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STABILITY ANALYSIS AND OPTIMAL CONTROL OF A STATIONARY STOKES HEMIVARIATIONAL INEQUALITY

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Dedicated to Professor Meir Shillor on the occasion of his 70th birthday

Abstract. In this paper, we provide stability analysis for a stationary Stokes hemivariational inequality where along the tangential direction of the slip boundary, an inclusion relation involving the generalized subdifferential of a superpotential is specified. With viscous incompressible fluid flows as application background, stability is analyzed for solutions with respect to perturbations in the superpotential and the density of external forces. We also present a result on the existence of a solution to an optimal control problem for the stationary Stokes hemivariational inequality.

1. **Introduction.** The mathematical theory of variational and hemivariational inequalities has emerged as an important tool to study in a unified framework a wide range of nonlinear problems arising in several branches of science and engineering. The history of variational inequalities is traced back to 1933 when Signorini ([36]) posed a mechanical problem on frictionless contact between a linearly elastic body and a rigid foundation, which was studied as a variational inequality in [9]. Systematic mathematical analysis of variational inequalities started in the sixties (cf. [28, 3]). Since then, there has been an increasing interest in studying variational inequality problems. For example, the study of variational inequalities and their applications can be found in several early monographs, such as [6, 26, 1]. Numerical methods for solving variational inequalities have been studied by many authors, and some comprehensive references include [15, 16, 20, 23].

Variational inequality problems are concerned with convex energy functionals (potentials), and their analysis relies on tools and techniques from convex analysis, such as monotonicity arguments. In contrast, hemivariational inequalities are mathematical problems which are concerned with nonsmooth and nonconvex energy

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functionals (superpotentials) and are particularly useful for analyzing and solving certain nonsmooth and nonconvex problems. The notion of hemivariational inequalities was first introduced by Panagiotopoulos in early 1980s ([33]) and it is closely related to the development of the concepts of the generalized directional derivative and subdifferential of a locally Lipschitz functional in the sense of Clarke ([4, 5]). Since that time many publications have appeared on hemivariational inequalities, see for example [34, 32, 31, 37]. The finite element method is a natural choice to solve hemivariational inequalities ([21]). Optimal order error estimates have been derived for the numerical solution of various hemivariational inequalities, starting with the paper [17]. See [18] for a recent summarized account of numerical analysis of hemivariational inequalities.

Fujita in [11, 12] investigated the boundary value problem for steady motions of viscous incompressible fluid, where he introduced slip or leak boundary conditions of friction type. Theoretical results on solution existence, uniqueness and other properties for Stokes problems can be found in [14, 13, 35]. In these references, the weak formulations of the problems are variational inequalities.

Recently, stability of hemivariational inequalities has attracted attention, since in applications, one cannot expect to know the problem data exactly; see [19, 41] on stability analysis of elliptic hemivariational inequalities. The stability is especially significant from the view-point of numerical approximations since a numerical solution is meaningful only if the problem being solved is stable with respect to the data. The optimal control theory has a wide range of applications in robotics, aviation and space technology, heat conduction, electromagnetic waves and fluid flows, to name a few. Optimal control problems for variational and hemivariational inequalities have been extensively studied; see, for example, [2, 8, 10, 22, 29, 30, 39]. In this paper, we study a hemivariational inequality problem for the stationary Stokes equations with a nonlinear slip boundary condition. Firstly, we present a stability result for solutions with respect to perturbations in the slip superpotential and the density of external forces $\boldsymbol{f}$. Then, we investigate an optimal control problem for the stationary Stokes hemivariational inequality.

Consider a viscous incompressible fluid that occupies a domain Ω in $\mathbb{R}^d$ $(d \le 3)$. Assume Ω is a Lipschitz domain, i.e., its boundary $\partial\Omega$ is Lipschitz continuous. The boundary is split into two parts: $\partial\Omega = \overline{\Gamma \cup S}$ with $\mathrm{meas}(\Gamma) > 0$ and $\mathrm{meas}(S) > 0$, and $\overline{\Gamma} \cap \overline{S} = \emptyset$. Denote by $\boldsymbol{n} = (n_1, \cdots, n_d)^T$ the unit outward normal on the boundary $\partial\Omega$. For a vector-valued function $\boldsymbol{v}$ on the boundary, let $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_n \boldsymbol{n}$ be the normal component and the tangential component, respectively. With the flow velocity field $\boldsymbol{u}$ and the pressure p, we define the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ and the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix and $\nu > 0$ is the viscosity coefficient. Let $\sigma_n = \boldsymbol{n} \cdot \boldsymbol{\sigma n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma n} - \sigma_n \boldsymbol{n}$ be the normal component and the tangential component of $\boldsymbol{\sigma}$.

In this paper, we consider the Stokes problem

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

with the following boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3}$$

$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on } S. \tag{4}$$

Here, $j : S \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be locally Lipschitz and ∂j is the subdifferential of $j(\boldsymbol{x}, \cdot)$ in the sense of Clarke, $j(\boldsymbol{u}_\tau)$ is a short-hand notation for $j(\boldsymbol{x}, \boldsymbol{u}_\tau)$, $\boldsymbol{f}$ is the density of external forces. In the literature, (4) is known as a slip boundary condition.

We recall here the definition of generalized directional derivative and generalized gradient in the sense of Clarke for a locally Lipschitz continuous function.

Definition 1.1. ([5]) Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a locally Lipschitz continuous function. The generalized directional derivative of f at $x \in X$ in the direction $v \in X$, denoted by $f^0(x; v)$, is defined by

$$f^0(x; v) = \limsup_{y \to x,\, \lambda \to 0+} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

The generalized gradient or subdifferential of f at x, denoted by $\partial f(x)$, is a subset of the dual space X^* given by

$$\partial f(x) = \{\zeta \in X^* \mid f^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \ \forall\, v \in X\}. \tag{5}$$

We will need the following properties of the generalized directional derivative and the generalized gradient.

Proposition 1. ([31]) *Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a locally Lipschitz continuous function. Then the following hold:*

(i) For every $x \in X$, the funcion $X \ni v \mapsto f^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $f^0(x; \lambda v) = \lambda f^0(x; v)$ for all $\lambda \geq 0, v \in X$ and $f^0(x; v_1 + v_2) \leq f^0(x; v_1) + f^0(x; v_2)$ for all $v_1, v_2 \in X$, respectively.

