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NUMERICAL ANALYSIS OF ELLIPTIC HEMIVARIATIONAL INEQUALITIES FOR SEMIPERMEABLE MEDIA*

Weimin Han
Program in Applied Mathematical and Computational Sciences (AMCS) & Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
Email: weimin-han@uiowa.edu
Ziping Huang
School of Mathematical Sciences, Tongji University, Shanghai 200092, China
Tongji Zhejiang College, Jiaxing 314051, China
Email: huangziping@tongji.edu.cn
Cheng Wang
School of Mathematical Sciences, Tongji University, Shanghai 200092, China
Email: wangcheng@tongji.edu.cn
Wei Xu
School of Mathematical Sciences, Tongji University, Shanghai 200092, China
Tongji Zhejiang College, Jiaxing 314051, China
Email: xuwei820824@163.com

Abstract

In this paper, we consider elliptic hemivariational inequalities arising in applications in semipermeable media. In its general form, the model includes both interior and boundary semipermeability terms. Detailed study is given on the hemivariational inequality in the case of isotropic and homogeneous semipermeable media. Solution existence and uniqueness of the problem are explored. Convergence of the Galerkin method is shown under the basic solution regularity available from the existence result. An optimal order error estimate is derived for the linear finite element solution under suitable solution regularity assumptions. The results can be readily extended to the study of more general hemivariational inequalities for non-isotropic and heterogeneous semipermeable media with interior semipermeability and/or boundary semipermeability. Numerical examples are presented to show the performance of the finite element approximations; in particular, the theoretically predicted optimal first order convergence in H^1 norm of the linear element solutions is clearly observed.

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1. Introduction

Variational inequalities for flow problems through porous media are studied in [9]. Such variational inequality problems adopt monotone semipermeability relations for the media. In [19], extension of the problems is made for semipermeable media to allow non-monotone semipermeability relations, leading to hemivariational inequalities. In both these references, semipermeability on the boundary or in the domain is considered.

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Since the pioneering work by Panagiotopoulos in early 1980s ([18]) on variational problems with nonconvex and generally nondifferentiable super-potentials, hemivariational inequalities have attracted steady attention from the research communities in mathematics, physical sciences and engineering. The formulation of hemivariational inequalities provides a useful framework to both theoretically and numerically treat application problems involving non-monotone, nonsmooth and multivalued constitutive laws, forces, and boundary conditions. Hemivariational inequalities have been shown very useful across a variety of subjects. Mathematical theory, numerical approximations and applications of hemivariational inequalities can be found in several monographs, e.g., [4, 15–17, 20]. The number of research papers on hemivariational inequalities is growing rapidly. The reference [15] discusses finite element approximations of hemivariational inequalities, including their convergence; however, no error estimates are provided. Recently, optimal order error estimates are derived for numerical solutions of hemivariational inequalities. The first paper along this direction is [12] where optimal order error estimates for the linear finite element solutions for some stationary hemivariational and variational-hemivariational inequalities are derived. This paper is followed by numerous papers on optimal order error estimates of the linear finite element solutions for various hemivariational inequalities of different form, e.g., [3] for the numerical solution of a hyperbolic hemivariational inequality, and [2] for the numerical solution of an evolutionary variational–hemivariational inequality. A general framework is presented on convergence analysis and error estimation for internal approximations of elliptic hemivariational inequalities in [13], and that for variational–hemivariational inequalities in [14]. In [11], a comprehensive convergence analysis and error estimation are given for both internal and external approximations of stationary variational–hemivariational inequalities and hemivariational inequalities. In all these references on numerical analysis of hemivariational inequalities, the application background is contact mechanics.

The purpose of this paper is to study and approximate elliptic hemivariational inequalities for the semipermeable media. The general hemivariational inequality incorporates both the interior and boundary semipermeability. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, i.e., Ω is an open, bounded and connected region in $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Here the positive integer d is the dimension of the problem under consideration. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on $\partial\Omega$. We split the boundary $\partial\Omega$ into two non-overlapping and measurable parts Γ_0 and Γ_1 with $\operatorname{meas}(\Gamma_0) > 0$:

$$\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}. \tag{1.1}$$

We will specify a Dirichlet boundary condition on Γ_0 and a Neumann inclusion condition on Γ_1. The pointwise formulation of the model problem is as follows:

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \qquad \text{on } \Gamma_0, \tag{1.3}$$

$$-\frac{\partial u}{\partial \nu} \in \partial j_2(u) \qquad \text{on } \Gamma_1. \tag{1.4}$$

The differential equation (1.2) corresponds to the case of isotropic and homogeneous media (cf. [9,19]). Here, ∂j_2 is the generalized subdifferential of a locally Lipschitz continuous function j_2 (cf. Section 2). For simplicity, we let the Dirichlet boundary value to be zero in (1.3). The problem with a nonzero Dirichlet boundary value on Γ_0 can be handled with the standard technique (cf. e.g., [1, Subsection 8.4.2]). To allow the interior semipermeability condition, we write the right hand side of (1.2) as

$$f = f_0 + f_1, \qquad f_0 \in L^2(\Omega),\ -f_1 \in \partial j_1(u). \tag{1.5}$$

To introduce the weak formulation of the boundary value problem (1.2)–(1.5), we will need a function space

$$V = \Big\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0 \Big\} \tag{1.6}$$

and a bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad u, v \in V. \tag{1.7}$$

By a standard procedure, we can derive from (1.2)–(1.5) the following weak formulation.

Problem (P_{model}). Find $u \in V$ such that

$$a(u,v) + \int_\Omega j_1^0(u;v)\, dx + \int_{\Gamma_1} j_2^0(u;v)\, ds \ge \int_\Omega f_0 v \, dx \qquad \forall\, v \in V. \tag{1.8}$$

In (1.8), j_1^0 and j_2^0 denote generalized directional derivatives of j_1 and j_2, cf. Section 2. We note that Problem (P_{model}) contains as special cases the model with the interior semipermeable term only (e.g. j_2 is a differentiable function in the classical sense), and the model with the boundary semipermeable term only (e.g. j_1 is a differentiable function in the classical sense).

The rest of the paper is organized as follows. In Section 2, we present necessary preliminary materials. In Section 3, we introduce an abstract hemivariational inequality, explore its solution existence and uniqueness. In Section 4, we consider a general Galerkin approximation of the abstract hemivariational inequality, prove the convergence of the Galerkin solution under the minimal solution regularity available from Section 3, and derive a Céa inequality for the numerical solution error. Compared with numerical analysis of hemivariational inequalities in the existing literature, in this paper, we provide a more thorough convergence analysis; moreover, the hemivariational inequality considered contains two generalized directional derivatives, one is defined through an integral in the problem domain, and the other an integral on part of the boundary. In Section 5, we apply the results for the abstract hemivariational inequality to Problem (P_{model}) and derive an optimal order error estimate for the linear finite element solution of the problem. In Section 6, we comment on the extension of the analysis to hemivariational inequalities for the case of non-isotropic and heterogeneous semipermeable media. Finally, we provide numerical examples to show the performance of the finite element method, and present numerical evidence of the theoretically predicted optimal first order convergence of the linear element solutions.

