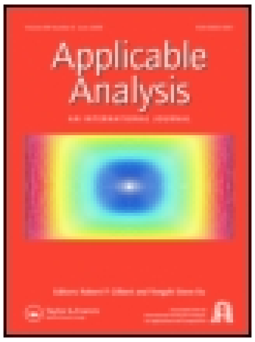

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Solving a nonlinear inverse Robin problem through a linear Cauchy problem

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ABSTRACT

Considered in this paper is an inverse Robin problem governed by a steady-state diffusion equation. By the Robin inverse problem, one wants to recover the unknown Robin coefficient on an inaccessible boundary from Cauchy data measured on the accessible boundary. In this paper, instead of reconstructing the Robin coefficient directly, we compute first the Cauchy data on the inaccessible boundary which is a linear inverse problem, and then compute the Robin coefficient through Newton's law. For the Cauchy problem, a parameter-dependent coupled complex boundary method (CCBM) is applied. The CCBM has its own merits, and this is particularly true when it is applied to the Cauchy problem. With the introduction of a positive parameter, we can prove the regularized solution is uniformly bounded with respect to the regularization parameter which is a very good property because the solution can now be reconstructed for a rather small value of the regularization parameter. For the problem of computing the Robin coefficient from the recovered Cauchy data, a least square output Tikhonov regularization method is applied to Newton's law to obtain a stable approximate Robin coefficient. Numerical results are given to show the feasibility and effectiveness of the proposed method.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d \leq 3$: space dimension) be an open bounded set with a Lipschitz boundary $\Gamma := \partial\Omega$, which is split into two measurable subsets: $\Gamma = \Gamma_a \bigcup \Gamma_u$ with $\Gamma_a \bigcap \Gamma_u = \emptyset$. In applications, Γ_a and Γ_u are known as accessible and unaccessible parts of the boundary for the object of interest, respectively. Denote by ν the unit outward normal to Γ. We consider the following inverse Robin problem governed by the steady-state heat conductivity equation.

Problem 1.1: *Given Cauchy data (Φ, T) on Γ_a, find γ on Γ_u such that the solution of the boundary value problem (BVP):*

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u) &= f \quad in\ \Omega, \\ \sigma \partial_\nu u &= \Phi \quad on\ \Gamma_a, \\ \sigma \partial_\nu u + \gamma\, u &= g \quad on\ \Gamma_u \end{aligned} \tag{1}$$

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satisfies

$$u = T \text{ on } \Gamma_a. \tag{2}$$

The Robin inverse problem above arises in many applications such as corrosion detection [1], the metal-to-silicon contact in semiconductor devices [2], designing gas turbine blades and nuclear reactors and analyzing quenching processes [3]. Both identifiability and stability of inverse Robin problems have been investigated intensively. For instance, if γ is (piecewise) continuous, it can be uniquely determined by T in all dimensions [1,4,5]. A uniqueness result is proved in [6] when $\sigma \in C^2(\bar{\Omega})$ and $\text{meas}(\{x \in \Gamma_u : u(\gamma)(x) = 0\}) = 0$. More recently, it is shown in [7] that for $d = 2$, an L^∞ Robin coefficient γ can be uniquely determined by T in the case $\Phi \in L^2(\Gamma_a)$ and $\sigma \in W^{1,r}(\Omega)$ with $r > 2$. Local Lipschitz and logarithmic stability are established in [4] and [8]. Stable numerical methods to compute the Robin coefficient are also studies intensively using e.g. variational approaches[9–11], the boundary integral method [12], the finite element methods [6,13].

The aforementioned work focuses on the ill-posedness and many regularization methods are designed to deal with it. In this paper, we propose a new method with the starting point on the treatment of the nonlinearity. Specifically, instead of reconstructing the Robin coefficient directly, we first solve a linear inverse problem to compute the Cauchy data on the inaccessible boundary, and then compute the Robin coefficient from Newton's law. The linear inverse problem for the Cauchy data is as follows.

Problem 1.2: *Given f in Ω, and Cauchy data (Φ, T) on Γ_a, find (ϕ, t) on Γ_u such that the following relations hold:*

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u) &= f && \text{in } \Omega, \\ \sigma \partial_\nu u = \Phi, \quad u &= T && \text{on } \Gamma_a, \\ \sigma \partial_\nu u = \phi, \quad u &= t && \text{on } \Gamma_u. \end{aligned} \tag{3}$$

Once Problem 3 is solved, the Robin coefficient γ is computed from the Newton's law [6,14]:

$$\gamma = \frac{g - \sigma \partial_\nu u|_{\Gamma_u}}{u|_{\Gamma_u}} = \frac{g - \phi}{t}. \tag{4}$$

