error estimate is derived for the combined effect of the Kačanov iteration and the regularization.

We will need the following well-known result (cf. [13]).

Theorem 1.1. *Let V be a reflexive Banach space, $K \subset V$ a nonempty, closed, convex subset. Assume $f : K \to R$ is convex, continuous and weakly coercive (i.e., $f(u) \to \infty$ as $\|u\| \to \infty$, $u \in K$). Then, f has a minimum on K. Furthermore, if f is strictly convex, then the minimum point of f on K is unique.*

§2. A Convergence Result

First, we introduce a general nonlinear variational inequality of the second kind. Let V be a Hilbert space, $K \subset V$ a nonempty closed convex subset. Let $E : K \to R$ be Gâteaux-differentiable, $D : K \to R$ be non-negative, convex and continuous, $l \in V^*$ a continuous linear functional on V. The functional D is not assumed to be differentiable. Then let us consider the constrained minimization problem:

$$\text{find } u \in K, \text{ such that } E(u) + D(u) - l(u) = \inf_{v \in K} \{E(v) + D(v) - l(v)\}. \tag{2.1}$$

Since E is Gâteaux-differentiable, a solution of (2.1) satisfies the variational inequality

$$u \in K, \quad \langle E'(u), v - u\rangle + D(v) - D(u) \geq l(v - u), \quad \forall\, v \in K. \tag{2.2}$$

When $E(v)$ is quadratic in v, the differential operator associated with the left-hand side of (2.2) is linear. When $E(v)$ is not quadratic in v, the differential operator associated with the left-hand side of (2.2) is nonlinear, which makes the variational inequality problem more difficult to solve.

Assume for $E'(u)$, we can find a functional $B : K \times V \times V \to R$, such that

$$\langle E'(u), v - w\rangle = B(u; u, v - w), \quad \forall\, u, v, w \in K, \tag{2.3}$$

and for fixed $u \in K$, $(v, w) \mapsto B(u; v, w)$ is a bilinear form on V. The problem (2.2) can be rewritten as $u \in K, \quad B(u; u, v - u) + D(v) - D(u) \geq l(v - u), \quad \forall\, v \in K$. Then the Kačanov method for (2.2) is defined as follows.

Let u_0 be an initial guess chosen from K. For $k = 0, 1, \ldots$, we find $u_{k+1} \in K$ such that

$$B(u_k; u_{k+1}, v - u_{k+1}) + D(v) - D(u_{k+1}) \geq l(v - u_{k+1}), \quad \forall\, v \in K. \tag{2.4}$$

The following theorem is on the unique solvability of the approximate problem (2.4) and of the original problem (2.2), and on the convergence of the Kačanov method.

Theorem 2.1. *We keep the conditions on the given data V, K, E, D, l and B.*

(a) *Assume for each $u \in K$, the bilinear form $(v,w) \mapsto B(u;v,w)$ is symmetric from $V \times V$ to R. Assume there are constants $\delta_1 > 0$ and $\delta_0 > 0$, such that*

$$|B(u;v,w)| \le \delta_1 \|v\| \, \|w\|, \quad \forall\, u \in K, \ \forall\, v, w \in V \tag{2.5}$$

and

$$B(u; v-w, v-w) \ge \delta_0 \|v-w\|^2, \quad \forall\, u, v, w \in K. \tag{2.6}$$

Then the problem (2.4) has a unique solution $u_{k+1} \in K$, which is also the unique solution of the minimization problem

$$\frac{1}{2} B(u_k; v, v) + D(v) - l(v) \to \inf, \quad v \in K. \tag{2.7}$$

(b) *Further assume $E' : K \to V^*$ is continuous and strongly monotone, i.e., for a constant $\rho_0 > 0$,*

$$\langle E'(u) - E'(v), u - v \rangle \ge \rho_0 \|u - v\|^2, \quad \forall\, u, v \in K. \tag{2.8}$$

Also assume the following key inequality holds:

$$E(v) - E(u) \le \frac{1}{2} \left(B(u;v,v) - B(u;u,u) \right), \quad \forall\, u, v \in K. \tag{2.9}$$

Then (2.1) has a unique solution $u \in K$, which is also the unique solution of the variational inequality problem (2.2). The Kačanov method defined in (2.4) converges, i.e., $u_k \to u$ in V, as $k \to \infty$.

