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A numerical method for generalized Fokker-Planck equations

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Abstract. Generalized Fokker-Planck (GFP) equations have been employed to approximate the radiative transfer equation in applications of highly forward peaked biological media. In this paper, we discuss a numerical method for solving GFP equations. The numerical method is based on a variational formulation involving even and odd components of the solution. We show the well-posedness of the variational formulation and develop a Galerkin method where spherical harmonics are used for the angular approximation and finite elements are used for spatial discretization. An iteration procedure is introduced to solve the problem and its convergence is shown.

1. Introduction

The radiative transfer equation (RTE) arises in a wide range of applications, such as neutron transport, heat transfer, radiation in atmosphere and ocean, optics. The RTE also describes the light propagation within biological media ([**NW01**]), and for this reason, in recent years, there has been active research in the community of medical optics on direct and inverse problems of the RTE (cf. e.g. [**A99**, **B09**, **HEHL11**]). The steady state monochromatic form of the RTE is ([**LM84**, **A98**])

$$\boldsymbol{\omega}\cdot\boldsymbol{\nabla} u + \mu_t u = \mu_s\, Su + f \quad \text{in } Q := X \times \Omega. \tag{1.1}$$

Here X is a bounded domain in $\mathbb{R}^3$, Ω is the unit sphere in $\mathbb{R}^3$, $\mu_t = \mu_a + \mu_s$, μ_a is the absorption coefficient, μ_s is the scattering coefficient, f is a source function, and S is an integral operator of the form

$$Su(\mathbf{x},\boldsymbol{\omega}) = \int_\Omega \eta(\mathbf{x},\boldsymbol{\omega}\cdot\boldsymbol{\omega}')\, u(\mathbf{x},\boldsymbol{\omega}')\, d\sigma(\boldsymbol{\omega}')$$

with the phase function $\eta(\mathbf{x},\boldsymbol{\omega}\cdot\boldsymbol{\omega}') \ge 0$ satisfying the normalization condition

$$\int_\Omega \eta(\mathbf{x},\boldsymbol{\omega}\cdot\boldsymbol{\omega}')\, d\sigma(\boldsymbol{\omega}') = 1 \quad \forall\,\boldsymbol{\omega}\in\Omega,\ \forall\,\mathbf{x}\in X.$$

A well-known example is the Henyey-Greenstein phase function (cf. [**HG41**])

$$\eta(t) = \frac{1-g^2}{4\pi(1+g^2-2gt)^{3/2}}, \quad t\in[-1,1]. \tag{1.2}$$

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The parameter $g \in (-1, 1)$ is the anisotropy factor of the scattering medium: $g > 0$ for forward scattering, and the bigger the value of g, the stronger the degree of forward scattering. In biomedical optics, g is close to 1.

We assume the boundary ∂X is C^1 and use $\boldsymbol{\nu}(\mathbf{x})$ to denote the unit outward normal vector at $\mathbf{x} \in \partial X$. Let Γ be the boundary of the set Q and define the inflow boundary $\Gamma_- = \{(\mathbf{x}, \boldsymbol{\omega}) \in \Gamma : \boldsymbol{\nu}(\mathbf{x})\cdot\boldsymbol{\omega} < 0\}$ and outgoing boundary $\Gamma_+ = \{(\mathbf{x}, \boldsymbol{\omega}) \in \Gamma : \boldsymbol{\nu}(\mathbf{x})\cdot\boldsymbol{\omega} > 0\}$. We assume

$$\mu_t, \mu_s \in L^\infty(X), \quad \mu_s \ge 0 \text{ a.e. in } X, \quad \mu_a \ge c_0 > 0 \text{ a.e. in } X, \tag{1.3}$$

$$f \in L^2(Q). \tag{1.4}$$

These assumptions are naturally valid in applications.