2. Preliminaries

We only use real spaces in this paper. As usual, for a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. When no confusion may arise, we simply write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_{X^*\times X}$. We use 2^{X^*} to denote the collection of all the subsets of X^*. Weak convergence is indicated by the symbol $\rightharpoonup$. Given two normed spaces X and Y, $\mathcal{L}(X,Y)$ is the space of all linear continuous operators from X to Y.

We deal with both single-valued and multivalued operators defined on a normed space X. We start by recalling several definitions for single-valued operators.

Definition 2.1. *An operator $A\colon X \to X^*$ is bounded if it maps bounded sets of X to bounded sets of X^*. The operator A is monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in X$. It is maximal monotone if it is monotone and $\langle Au - w, u - v\rangle \geq 0$ for any $u \in X$ implies that $w = Av$. It is pseudomonotone if it is bounded and $u_n \rightharpoonup u$ in X with $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ imply $Au_n \rightharpoonup Au$ in X^* and $\lim_{n\to\infty}\langle Au_n, u_n\rangle = \langle Au, u\rangle$.*

For a multivalued operator $T\colon X \to 2^{X^*}$, its graph $\mathcal{G}(T)$ is

$$\mathcal{G}(T) := \{(x, x^*) \in X \times X^* \mid x^* \in Tx\}.$$

Definition 2.2. *An operator $T\colon X \to 2^{X^*}$ is monotone if $\langle u^* - v^*, u - v\rangle \geq 0$ for all (u, u^*), $(v, v^*) \in \mathcal{G}(T)$. It is maximal monotone if it is monotone and maximal in the sense of inclusion of graphs in the family of monotone operators from X to 2^{X^*}.*

Definition 2.3. *Let X be a reflexive Banach space. A multivalued operator $T\colon X \to 2^{X^*}$ is* generalized pseudomonotone *if for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \rightharpoonup u$ in X, $u_n^* \in Tu_n$ for $n \geq 1$, $u_n^* \rightharpoonup u^*$ in X^* and $\limsup\langle u_n^*, u_n - u\rangle \leq 0$, we have $u^* \in Tu$ and*

$$\lim_{n\to\infty}\langle u_n^*, u_n\rangle = \langle u^*, u\rangle.$$

The following surjectivity result, derived from [17, Theorem 2.12], will be applied in studying the elliptic hemivariational inequalities.

Theorem 2.1. *Let X be a reflexive Banach space, $T_1\colon X \to 2^{X^*}$ bounded and generalized pseudomonotone, and $T_2\colon X \to 2^{X^*}$ maximal monotone. Assume $u_0 \in X$ is such that $T_2(u_0) \neq \emptyset$ and there exists a function $c_0\colon \mathbb{R}_+ \to \mathbb{R}$ with $c_0(r) \to \infty$ as $r \to \infty$ and*

$$\langle u^*, u - u_0\rangle \geq c_0(\|u\|_X)\,\|u\|_X \qquad \forall\,(u, u^*) \in \mathcal{G}(T_1). \tag{2.1}$$

Then $T_1 + T_2$ is surjective.

Finally, we recall the definitions of the convex and the Clarke subdifferentials.

Definition 2.4. *Let $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. The mapping $\partial_c\varphi\colon X \to 2^{X^*}$ defined by*

$$\partial_c\varphi(x) := \{x^* \in X^* \mid \langle x^*, v - x\rangle \leq \varphi(v) - \varphi(x)\ \forall\, v \in X\}$$

is called the (convex) subdifferential *of φ. An element $x^* \in \partial_c\varphi(x)$ (if it is non-empty) is called a* subgradient *of φ at x.*

Definition 2.5. *Let $\psi\colon X \to \mathbb{R}$ be a locally Lipschitz functional. The* generalized (Clarke) directional derivative *of ψ at $x \in X$ in the direction $v \in X$ is defined by*

$$\psi^0(x; v) := \limsup_{y\to x,\ \lambda\downarrow 0} \frac{\psi(y + \lambda v) - \psi(y)}{\lambda}.$$

The generalized gradient (subdifferential) *of ψ at x is defined by*

$$\partial\psi(x) := \Big\{\zeta \in X^* \mid \psi^0(x; v) \geq \langle\zeta, v\rangle\ \forall\, v \in X\Big\}.$$

Properties of the subdifferential mappings, both in the convex and Clarke sense, can be found in several books, e.g. [6–8, 16, 17, 20]. In particular, knowing the generalized subdifferential, we can compute the generalized directional derivative through the formula ([6])

$$\psi^0(x; v) = \max\{\langle\zeta, v\rangle \mid \zeta \in \partial\psi(x)\}. \tag{2.2}$$

3. An Abstract Elliptic Hemivariational Inequality

In this section, we study an abstract elliptic hemivariational inequality of a general form. In the discussion of the abstract hemivariational inequality, we use X, X_1, X_2 for function spaces. We first introduce the following data and assumptions.

(A_1) X is a reflexive Banach space, and K is a non-empty, closed and convex subset of X.

(A_2) For $i = 1, 2$, X_i is a Banach space, $\gamma_i \in \mathcal{L}(X, X_i)$: for a constant $c_i > 0$,

$$\|\gamma_i v\|_{X_i} \le c_i \|v\|_X \qquad \forall\, v \in X. \tag{3.1}$$

(A_3) $A\colon X \to X^*$ is bounded, continuous and strongly monotone: for a constant $m_A > 0$,

$$\langle Av_1 - Av_2, v_1 - v_2 \rangle \ge m_A \|v_1 - v_2\|_X^2 \qquad \forall\, v_1, v_2 \in X. \tag{3.2}$$

(A_4) For $i = 1, 2$, $J_i\colon X_i \to \mathbb{R}$ is locally Lipschitz continuous, and there exist constants $c_{i,0}, c_{i,1}, \alpha_i \ge 0$ such that

$$\|\partial J_i(z)\|_{X_j^*} \le c_{i,0} + c_{i,1} \|z\|_{X_i} \qquad \forall\, z \in X_i, \tag{3.3}$$

$$J_i^0(z_1; z_2 - z_1) + J_i^0(z_2; z_1 - z_2) \le \alpha_i \|z_1 - z_2\|_{X_i}^2 \qquad \forall\, z_1, z_2 \in X_i. \tag{3.4}$$

(A_5)

$$\alpha_1 c_1^2 + \alpha_2 c_2^2 < m_A. \tag{3.5}$$

(A_6)

$$f \in X^*. \tag{3.6}$$

The inequality (3.4) is usually called the relaxed monotonicity condition, so named since if J_i is convex, then (3.4) holds with $\alpha_i = 0$. The assumption (A_5) is known as the smallness condition; it imposes a restriction on the sizes of the relaxed monotonicity coefficients α_1 and α_2. This assumption is essential in ensuring the uniqueness of a solution of the problem below.

Problem (P_{abs}). *Find an element $u \in K$ such that*

$$\langle Au, v - u \rangle + J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) \ge \langle f, v - u \rangle \qquad \forall\, v \in K. \tag{3.7}$$

We have the following existence and uniqueness result on Problem (P_{abs}).