(ii) The function $X \times X \ni (x, v) \mapsto f^0(x; v) \in \mathbb{R}$ is upper semicontinuous, i.e., for all $x \in X$, $v \in X$, $\{x_n\} \subset X$, $\{v_n\} \subset X$ such that $x_n \to x$ in X and $v_n \to v$ in X, we have $\limsup f^0(x_n; v_n) \leq f^0(x; v)$.

(iii) For all $v \in X$, $f^0(x; v) = \max\{\langle \zeta, v \rangle_{X^* \times X} \mid \zeta \in \partial f(x)\}$.

The rest of this paper is organized as follows. In Section 2, we introduce the weak formulation of the Stokes problem, and recall an existence and uniqueness result of the solution. In Section 3, we present the stability result for the stationary Stokes hemivariational inequality. Finally, in Section 4, we consider a class of optimal control problems associated to the stationary Stokes hemivariational inequality. Existence of an optimal solution to the control problem is shown.

2. The Stokes hemivariational inequality.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\sigma}\|_{\mathbb{S}^d} = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Here and below the indices i and j run between 1 and d, and the summation convention over repeated indices is used.

The space $(H^m(\Omega))^d$ $(m \geq 1)$ is denoted by $\boldsymbol{H}^m(\Omega)$. We use

$$\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v}|_\Gamma = \boldsymbol{0},\, v_n|_S = 0\}$$

as the space of the velocity variable,

$$M := L_0^2(\Omega) = \Big\{q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0\Big\}$$

as the space for the pressure variable,

$$\mathcal{H} := \{\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d} \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\, 1 \le i,j \le d\}$$

as the space of the strain or stress fields. Let

$$\boldsymbol{V}_0 := H_0^1(\Omega)^d$$

and

$$\mathcal{H}_1 := \{\boldsymbol{\sigma} \in \mathcal{H} \mid \mathrm{Div}\boldsymbol{\sigma} \in \boldsymbol{H}\},$$

where

$$\boldsymbol{H} := L^2(\Omega;\mathbb{R}^d).$$

Define $\boldsymbol{\varepsilon} : \boldsymbol{H}^1(\Omega) \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to \boldsymbol{H}$ by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\ \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{i,j}),\ \mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})_d.$$

Here and below an index that follows a comma indicates a derivative with respect to the corresponding component of the variable. Therefore, since the summation convention over repeated indices is adopted, the divergence of the stress field is given by

$$\sigma_{ij,j} = \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}.$$

Recall the following formulas ([31, Chapter 2]):

$$\int_\Omega (u\,\mathrm{div}\boldsymbol{v} + \nabla u \cdot \boldsymbol{v})\,dx = \int_{\partial\Omega} u\,v_n ds \quad \forall\, u \in H^1(\Omega),\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \tag{6}$$

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})dx + \int_\Omega \mathrm{Div}\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx = \int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{v}\,ds \quad \forall\,\boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\, \boldsymbol{\sigma} \in H^1(\Omega;\mathbb{S}^d). \tag{7}$$

It is well known that the spaces $\boldsymbol{H}$ and $\mathcal{H}$ are Hilbert spaces equipped with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{H}} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\,dx.$$

Let $\|\cdot\|_1$ be the standard norm in the Hilbert space $\boldsymbol{H}^1(\Omega)$. Since $\mathrm{meas}(\Gamma) > 0$, the following Korn's inequality (cf. [25, Lemma 6.2]) holds:

$$\|\boldsymbol{v}\|_1 \le c\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}, \tag{8}$$

where the constant c depends only on Ω and Γ. This implies that the norm $\|\cdot\|_V = \|\boldsymbol{\varepsilon}(\cdot)\|_{\mathcal{H}}$ is equivalent on $\boldsymbol{V}$ with the norm $\|\cdot\|_1$. Therefore, $(\boldsymbol{V},\|\cdot\|_V)$ is a Hilbert space.

The duality pairing between $\boldsymbol{V}$ and $\boldsymbol{V}^*$ is denoted by $\langle\cdot,\cdot\rangle$. Identifying $\boldsymbol{H}$ with its dual, we have an evolution triple $\boldsymbol{V} \subset \boldsymbol{H} \subset \boldsymbol{V}^*$ with dense, continuous and compact embeddings. We denote by $\gamma\colon \boldsymbol{V} \to \boldsymbol{L}^2(S) := L^2(S;\mathbb{R}^d)$ the tangential component trace operator, defined by $\gamma\boldsymbol{v} = \boldsymbol{v}_\tau$ on S for $\boldsymbol{v} \in \boldsymbol{V}$. By the Sobolev trace theorem and (8), γ is a linear continuous operator from $\boldsymbol{V}$ to $\boldsymbol{L}^2(S)$; we denote its operator norm by $\|\gamma\|$.

Introduce two bilinear forms:

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\,\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}, \tag{9}$$

$$b(\boldsymbol{v},q) = \int_\Omega q\,\mathrm{div}\boldsymbol{v}\,dx \quad \forall\,\boldsymbol{v} \in \boldsymbol{V},\, q \in M. \tag{10}$$

Obviously, $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}$; indeed,

$$a(\boldsymbol{v},\boldsymbol{v}) = 2\nu\,\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}. \tag{11}$$

Concerning the superpotential j, we assume the following properties:

$H(j)$. $j : S\times\mathbb{R}^d\to\mathbb{R}$ is such that

(i) $j(\cdot,\boldsymbol{\xi})$ is measurable on S for all $\boldsymbol{\xi}\in\mathbb{R}^d$ and $j(\cdot,\boldsymbol{0})\in L^1(S)$;

(ii) $j(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x}\in S$;

(iii) $\|\boldsymbol{\eta}\|_{\mathbb{R}^d}\le c_0+c_1\|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ for all $\boldsymbol{\xi}\in\mathbb{R}^d,\boldsymbol{\eta}\in\partial j(\boldsymbol{x},\boldsymbol{\xi})$, a.e. $\boldsymbol{x}\in S$ with constants $c_0,c_1\ge 0$;

(iv) $(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\ge -m_\tau\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathbb{R}^d}^2$ for all $\boldsymbol{\eta}_i,\boldsymbol{\xi}_i\in\mathbb{R}^d$, $\boldsymbol{\eta}_i\in\partial j(\boldsymbol{x},\boldsymbol{\xi}_i)$, $i=1,2$, a.e. $\boldsymbol{x}\in S$ with a constant $m_\tau\ge 0$.