Due to the high dimensionality and integro-differential form of the equation, it is very challenging to numerically solve the RTE accurately. Moreover, in applications in biomedical optics, the light propagation within the biological media is highly forward peaked, leading to additional numerical difficulties for solving the RTE. It is shown in the literature (e.g. [**KimK03**]) that for applications in highly forward peaked biological media, the following generalized Fokker-Planck equation (GFPE) is a good approximation to RTE:

$$\boldsymbol{\omega}\cdot\boldsymbol{\nabla} u + \mu_t u = \mu_s (I - \alpha\, \Delta^*)^{-1} u + f \quad \text{in } Q. \tag{1.5}$$

Here, $\alpha(\mathbf{x}) \ge 0$ and Δ^* is the Laplace–Beltrami operator, i.e., the restriction of the Laplace operator on the unit sphere ([**AH12**]). For the Henyey-Greenstein phase function (1.2), $\alpha = (1 - g)/(2g)$ is a constant. In this paper, we consider any GFPE of the form (1.5) with α bounded and positively-valued. The equation (1.5) is supplemented by the following boundary condition:

$$u = u_{\text{in}} \quad \text{on } \Gamma_-. \tag{1.6}$$

Introduce a function space

$$H^{1,2}(Q) := \{v \in L^2(Q) \mid \boldsymbol{\omega}\cdot\boldsymbol{\nabla} v \in L^2(Q)\},$$

$\boldsymbol{\omega}\cdot\boldsymbol{\nabla} v$ being the generalized directional derivative of v in the direction $\boldsymbol{\omega}$. We denote by $(u, v)_Q$ the integral of uv on Q, and similarly define $(\cdot,\cdot)_\Gamma$, $(\cdot,\cdot)_{\Gamma_-}$, $(\cdot,\cdot)_\Omega$. We assume

$$u_{\text{in}} \in L^2_*(\Gamma_-), \tag{1.7}$$

where $L^2_*(\Gamma_-)$ denotes the space of measurable functions v on Γ_- such that the norm $\|v\|_{L^2_*(\Gamma_-)} := (|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, v, v)^{1/2}_{\Gamma_-} < \infty$. It is shown in [**HECW11**] that under the assumptions (1.3)–(1.4) and (1.7), the problem (1.5)–(1.6) has a unique solution $u \in H^{1,2}(Q)$ and this solution is Lipschitz continuous with respect to the source function f and the boundary condition u_{in}. Moreover, if $f \ge 0$ and $u_{\text{in}} \ge 0$, then $u \ge 0$. This property is desirable for the GFPE to be a physically meaningful model.

In [**ES12**], a mixed weak framework is studied for RTE. In this paper, we introduce a similar weak formulation for the problem (1.5)–(1.6) and study its numerical approximation. Although it is possible to give the presentation for any spatial dimension, for definiteness and due to the importance and relevance in applications, we focus on the case of three spatial dimension.

2. Spaces and operators

We introduce additional function spaces and operators that will be needed later. Let $V_0 := L^2(Q)$ and let V_1 be the subspace of $H^{1,2}(Q)$ with traces on Γ in $L^2_*(\Gamma)$.

Define the transport operator from V_1 to V_0 by $v(\mathbf{x},\boldsymbol{\omega}) \mapsto \boldsymbol{\omega}\cdot\nabla v(\mathbf{x},\boldsymbol{\omega})$, and the removal operator $R : V_0 \to V_0$ by

$$(Rv)(\mathbf{x},\boldsymbol{\omega}) := \mu_t(\mathbf{x})\, v(\mathbf{x},\boldsymbol{\omega}) - \mu_s(\mathbf{x})\,(I-\alpha\,\Delta^*)^{-1}\, v(\mathbf{x},\boldsymbol{\omega}).$$

We have the following properties for the removal operator R.

PROPOSITION 2.1. The operator $R : V_0 \to V_0$ is linear, and is
(a) self-adjoint: $(Ru,v)_Q = (u,Rv)_Q\ \forall\, u,v\in V_0$,
(b) bounded: $\|Rv\|_{V_0} \le c_1\|v\|_{V_0}\ \forall\, v\in V_0$,
(c) elliptic: $(Rv,v)_Q \ge c_0\|v\|^2_{V_0},\ \ c_0>0,\ \forall\, v\in V_0$.