Theorem 3.1. *Under Assumptions (A_1)–(A_6), Problem (P_{abs}) has a unique solution $u \in K$.*

The proof of the result is similar to and extends the argument found in [13]. For this reason, we will only sketch the main points of the proof, emphasizing the part that is different from the proof in [13].

Proof. Recall the definition of the indicator function

$$I_K(v) = \begin{cases} 0 & \text{if } v \in K, \\ +\infty & \text{if } v \in X \backslash K. \end{cases}$$

We can express Problem (P_{abs}) equivalently as the problem of finding $u \in X$ such that

$$\langle Au, v - u \rangle + I_K(v) - I_K(u) + J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) \ge \langle f, v - u \rangle \quad \forall\, v \in X, \tag{3.8}$$

or

$$u \in X, \quad T_1 u + T_2 u \ni f, \tag{3.9}$$

where

$$T_1 v = Av + \gamma_1^* \partial J_1(\gamma_1 v) + \gamma_2^* \partial J_2(\gamma_2 v), \quad T_2 v = \partial_c I_K(v)$$

for $v \in X$, ∂J_1 and ∂J_2 are the generalized gradients of J_1 and J_2, $\partial_c I_K$ is the convex subdifferential of I_K, and $\gamma_1^* \in \mathcal{L}(X_1^*, X^*)$, $\gamma_2^* \in \mathcal{L}(X_2^*, X^*)$ are the adjoint operators of γ_1 and γ_2.

The operator T_1 is bounded by (A_2), (A_3) and (A_4). By [21, Proposition 27.6], (A_3) implies that $A\colon X \to X^*$ is pseudomonotone. Following the argument in [13, p. 646], we can show that T_1 is generalized pseudomonotone. Fix an element $u_0 \in K$. Then from (3.2),

$$\langle Av, v - u_0 \rangle \geq m_A \|v - u_0\|_X^2 + \langle Au_0, v - u_0 \rangle.$$

This, together with (3.4) and (3.5), implies (2.1). Since I_K is proper, convex and lower semicontinuous due to the assumptions on K from (A_1), the operator $T_2 : X \to 2^{X^*}$ is maximal monotone with its effective domain $\mathcal{D}(T_2) = K$ (cf. [8, Theorem 1.3.19]). Thus, we can apply Theorem 2.1 to deduce the existence of an element $u \in X$ such that

$$Au + \xi_1^* + \xi_2^* + \zeta^* = f \tag{3.10}$$

with $\xi_1^* \in \gamma_1^* \partial J_1(\gamma_1 u)$, $\xi_2^* \in \gamma_2^* \partial J_2(\gamma_2 u)$ and $\zeta^* \in \partial_c I_K(u)$. For all $v \in X$, we have

$$\langle \xi_1^*, v - u \rangle \leq J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u), \tag{3.11}$$

$$\langle \xi_2^*, v - u \rangle \leq J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u), \tag{3.12}$$

$$\langle \zeta^*, v - u \rangle \leq I_K(v) - I_K(u). \tag{3.13}$$

Combine (3.10) with (3.11)–(3.13),

$$\langle Au, v - u \rangle + J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) + I_K(v) - I_K(u) \geq \langle f, v - u \rangle.$$

Therefore, $u \in K$ is a solution of Problem (P_{abs}).

For uniqueness, let $\tilde{u} \in K$ be another solution of Problem (P_{abs}). Take $v = \tilde{u}$ in (3.7),

$$\langle Au, \tilde{u} - u \rangle + J_1^0(\gamma_1 u; \gamma_1 \tilde{u} - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 \tilde{u} - \gamma_2 u) \geq \langle f, \tilde{u} - u \rangle.$$

Switch the roles played by $\tilde{u}$ and u,

$$\langle A\tilde{u}, u - \tilde{u} \rangle + J_1^0(\gamma_1 \tilde{u}; \gamma_1 u - \gamma_1 \tilde{u}) + J_2^0(\gamma_2 \tilde{u}; \gamma_2 u - \gamma_2 \tilde{u}) \geq \langle f, u - \tilde{u} \rangle.$$

Add the two inequalities,

$$\begin{aligned} \langle Au - A\tilde{u}, u - \tilde{u} \rangle \leq\ & J_1^0(\gamma_1 u; \gamma_1 \tilde{u} - \gamma_1 u) + J_1^0(\gamma_1 \tilde{u}; \gamma_1 u - \gamma_1 \tilde{u}) \\ & + J_2^0(\gamma_2 u; \gamma_2 \tilde{u} - \gamma_2 u) + J_2^0(\gamma_2 \tilde{u}; \gamma_2 u - \gamma_2 \tilde{u}). \end{aligned}$$

Apply (3.2), (3.4) and (3.1):

$$m_A \|u - \tilde{u}\|_X^2 \leq \left(\alpha_1 c_1^2 + \alpha_2 c_2^2\right) \|u - \tilde{u}\|_X^2.$$

By the condition (3.5), we conclude the uniqueness $\tilde{u} = u$. □

Similar to [11, Theorem 3.2], we have the following result, useful in convergence analysis of numerical solutions. Recall that the operator $A\colon X \to X^*$ is said to be radially continuous if the function $t \mapsto \langle A(u + tv), v \rangle$ is continuous on $[0, 1]$ for any $u, v \in X$.

Theorem 3.2. *Assume $K \subset X$ is convex, $A\colon X \to X^*$ is monotone and radially continuous. Then $u \in K$ is a solution of Problem (P_{abs}) if and only if it satisfies*

$$\langle Av, v-u\rangle + J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{3.14}$$

4. Numerical Analysis of the Abstract Hemivariational Inequality

In this section, we assume (A_1)–(A_6) so that Problem (P_{abs}) has a unique solution $u \in K$. We now introduce and analyze a Galerkin method to solve Problem (P_{abs}). In the rest of the paper, we will use c for a generic positive constant that is independent of the discretization parameter h, and its value may vary from one place to another.

Let $X^h \subset X$ be a finite dimensional subspace with $h > 0$ denoting a spatial discretization parameter. We use some convex subset $K^h \subset X^h$ to approximate the convex set K in the following sense:

$$v^h \in K^h \text{ and } v^h \rightharpoonup v \text{ in } X \text{ imply } v \in K; \tag{4.1}$$

$$\forall\, v \in K,\ \exists\, v^h \in K^h \text{ such that } v^h \to v \text{ in } X \text{ as } h \to 0. \tag{4.2}$$

We remark that the assumptions (4.1)–(4.2) are standard in the finite element approximations of inequality problems (cf. [10]).

The Galerkin method for Problem (P_{abs}) is the following.

Problem (P^h_{abs}). *Find $u^h \in K^h$ such that*

$$\langle Au^h, v^h - u^h\rangle + J_1^0(\gamma_1 u^h; \gamma_1 v^h - \gamma_1 u^h) + J_2^0(\gamma_2 u^h; \gamma_2 v^h - \gamma_2 u^h) \geq \langle f, v^h - u^h\rangle \quad \forall\, v^h \in K^h. \tag{4.3}$$

By a discrete analogue of Theorem 3.1, we know that under the assumptions (A_1)–(A_6), Problem (P^h_{abs}) has a unique solution $u^h \in K^h$. Let us show that the numerical solutions u^h are uniformly bounded.