Proof. (a) The equivalence of the problems (2.4) and (2.7) can be established in the standard way. From the assumptions, it can be verified that the functional $(1/2)\, B(u_k; v, v) + D(v) - l(v)$ is strictly convex, continuous and weakly coercive on K. Thus, from Theorem 1.1, we have the existence of a unique solution to the minimization problem (2.7).

(b) Using the assumption (2.8), we know that the functional $G(v) \equiv E(v) + D(v) - l(v)$ is strictly convex and weakly coercive on K. The other conditions in Theorem 1.1 can be easily verified. Hence, by Theorem 1.1, we have the existence of a unique solution to the original problem (2.1), or equivalently, (2.2).

Now we prove the convergence of the Kačanov method for solving the problem (2.2). We have

$$\begin{aligned}
&G(u_k) - G(u_{k+1}) \\
&= E(u_k) - E(u_{k+1}) + D(u_k) - D(u_{k+1}) - l(u_k - u_{k+1}) \\
&\quad \text{(using the key inequality (2.9) with } v = u_{k+1} \text{ and } u = u_k) \\
&\ge \frac{1}{2} \left(B(u_k; u_k, u_k) - B(u_k; u_{k+1}, u_{k+1}) \right) + D(u_k) - D(u_{k+1}) - l(u_k - u_{k+1}) \\
&\quad \text{(using the inequality (2.4) with } v = u_k) \\
&\ge \frac{1}{2} \left(B(u_k; u_k, u_k) - B(u_k; u_{k+1}, u_{k+1}) \right) - B(u_k; u_{k+1}, u_k - u_{k+1}) \\
&= \frac{1}{2} B(u_k; u_k - u_{k+1}, u_k - u_{k+1}).
\end{aligned}$$

Thus, using the assumption (2.6), we obtain the inequality

$$\frac{\delta_0}{2}\|u_{k+1}-u_k\|^2 \le G(u_k)-G(u_{k+1}). \tag{2.10}$$

A simple consequence of the above inequality is that the sequence $\{G(u_k)\}$ is decreasing. Since the sequence $\{G(u_k)\}$ is bounded below by the minimum value of $G(v)$ on K, we find that $G(u_k)-G(u_{k+1})\to 0$ as $k\to\infty$. Hence, from (2.10), we have

$$\|u_{k+1}-u_k\|\to 0 \quad \text{as } k\to\infty. \tag{2.11}$$

Now, using the assumption (2.8), we have

$$\begin{aligned}
\rho_0\|u_k-u\|^2 &\le \langle E'(u_k)-E'(u),u_k-u\rangle\\
&= \langle E'(u_k),u_k-u\rangle-\langle E'(u),u_k-u\rangle\\
&\quad \text{(using the inequality (2.2) with } v=u_k\text{, and (2.3) at } u_k)\\
&\le B(u_k;u_k,u_k-u)+D(u_k)-D(u)-l(u_k-u)\\
&\quad \text{(using the inequality (2.4) with } v=u)\\
&\le B(u_k;u_k,u_k-u)+D(u_k)-D(u)-l(u_k-u)\\
&\quad +B(u_k;u_{k+1},u-u_{k+1})+D(u)-D(u_{k+1})-l(u-u_{k+1})\\
&= B(u_k;u_k,u_k-u)+B(u_k;u_{k+1},u-u_{k+1})\\
&\quad +D(u_k)-D(u_{k+1})-l(u_k-u_{k+1})\\
&= B(u_k;u_k-u_{k+1},u_k-u)+B(u_k;u_{k+1},u_k-u_{k+1})\\
&\quad +D(u_k)-D(u_{k+1})-l(u_k-u_{k+1})\\
&\to 0 \quad \text{as } k\to\infty,
\end{aligned}$$

where, in the last step, we used the continuity properties of B, D and l, and the fact that $\{u_k\}$ is bounded.