It can be verified that the assumption $H(j)$ (iv) is equivalent to

$$j^0(\boldsymbol{x},\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)+j^0(\boldsymbol{x},\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\le m_\tau\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathbb{R}^d}^2 \quad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^d, \text{a.e. } \boldsymbol{x}\in S. \tag{12}$$

Introduce the functional $J:\boldsymbol{L}^2(S)\to\mathbb{R}$ defined by

$$J(\boldsymbol{v})=\int_S j(\boldsymbol{x},\boldsymbol{v})\,ds,\quad \boldsymbol{v}\in\boldsymbol{L}^2(S). \tag{13}$$

Using arguments similar to those in the proof of Theorem 3.47 in [31], we have the following result.

Lemma 2.1. *Assume $H(j)$. Then the functional J defined by (13) has the following properties:*

(i) *$J(\cdot)$ is locally Lipschitz on $\boldsymbol{L}^2(S)$;*

(ii) *$\|\boldsymbol{z}\|_{\boldsymbol{L}^2(S)}\le\bar{c}_0+\bar{c}_1\|\boldsymbol{v}\|_{\boldsymbol{L}^2(S)}$ for all $\boldsymbol{z}\in\partial J(\boldsymbol{v})$, $\boldsymbol{v}\in L^2(S)^d$ with $\bar{c}_0=\sqrt{3\,|S|}c_0$ and $\bar{c}_1=\sqrt{3}\,c_1$;*

(iii) *$\langle\boldsymbol{z}_1-\boldsymbol{z}_2,\boldsymbol{v}_1-\boldsymbol{v}_2\rangle_{L^2(S)^d}\ge -m_\tau\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{\boldsymbol{L}^2(S)}^2$ for all $\boldsymbol{z}_i\in\partial J(\boldsymbol{v}_i)$, $\boldsymbol{v}_i\in\boldsymbol{L}^2(S)$, $i=1,2$.*

From [7], we have the following weak formulation of the boundary value problem (1)–(4):

Problem (P). Find $(\boldsymbol{u},p)\in\boldsymbol{V}\times M$ such that

$$a(\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{v},p)+\int_S j^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,ds\ge(\boldsymbol{f},\boldsymbol{v})_{\boldsymbol{H}} \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}, \tag{14}$$

$$b(\boldsymbol{u},q)=0 \quad \forall\,q\in M. \tag{15}$$

Let us recall the well-known inf-sup condition ([38]): for a constant $\beta_1>0$,

$$\beta_1\|q\|_{L^2(\Omega)}\le\sup_{\boldsymbol{v}\in V_0}\frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_V} \quad \forall\,q\in M. \tag{16}$$

The following existence and uniqueness result for Problem (P) can be found in [7].

Theorem 2.2. *Assume $\boldsymbol{f}\in\boldsymbol{H}$, $H(j)$ and*

$$2\nu>m_\tau\|\gamma\|^2. \tag{17}$$

Then Problem (P) has a unique solution.

2.1. **A stability result.** We now consider a perturbed stationary Stokes hemivariational inequality where the superpotential j and the external force density $\boldsymbol{f}$ are replaced by their perturbations j_n and $\boldsymbol{f}_n \in \boldsymbol{H}$.

Similar to $H(j)$, we introduce assumptions on j_n.

$H(j_n)$. $j_n : S \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j_n(\cdot, \boldsymbol{\xi})$ is measurable on S for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $j(\cdot, \boldsymbol{0}) \in L^1(S)$;

(ii) $j_n(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in S$;

(iii) $\|\boldsymbol{\eta}_n\|_{\mathbb{R}^d} \le c_{0n} + c_{1n}\|\boldsymbol{\xi}_n\|_{\mathbb{R}^d}$ for all $\boldsymbol{\xi}_n \in \mathbb{R}^d$, $\boldsymbol{\eta}_n \in \partial j_n(\boldsymbol{x}, \boldsymbol{\xi}_n)$ a.e. $\boldsymbol{x} \in S$ with $c_{0n}, c_{1n} \ge 0$, and the sequences $\{c_{0n}\}$ and $\{c_{1n}\}$ are bounded;

(iv) $j_n^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j_n^0(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le m_{\tau n}\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^d}$ $\forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$ with $m_{\tau n} \ge 0$, and there exists a constant $m_0 > 0$ such that $m_{\tau n} \le m_0$ for each $n \in \mathbb{N}$.

Then the perturbed problem is as follows.

Problem (P_n). Find $(\boldsymbol{u}_n, p_n) \in \boldsymbol{V} \times M$ such that

$$a(\boldsymbol{u}_n, \boldsymbol{v}) - b(\boldsymbol{v}, p_n) + \int_S j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau)\, ds \ge (\boldsymbol{f}_n, \boldsymbol{v})_{\boldsymbol{H}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{18}$$

$$b(\boldsymbol{u}_n, q) = 0 \quad \forall\, q \in M. \tag{19}$$

Similar to Theorem 2.2, we have the existence and uniqueness result for the perturbed stationary Stokes hemivariational inequality.

Theorem 2.3. *Assume $H(j_n)$, $2\nu > m_0\|\gamma\|^2$ and $\boldsymbol{f}_n \in \boldsymbol{H}$. Then Problem (P_n) has a unique solution.*

To measure the closeness of the problem data, we introduce the next two assumptions.

$(H_{j_n \to j})$. If $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ and $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ in $\mathbb{R}^d$, then

$$\limsup_{n\to\infty} j_n^0(\boldsymbol{\xi}_n; \boldsymbol{\eta}_n) \le j^0(\boldsymbol{\xi}; \boldsymbol{\eta}). \tag{20}$$

$(H_{f_n \to f}) : \boldsymbol{f}_n \rightharpoonup \boldsymbol{f}$ in $\boldsymbol{H}$.