Proposition 4.1. *The numerical solutions are uniformly bounded in X independent of h.*

Proof. Since K is non-empty, we can find an element $u_0 \in K$. By (4.2), there exists $u_0^h \in K^h$ such that $u_0^h \to u_0$ in X as $h \to 0$. By (3.2),

$$m_A \|u^h - u_0^h\|_X^2 \leq \langle Au^h, u^h - u_0^h\rangle - \langle Au_0^h, u^h - u_0^h\rangle.$$

We let $v^h = u_0^h$ in (4.3) to get

$$\langle Au^h, u_0^h - u^h\rangle + J_1^0(\gamma_1 u^h; \gamma_1 u_0^h - \gamma_1 u^h) + J_2^0(\gamma_2 u^h; \gamma_2 u_0^h - \gamma_2 u^h) \geq \langle f, u_0^h - u^h\rangle.$$

Then,

$$\begin{aligned} m_A \|u^h - u_0^h\|_X^2 \leq{}& J_1^0(\gamma_1 u^h; \gamma_1 u_0^h - \gamma_1 u^h) + J_2^0(\gamma_2 u^h; \gamma_2 u_0^h - \gamma_2 u^h) \\ &+ \langle f - Au_0^h, u^h - u_0^h\rangle. \end{aligned}$$

By (3.4), (3.3) and (2.2), for $i = 1, 2$, we have

$$\begin{aligned} J_i^0(\gamma_i u^h; \gamma_i u_0^h - \gamma_i u^h) &\leq \alpha_i \|\gamma_i u_0^h - \gamma_i u^h\|_{X_i}^2 - J_i^0(\gamma_i u_0^h; \gamma_i u^h - \gamma_i u_0^h), \\ -J_i^0(\gamma_i u_0^h; \gamma_i u^h - \gamma_i u_0^h) &\leq \left(c_{i,0} + c_{i,1}\|\gamma_i u_0^h\|_{X_i}\right) \|\gamma_i u^h - \gamma_i u_0^h\|_{X_i}. \end{aligned}$$

Consequently,

$$m_A\|u^h-u_0^h\|_X^2 \le \alpha_1\|\gamma_1 u_0^h-\gamma_1 u^h\|_{X_1}^2+\alpha_2\|\gamma_2 u_0^h-\gamma_2 u^h\|_{X_2}^2 \\ +\sum_{i=1}^2\left(c_{i,0}+c_{i,1}\|\gamma_i u_0^h\|_{X_i}\right)\|\gamma_i u^h-\gamma_i u_0^h\|_{X_i}+\langle f-Au_0^h,u^h-u_0^h\rangle.$$

Thus, recalling (3.1),

$$\left(m_A-\alpha_1c_1^2-\alpha_2c_2^2\right)\|u^h-u_0^h\|_X^2 \\ \le\left(\sum_{i=1}^2 c_i\left(c_{i,0}+c_{i,1}c_i\|u_0^h\|_X\right)+\|f-Au_0^h\|_{X^*}\right)\|u^h-u_0^h\|_X,$$

or

$$\left(m_A-\alpha_1c_1^2-\alpha_2c_2^2\right)\|u^h-u_0^h\|_X\le\sum_{i=1}^2 c_i\left(c_{i,0}+c_{i,1}c_i\|u_0^h\|_X\right)+\|f-Au_0^h\|_{X^*}. \tag{4.4}$$

Since $u_0^h\to u_0$ in X and $A\colon X\to X^*$ is bounded, we know that $\|u_0^h\|_X$ and $\|Au_0^h\|_{X^*}$ are uniformly bounded with respect to h. Finally, by the smallness condition, we conclude from (1.2) that $\{\|u^h-u_0^h\|_X\}$, and then also $\{\|u^h\|_X\}$ is uniformly bounded in h. □

For convergence and error analysis of Problem (P_{abs}^h), we will assume $A:X\to X^*$ is Lipschitz continuous, i.e. for some constant $L_A>0$,

$$\|Au-Av\|_{X^*}\le L_A\|u-v\|_X\quad\forall\,u,v\in X. \tag{4.5}$$

Obviously, (4.5) implies that A is radially continuous. For convergence but not for error estimation, we will further assume that

$$\gamma_i\in\mathcal{L}(X;X_i)\text{ is compact, }\ i=1,2. \tag{4.6}$$

In our applications, (4.6) is trivially satisfied since γ_1 is an embedding operator from a subspace of $H^1(\Omega)$ to $L^2(\Omega)$ and γ_2 is a trace operator from a subspace of $H^1(\Omega)$ to $L^2(\Gamma_1)$, cf. Sect. 5.

We now prove the convergence of the numerical solutions to $u\in K$ without assuming any solution regularity.

Theorem 4.1. *Assume* (A_1)–(A_6), (4.1), (4.2), (4.5), *and* (4.6). *Then,*

$$u^h\to u\quad\text{in } X\text{ as } h\to 0. \tag{4.7}$$

Proof. We first show the weak convergence of the numerical solutions. By a discrete analogue of Theorem 3.2, $u^h\in K^h$ is a solution of Problem (P_{abs}^h) if and only if

$$\langle Av^h,v^h-u^h\rangle+J_1^0(\gamma_1u^h;\gamma_1v^h-\gamma_1u^h)+J_2^0(\gamma_2u^h;\gamma_2v^h-\gamma_2u^h)\ge\langle f,v^h-u^h\rangle \\ \forall\,v^h\in K^h. \tag{4.8}$$

By Proposition 4.1, $\{\|u^h\|_X\}_h$ is bounded. Since X is reflexive and the operators $\gamma_i\colon X\to X_i$, $i=1,2$, are compact, there exist a subsequence $\{u^{h'}\}\subset\{u^h\}$ and an element $w\in X$ such that

$$u^{h'}\rightharpoonup w\text{ in }X,\quad\gamma_iu^{h'}\to\gamma_iw\text{ in }X_i,\ i=1,2.$$

We have $w \in K$ following (4.1).

Let $v \in K$ be an arbitrarily fixed element. By (4.2), there exists a sequence $\{v^{h'}\}$, $v^{h'} \in K^{h'}$, such that $v^{h'} \to v$ in X as $h' \to 0$. Then, as $h' \to 0$,

$$\begin{aligned} &Av^{h'} \to Av, \quad \langle Av^{h'}, v^{h'} - u^{h'} \rangle \to \langle Av, v - w \rangle, \\ &J_i^0(\gamma_i w; \gamma_i v - \gamma_i w) \geq \limsup_{h' \to 0} J_i^0(\gamma_i u^{h'}; \gamma_i v^{h'} - \gamma_i u^{h'}), \quad i = 1, 2, \\ &\langle f, v^{h'} - u^{h'} \rangle \to \langle f, v - w \rangle. \end{aligned}$$

From (4.8) with $h = h'$,

$$\langle Av^{h'}, v^{h'} - u^{h'} \rangle + J_1^0(\gamma_1 u^{h'}; \gamma_1 v^{h'} - \gamma_1 u^{h'}) + J_2^0(\gamma_2 u^{h'}; \gamma_2 v^{h'} - \gamma_2 u^{h'}) \geq \langle f, v^{h'} - u^{h'} \rangle. \quad (4.9)$$

Taking the upper limit in (4.9) as $h' \to 0$, we find

$$\langle Av, v - w \rangle + J_1^0(\gamma_1 w; \gamma_1 v - \gamma_1 w) + J_2^0(\gamma_2 w; \gamma_2 v - \gamma_2 w) \geq \langle f, v - w \rangle. \quad (4.10)$$

Since (4.10) holds for an arbitrary $v \in K$, by Theorem 3.2, we know that w is a solution of Problem (P_{abs}). Since a solution of Problem (P_{abs}) is unique, $w = u$. So $u^{h'} \rightharpoonup u$ in X. Since the limit u does not depend on the subsequence $\{u^{h'}\}$, the entire family of numerical solutions converges weakly to u.