PROOF. The linearity and self-adjointness of R are easily seen.

Let $w = (I-\alpha\,\Delta^*)^{-1}\, v$. Then $w-\alpha\,\Delta^* w = v$, and

$$(w,z)_\Omega + \alpha\,(\nabla^* w,\nabla^* z)_\Omega = (v,z)_\Omega \quad \forall\, z\in H^1(\Omega),$$

where $H^1(\Omega) := \left\{z\in L^2(\Omega) : |\nabla^* z|\in L^2(\Omega)\right\}$. Take $z=w$ to obtain

$$\|w\|^2_{L^2(\Omega)} + \alpha\,\||\nabla^* w|\|^2_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \le \frac{1}{2}\,\|w\|^2_{L^2(\Omega)} + \frac{1}{2}\,\|v\|^2_{L^2(\Omega)}.$$

Thus,

$$\|w\|^2_{L^2(\Omega)} \le \|v\|^2_{L^2(\Omega)}. \tag{2.1}$$

Therefore, $\|w\|_{V_0}\le\|v\|_{V_0}$, and the boundedness (b) is valid:

$$\|Rv\|_{V_0} \le \|\mu_t v\|_{V_0} + \|\mu_s w\|_{V_0} \le c_1\|v\|_{V_0},\quad c_1 = \|\mu_t\|_{L^\infty(X)} + \|\mu_s\|_{L^\infty(X)}.$$

By the Cauchy-Schwarz inequality,

$$(\mu_s w,v)_Q \le (\mu_s w,w)_Q^{1/2}\,(\mu_s v,v)_Q^{1/2} \le (\mu_s v,v)_Q\,.$$

Hence,

$$\begin{aligned}(Rv,v)_Q &= (\mu_t v,v)_Q - (\mu_s w,v)_Q \ge (\mu_t v,v)_Q - (\mu_s v,v)_Q\\ &= (\mu_a v,v)_Q \ge c_0\|v\|^2_{V_0},\end{aligned}$$

i.e., the ellipticity (c) holds. □

Given the properties stated in Proposition 2.1, we can apply the Lax-Milgram Lemma to conclude the following result.

COROLLARY 2.2. The operator R has an inverse $R^{-1} : V_0\to V_0$ that is linear, self-adjoint, bounded and elliptic. Moreover, operators $R^{\pm 1/2}$ are well-defined. The expressions $\|v\|_R := (Rv,v)_Q^{1/2}$ and $\|v\|_{R^{-1}} := \left(R^{-1}v,v\right)_Q^{1/2}$ define norms on V_0 that are equivalent to the standard norm $\|v\|_{V_0}$.

The weak formulations studied in this paper involve the splitting of a function v into even part v^+ and odd part v^-, defined by

$$v^+(\mathbf{x},\boldsymbol{\omega}) := \frac{1}{2}\left(v(\mathbf{x},\boldsymbol{\omega}) + v(\mathbf{x},-\boldsymbol{\omega})\right), \qquad v^-(\mathbf{x},\boldsymbol{\omega}) := \frac{1}{2}\left(v(\mathbf{x},\boldsymbol{\omega}) - v(\mathbf{x},-\boldsymbol{\omega})\right).$$

Then given a function space V, we define $V^{\pm}$ to be the subspaces of V consisting of even and odd functions in V. In particular, we will use the space $W := V_1^+ \oplus V_0^-$. The norm in the space W is

$$\|v\|_W := \left(\|\boldsymbol{\omega}\cdot\nabla v^+\|_{R^{-1}}^2 + \|v\|_R^2 + \|v^+\|_{L_*^2(\Gamma)}^2\right)^{1/2},$$

where $\|v\|_R^2 = (Rv, v)_Q$, $\|v\|_{R^{-1}}^2 = (R^{-1}v, v)_Q$.