Next we examine an example that satisfies the assumptions $H(j), H(j_n)$ and $H(j_n \to j)$. For simplicity, we consider only the case in which the function is independent on the spatial variable $\boldsymbol{x}$.

Example 2.4. Let $j : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j(r) = \begin{cases} -r, & r < 0, \\ 0, & 0 \le r \le 1, \\ -(r-1)^2, & r > 1. \end{cases} \tag{21}$$

Then j satisfies the assumptions $H(j)(i)$ and $H(j)(ii)$. It follows from the definition of the Clarke subdifferential that

$$\partial j(r) = \begin{cases} -1, & r \ge 0, \\ [-1, 0], & r = 0, \\ 0, & 0 < r \le 1, \\ -2(r-1), & r > 1. \end{cases} \tag{22}$$

Thus, $|\xi| \le 1 + 2|r|$ for all $\xi \in \partial j(r)$ and $r \in \mathbb{R}$, and hence the assumption $H(j)$ (iii) holds with $c_0 = 1$, $c_1 = 2$. The assumption $H(j)$ (iv) holds with $m_\tau = 2$.

We approximate non-differentiable function j by a sequence of continuously differentiable functions $\{j_n\}$. Note that for numerical simulations, it will be advantageous to have continuously differentiable superpotentials. For each $n \in \mathbb{N}$, let

$j_n : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j_n(r) = \begin{cases} -r, & r < -\frac{1}{n}, \\ \frac{n}{4}(r - \frac{1}{n})^2, & -\frac{1}{n} \le r < \frac{1}{n}, \\ 0, & \frac{1}{n} \le r \le 1, \\ -(r-1)^2, & r > 1. \end{cases} \tag{23}$$

Then $j_n(r)$ is continuously differentiable approximation of $j(r)$. It is easy to see that j_n satisfies the assumptions $H(j_n)(i)$ and $H(j_n)(ii)$. From the formula

$$\partial j_n(r) = j_n'(r) = \begin{cases} -1, & r < -\frac{1}{n}, \\ \frac{n}{2}(r - \frac{1}{n}), & -\frac{1}{n} \le r < \frac{1}{n}, \\ 0, & \frac{1}{n} \le r \le 1, \\ -2(r-1), & r > 1. \end{cases} \tag{24}$$

We see that $|\xi_n| \le 1 + 2|r|$ for all $\xi_n \in \partial j_n(r)$ and $r \in \mathbb{R}$, and hence the assumption $H(j_n)$ (iii) holds with $c_{0n} = 1$, $c_{1n} = 2$. Assumption $H(j_n)$ (iv) holds with $m_0 = 2$.

Assume that $\xi_n \to \xi$ and $\eta_n \to \eta$ in $\mathbb{R}$. Note that $j_n^0(\xi_n; \eta_n) = j_n'(\xi_n)\eta_n$. We distinguish two cases. First, for $\xi \ne 0$, $j^0(\xi; \eta) = j'(\xi)\eta$. Since

$$j_n^0(\xi_n; \eta_n) - j^0(\xi; \eta) = j_n'(\xi_n)\eta_n - j'(\xi)\eta = j_n'(\xi_n)(\eta_n - \eta) + (j_n'(\xi_n) - j'(\xi))\eta,$$

$$\limsup_{n\to\infty}[j_n^0(\xi_n; \eta_n) - j^0(\xi; \eta)] \le \limsup_{n\to\infty} |j_n'(\xi_n)||\eta_n - \eta| + \limsup_{n\to\infty} |j_n'(\xi_n) - j'(\xi)||\eta| \le 0.$$

Thus,

$$\limsup_{n\to\infty} j_n^0(\xi_n; \eta_n) \le j^0(\xi; \eta).$$

Then consider the case $\xi = 0$. Notice that

$$j^0(0; v) = \begin{cases} 0, & v \ge 0, \\ -v, & v < 0. \end{cases} \tag{25}$$

If $\eta \ge 0$, then

$$\limsup_{n\to\infty} j_n^0(\xi_n; \eta_n) = \limsup_{n\to\infty}[j_n'(\xi_n)\eta_n] \le 0 = j^0(\xi; \eta).$$

If $\eta < 0$, then

$$\limsup_{n\to\infty} j_n^0(\xi_n; \eta_n) = \limsup_{n\to\infty}[j_n'(\xi_n)\eta_n] = -\frac{1}{2}\eta < -\eta = j^0(\xi; \eta).$$

The stability result is presented next.

Theorem 2.5. *Keep the assumptions stated in Theorems 2.2 and 2.3. Assume $(H_{j_n\to j})$ and $(H_{f_n\to f})$. Then for the solution pair $(\boldsymbol{u}, p)$ of Problem (P) and the solution pair $(\boldsymbol{u}_n, p_n)$ of Problem (P_n), we have the convergence:*

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } \boldsymbol{V}, \quad p_n \to p \text{ in } M, \quad \text{as } n \to \infty.$$

Proof. The proof is divided into three steps.

Step 1. We prove that the sequences $\{\|\boldsymbol{u}_n\|_V\}_n$ and $\{\|p_n\|_{L^2(\Omega)}\}_n$ are bounded.

Set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_n$ in (18),

$$a(\boldsymbol{u}_n, \boldsymbol{u} - \boldsymbol{u}_n) - b(\boldsymbol{u} - \boldsymbol{u}_n, p_n) + \int_S j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{n\tau})\, ds \ge (\boldsymbol{f}_n, \boldsymbol{u} - \boldsymbol{u}_n)_{\boldsymbol{H}}.$$

Then, since $b(\boldsymbol{u},p_n)=0$ and $b(\boldsymbol{u}_n,p_n)=0$,

$$\int_S j_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds+(\boldsymbol{f}_n,\boldsymbol{u}_n-\boldsymbol{u})_{\boldsymbol{H}}\geq a(\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{u}). \tag{26}$$

Using (11), we have

$$a(\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{u})=2\nu\|\boldsymbol{u}_n-\boldsymbol{u}\|_V^2+a(\boldsymbol{u},\boldsymbol{u}_n-\boldsymbol{u}). \tag{27}$$

Write

$$\int_S j_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds=\int_S\big[j_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})+j_n^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau)\\ -j_n^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau)\big]\,ds.$$