Contrary to Problem 1.1 which is nonlinear, Problem 1.2 is linear. Note that Problem 1.2, also known as data completion [15], itself has wide applications in physics and engineering such as linear elasticity [16], thermostatics [17], plasma physics [18], mechanical engineering [19] and electrocardiography [20] etc., and thus has attracted a large amount of attention from mathematicians, physicists and engineers. It is well-known that Problem 1.2 is also ill-posed. A rigorous proof of the ill-posedness was given in [21] for a general domain. Moreover, after reformulating the Cauchy problem as a variational equation, Ben Belgacem showed in [22] that the Cauchy problem is exponentially ill-posed for both smooth and non-smooth domains. Lavrent'ev demonstrated in [23] that the solution of the Cauchy problem for the Laplace equation is stable given a supplementary condition. Payne in [24] generalized the work of [23] and deduced a pointwise bound for the problem in n-dimensions. We also refer to [25] for an overview on the stability of the Cauchy problem for general elliptic equations under rather weak assumptions on the problem domain. Due to the severe ill-posedness of the Cauchy problem, regularization strategies are needed to obtain a stable approximate solution, especially when the measured data (Φ, T) are polluted inevitably by random noise. These regularization strategies include quasi-reversibility method [26,27], iterative regularization [28,29], Lavrentiev regularization [30,31], truncation regularization method [32,33], discretization method [34,35], moment problem method [36,37], and perturbation regularization method [33,38] etc.. Among them, the Tikhonov regularization methods [17,39–41] are the most popular and frequently used ones which convert Problem 1.2 to data-fitting minimization problems with regularization terms.

Recently, Cheng et al. [42] proposed a coupled complex boundary method (CCBM) for an inverse source problem, where a complex Robin boundary condition is used to treat simultaneously both Dirichlet and Neumann boundary conditions. As is shown in [42], the CCBM provides a more robust and more efficient approach to solve inverse source problems. In the sequel [43], the method is applied successfully to the Cauchy problem. With CCBM, the data to fit is transferred from Γ_a to Ω; the missing data (ϕ, t) on Γ_u can be reconstructed simultaneously; weaker regularity is sufficient on the Dirichlet data for the forward problem to have an H^1 solution. However, like other Tikhonov regularization methods, in the CCBM-based regularization framework, it is crucial to choose a proper regularization parameter for the trade off between the accuracy and the stability of approximate solutions. In [44], a new parameter-dependent CCBM is proposed for an inverse source problem arising from bioluminescence tomography. With the introduction of a positive parameter α (see Section 2 below), the regularized solution is uniformly bounded with respect to the regularization parameter which is a very good property because the solution can now be reconstructed with a rather small value of the regularization parameter. In this paper, this parameter-dependent CCBM is applied to Problem 1.2, and a Tikhonov regularization framework is proposed for solving the reduced inverse problem. Moreover, different from using iterative schemes for the Robin inverse problem because of this nonlinearity, the Cauchy problem is linear, and thus with the help of the adjoint equation, the solution of the regularized reconstruction framework can be computed through a system of BVPs. As a result, no iteration is needed and the computation is effective.

The structure of the rest of the paper is as follows. Applying CCBM, we present in Section 2 a reformulation of Problem 1.2 and then a Tikhonov regularization reconstruction framework. Convergence and uniform boundedness of the regularized solutions of Problem 1.2, are shown in *Section* 3. Section 4 is devoted to a regularization method for implementing Newton's law (4) numerically and stably. A simple numerical scheme is given in Section 5. Several numerical examples are presented in Section 6 to demonstrate the feasibility and efficiency of the proposed method. Finally, concluding remarks are given in Section 7.

2. Parameter-dependent CCBM for Cauchy problem

We first introduce some notation. For a set G (e.g. Ω, Γ, Γ_a or Γ_u), we denote by $W^{m,s}(G)$ the standard Sobolev spaces with the norms $\|\cdot\|_{m,s,G}$ [45]. In particular, $L^s(G) := W^{0,s}(G)$, and $H^m(G) := W^{m,2}(G)$ with the corresponding inner products $(\cdot,\cdot)_{m,G}$ and norms $\|\cdot\|_{m,G}$. Let $\mathbf{H}^m(G)$ be the complex version of $H^m(G)$ with the inner product $((\cdot,\cdot))_{m,G}$ and norm $|||\cdot|||_{m,G}$ defined as follows: $\forall\, u, v \in \mathbf{H}^m(G)$, $((u,v))_{m,G} = (u,\bar{v})_{m,G}$, $|||v|||_{m,G}^2 = ((v,v))_{m,G}$, where $\bar{v}$ is the complex conjugate of v. Moreover, denote $V = H^1(\Omega)$, $Q = L^2(\Omega)$, $Q_\Gamma = L^2(\Gamma)$, $Q_{\Gamma_a} = L^2(\Gamma_a)$, $Q_{\Gamma_u} = L^2(\Gamma_u)$, $\mathbf{V} = \mathbf{H}^1(\Omega)$. In addition, suppose the exact Cauchy data $\Phi \in Q_{\Gamma_a}$ and $T \in H^{1/2}(\Gamma_a)$. In the following, c denotes a constant which may have different values at different places.