It is easy to see that $\boldsymbol{\omega}\cdot\nabla^* v \in V_0^{\mp}$ for $v \in V_1^{\pm}$. Moreover, the removal operator R is parity preserving, i.e., $R : V_1^+ \to V_1^+$, and $V_0^- \to V_0^-$.

3. A weak formulation

To derive the weak formulation, rewrite the equation (1.5) as

$$\boldsymbol{\omega}\cdot\nabla u + Ru = f.$$

Multiply the equation by a smooth function v and integrate,

$$(\boldsymbol{\omega}\cdot\nabla u, v)_Q + (Ru, v)_Q = (f, v)_Q\,.$$

Use the decomposition $u = u^+ + u^-$ and $v = v^+ + v^-$ in terms of the even and odd components to obtain

$$(\boldsymbol{\omega}\cdot\nabla u, v)_Q = \left(\boldsymbol{\omega}\cdot\nabla u^+, v^-\right)_Q + \left(\boldsymbol{\omega}\cdot\nabla u^-, v^+\right)_Q.$$

Then perform an integration by parts,

$$\left(\boldsymbol{\omega}\cdot\nabla u^-, v^+\right)_Q = \left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u^-, v^+\right)_\Gamma - \left(u^-, \boldsymbol{\omega}\cdot\nabla v^+\right)_Q.$$

Note that

$$\begin{aligned}\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u^-, v^+\right)_\Gamma &= 2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u^-, v^+\right)_{\Gamma_-} = 2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\,(u_{\text{in}} - u^+), v^+\right)_{\Gamma_-}\\ &= 2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u_{\text{in}}, v^+\right)_{\Gamma_-} + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, u^+, v^+\right)_\Gamma,\end{aligned}$$

where the boundary condition (1.6) is applied. Then for a solution of the problem (1.5)–(1.6),

$$\begin{aligned}&(Ru, v)_Q - \left(u^-, \boldsymbol{\omega}\cdot\nabla v^+\right)_Q + \left(\boldsymbol{\omega}\cdot\nabla u^+, v^-\right)_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, u^+, v^+\right)_\Gamma\\ &\quad = (f, v)_Q - 2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u_{\text{in}}, v^+\right)_{\Gamma_-}\end{aligned}$$

for any smooth function v. Define a bilinear form and a linear form over W as follows:

$$b(u, v) := (Ru, v)_Q - \left(u^-, \boldsymbol{\omega}\cdot\nabla v^+\right)_Q + \left(\boldsymbol{\omega}\cdot\nabla u^+, v^-\right)_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, u^+, v^+\right)_\Gamma, \tag{3.1}$$

$$\ell(v) := (f, v)_Q - 2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u_{\text{in}}, v^+\right)_{\Gamma_-}. \tag{3.2}$$

Then the weak formulation is

$$u \in W, \qquad b(u, v) = \ell(v) \quad \forall\, v \in W. \tag{3.3}$$

THEOREM 3.1. *Under the assumptions* (1.3)–(1.4) *and* (1.7)*, the problem* (3.3) *has a unique solution, and the solution depends continuously on the data.*

As in [**ES12**], Theorem 3.1 is proved by employing the following result adapted from [**B71**].

THEOREM 3.2. *Assume $b(\cdot,\cdot)$ is a bounded bilinear form on W and there exists a constant $b_0 > 0$ such that*

$$\inf_{u\in W}\sup_{v\in W}\frac{b(u,v)}{\|u\|_W\|v\|_W}\ge b_0,\qquad \inf_{v\in W}\sup_{u\in W}\frac{b(u,v)}{\|u\|_W\|v\|_W}\ge b_0. \tag{3.4}$$

Then for any $\ell\in W'$, the problem (3.3) *has a unique solution $u\in W$ and for some constant c, $\|u\|_W\le c\,\|\ell\|_{W'}$.*

In applying Theorem 3.2 to prove Theorem 3.1, the crucial part is to show (3.4). For the bilinear form defined by (3.1), let us prove below the first inequality of (3.4); the second inequality can be proved similarly.