By $H(j_n)$ (iv),

$$j_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})+j_n^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau)\leq m_{\tau n}\|\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau\|_{\mathbb{R}^d}^2.$$

By $H(j_n)$ (iii),

$$-j_n^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau)\leq(c_{0n}+c_{1n}\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d})\,\|\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau\|_{\mathbb{R}^d}.$$

Thus,

$$\int_S j_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds\leq m_{\tau n}\int_S\|\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau\|_{\mathbb{R}^d}^2ds\\ +\int_S[(c_{0n}+c_{1n}\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d})\|\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau\|_{\mathbb{R}^d}]\,ds.$$

Hence,

$$\int_S j_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})ds\leq m_{\tau n}\|\gamma\|^2\|\boldsymbol{u}_n-\boldsymbol{u}\|_V^2\\ +(\sqrt{2\,|S|}c_{0n}+\sqrt{2}c_{1n}\|\gamma\|\|\boldsymbol{u}\|_V)\|\gamma\|\|\boldsymbol{u}_n-\boldsymbol{u}\|_V. \tag{28}$$

In addition,

$$a(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}_n)\leq 2\nu\|\boldsymbol{u}\|_V\|\boldsymbol{u}_n-\boldsymbol{u}\|_V. \tag{29}$$

Therefore, it follows from (26)-(29) that

$$(2\nu-m_{\tau n}\|\gamma\|^2)\|\boldsymbol{u}_n-\boldsymbol{u}\|_V\leq\|\boldsymbol{f}_n\|_{V^*}+(\sqrt{2\,|S|}c_{0n}+\sqrt{2}c_{1n}\|\gamma\|\|\boldsymbol{u}\|_V)\|\gamma\|+2\nu\|\boldsymbol{u}\|_V.$$

Thus, it follows from $H(j_n)$ (iv) that

$$(2\nu-m_0\|\gamma\|^2)\|\boldsymbol{u}_n-\boldsymbol{u}\|_V\leq(\sqrt{2\,|S|}c_{0n}+\sqrt{2}c_{1n}\|\gamma\|\|\boldsymbol{u}\|_V)\|\gamma\|+\|\boldsymbol{f}_n\|_{V^*}+2\nu\|\boldsymbol{u}\|_V. \tag{30}$$

In view of the assumption $(H_{f_n\to f})$, we know that the sequence $\{\|\boldsymbol{f}_n\|_{V^*}\}$ is bounded. Since the sequences $\{c_{0n}\}$ and $\{c_{1n}\}$ are bounded, it follows from (30) that $\{\|\boldsymbol{u}_n\|_V\}$ is bounded.

From (14), we have

$$a(\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{v},p)=(\boldsymbol{f},\boldsymbol{v})_{\boldsymbol{H}}\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}_0. \tag{31}$$

From (18), we have

$$a(\boldsymbol{u}_n,\boldsymbol{v})-b(\boldsymbol{v},p_n)=(\boldsymbol{f}_n,\boldsymbol{v})_{\boldsymbol{H}}\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}_0. \tag{32}$$

Hence, it follows from (31) and (32) that

$$b(\boldsymbol{v},p_n-p)=a(\boldsymbol{u}_n-\boldsymbol{u},\boldsymbol{v})+(\boldsymbol{f}-\boldsymbol{f}_n,\boldsymbol{v})_{\boldsymbol{H}}\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}_0. \tag{33}$$

From (33) and the discrete inf-sup condition (16) we have

$$\begin{aligned}\beta_1\|p_n - p\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{v}\in\boldsymbol{V}_0}\frac{b(\boldsymbol{v}, p_n - p)}{\|\boldsymbol{v}\|_V}\\ &= \sup_{\boldsymbol{v}\in\boldsymbol{V}_0}\frac{a(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{f} - \boldsymbol{f}_n, \boldsymbol{v})_{\boldsymbol{H}}}{\|\boldsymbol{v}\|_V}\\ &\le 2\nu\|\boldsymbol{u}_n - \boldsymbol{u}\|_V + \|\boldsymbol{f}_n - \boldsymbol{f}\|_{V^*}.\end{aligned} \tag{34}$$

Since $\{\|\boldsymbol{u}_n\|_V\}$ and $\{\|\boldsymbol{f}_n\|_{V^*}\}$ are bounded, $\{\|p_n\|_{L^2(\Omega)}\}$ is bounded.

Step 2. We prove the weak convergence:

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{V} \text{ and } p_n \rightharpoonup p \text{ in } M \text{ as } n \to \infty.$$

Since $\{\|\boldsymbol{u}_n\|_V\}$ and $\{\|p_n\|_{L^2(\Omega)}\}$ are bounded, there exist subsequences of $\{\boldsymbol{u}_n\}$ and $\{p_n\}$, still denoted by $\{\boldsymbol{u}_n\}$ and $\{p_n\}$, and two elements $\overline{\boldsymbol{u}} \in \boldsymbol{V}$ and $\overline{p} \in M$ such that

$$\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}} \text{ in } \boldsymbol{V} \text{ and } p_n \rightharpoonup \overline{p} \text{ in } M \text{ as } n \to \infty.$$

By resorting to a subsequence if necessary, we have $\boldsymbol{u}_n \to \overline{\boldsymbol{u}}$ a.e. on S.