§3. Application in Solving an Elastoplasticity Problem

We apply the Kačanov method to solve a nonlinear variational inequality of the second kind arising in elastoplasticity. The problem is discussed in detail in [11,7]. Here, we will briefly review one mathematical formulation of the problem. We adopt the summation convention over repeated indices. However, no summation is implied over the repeated index k for the Kačanov iterates.

Consider the quasistatic behavior of an elastoplastic body which occupies a bounded domain $\Omega\subset R^d$ ($d=3$ in practice) with Lipschitz boundary. The plastic behavior of the material is described in terms of a dissipation function, and we assume the material undergoes nonlinear kinematic hardening, the nonlinear part of which takes the form of an exponential decay. The boundary value problem we are going to present arises in a typical time-step in approximating the rate of change of the plastic strain by a backward Euler difference (cf. [11]). We assume the material is subject to the action of a body force with density b. For simplicity in presentation, we assume the boundary of the body is fixed. The unknown variables of the problem are the displacement u and the plastic strain tensor p.

We seek the displacement in the space

$$V = [H_0^1(\Omega)]^d.$$

Let

$$Q = \{q = (q_{ij}) : q_{ij} \in L^2(\Omega),\ q_{ji} = q_{ij},\ 1 \le i, j \le d\}.$$

For the plastic strain, we use the space $Q_0 = \{q \in Q : \operatorname{tr} q = 0\}$, where, $\operatorname{tr} p = p_{ii}$. We require the restriction $\operatorname{tr} p = 0$ on the plastic strain p by the conventional assumption of no volume change accompanying the plastic deformation.

Both V and Q are Hilbert spaces with inner products

$$(u, v)_V = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx \quad \text{and} \quad (p, q)_Q = \int_\Omega p \cdot q \; dx = \int_\Omega p_{ij} q_{ij}\, dx,$$

and norms $\|v\|_V = (v, v)^{1/2}$, $\|q\|_Q = (q, q)^{1/2}$. Furthermore, Q_0 is a closed subspace of Q.

To formulate the problem, we need to use the product space $\overline{V} = V \times Q$ which is a Hilbert space with the inner product $(\overline{u}, \overline{v})_{\overline{V}} = (u, v)_V + (p, q)_Q$ and norm $\|\overline{u}\|_{\overline{V}} = (\overline{u}, \overline{u})_{\overline{V}}^{1/2}$, for $\overline{u} = (u, p), \overline{v} = (v, q) \in \overline{V}$. We also define $\overline{V}_0 = V \times Q_0$, a closed subspace of $\overline{V}$. The topological dual of a Hilbert space X is denoted by X^*.

Define an operator $A : \bar{V} \to \bar{V}^*$ by

$$\langle A\bar{u}, \bar{v}\rangle = \int_\Omega [C\,(\epsilon(u) - p) \cdot (\epsilon(v) - q) + h(|p|) p \cdot q]\; dx, \tag{3.1}$$

where $\epsilon(u) = (\epsilon_{ij}(u))$ is the (linearized) strain tensor with the components

$$\epsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad 1 \le i, j \le d,$$
$$C\,(\epsilon(u) - p) \cdot (\epsilon(v) - q) = C_{ijkl}(\epsilon_{ij}(u) - p_{ij})(\epsilon_{kl}(v) - q_{kl}),$$
$$p \cdot q = p_{ij} q_{ij}, \quad |p| = \{p \cdot p\}^{1/2}$$

and

$$h(\alpha) = h_0 + h_1 e^{-\nu\alpha}.$$

Assume that the coefficient $\nu > 0$ and the functions h_0 and h_1 satisfy the conditions

$$h_0(x) \ge \eta_0 > 0, \quad h_1(x) \ge 0, \quad h_1(x) < \theta\, h_0(x)\, e^2 \text{ for some } \theta \in (0, 1).$$