For $u\ne 0$, let $\overline{u}=u+R^{-1}(\boldsymbol{\omega}\cdot\nabla u^+)$. Then $\|\overline{u}\|_W\le c\,\|u\|_W$ and

$$b(u,\overline{u})=(Ru,u)_Q+\big(u,\boldsymbol{\omega}\cdot\nabla u^+\big)_Q+\big(\boldsymbol{\omega}\cdot\nabla u^+,R^{-1}(\boldsymbol{\omega}\cdot\nabla u^+)\big)_Q+\big(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\,u^+,u^+\big)_\Gamma.$$

Now

$$\begin{aligned}\big(u,\boldsymbol{\omega}\cdot\nabla u^+\big)_Q&=\big(u^-,\boldsymbol{\omega}\cdot\nabla u^+\big)_Q=\Big(R^{1/2}u^-,R^{-1/2}(\boldsymbol{\omega}\cdot\nabla u^+)\Big)_Q\\&\ge-\frac12\,\|\boldsymbol{\omega}\cdot\nabla u^+\|_{R^{-1}}^2-\frac12\,\|u^-\|_{R^{-1}}^2.\end{aligned}$$

Hence,

$$b(u,\overline{u})\ge\frac12\,\|\boldsymbol{\omega}\cdot\nabla u^+\|_{R^{-1}}^2+\frac12\,\|u\|_R^2+\|u^+\|_{L^2_*(\Gamma)}^2\ge\frac12\,\|u\|_W^2.$$

This inequality, combined with $\|\overline{u}\|_W\le c\,\|u\|_W$, implies the first inequality of (3.4) for some constant $b_0>0$.

The rest of the assumptions of Theorem 3.2 can be verified easily. Thus, Theorem 3.1 holds. As in [**ES12**], it can be further proved that the solution u of the problem (3.3) satisfies the equation (1.5) a.e. in Q and the boundary condition (1.6) a.e. on Γ_-.

Using $w:=(I-\alpha\,\Delta^*)^{-1}u$ as an unknown, we can rewrite the problem (3.3) as: Find $u^+\in V_1^+$, $u^-\in V_0^-$ and $w(\mathbf{x},\cdot)\in H^1(\Omega)$ for a.e. $\mathbf{x}\in X$ such that

$$\begin{aligned}&\big(\mu_t u^+-\mu_s w^+,v^+\big)_Q-\big(u^-,\boldsymbol{\omega}\cdot\nabla v^+\big)_Q+\big(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\,u^+,v^+\big)_\Gamma\\&\qquad=\big(f^+,v^+\big)_Q-2\,\big(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\,u_{\rm in},v^+\big)_Q\quad\forall\,v^+\in V_1^+,\end{aligned}\tag{3.5}$$

$$\big(\mu_t u^--\mu_s w^-,v^-\big)_Q+\big(\boldsymbol{\omega}\cdot\nabla u^+,v^-\big)_Q=\big(f^-,v^-\big)_Q\quad\forall\,v^-\in V_0^-,\tag{3.6}$$

$$(w,v)_\Omega+\alpha\,(\nabla^*w,\nabla^*v)_\Omega=(u,v)_\Omega\quad\forall\,v\in H^1(\Omega).\tag{3.7}$$

4. Galerkin approximation

For a discretization of the problem (3.3), we use spherical harmonics of an arbitrary order for the angular approximation and finite elements for spatial discretization. For this purpose, denote by h the meshsize of a finite element partition of the domain X. Let $V_{1,h}$ be the linear element space of continuous piecewise linear functions and $V_{0,h}$ be the space of piecewise constant functions. For a positive integer n, let $\{Y_{n,m}\}_{m=-n}^n$ be a basis of spherical harmonics of order n, e.g., in spherical coordinates,