Taking upper limit in (18), from $H(j_n)$ we have for any $\boldsymbol{v} \in \boldsymbol{V}$,

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{H}} &\le a(\overline{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, \overline{p}) + \limsup_{n\to\infty}\int_S j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau)\, ds\\ &\le a(\overline{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, \overline{p}) + \int_S \limsup_{n\to\infty} j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau)\, ds.\end{aligned} \tag{35}$$

By $(H_{j_n\to j})$, since $\boldsymbol{u}_{n\tau} \to \overline{\boldsymbol{u}}_\tau$ a.e. on S,

$$\limsup_{n\to\infty} j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau) \le j^0(\overline{\boldsymbol{u}}_\tau; \boldsymbol{v}_\tau).$$

Thus, from (35) we have for any $\boldsymbol{v} \in \boldsymbol{V}$,

$$(\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{H}} \le a(\overline{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, \overline{p}) + \int_S j^0(\overline{\boldsymbol{u}}_\tau; \boldsymbol{v}_\tau)\, ds. \tag{36}$$

Letting $n \to \infty$ in (19) we have

$$b(\overline{\boldsymbol{u}}, q) = 0 \quad \forall\, q \in M. \tag{37}$$

Therefore, it follows from (36)–(37) that $(\overline{\boldsymbol{u}}, \overline{p}) \in \boldsymbol{V} \times M$ is a solution of Problem (P). Since Problem (P) has a unique solution by Theorem 2.2, $\boldsymbol{u} = \overline{\boldsymbol{u}}$ and $p = \overline{p}$. This implies that every subsequence of the sequence $\{(\boldsymbol{u}_n, p_n)\}$ which converges weakly in $\boldsymbol{V} \times M$ has the same limit and hence the whole sequence $\{(\boldsymbol{u}_n, p_n)\}$ converges weakly in $\boldsymbol{V} \times M$ to $(\boldsymbol{u}, p)$, as $n \to \infty$.

Step 3. We prove the strong convergence:

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } \boldsymbol{V} \text{ and } p_n \to p \text{ in } M \text{ as } n \to \infty.$$

Again, without loss of generality, we can assume $\boldsymbol{u}_n \to \boldsymbol{u}$ a.e. on S for the solution sequence $\{\boldsymbol{u}_n\}$. Taking upper limit in (26) and using the assumption $(H_{f_n\to f})$, we have

$$\limsup_{n\to\infty} a(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) \le \limsup_{n\to\infty}\int_S j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{n\tau})\, ds. \tag{38}$$

Applying $(H_{j_n \to j})$ and Proposition 1(i),

$$\begin{aligned}\limsup_{n\to\infty} \int_S j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{n\tau})\, ds &\le \int_S \limsup_{n\to\infty} j_n^0(\boldsymbol{u}_{n\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{n\tau})\, ds \\ &\le \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{u}_\tau)\, ds \\ &= \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{0})\, ds \\ &= 0.\end{aligned}$$

Thus, from (38),

$$\limsup_{n\to\infty} a(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) \le 0.$$

Since

$$2\nu \|\boldsymbol{u}_n - \boldsymbol{u}\|_V^2 = a(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}) = a(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) - a(\boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}),$$

we then have

$$2\nu \limsup_{n\to\infty} \|\boldsymbol{u}_n - \boldsymbol{u}\|_V^2 \le \limsup_{n\to\infty} a(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) \le 0.$$

Hence,

$$\lim_{n\to\infty} \|\boldsymbol{u}_n - \boldsymbol{u}\|_V^2 = 0.$$

Therefore, $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{V}$ as $n \to \infty$. Since $p_n \rightharpoonup p$ in M as $n \to \infty$, from (34) we have that $p_n \to p$ as $n \to \infty$. This completes the proof. □

2.2. **An optimal control problem.** The control of electrically conducting flows for the purpose of achieving some desired objective is crucial to many technological applications such as fusion technology, design of novel submarine propulsion devices, and modeling of nuclear reactors or molten metal string. The model on stationary flows of incompressible media is closely related to electrically conducting fluids that can be influenced by magnetic fields ([24, 40]). A possible cost functional is of the form

$$Q(\boldsymbol{f}) = \frac{1}{2\lambda} \int_\Omega \|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_d(\boldsymbol{x})\|_{\mathbb{R}^d}^2 dx + \frac{\theta}{2} \int_\Omega \|\boldsymbol{f}(\boldsymbol{x})\|_{\mathbb{R}^d}^2 dx, \quad \boldsymbol{f} \in \boldsymbol{H},\ \boldsymbol{u} \in \boldsymbol{V}, \tag{39}$$

where $\boldsymbol{f}$ is the control, $\boldsymbol{u}$ is the velocity field determined from the Stokes hemivariational inequality, and $\boldsymbol{u}_d$ denotes a desired velocity field. The constants $\lambda > 0$ and $\theta > 0$ are positive parameters that adjust the relative weight of the two terms in the functional. The goal is to choose the control $\boldsymbol{f}$ in such a way that the corresponding velocity field $\boldsymbol{u}$ is the best possible approximation to the desired velocity field $\boldsymbol{u}_d$.

In this section we study an optimal control problem for a system described by the stationary Stokes hemivariational inequality in Problem (P). The control quantity is the density of the external force.

We use $\boldsymbol{H}$ for the control space. Let $\mathcal{U}_{ad} \subset \boldsymbol{H}$ be the set of admissible controls. Consider a general objective functional $Q : \boldsymbol{H} \to \mathbb{R}$ of the form

$$Q(\boldsymbol{f}) = \int_\Omega L(\boldsymbol{x}, p(\boldsymbol{f}; \boldsymbol{x}), \boldsymbol{u}(\boldsymbol{f}; \boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}))\, dx,$$

where $L : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $(\boldsymbol{u}, p) = (\boldsymbol{u}(\boldsymbol{f}), p(\boldsymbol{f}))$ is the solution of the problem: $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \ge (\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{H}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{40}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in M. \tag{41}$$

For simplicity, we will denote the objective functional as

$$Q(\boldsymbol{f}) = \int_\Omega L(\boldsymbol{x}, p(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}))\, dx, \tag{42}$$

with the understanding that $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ is the solution of the problem (40)–(41). Then the optimal control problem is

$$\inf \{Q(\boldsymbol{f}) \mid \boldsymbol{f} \in \mathcal{U}_{ad}\}. \tag{43}$$

In the study of the problem (43), we introduce the following hypotheses:

$H(\mathcal{U}_{ad})$: $\mathcal{U}_{ad}$ is a bounded and weakly closed subset of $\boldsymbol{H}$.

$H(L)$: $L : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a measurable function and

(i) $L(\boldsymbol{x}, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega$;

(ii) there exist $c > 0$ and $\varphi \in L^1(\Omega)$ such that for all $q \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{R}^d$, $\boldsymbol{h} \in \mathbb{R}^d$, and a.e. $\boldsymbol{x} \in \Omega$, we have

$$L(\boldsymbol{x}, q, \boldsymbol{v}, \boldsymbol{h}) \geq \varphi(\boldsymbol{x}) - c\left(|q|^2 + \|\boldsymbol{v}\|^2_{\mathbb{R}^d} + \|\boldsymbol{h}\|^2_{\mathbb{R}^d}\right).$$

The condition (ii) above is used to ensure that the integral in the definition of $Q(\boldsymbol{f})$ in (42) is well defined for $p \in L^2(\Omega)$ and $\boldsymbol{u}, \boldsymbol{f} \in \boldsymbol{H}$. It can certainly be replaced by other similar conditions.