The smallness assumption on h_1 (relative to h_0) is needed in proving the unique solvability of the elastoplasticity problem (3.3) below (cf. [11]). Then, for $b \in V^*$, the density of body force, define the linear functional

$$l : \overline{V} \to R,\ \langle l, \overline{v}\rangle = \int_\Omega b \cdot v\, dx$$

and, for some material parameter $g > 0$, define the functional

$$j : \overline{V} \to R,\ j(\overline{v}) = \int_\Omega g\, |q(x)|\, dx, \tag{3.2}$$

where as before $\overline{u} = (u, p)$ and $\overline{v} = (v, q)$. The functional j is known as the dissipation function in plasticity. The functionals $l(\cdot)$ and $j(\cdot)$ are easily shown to be bounded, $j(\cdot)$ is a convex, positively homogeneous (i.e., $j(\alpha\overline{v}) = \alpha\, j(\overline{v})$ for $\alpha > 0$), non-negative Lipschitz continuous functional. Note that, however, j is not differentiable.

The variational form of the elastoplasticity problem is: find $\overline{u} = (u,p) \in \overline{V}_0$ such that

$$\langle A\overline{u}, \overline{v} - \overline{u}\rangle + j(\overline{v}) - j(\overline{u}) - \langle l, \overline{v} - \overline{u}\rangle \geq 0, \quad \forall \overline{v} \in \overline{V}_0. \tag{3.3}$$

It is shown in [11] that the problem has a unique solution.

The Kačanov method for the problem (3.3) is: choose an initial guess $\overline{u}_0 \in \overline{V}_0$; for $k = 0, 1, \cdots$, find $\overline{u}_{k+1} \in \overline{V}_0$, such that

$$B(\overline{u}_k; \overline{u}_{k+1}, \overline{v} - \overline{u}_{k+1}) + j(\overline{v}) - j(\overline{u}_{k+1}) - \langle l, \overline{v} - \overline{u}_{k+1}\rangle \geq 0, \quad \forall \overline{v} \in \overline{V}_0, \tag{3.4}$$

where

$$B(\overline{u}_k; \overline{u}_{k+1}, \overline{v}) = \int_\Omega \left[C\left(\epsilon(u_{k+1}) - p_{k+1}\right) \cdot (\epsilon(v) - q) + h(|p_k|)p_{k+1} \cdot q\right] dx. \tag{3.5}$$

To apply Theorem 2.1 for a convergence analysis of the method, we notice that the problem (3.3) is equivalent to the minimization problem

$$\overline{u} = (u,p) \in \overline{V}_0, \quad E(\overline{u}) + j(\overline{u}) - \langle l, \overline{u}\rangle = \inf\{E(\overline{v}) + j(\overline{v}) - \langle l, \overline{v}\rangle : \overline{v} \in \overline{V}_0\}, \tag{3.6}$$

where

$$E(\overline{v}) = \int_\Omega \left[\frac{1}{2} C\left(\epsilon(v) - q\right) \cdot (\epsilon(v) - q) + H(|q|)\right] dx \tag{3.7}$$

with

$$H(\alpha) = \frac{1}{2} h_0 \alpha^2 + \frac{1}{\nu^2} h_1 (1 - e^{-\nu\alpha}) - \frac{1}{\nu} h_1 \alpha e^{-\nu\alpha}. \tag{3.8}$$

A crucial step in applying Theorem 2.1 for the convergence of the method (3.5) is to prove the key inequality (2.9), which in the present case is equivalent to (3.9) in the next lemma.