We proceed to prove the strong convergence (4.7). By (4.2), there exists a sequence $\{\bar{u}^h\}$, $\bar{u}^h \in K^h$, such that $\bar{u}^h \to u$ in X as $h \to 0$. Using (3.2),

$$\begin{aligned} m_A \|u - u^h\|_X^2 &\leq \langle Au - Au^h, u - u^h \rangle \\ &= \langle Au, u - u^h \rangle - \langle Au^h, u - \bar{u}^h \rangle - \langle Au^h, \bar{u}^h - u^h \rangle. \end{aligned}$$

Take $v^h = \bar{u}^h$ in (4.3),

$$-\langle Au^h, \bar{u}^h - u^h \rangle \leq J_1^0(\gamma_1 u^h; \gamma_1 \bar{u}^h - \gamma_1 u^h) + J_2^0(\gamma_2 u^h; \gamma_2 \bar{u}^h - \gamma_2 u^h) - \langle f, \bar{u}^h - u^h \rangle.$$

Combining the above inequalities, we have

$$\begin{aligned} m_A \|u - u^h\|_X^2 \leq\ & \langle Au, u - u^h \rangle - \langle Au^h, u - \bar{u}^h \rangle + J_1^0(\gamma_1 u^h; \gamma_1 \bar{u}^h - \gamma_1 u^h) \\ & + J_2^0(\gamma_2 u^h; \gamma_2 \bar{u}^h - \gamma_2 u^h) - \langle f, \bar{u}^h - u^h \rangle. \end{aligned} \quad (4.11)$$

Since $u^h \rightharpoonup u$ in X and $\bar{u}^h \to u$ in X, we have $\bar{u}^h - u^h \rightharpoonup 0$ and for $i = 1, 2$, $\|\gamma_i u^h\|_{X_i}$ is uniformly bounded and $\gamma_i \bar{u}^h - \gamma_i u^h \to 0$ in X_i. Thus, from (4.11),

$$\limsup_{h \to 0} \|u - u^h\|_X^2 \leq 0.$$

This implies the strong convergence $u^h \to u$ in X. □

For error estimation, we will derive a Céa type inequality. Let $v \in K$ and $v^h \in K^h$ be arbitrary. By (3.2),

$$m_A \|u - u^h\|_X^2 \leq \langle Au - Au^h, u - u^h \rangle.$$

Then we have

$$\begin{aligned} m_A \|u - u^h\|_X^2 \leq\ & \langle Au - Au^h, u - v^h \rangle + \langle Au, v^h - u \rangle + \langle Au, v - u^h \rangle \\ & + \langle Au, u - v \rangle + \langle Au^h, u^h - v^h \rangle. \end{aligned} \quad (4.12)$$

Using (3.7) gives

$$\langle Au, u - v\rangle \le J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) - \langle f, v - u\rangle.$$

Using (4.3) gives

$$\langle Au^h, u^h - v^h\rangle \le J_1^0(\gamma_1 u^h; \gamma_1 v^h - \gamma_1 u^h) + J_2^0(\gamma_2 u^h; \gamma_2 v^h - \gamma_2 u^h) - \langle f, v^h - u^h\rangle.$$

Using these inequalities in (4.12), after some rearrangement of the terms, we have

$$m_A \|u - u^h\|_X^2 \le \langle Au - Au^h, u - v^h\rangle + R_u(v^h - u) + R_u(v - u^h) + I_u(u^h, v, v^h), \tag{4.13}$$

where

$$R_u(v) := \langle Au, v\rangle + J_1^0(\gamma_1 u; \gamma_1 v) + J_2^0(\gamma_2 u; \gamma_2 v) - \langle f, v\rangle, \tag{4.14}$$

$$\begin{aligned} I_u(u^h, v, v^h) := {} & J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_1^0(\gamma_1 u^h; \gamma_1 v^h - \gamma_1 u^h) - J_1^0(\gamma_1 u; \gamma_1 v^h - \gamma_1 u) \\ & - J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u^h) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) + J_2^0(\gamma_2 u^h; \gamma_2 v^h - \gamma_2 u^h) \\ & - J_2^0(\gamma_2 u; \gamma_2 v^h - \gamma_2 u) - J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u^h). \end{aligned} \tag{4.15}$$

The first term on the right hand side of (4.13) is bounded through an application of (4.5),

$$\langle Au - Au^h, u - v^h\rangle \le L_A \|u - u^h\|_X \|u - v^h\|_X.$$

So for any $\varepsilon > 0$ arbitrarily small,

$$\langle Au - Au^h, u - v^h\rangle \le \varepsilon \,\|u - u^h\|_X^2 + c\,\|u - v^h\|_X^2 \tag{4.16}$$

for some constant c depending on ε. To bound the term $I_u(u^h, v, v^h)$ of (4.15), we apply the subadditivity of the generalized directional derivative to write, for $i = 1, 2$,

$$\begin{aligned} J_i^0(\gamma_i u; \gamma_i v - \gamma_i u) &\le J_i^0(\gamma_i u; \gamma_i v - \gamma_i u^h) + J_i^0(\gamma_i u; \gamma_i u^h - \gamma_i u), \\ J_i^0(\gamma_i u^h; \gamma_i v^h - \gamma_i u^h) &\le J_i^0(\gamma_i u^h; \gamma_i v^h - \gamma_i u) + J_i^0(\gamma_i u^h; \gamma_i u - \gamma_i u^h). \end{aligned}$$

Thus,

$$\begin{aligned} I_u(u^h, v, v^h) \le {} & J_1^0(\gamma_1 u^h; \gamma_1 v^h - \gamma_1 u) - J_1^0(\gamma_1 u; \gamma_1 v^h - \gamma_1 u) + J_1^0(\gamma_1 u; \gamma_1 u^h - \gamma_1 u) \\ & + J_1^0(\gamma_1 u^h; \gamma_1 u - \gamma_1 u^h) + J_2^0(\gamma_2 u^h; \gamma_2 v^h - \gamma_2 u) - J_2^0(\gamma_2 u; \gamma_2 v^h - \gamma_2 u) \\ & + J_2^0(\gamma_2 u; \gamma_2 u^h - \gamma_2 u) + J_2^0(\gamma_2 u^h; \gamma_2 u - \gamma_2 u^h). \end{aligned}$$

By (3.4), for $i = 1, 2$,

$$J_i^0(\gamma_i u; \gamma_i u^h - \gamma_i u) + J_i^0(\gamma_i u^h; \gamma_i u - \gamma_i u^h) \le \alpha_i \|\gamma_i u - \gamma_i u^h\|_{X_j}^2.$$