Corresponding to the cost function (39),

$$L(\boldsymbol{x}, p, \boldsymbol{u}, \boldsymbol{f}) = \frac{1}{2\lambda} \|\boldsymbol{u} - \boldsymbol{u}_d(\boldsymbol{x})\|^2_{\mathbb{R}^d} + \frac{\theta}{2} \|\boldsymbol{f}\|^2_{\mathbb{R}^d}.$$

It is easy to see that $H(L)$ is satisfied. Another example is the following (cf. [27]):

$$L(\boldsymbol{x}, p, \boldsymbol{u}, \boldsymbol{f}) = |p - \widetilde{p}(\boldsymbol{x})|^2 + \boldsymbol{u}^T M_1 \boldsymbol{u} + \boldsymbol{f}^T M_2 \boldsymbol{f},$$

where $M_1, M_2 \in \mathbb{R}^{d\times d}$ are two $d \times d$ real, symmetric, positive definite matrices, and $\widetilde{p}(\boldsymbol{x})$ is a desired pressure distribution function. Then the hypothesis $H(L)$ is satisfied.

Let us show that the optimal control problem (43) has a solution.

Theorem 2.6. *Assume $H(j)$, $H(\mathcal{U}_{ad})$, $H(L)$ and (17). Then the control problem (43) has a solution $\boldsymbol{f} \in \mathcal{U}_{ad}$.*

Proof. Let $\{\boldsymbol{f}_k\} \subset \mathcal{U}_{ad}$ be a minimizing sequence, i.e.,

$$\lim_{k\to\infty} Q(\boldsymbol{f}_k) = m := \inf \{Q(\boldsymbol{f}) \mid \boldsymbol{f} \in \mathcal{U}_{ad}\}.$$

We write $(\boldsymbol{u}_k, p_k) \in \boldsymbol{V} \times M$ for the solution of the problem (40)–(41) with $\boldsymbol{f}$ replaced by $\boldsymbol{f}_k$, i.e.,

$$a(\boldsymbol{u}_k, \boldsymbol{v}) - b(\boldsymbol{v}, p_k) + \int_S j^0(\boldsymbol{u}_{k\tau}; \boldsymbol{v}_\tau)\, ds \geq (\boldsymbol{f}_k, \boldsymbol{v})_{\boldsymbol{H}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{44}$$

$$b(\boldsymbol{u}_k, q) = 0 \quad \forall\, q \in M. \tag{45}$$

Note that $b(\boldsymbol{u}, q) = b(\boldsymbol{u}_k, q) = 0$ for all $q \in M$. We take $\boldsymbol{v} = \boldsymbol{u}_k - \boldsymbol{u}$ in (14) and $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_k$ in (44) to get

$$a(\boldsymbol{u}, \boldsymbol{u}_k - \boldsymbol{u}) + \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_{k\tau} - \boldsymbol{u}_\tau)\, ds \geq (\boldsymbol{f}, \boldsymbol{u}_k - \boldsymbol{u})_{\boldsymbol{H}},$$

$$a(\boldsymbol{u}_k, \boldsymbol{u} - \boldsymbol{u}_k) + \int_S j^0(\boldsymbol{u}_{k\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{k\tau})\, ds \geq (\boldsymbol{f}_k, \boldsymbol{u} - \boldsymbol{u}_k)_{\boldsymbol{H}}.$$

Add the two inequalities,

$$a(\boldsymbol{u}_k - \boldsymbol{u}, \boldsymbol{u}_k - \boldsymbol{u}) \le \int_S \left[j^0(\boldsymbol{u}_{k\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{k\tau}) + j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_{k\tau} - \boldsymbol{u}_\tau) \right] ds \\ + (\boldsymbol{f}_k - \boldsymbol{f}, \boldsymbol{u}_k - \boldsymbol{u})_{\boldsymbol{H}}. \tag{46}$$

Then,

$$\begin{aligned} 2\nu \|\boldsymbol{u}_k - \boldsymbol{u}\|_V^2 &\le m_\tau \int_S \|\boldsymbol{u}_{k\tau} - \boldsymbol{u}_\tau\|_{\mathbb{R}^d}^2 ds + \|\boldsymbol{f}_k - \boldsymbol{f}\|_{V^*} \|\boldsymbol{u}_k - \boldsymbol{u}\|_V \\ &\le m_\tau \|\gamma\|^2 \|\boldsymbol{u}_k - \boldsymbol{u}\|_V^2 + \|\boldsymbol{f}_k - \boldsymbol{f}\|_{V^*} \|\boldsymbol{u}_k - \boldsymbol{u}\|_V, \end{aligned} \tag{47}$$

where the first inequality follows from (11), (12) and the Cauchy inequality, and the second one follows from trace inequality. Hence,

$$(2\nu - m_\tau \|\gamma\|^2) \|\boldsymbol{u}_k - \boldsymbol{u}\|_V \le \|\boldsymbol{f}_k - \boldsymbol{f}\|_{V^*}. \tag{48}$$

From $H(\mathcal{U}_{ad})$ we know that the sequence $\{\|\boldsymbol{f}_k\|_{V^*}\}$ is bounded. Thus, it follows from (48) that $\{\|\boldsymbol{u}_k\|_V\}$ is bounded. Similar to the proof of Step 1 in Theorem 2.5, we know that $\{\|p_k\|_{L^2(\Omega)}\}$ is also bounded.