Lemma 3.1. *The following inequality holds:*

$$H(\beta) - H(\alpha) \leq \frac{1}{2} h(\alpha)(\beta^2 - \alpha^2), \quad \forall \alpha, \beta \geq 0. \tag{3.9}$$

Proof. From the definitions of $H(\alpha)$ and $h(\alpha)$, we have

$$\begin{aligned} &H(\beta) - H(\alpha) - \frac{1}{2} h(\alpha)(\beta^2 - \alpha^2) \\ &= \frac{h_1}{\nu^2}\left[e^{-\nu\alpha} - e^{-\nu\beta} + \nu(\alpha e^{-\nu\alpha} - \beta e^{-\nu\beta}) - \frac{\nu^2}{2} e^{-\nu\alpha}(\beta^2 - \alpha^2)\right]. \end{aligned}$$

For a fixed $\alpha \geq 0$, let us consider a function of β defined by

$$f(\beta) = e^{-\nu\alpha} - e^{-\nu\beta} + \nu(\alpha e^{-\nu\alpha} - \beta e^{-\nu\beta}) - \frac{\nu^2}{2} e^{-\nu\alpha}(\beta^2 - \alpha^2).$$

We have

$$f'(\beta) = \nu^2 \beta (e^{-\nu\beta} - e^{-\nu\alpha}).$$

Obviously, $f'(\beta) \geq 0$ for $\beta \in [0, \alpha]$, $f'(\beta) \leq 0$ for $\beta \geq \alpha$. We conclude that the function $f(\beta)$, $\beta \geq 0$, attains its maximum value at $\beta = \alpha$. Thus, $f(\beta) \leq f(\alpha) = 0, \quad \forall \beta \geq 0$. Hence, the inequality (3.9) is proved.

The validity of the inequality (2.8) for the problem (3.3) is proved in [11] (see Lemma 3 there). The other conditions of Theorem 2.1 are easily verified. Therefore, the Kačanov method (3.5) converges.

§4. A Posteriori Error Estimate

We have shown the convergence of the Kačanov method (3.4) for the problem (3.3) at the end of last section. From the viewpoint of practical implementation of the method, it is highly desirable to derive some a posteriori error estimate so that once a Kačanov iterate is computed, we can compute a (presumably efficient) error bound for the approximate solution with ease. In this section, we present such an a posteriori error estimate for the Kačanov method (3.4). A general framework for deriving a posteriori error estimates for various mathematical procedures is given in [5].

To derive an a posteriori error estimate for the approximation error $\overline{u}_{k+1} - \overline{u}$, we apply the duality theory in convex analysis (cf. [1]). Let $s = \epsilon(v)$ be a dual variable, and define the operator

$$\Lambda : \overline{V}_0 \to S, \quad \Lambda\overline{v} = \epsilon(v),$$

where

$$S = \{s = (s_{ij}) : s_{ij} = s_{ji} \in L^2(\Omega),\ 1 \le i, j \le d\}.$$

We identify S^* with S. Now define the functional

$$J(\overline{v}, s) = \int_\Omega \left[\frac{1}{2} C(s-q)\cdot(s-q) + H(|q|) + g\,|q| - b\cdot v\right] dx.$$

Then, the problem (3.6) can be rewritten as

$$\overline{u} \in \overline{V}_0 : \quad J(\overline{u}, \Lambda\overline{u}) = \inf\left\{J(\overline{v}, \Lambda\overline{v}) : \overline{v} \in \overline{V}_0\right\}. \tag{4.1}$$

Following [7], we have, for the conjugate functional J^* of J,

$$J^*(\Lambda^* s^*, -s^*) = \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{2} C^{-1} s^* \cdot s^* + K(|s^*|)\right] dx, \\ \qquad \text{if } \displaystyle\int_\Omega [\epsilon(v)\cdot s^* + b\cdot v]\, dx = 0,\ \forall v \in V, \\ \infty, \quad \text{otherwise.} \end{cases} \tag{4.2}$$