By (3.3), for $i = 1, 2$,

$$\begin{aligned} \left|J_i^0(\gamma_i u^h; \gamma_i v^h - \gamma_i u)\right| &\le \left(c_{i,0} + c_{i,1}\|\gamma_i u^h\|_{X_i}\right) \|\gamma_i v^h - \gamma_i u\|_{X_i}, \\ \left|J_i^0(\gamma_i u; \gamma_i v^h - \gamma_i u)\right| &\le \left(c_{i,0} + c_{i,1}\|\gamma_i u\|_{X_i}\right) \|\gamma_i v^h - \gamma_i u\|_{X_i}. \end{aligned}$$

Recall that $\|\gamma_i u^h\|_{X_i}$ is uniformly bounded by Proposition 4.1. We combine the above four inequalities to find a constant $c > 0$ independent of h such that

$$I_u(u^h, v, v^h) \le \sum_{i=1}^{2} \alpha_i \|\gamma_i u - \gamma_i u^h\|_{X_i}^2 + c \sum_{i=1}^{2} \|\gamma_i u - \gamma_i v^h\|_{X_i}. \tag{4.17}$$

Using (4.16) and (4.17) in (4.13), we have

$$\big(m_A-\alpha_1c_1^2-\alpha_2c_2^2-\varepsilon\big)\,\|u-u^h\|_X^2\le c\,\|u-v^h\|_X^2+c\sum_{i=1}^2\|\gamma_iu-\gamma_iv^h\|_{X_i}$$
$$+R_u(v^h-u)+R_u(v-u^h). \tag{4.18}$$

By (3.5), $\alpha_1c_1^2+\alpha_2c_2^2<m_A$. We take $\varepsilon=(m_A-\alpha_1c_1^2-\alpha_2c_2^2)/2>0$ in (4.18) and get the inequality

$$\|u-u^h\|_X^2\le c\left[\|u-v^h\|_X^2+\|\gamma_1u-\gamma_1v^h\|_{X_1}+\|\gamma_2u-\gamma_2v^h\|_{X_2}+R_u(v^h-u)+R_u(v-u^h)\right].$$

In summary, we have proved the following result.

Theorem 4.2. *Assume* (A_1)–(A_6) *and* (4.5). *Then for the numerical solution* u^h *of Problem* (P^h_{abs}), *we have the Céa type inequality*

$$\|u-u^h\|_X\le c\inf_{v^h\in K^h}\Big[\|u-v^h\|_X+\|\gamma_1u-\gamma_1v^h\|_{X_1}^{1/2}+\|\gamma_2u-\gamma_2v^h\|_{X_2}^{1/2}$$
$$+|R_u(v^h-u)|^{1/2}\Big]+c\inf_{v\in K}|R_u(v-u^h)|^{1/2}. \tag{4.19}$$

In the special case where $K=X$, we let $K^h=X^h$. Then we have the simplified Céa type inequality:

$$\|u-u^h\|_X\le c\inf_{v^h\in X^h}\Big[\|u-v^h\|_X+\|\gamma_1u-\gamma_1v^h\|_{X_1}^{1/2}+\|\gamma_2u-\gamma_2v^h\|_{X_2}^{1/2}+|R_u(v^h-u)|^{1/2}\Big]. \tag{4.20}$$

This is the starting point of error estimation for the numerical solution of the model problem (P_{model}). To get an actual convergence order, we need to bound the residual-type term defined by (4.14). We do this for (P_{model}) in the next section.

5. Analysis of the Model Problem for Semipermeable Media

Now we apply the results of the previous sections to the model problem (P_{model}). We use the function space V of (1.6) for X (and for K as well). In addition, choose $V_1=L^2(\Omega)$ for X_1, $V_2=L^2(\Gamma_1)$ for X_2, and let $\gamma_1\colon V\to V_1$ be the embedding operator, $\gamma_2\colon V\to V_2$ the trace operator. Corresponding to the bilinear form $a(\cdot,\cdot)$ of (1.7), we define an operator

$$A\colon V\to V^*,\quad \langle Au,v\rangle=a(u,v)\quad\forall\,u,v\in V. \tag{5.1}$$

Obviously, (4.5) holds with $L_A=1$. On the functions $j_i\colon\mathbb{R}\to\mathbb{R}$, $i=1,2$, we assume

$$\left\{\begin{array}{l}\text{(a) } j_1 \text{ and } j_2 \text{ are locally Lipschitz continuous;}\\ \text{(b) there exist constants } c_{i,0},\ c_{i,1}\in\mathbb{R} \text{ such that}\\ \qquad |\partial j_i(t)|\le\bar c_{i,0}+\bar c_{i,1}|t|\quad\forall\,t\in\mathbb{R},\ i=1,2;\\ \text{(c) there exist constants } \alpha_i \text{ such that}\\ \qquad j_i^0(t_1;t_2-t_1)+j_i^0(t_2;t_1-t_2)\le\alpha_i|t_1-t_2|^2\quad\forall\,t\in\mathbb{R},\ i=1,2.\end{array}\right. \tag{5.2}$$

We observe that (A_1) is valid. The assumption (A_2) is satisfied with

$$c_1=\lambda_1^{-1/2},\quad c_2=\mu_1^{-1/2}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of the problem

$$\begin{aligned} -\Delta u &= \lambda\, u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma_1, \end{aligned}$$

whereas $\mu_1 > 0$ is the smallest eigenvalue of the problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} &= \mu\, u && \text{on } \Gamma_1. \end{aligned}$$

For the assumption (A_3), we have

$$m_A = 1.$$

Introduce two functionals

$$J_1\colon \quad V_1 \to \mathbb{R}, \quad J_1(v) = \int_\Omega j_1(v)\, dx, \tag{5.3}$$

$$J_2\colon \quad V_2 \to \mathbb{R}, \quad J_2(v) = \int_{\Gamma_1} j_2(v)\, ds. \tag{5.4}$$

Notice that (cf. e.g., [16, Theorem 3.47]),

$$J_1^0(v; w) \le \int_\Omega j_1^0(v; w)\, dx, \qquad v, w \in V_1, \tag{5.5}$$

$$J_2^0(v; w) \le \int_{\Gamma_1} j_2^0(v; w)\, ds, \qquad v, w \in V_2. \tag{5.6}$$

Then, J_1 and J_2 satisfy (A_4) with constants $c_{i,0}$ depending on $\bar{c}_{i,0}$ and $|\Omega|$, and $c_{i,1}$ depending on $\bar{c}_{i,1}$ and $|\Gamma_1|$. The assumption (A_5) becomes

$$\alpha_1 \lambda_1^{-1} + \alpha_2 \mu_1^{-1} < 1. \tag{5.7}$$

Applying Theorem 3.1, we know that under the condition (5.7), there is a unique solution to the problem

$$u \in V, \quad \langle Au, v\rangle + J_1^0(\gamma_1 u; \gamma_1 v) + J_2^0(\gamma_2 u; \gamma_2 v) \ge \int_\Omega f_0 v\, dx \qquad \forall\, v \in V. \tag{5.8}$$

By (5.5)–(5.6), a solution of (5.8) is also a solution of Problem (P_{model}). The same kind of argument as in the proof of Theorem 3.1 shows that a solution of Problem (P_{model}) is unique.