By Theorem 2.5, we have $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{V}$, $p_n \to p$ in M, and $(\boldsymbol{u}, p) \in V \times M$ is the solution of Problem (P) with the source density function $\boldsymbol{f}$. Apply the condition $H(L)$ (i), resorting to a subsequence if necessary,

$$Q(\boldsymbol{f}) = \int_\Omega L(\boldsymbol{x}, p(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}))\, dx \le \int_\Omega \liminf_{k\to\infty} L(\boldsymbol{x}, p_k(\boldsymbol{x}), \boldsymbol{u}_k(\boldsymbol{x}), \boldsymbol{f}_k(\boldsymbol{x}))\, dx,$$

Then,

$$Q(\boldsymbol{f}) \le \liminf_{k\to\infty} \int_\Omega L(\boldsymbol{x}, p_k(\boldsymbol{x}), \boldsymbol{u}_k(\boldsymbol{x}), \boldsymbol{f}_k(\boldsymbol{x}))\, dx = m.$$

Hence,

$$Q(\boldsymbol{f}) = m$$

and the proof is completed. □

REFERENCES

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1984.

[2] V. Barbu, *Optimal Control of Variational Inequalities*, Research Notes in Mathematics, 100. Pitman, Boston, MA, 1984.

[3] H. Brézis, Équations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier*, **18** (1968), 115–175.

[4] F. H. Clarke, Generalized gradients and applications, *Trans. Amer. Math. Soc.*, **205** (1975), 247–262.

[5] F. H. Clarke, *Optimization and Nonsmooth Analysis*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983.

[6] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren der Mathematischen Wissenschaften, 219. Springer-Verlag, Berlin-New York, 1976.

[7] C. Fang, K. Czuprynski, W. Han, X.-L. Cheng and X. Dai, Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition, *IMA Journal of Numerical Analysis*, (2019).

[8] C. Fang and W. Han, Well-posedness and optimal control of a hemivariational inequality for nonstationary Stokes fluid flow, *Discrete Contin. Dyn. Syst.*, **36** (2016), 5369–5386.

[9] G. Fichera, Problemi elastostatici con vincoli unilaterali. II. Problema di Signorini con ambique condizioni al contorno, *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. Ia*, **7** (1963/64), 91–140.

[10] A. Friedman, Optimal control for variational inequalities, *SIAM J. Control Optim.*, **24** (1986), 439–451.

[11] H. Fujita, *Flow Problems with Unilateral Boundary Conditions*, College de France, Lecons, 1993.

[12] H. Fujita, A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions, *RIMS Kôkyûroku*, **888** (1994), 199–216.

[13] H. Fujita and H. Kawarada, Variational inequalities for the Stokes equation with boundary conditions of friction type, *Recent Developments in Domain Decomposition Methods and Flow Problems, GAKUTO Internat. Ser. Math. Sci. Appl., Gakkōtosho, Tokyo*, **11** (1998), 15–33.

[14] H. Fujita, H. Kawarada and A. Sasamoto, Analytical and numerical approaches to stationary flow problems with leak and slip boundary conditions, *Lecture Notes Numer. Appl. Anal., Kinokuniya, Tokyo*, **14** (1995), 17–31.

[15] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer Series in Computational Physics. Springer-Verlag, New York, 1984.

[16] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[17] W. Han, S. Migórski and M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, *SIAM J. Math. Anal.*, **46** (2014), 3891–3912.

[18] W. Han and M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, *Acta Numerica*, **28** (2019), 175–286.

[19] W. Han and Y. Li, Stability analysis of stationary variational and hemivariational inequalities with applications, *Nonlinear Anal. Real World Appl.*, **50** (2019), 171–191.

[20] J. Haslinger, I. Hlaváček and J. Nečas, Numerical methods for unilateral problems in solid mechanics, *Handbook of Numerical Analysis, Handb. Numer. Anal., North-Holland, Amsterdam*, **4** (1996), 313–485.

[21] J. Haslinger, M. Miettinen and P. D. Panagiotopoulos, *Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications*, Nonconvex Optimization and its Applications, 35. Kluwer Academic Publishers, Dordrecht, 1999.

[22] J. Haslinger and P. D. Panagiotopoulos, Optimal control of systems governed by hemivariational inequalities. Existence and approximation results, *Nonlinear Anal.*, **24** (1995), 105–119.

[23] I. Hlaváček, J. Haslinger, J. Nečas and J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Applied Mathematical Sciences, 66. Springer-Verlag, New York, 1988.

[24] L. S. Hou and S. S. Ravindran, Computations of boundary optimal control problems for an electrically conducting fluid, *Journal of Computational Physics*, **128** (1996), 319–330.

[25] N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Studies in Applied Mathematics, 8. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988.

[26] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Pure and Applied Mathematics, 88. Academic Press, Inc., New York-London, 1980.

[27] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Die Grundlehren der Mathematischen Wissenschaften, Band 170 Springer-Verlag, New York-Berlin, 1971.

[28] J.-L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.*, **20** (1967), 493–519.

[29] S. Migórski, A note on optimal control problem for a hemivariational inequality modeling fluid flow, *Discrete and Continuous Dynam. Systems - Supplement*, (2013), 545–554.

[30] S. Migórski and A. Ochal, Optimal control of parabolic hemivariational inequalities, *J. Global Optim.*, **17** (2000), 285–300.

[31] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, 26. Springer, New York, 2013.

[32] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, 188. Marcel Dekker, Inc., New York, 1995.

[33] P. D. Panagiotopoulos, Nonconvex energy functions. Hemivariational inequalities and substationary principles, *Acta Mech.*, **42** (1983), 111–130.

[34] P. D. Panagiotopoulos, *Hemivariational Inequalities: Applications in Mechanics and Engineering*, SpringerVerlag, Berlin, 1993.

[35] F. Saidi, Non-Newtonian Stokes flow with frictional boundary conditions, *Math. Model. Anal.*, **12** (2007), 483–495.

[36] A. Signorini, Sopra a une questioni di elastostatica, *Attidella Società Italiana per il Progresso delle Scienze*, (1933).

[37] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, CRC Press, Boca Raton, FL, 2018.

[38] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[39] D. Tiba, *Optimal Control of Nonsmooth Distributed Parameter Systems*, Lecture Notes in Mathematics, 1459. Springer-Verlag, Berlin, 1990.

[40] F. Tröltzsch, *Optimal Control of Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 2010.

[41] Y.-B. Xiao and M. Sofonea, On the optimal control of variational-hemivariational inequalities, *J. Math. Anal. Appl.*, **475** (2019), 364–384.

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