Here,

$$K(|s^*|) = T(t(|s^{*D}|)), \tag{4.3}$$

with the deviatoric strain tensor defined as $s^{*D} = s^* - \frac{1}{d}\mathrm{tr}\,(s^*)I$, I being the second-order identity tensor, $T(t) = (|s^{*D}| - g)\,t - H(t)$, and $t(|s^{*D}|) = 0$ if $|s^{*D}| \le g$, $t(|s^{*D}|) > 0$ being the unique solution of the equation (the unique solvability is guaranteed by the assumption $h_0 > 0$)

$$(h_0 + h_1 e^{-\nu t})\, t = |s^{*D}| - g \quad \text{if } |s^{*D}| > g.$$

We will say an $s^* \in S^*$ is admissible, if it satisfies the relation

$$\int_\Omega [\epsilon(v)\cdot s^* + b\cdot v]\, dx = 0, \quad \forall v \in V. \tag{4.4}$$

Notice that, from (4.2), the value of $J^*(\Lambda^* s^*, -s^*)$ is infinite if s^* is not admissible.

Now consider the difference

$$J(\overline{u}_{k+1}, \Lambda\overline{u}_{k+1}) - J(\overline{u}, \Lambda\overline{u}).$$

Following the argument in [7], we have

$$J(\overline{u}_{k+1}, \Lambda\overline{u}_{k+1}) - J(\overline{u}, \Lambda\overline{u}) \ge \overline{\alpha}\,\left(\|u_{k+1} - u\|_V^2 + \|p_{k+1} - p\|_Q^2\right), \tag{4.5}$$

for some constant $\overline{\alpha} > 0$ (for an expression to calculate $\overline{\alpha} > 0$, see [7]).

To have an upper bound for the difference, we need to choose a suitable admissible variable s^*. We notice that the problem (3.4) is equivalent to

$$\int_\Omega C\left(\epsilon(u_{k+1})-p_{k+1}\right)\cdot\epsilon(v)\,dx=\int_\Omega b\cdot v\,dx,\quad \forall\, v\in V, \tag{4.6}$$

and

$$\int_\Omega\left[-C\left(\epsilon(u_{k+1})-p_{k+1}\right)\cdot(q-p_{k+1})+h(|p_k|)\,p_{k+1}\cdot(q-p_{k+1})+g\left(|q|-|p_{k+1}|\right)\right]dx\geq 0,$$
$$\forall\, q\in Q_0. \tag{4.7}$$

From (4.6), it follows that

$$s^*=-C\left(\epsilon(u_{k+1})-p_{k+1}\right) \tag{4.8}$$

is admissible. With this choice of s^*, we have the following upper bound on $J(\overline{u}_{k+1},\Lambda\overline{u}_{k+1})-J(\overline{u},\Lambda\overline{u})$.

$$\begin{aligned}&J(\overline{u}_{k+1},\Lambda\overline{u}_{k+1})-J(\overline{u},\Lambda\overline{u})\\&\leq J(\overline{u}_{k+1},\Lambda\overline{u}_{k+1})+J^*(\Lambda^*s^*,-s^*)\\&=\int_\Omega[C(\epsilon(u_{k+1})-p_{k+1})\cdot(\epsilon(u_{k+1})-p_{k+1})\\&\quad+H(|p_{k+1}|)+g\,|p_{k+1}|-b\cdot u_{k+1}+T(t(|s^{*D}|))]\,dx.\end{aligned}$$

Applying (4.6) with $v=u_{k+1}$, we obtain

$$\begin{aligned}&J(\overline{u}_{k+1},\Lambda\overline{u}_{k+1})-J(\overline{u},\Lambda\overline{u})\\&\leq\int_\Omega\left[-C(\epsilon(u_{k+1})-p_{k+1})\cdot p_{k+1}+H(|p_{k+1}|)+g\,|p_{k+1}|+T(t(|s^{*D}|))\right]\,dx.\end{aligned}$$

Combined with (4.5), this implies the following a posteriori error estimate

$$\begin{aligned}&\overline{\alpha}\,\left(\|u_{k+1}-u\|_V^2+\|p_{k+1}-p\|_Q^2\right)\\&\leq\int_\Omega\left[-C(\epsilon(u_{k+1})-p_{k+1})\cdot p_{k+1}+H(|p_{k+1}|)+g\,|p_{k+1}|+T(t(|s^{*D}|))\right]\,dx,\end{aligned} \tag{4.9}$$

where s^* is defined in (4.8).