$$(-1)^{(m+|m|)/2}\left[\frac{(2\,n+1)\,(n-|m|)!}{4\,\pi\,(n+|m|)!}\right]^{\frac12}(\sin\theta)^m P_n^{(|m|)}(\cos\theta)\,e^{im\phi},\quad -n\le m\le n,$$

where $P_n^{(|m|)}(t)$ is the $|m|^{th}$ derivative of $P_n(t)$, the Legendre polynomial of degree n. In real valued form, the following basis functions are used:

$$\left[\frac{(2\,n+1)\,(n-m)!}{2\,\pi\,(n+m)!}\right]^{\frac{1}{2}}\ (\sin\theta)^m P_n^{(m)}(\cos\theta)\,\cos(m\phi),\quad 0\le m\le n,$$

$$\left[\frac{(2\,n+1)\,(n-m)!}{2\,\pi\,(n+m)!}\right]^{\frac{1}{2}}\ (\sin\theta)^m P_n^{(m)}(\cos\theta)\,\sin(m\phi),\quad 1\le m\le n.$$

See [**AH12**] for an introduction of spherical harmonics. We then define the following finite dimensional spaces of combined finite elements and spherical harmonics:

$$V_{1,h,n}^{+}=\left\{v_{h,n}^{+}(\mathbf{x},\boldsymbol{\omega})=\sum_{j=-n}^{n}\sum_{i=-2j}^{2j}v_{h,2j,i}(\mathbf{x})\,Y_{2j,i}(\boldsymbol{\omega}):\ v_{h,2j,i}\in V_{1,h}\right\},$$

$$V_{0,h,n}^{-}=\left\{v_{h,n}^{-}(\mathbf{x},\boldsymbol{\omega})=\sum_{j=-n-1}^{n}\sum_{i=-(2j+1)}^{2j+1}v_{h,2j+1,i}(\mathbf{x})\,Y_{2j+1,i}(\boldsymbol{\omega}):\ v_{h,2j+1,i}\in V_{0,h}\right\},$$

and $W_{h,n}=V_{1,h,n}^{+}\oplus V_{0,h,n}^{-}$. Then any function $v_{h,n}\in W_{h,n}$ can be expressed as $v_{h,n}=v_{h,n}^{+}+v_{h,n}^{-}$ with $v_{h,n}^{+}\in V_{1,h,n}^{+}$ and $v_{h,n}^{-}\in V_{0,h,n}^{-}$. Note that $\boldsymbol{\omega}\cdot\nabla v_{h,n}^{+}\in V_{0,h,n}^{-}$ for any $v_{h,n}^{+}\in V_{1,h,n}^{+}$. This property ensures the discrete version of (3.4): For the same constant $b_0>0$,

$$\inf_{u_{h,n}\in W_{h,n}}\ \sup_{v_{h,n}\in W_{h,n}}\frac{b(u_{h,n},v_{h,n})}{\|u_{h,n}\|_W\|v_{h,n}\|_W}\ge b_0,$$

$$\inf_{v_{h,n}\in W_{h,n}}\ \sup_{u_{h,n}\in W_{h,n}}\frac{b(u_{h,n},v_{h,n})}{\|u_{h,n}\|_W\|v_{h,n}\|_W}\ge b_0.$$

Then the Galerkin approximation of the weak formulation (3.3)

$$u_{h,n}\in W_{h,n},\qquad b(u_{h,n},v_{h,n})=\ell(v_{h,n})\quad\forall\,v_{h,n}\in W_{h,n}\tag{4.1}$$

has a unique solution and for the error,

$$\|u-u_{h,n}\|_W\le 2\,b_0\inf_{v_{h,n}\in W_{h,n}}\|u-v_{h,n}\|_W.\tag{4.2}$$

It can be verified that similar to (3.5)–(3.7), the discrete problem (4.1) can be rewritten as: Find $u_{h,n}^{+}\in V_{1,h,n}^{+}$, $u_{h,n}^{-}\in V_{0,h,n}^{-}$ and $w_{h,n}\in W_{h,n}$ such that