Turn now to the finite element approximation of Problem (P_{model}). For simplicity in exposition, assume Ω is a polygonal/polyhedral domain. We express $\overline{\Gamma_0}$ and $\overline{\Gamma_1}$ as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \qquad k = 0, 1.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $k = 0, 1$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. We use the linear element space corresponding to $\mathcal{T}^h$:

$$V^h = \left\{v^h \in C(\overline{\Omega}) \mid v^h|_T \in \mathbb{P}_1(T),\ T \in \mathcal{T}^h,\ v^h = 0 \text{ on } \Gamma_1\right\},$$

where $\mathbb{P}_1(T)$ is the space of polynomials of degree less than or equal to 1 on T. The finite element method for Problem (P_{model}) is: Find $u^h \in V^h$, such that

$$a(u^h, v^h) + \int_\Omega j_1^0(u^h; v^h)\, dx + \int_{\Gamma_1} j_2^0(u^h; v^h)\, ds \ge \int_\Omega f_0 v^h dx \quad \forall\, v^h \in V^h. \tag{5.9}$$

Under the stated assumptions on the data, like for Problem (P_{model}), we have a unique solution $u^h \in V^h$ of (5.9). All the arguments in Section 4 can be applied on the finite element approximation (5.9) of Problem (P_{model}), with $J_1^0(\cdot;\cdot)$ replaced by $\int_\Omega j_1^0(\cdot;\cdot)\, dx$ and $J_2^0(\cdot;\cdot)$ replaced by $\int_{\Gamma_1} j_2^0(\cdot;\cdot)\, ds$. Thus, similar to Theorem 4.1, we have the convergence of the finite element method (5.9):

$$\|u^h - u\|_V \to 0 \quad \text{as } h \to 0. \tag{5.10}$$

For error estimation, similar to (4.20), we have

$$\|u - u^h\|_V \le c \inf_{v^h \in V^h} \left[\|u - v^h\|_V + \|u - v^h\|_{L^2(\Omega)}^{1/2} + \|u - v^h\|_{L^2(\Gamma_1)}^{1/2} + |R_u(v^h - u)|^{1/2}\right], \tag{5.11}$$

where

$$R_u(v) = \int_\Omega \left(\nabla u \cdot \nabla v - f_0 v + j_1^0(u; v)\right) dx + \int_{\Gamma_1} j_2^0(u; v)\, ds.$$

To proceed further, we make the solution regularity assumption

$$u \in H^2(\Omega), \quad u|_{\Gamma_{1,i}} \in H^2(\Gamma_{1,i}),\ 1 \le i \le i_1. \tag{5.12}$$

Note that (5.12) implies $\|\partial u/\partial \nu\|_{L^2(\Gamma_1)} \le c\, \|u\|_{H^2(\Omega)} < \infty$ and for any $v \in V$,

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_{\Gamma_1} \frac{\partial u}{\partial \nu}\, v\, ds - \int_\Omega \Delta u\, v\, dx.$$

Thus,

$$R_u(v) = \int_\Omega \left(-\Delta u\, v - f_0 v + j_1^0(u; v)\right) dx + \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu}\, v + j_2^0(u; v)\right) ds.$$

From this formula and (5.2), we can bound $R_u(v)$ as follows:

$$|R_u(v)| \le c(u) \left(\|v\|_{L^2(\Omega)} + \|v\|_{L^2(\Gamma_1)}\right) \tag{5.13}$$

for a constant $c(u)$ depending on $\|u\|_{H^2(\Omega)}$. Using (5.13) in (5.11), we find that

$$\|u - u^h\|_V \le c(u) \inf_{v^h \in V^h} \left[\|u - v^h\|_V + \|u - v^h\|_{L^2(\Omega)}^{1/2} + \|u - v^h\|_{L^2(\Gamma_1)}^{1/2}\right], \tag{5.14}$$

where the constant $c(u)$ depends on $\|u\|_{H^2(\Omega)}$. Recalling the solution regularity condition (5.12), we can apply the finite element interpolation error estimates ([1,5]) to find that, with $v^h = \Pi^h u$ being the finite element interpolant of u,

$$\begin{aligned}
&\|u - v^h\|_V \le c\, h\, \|u\|_{H^2(\Omega)}, \\
&\|u - v^h\|_{L^2(\Omega)} \le c\, h^2 \|u\|_{H^2(\Omega)}, \\
&\|u - v^h\|_{L^2(\Gamma_1)} \le c\, h^2 \left(\sum_{i=1}^{i_1} \|u\|_{H^2(\Gamma_{1,i})}^2\right)^{1/2}.
\end{aligned}$$

Therefore, we deduce from (5.14) the optimal order error estimate

$$\|u - u^h\|_V \le c(u)\, h, \tag{5.15}$$

where the constant $c(u)$ depends on $\|u\|_{H^2(\Omega)}$ and $\|u\|_{H^2(\Gamma_{1,i})}$, $1 \le i \le i_1$.

6. General Hemivariational Inequality for Non-isotropic and Heterogeneous Semipermeable Media

For non-isotropic and heterogeneous semipermeable media, we use a general second-order differential operator

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + a_0 u. \tag{6.1}$$

The coefficients are allowed to be functions of $\boldsymbol{x} \in \Omega$ and we assume (cf. [1, Subsection 8.4.5])

$$a_{ij}, a_0 \in L^\infty(\Omega), \tag{6.2}$$

$$\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \ge \theta\,|\boldsymbol{\xi}|^2 \quad \forall\,\boldsymbol{\xi} = (\xi_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega, \tag{6.3}$$

$$a_0 \ge 0 \text{ a.e. in } \Omega, \tag{6.4}$$

where the constant $\theta > 0$. In the case of isotropic and homogeneous media, after a scaling argument,

$$Lu = -\Delta u.$$

Corresponding to the differential operator L of (6.1), we define the co-normal operator on the boundary

$$\frac{\partial u}{\partial \nu_L} = \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\,\nu_j. \tag{6.5}$$

The pointwise formulation of the boundary value problem we consider is the following:

$$Lu = f \qquad \text{in } \Omega, \tag{6.6}$$

$$u = 0 \qquad \text{on } \Gamma_0, \tag{6.7}$$

$$-\frac{\partial u}{\partial \nu_L} \in \partial j_2(u) \qquad \text{on } \Gamma_1. \tag{6.8}$$

Again, to allow the interior semipermeability condition, we write the right hand side of (1.2) as

$$f = f_0 + f_1, \qquad f_0 \in L^2(\Omega),\ -f_1 \in \partial j_1(u). \tag{6.9}$$

To study the problem (6.6)–(6.9), we continue to use the function space V defined by (1.6). The bilinear form of (1.7) is changed to a general one:

$$a_g(u,v) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 u\, v\right) dx, \quad u, v \in V. \tag{6.10}$$

The weak formulation of the problem is then the following.