For the purpose of comparing with the a posteriori error estimate to be derived in the next section for the combined effect of a regularization procedure and the Kačanov iteration, we derive another a posteriori error estimate for the Kačanov method (3.4) as a consequence of the estimate (4.9). We take $q=0$ in (4.7) to obtain

$$\int_\Omega\left[C(\epsilon(u_{k+1}-p_{k+1})\cdot p_{k+1}-h(|p_k|)\,|p_{k+1}|^2-g\,|p_{k+1}|\right]\,dx\geq 0.$$

Applying this inequality to the right-hand side of (4.9), we get the estimate

$$\begin{aligned}&\overline{\alpha}\,\left(\|u_{k+1}-u\|_V^2+\|p_{k+1}-p\|_Q^2\right)\\&\leq\int_\Omega\left[H(|p_{k+1}|)-h(|p_k|)\,|p_{k+1}|^2+T(t(|s^{*D}|))\right]\,dx.\end{aligned} \tag{4.10}$$

§5. Regularization, A Posteriori Error Estimate

We observe that in each step of the Kačanov iteration one has a (linear) variational inequality of the second kind, (3.4). The difficulty in solving (3.4) directly lies in the fact that term j is non-differentiable. One approach used in practice for overcoming this difficulty is to use a regularization technique. In a regularization technique, j is replaced by a differentiable function j^ε, such that $j^\varepsilon \to j$ as $\varepsilon \to 0$. There are many choices for the regularization function j^ε. Here, for definiteness, we take

$$j^\varepsilon(\overline{v}) = \int_\Omega g\sqrt{|q(x)|^2 + \varepsilon^2}\, dx. \tag{5.1}$$

Once the regularization function j^ε is chosen, the variational inequality (3.4) is approximated by the following

$$B(\overline{u}_k; \overline{u}^\varepsilon_{k+1}, \overline{v} - \overline{u}^\varepsilon_{k+1}) + j^\varepsilon(\overline{v}) - j^\varepsilon(\overline{u}^\varepsilon_{k+1}) - \langle l, \overline{v} - \overline{u}^\varepsilon_{k+1}\rangle \geq 0, \quad \forall\, \overline{v} \in \overline{V}_0.$$

Owing to the fact that j^ε is differentiable, the regularized problem can be rewritten as

$$B(\overline{u}_k; \overline{u}^\varepsilon_{k+1}, \overline{v}) + \langle Dj^\varepsilon(\overline{u}^\varepsilon_{k+1}), \overline{v}\rangle - \langle l, \overline{v}\rangle = 0, \quad \forall\, \overline{v} \in \overline{V}_0,$$

or, in detail,

$$\int_\Omega C\left(\epsilon(u^\varepsilon_{k+1}) - p^\varepsilon_{k+1}\right) \cdot \epsilon(v)\, dx = \int_\Omega b \cdot v\, dx, \quad \forall\, v \in V, \tag{5.2}$$

and

$$\begin{aligned}&\int_\Omega \Big[-C\left(\epsilon(u^\varepsilon_{k+1}) - p^\varepsilon_{k+1}\right) \cdot q + \Big(h(|p_k|) + \frac{g}{\sqrt{|p^\varepsilon_{k+1}|^2 + \varepsilon^2}}\Big) p^\varepsilon_{k+1} \cdot q\Big]\, dx \\ &= 0, \quad \forall\, q \in Q_0.\end{aligned} \tag{5.3}$$

The convergence of the regularization method can be established by a standard procedure. Indeed, it is not difficult to prove that (see [12], e.g.)

$$\|u_{k+1} - u^\varepsilon_{k+1}\|^2_V + \|p_{k+1} - p^\varepsilon_{k+1}\|^2_Q \leq c\,\varepsilon.$$