(4.3)
$$\begin{aligned}&\left(\mu_t u_{h,n}^{+}-\mu_s w_{h,n}^{+},v_{h,n}^{+}\right)_Q-\left(u_{h,n}^{-},\boldsymbol{\omega}\cdot\nabla v_{h,n}^{+}\right)_Q+\left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\,u_{h,n}^{+},v_{h,n}^{+}\right)_\Gamma\\&\qquad=\left(f^{+},v_{h,n}^{+}\right)_Q-2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\,u_{\rm in},v_{h,n}^{+}\right)_Q\quad\forall\,v_{h,n}^{+}\in V_{1,h,n}^{+},\end{aligned}$$

(4.4)
$$\left(\mu_t u_{h,n}^{-}-\mu_s w_{h,n}^{-},v_{h,n}^{-}\right)_Q+\left(\boldsymbol{\omega}\cdot\nabla u_{h,n}^{+},v_{h,n}^{-}\right)_Q=\left(f^{-},v_{h,n}^{-}\right)_Q\quad\forall\,v_{h,n}^{-}\in V_{0,h,n}^{-},$$

(4.5)
$$(w_{h,n},v_n)_\Omega+\alpha\,(\nabla^* w_{h,n},\nabla^* v_n)_\Omega=(u_{h,n},v_n)_\Omega\quad\forall\,v_n\in V_n.$$

Here,

$$V_n=\operatorname{span}\{Y_{j,i}(\boldsymbol{\omega}):\ -j\le i\le j,\ -2n-1\le j\le 2n+1\}.$$

5. An iteration procedure

The form (3.5)–(3.7) or the discrete version (4.3)–(4.5) naturally suggests an iteration method for solving the problems. Here, to simplify the notation, we discuss the iteration method for solving (3.5)–(3.7) as an example. With an initial guess w_0, say $w_0 = 0$, we define a sequence $\{(u_k, w_k)\}_{k\geq 1}$ by the following: $u_k^+ \in V_1^+$, $u_k^- \in V_0^-$ and $w_k(\mathbf{x},\cdot) \in H^1(\Omega)$ for a.e. $\mathbf{x} \in X$ such that

$$(5.1)\qquad \left(\mu_t u_k^+ - \mu_s w_{k-1}^+, v^+\right)_Q - \left(u_k^-, \boldsymbol{\omega}\cdot\nabla v^+\right)_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, u_k^+, v^+\right)_\Gamma = \left(f^+, v^+\right)_Q - 2\left(\boldsymbol{\nu}\cdot\boldsymbol{\omega}\, u_{\rm in}, v^+\right)_Q \quad \forall\, v^+ \in V_1^+,$$

$$(5.2)\qquad \left(\mu_t u_k^- - \mu_s w_{k-1}^-, v^-\right)_Q + \left(\boldsymbol{\omega}\cdot\nabla u_k^+, v^-\right)_Q = \left(f^-, v^-\right)_Q \quad \forall\, v^- \in V_0^-,$$

$$(5.3)\qquad (w_k, v)_\Omega + \alpha\,(\nabla^* w_k, \nabla^* v)_\Omega = (u_k, v)_\Omega \quad \forall\, v \in H^1(\Omega).$$

The sequence is well defined. Here we focus on convergence of the iteration method.

THEOREM 5.1. *Under the assumptions* (1.3)–(1.4) *and* (1.7)*, the iteration method converges:*

$$\|u - u_k\|_{L^2(Q)} + \|w - w_k\|_{L^2(Q)} + \|u^+ - u_k^+\|_{L^2_*(\Gamma)} \to 0 \quad \text{as } k \to \infty.$$