Problem (P_g). Find $u \in V$ such that

$$a_g(u,v) + \int_\Omega j_1^0(u;v)\,dx + \int_{\Gamma_1} j_2^0(u;v)\,ds \ge \int_\Omega f_0 v\,dx \quad \forall\, v \in V. \tag{6.11}$$

For this more general hemivariational inequality problem, we have $m_A = \theta$. All the discussions and results from Sections 4 and 5 can be extended to Problem (P_g) in a straight-forward fashion, and are hence omitted.

7. Numerical Examples

In this section, we report simulation results on numerical examples, paying particular attention on the numerical convergence orders. The differential equation is the Possion equation over the square $\Omega = (0,1) \times (0,1)$. In the numerical examples, we use uniform triangulation finite element partitions of Ω with the unit interval $[0,1]$ being divided into $1/h$ equal parts; one representative finite element mesh is shown in Figure 7.1. We use the corresponding continuous linear finite element space for V^h. To compute the numerical solution errors, we use the numerical solution with $h = \frac{1}{512}$ as the reference "true" solution.

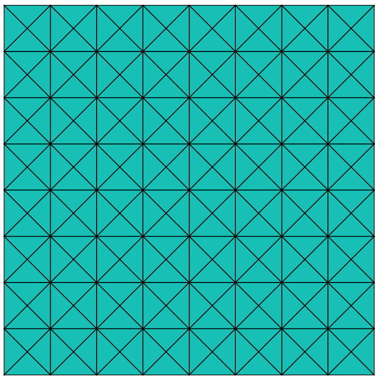


Fig. 7.1. A representative finite element mesh for numerical examples

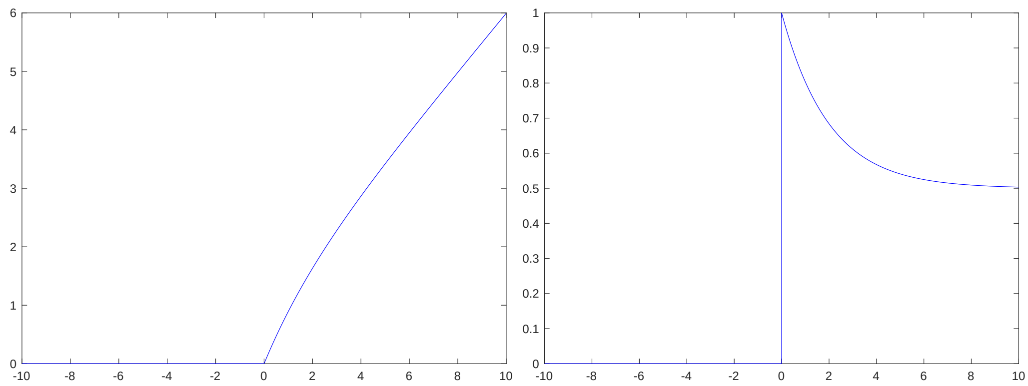
For positive parameters a and b, we let

$$j(t) = \begin{cases} 0 & \text{if } t < 0, \\ -e^{-a\,t} + b\,t + 1 & \text{if } t \ge 0. \end{cases}$$

Its generalized subdifferential is

$$\partial j(t) = \begin{cases} 0 & \text{if } t < 0, \\ [0, a+b] & \text{if } t = 0, \\ a\,e^{-a\,t} + b & \text{if } t > 0. \end{cases}$$

Observe that (5.2) (b) is valid. It can be verified that for this choice of j, (5.2) (c) is satisfied with $\alpha = a$. Graphs of j and ∂j for $a = 0.5$ and $b = 0.5$ are shown in Figure 7.2.



(a) $j(t)$ (b) $\partial j(t)$

Fig. 7.2. Sample graphs of $j(t)$ and $\partial j(t)$.

Example 7.1. The pointwise formulation of the problem is

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ -\frac{\partial u}{\partial n} &\in \partial j(u) && \text{on } \Gamma_1, \end{aligned}$$

where $f(x,y) = -40\,\sin(3\pi x)\,e^{2y}$. Here the boundary of the domain is decomposed into $\Gamma_1 = (0,1) \times \{0\}$ and $\Gamma_0 = \Gamma \backslash \Gamma_1$. We take $a = b = 0.5$ for the function j. The smallness condition (5.7) for this problem is

$$a < \lambda_1 = \frac{5}{4}\,\pi^2,$$

which is satisfied with $a = 0.5$. The numerical results are reported in Table 7.1. Clearly, the numerical convergence order in $H^1(\Omega)$-norm is close to one.

Example 7.2. The pointwise formulation of the problem is

$$\begin{aligned} -\Delta u &= f + f_0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f(x,y) = -40\,\sin(3\pi x)\,e^{2y}$ and $-f_0 \in \partial j(u)$. We take $a = b = 0.5$ for the function j. The smallness condition (5.7) for this problem is reduced to

$$a < \tilde{\lambda}_1 = 8\,\pi^2,$$

which is satisfied with $a = 0.5$. Here $\tilde{\lambda}_1$ is the smallest eigenvalue of the problem

$$\begin{aligned} -\Delta u &= \lambda\, u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Table 7.1: Numerical convergence orders for Example 7.1.

h	1/8	1/16	1/32	1/64	1/128
$\Vert u - u^h \Vert_{H^1(\Omega)}$	9.7614e-01	5.2665e-01	2.6811e-01	1.3495e-01	6.7440e-02
order	-	0.89	0.97	0.99	1.00

Table 7.2: Numerical convergence orders for Example 7.2.

h	1/8	1/16	1/32	1/64	1/128
$\|u-u^h\|_{H^1(\Omega)}$	9.6750e-01	5.2402e-01	2.6728e-01	1.3431e-01	6.7242e-02
order	-	0.88	0.97	0.99	1.00

Table 7.3: Numerical convergence orders for Example 7.3.

h	1/8	1/16	1/32	1/64	1/128
$\|u-u^h\|_{H^1(\Omega)}$	1.0163e+00	5.3127e-01	2.6980e-01	1.3494e-01	6.7434e-02
order	-	0.94	0.98	1.00	1.00

The numerical results are reported in Table 7.2. Again, the numerical convergence order in $H^1(\Omega)$-norm is close to one.

Example 7.3. The pointwise formulation of the problem is

$$\begin{aligned} -\Delta u &= f + f_0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ -\frac{\partial u}{\partial n} &\in \partial j_2(u) && \text{on } \Gamma_1, \end{aligned}$$

where $f(x,y) = -40\,\sin(2\pi x)\,e^{2y}$ and $-f_0 \in \partial j_1(u)$. The boundary decomposition $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is the same as in Example 7.1. We take $a = b = 0.5$ for the function j_1 and $a = b = 1$ for the function j_2. It can be shown that

$$\lambda_1 = \frac{5}{4}\,\pi^2, \quad \mu_1 = \frac{\pi\left(e^{2\pi}+1\right)}{e^{2\pi}-1}.$$

The smallness condition (5.7) for this problem

$$\alpha_1 \frac{4}{5\,\pi^2} + \alpha_2 \frac{e^{2\pi}-1}{\pi\left(e^{2\pi}+1\right)} < 1$$

is satisfied since $\alpha_1 = 0.5$ and $\alpha_2 = 1$. The numerical results are reported in Table 7.3. Once more, we observe that the numerical convergence order in $H^1(\Omega)$-norm is close to one.

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