The rest of the section is devoted to the derivation of a posteriori error estimates for the approximation error $\overline{u}^\varepsilon_{k+1} - \overline{u}$. As in the last section, we consider the difference

$$J(\overline{u}^\varepsilon_{k+1}, \Lambda\overline{u}^\varepsilon_{k+1}) - J(\overline{u}, \Lambda\overline{u}).$$

It is proved in [7] that

$$J(\overline{u}^\varepsilon_{k+1}, \Lambda\overline{u}^\varepsilon_{k+1}) - J(\overline{u}, \Lambda\overline{u}) \geq \overline{\alpha}\left(\|u^\varepsilon_{k+1} - u\|^2_V + \|p^\varepsilon_{k+1} - p\|^2_Q\right), \tag{5.4}$$

for the same constant $\overline{\alpha} > 0$ as in the last section.

By (5.2),

$$s^* = -C\left(\epsilon(u^\varepsilon_{k+1}) - p^\varepsilon_{k+1}\right) \tag{5.5}$$

is admissible. With this choice of s^*, we have the following upper bound on

$$J(\overline{u}^\varepsilon_{k+1}, \Lambda\overline{u}^\varepsilon_{k+1}) - J(\overline{u}, \Lambda\overline{u}).$$

$$\begin{aligned}&J(\overline{u}^\varepsilon_{k+1}, \Lambda\overline{u}^\varepsilon_{k+1}) - J(\overline{u}, \Lambda\overline{u}) \\ &\leq J(\overline{u}^\varepsilon_{k+1}, \Lambda\overline{u}^\varepsilon_{k+1}) + J^*(\Lambda^* s^*, -s^*) \\ &= \int_\Omega \Big[H(|p^\varepsilon_{k+1}|) - h(|p_k|)\,|p^\varepsilon_{k+1}|^2 \\ &\quad + \frac{g\,|p^\varepsilon_{k+1}|\,\varepsilon^2}{\sqrt{|p^\varepsilon_{k+1}|^2 + \varepsilon^2}\left(\sqrt{|p^\varepsilon_{k+1}|^2 + \varepsilon^2} + |p^\varepsilon_{k+1}|\right)} + T(t(|s^{*D}|))\Big]\, dx.\end{aligned}$$

Combined with (5.4), this implies the following a posteriori error estimate

$$\begin{aligned}
&\overline{\alpha}\left(\|u_{k+1}^{\varepsilon}-u\|_V^2+\|p_{k+1}^{\varepsilon}-p\|_Q^2\right)\\
&\le\int_\Omega\Big[H(|p_{k+1}^{\varepsilon}|)-h(|p_k|)\,|p_{k+1}^{\varepsilon}|^2\\
&\quad+\frac{g\,|p_{k+1}^{\varepsilon}|\,\varepsilon^2}{\sqrt{|p_{k+1}^{\varepsilon}|^2+\varepsilon^2}\left(\sqrt{|p_{k+1}^{\varepsilon}|^2+\varepsilon^2}+|p_{k+1}^{\varepsilon}|\right)}+T(t(|s^{*D}|))\Big]\,dx. \qquad (5.6)
\end{aligned}$$

To see the efficiency of the estimate, we have, as is done in [7], that in the linear case (i.e., $h_1=0$) the estimate (5.6) implies

$$\begin{aligned}
&\overline{\alpha}\left(\|u_{k+1}^{\varepsilon}-u\|_V^2+\|p_{k+1}^{\varepsilon}-p\|_Q^2\right)\\
&\le\int_\Omega\frac{g\,|p_{k+1}^{\varepsilon}|\,\varepsilon^2}{\sqrt{|p_{k+1}^{\varepsilon}|^2+\varepsilon^2}\left(\sqrt{|p_{k+1}^{\varepsilon}|^2+\varepsilon^2}+|p_{k+1}^{\varepsilon}|\right)}\,dx. \qquad (5.7)
\end{aligned}$$

It is easy to see that, at least formally, the error estimate (5.6) reduces to (4.10) when $\varepsilon\to 0$.

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