PROOF. Define the errors $e_{u,k} := u - u_k$, $e_{w,k} := w - w_k$, and their even and odd components $e_{u^+,k} = e_{u,k}^+ = u^+ - u_k^+$, etc. By subtracting the equations (5.1)–(5.3) from the corresponding equations (3.5)–(3.7), we obtain the error relations

$$(5.4)\qquad \left(\mu_t e_{u^+,k}, v^+\right)_Q - \left(e_{u^-,k}, \boldsymbol{\omega}\cdot\nabla v^+\right)_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, e_{u^+,k}, v^+\right)_\Gamma = \left(\mu_s e_{w^+,k-1}, v^+\right)_Q \quad \forall\, v^+ \in V_1^+,$$

$$(5.5)\qquad \left(\mu_t e_{u^-,k}, v^-\right)_Q + \left(\boldsymbol{\omega}\cdot\nabla e_{u^+,k}, v^-\right)_Q = \left(\mu_s e_{w^-,k-1}, v^-\right)_Q \quad \forall\, v^- \in V_0^-,$$

$$(5.6)\qquad (e_{w,k}, v)_\Omega + \alpha\,(\nabla^* e_{w,k}, \nabla^* v)_\Omega = (e_{u,k}, v)_\Omega \quad \forall\, v \in H^1(\Omega).$$

Take $v^+ = e_{u^+,k}$ in (5.4), $v^- = e_{u^-,k}$ in (5.5), and add the two resulting inequalities,

$$(5.7)\qquad (\mu_t e_{u,k}, e_{u,k})_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, e_{u^+,k}, e_{u^+,k}\right)_\Gamma = (\mu_s e_{w,k-1}, e_{u,k})_Q\,.$$

For a.e. $\mathbf{x} \in X$, take $v = e_{w,k}$ in (5.6) to obtain

$$(5.8)\qquad (e_{w,k}, e_{w,k})_\Omega + (\alpha\,\nabla^* e_{w,k}, \nabla^* e_{w,k})_\Omega = (e_{u,k}, e_{w,k})_\Omega\,.$$

From the assumption (1.3), we know that

$$\kappa := \sup_{\mathbf{x}\in X} \frac{\mu_s(\mathbf{x})}{\mu_t(\mathbf{x})} < 1.$$

By (5.7), we then have

$$(5.9)\qquad (\mu_t e_{u,k}, e_{u,k})_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, e_{u^+,k}, e_{u^+,k}\right)_\Gamma \le \kappa\,(\mu_t e_{w,k-1}, e_{w,k-1})_Q^{1/2}\,(\mu_t e_{u,k}, e_{u,k})_Q^{1/2}\,.$$

Then,

$$(\mu_t e_{u,k}, e_{u,k})_Q \le \kappa^2\,(\mu_t e_{w,k-1}, e_{w,k-1})_Q\,.$$

Using (5.9) again,

$$(5.10)\qquad (\mu_t e_{u,k}, e_{u,k})_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, e_{u^+,k}, e_{u^+,k}\right)_\Gamma \le \kappa^2\,(\mu_t e_{w,k-1}, e_{w,k-1})_Q\,.$$

From (5.8),

$$(5.11)\qquad (\mu_t e_{w,k}, e_{w,k})_Q \le (\mu_t e_{u,k}, e_{w,k})_Q\,.$$

Combining (5.10) and (5.11),

$$(\mu_t e_{u,k}, e_{u,k})_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, e_{u^+,k}, e_{u^+,k}\right)_\Gamma \le \kappa^2 \left(\mu_t e_{u,k-1}, e_{u,k-1}\right)_Q . \tag{5.12}$$

An induction on (5.12) shows that

$$(\mu_t e_{u,k}, e_{u,k})_Q + \left(|\boldsymbol{\nu}\cdot\boldsymbol{\omega}|\, e_{u^+,k}, e_{u^+,k}\right)_\Gamma \le \kappa^{2k} \left(\mu_t e_{u,0}, e_{u,0}\right)_Q \to 0 \text{ as } k \to \infty.$$

Moreover, by (5.11), we also have

$$(\mu_t e_{w,k}, e_{w,k})_Q \to 0 \text{ as } k \to \infty.$$

Thus, the stated convergence result holds. □

Implementation and simulation of the Galerkin method (4.1), as well as studies of related inverse problems, are future research topics.

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