With a constant parameter $\alpha > 0$, we consider a complex BVP

$$\begin{aligned} -\nabla\cdot(\sigma\nabla u) &= f && \text{in } \Omega,\\ \sigma\partial_\nu u + i\alpha\, u &= \Phi + i\alpha\, T && \text{on } \Gamma_a,\\ \sigma\partial_\nu u + i\alpha\, u &= \phi + i\alpha\, t && \text{on } \Gamma_u, \end{aligned} \tag{5}$$

where $i = \sqrt{-1}$ is the imaginary unit. Obviously, if (u, ϕ, t) satisfy (3), then (5) holds. Conversely, let (u, ϕ, t) satisfy (5) and write $u = u_1 + i\,u_2$, u_1 and u_2 being the real and imaginary parts of u. Then the real-valued functions u_1, u_2 satisfy

$$\begin{aligned} -\nabla\cdot(\sigma\nabla u_1) &= f && \text{in } \Omega,\\ \sigma\partial_\nu u_1 - \alpha\, u_2 &= \Phi && \text{on } \Gamma_a,\\ \sigma\partial_\nu u_1 - \alpha\, u_2 &= \phi && \text{on } \Gamma_u, \end{aligned} \tag{6}$$

and

$$\begin{aligned} -\nabla\cdot(\sigma\nabla u_2) &= 0 && \text{in } \Omega, \\ \sigma\,\partial_\nu u_2 + \alpha\, u_1 &= \alpha\, T && \text{on } \Gamma_a, \\ \sigma\,\partial_\nu u_2 + \alpha\, u_1 &= \alpha\, t && \text{on } \Gamma_u, \end{aligned} \tag{7}$$

respectively. If $u_2 = 0$ in Ω, then $u_2 = 0, \partial_\nu u_2 = 0$ on Γ. As a result, from the BVPs (6)–(7) and recalling $\alpha \neq 0$, there holds that (u_1, ϕ, t) satisfy (3).

To sum up, we get an equivalent form of Problem 1.2:

Problem 2.1: *Given f in Ω, Φ, T on Γ_a, find ϕ, t on Γ_u such that*

$$u_2 = 0 \text{ in } \Omega,$$

where u_2 is the imaginary part of the solution $u = u_1 + i\,u_2$ of the BVP (5).

Suppose that instead of knowing the exact Cauchy data (Φ, T), we only have polluted ones: $\Phi^\delta(x) = \Phi(x) + n_1(x), T^\delta(x) = T(x) + n_2(x),\ x \in \Gamma_a$ with $n_j(x), j = 1, 2$ being random noise of some distributions. Then Problem 2.1 is modified to

Problem 2.2: *Given f in Ω, Φ^δ, T^δ on Γ_a, find ϕ, t on Γ_u such that*

$$u_2^\delta = 0 \text{ in } \Omega,$$

where u_2^δ is the imaginary part of the solution of the BVP (5), *with Φ, T being replaced by Φ^δ, T^δ, that is, u^δ solves*

$$\begin{aligned} -\nabla\cdot(\sigma\nabla u^\delta) &= f && \text{in } \Omega, \\ \sigma\,\partial_\nu u^\delta + i\alpha\, u^\delta &= \Phi^\delta + i\alpha\, T^\delta && \text{on } \Gamma_a, \\ \sigma\,\partial_\nu u^\delta + i\alpha\, u^\delta &= \phi + i\alpha\, t && \text{on } \Gamma_u, \end{aligned} \tag{8}$$

Remark 2.1: Note that $T^\delta \in H^{1/2}(\Gamma_a)$ is required for the equivalence of Problem 1.2 with polluted data and Problem 2.2. However, for Problem 2.2, $T^\delta \in H^{-1/2}(\Gamma_a)$ is enough to have the solvability and classical H^1 regularity. Even if the exact Cauchy data (Φ, T) are smooth, since the noises n_1 and n_2 are typically non-smooth, $T^\delta \in H^{1/2}(\Gamma_a)$ is not a realistic assumption for applications. In the case where this regularity assumption is not satisfied, e.g. $T^\delta \in Q_{\Gamma_a}$ or $H^{-1/2}(\Gamma_a)$, the reformulation above provides a way of an approximate resolution of Problem 1.2 which may be unsolvable.

The weak form of the BVP (8) is

$$\text{find } u^\delta \in \mathbf{V}, \quad a(u^\delta, v) = F^\delta(\phi, t; v) \quad \forall\, v \in \mathbf{V}. \tag{9}$$

Here

$$\begin{aligned} a(u, v) &= \int_\Omega \sigma\,\nabla u\cdot\nabla\bar v\,\mathrm{d}x + i\alpha\int_\Gamma u\,\bar v\,\mathrm{d}s \quad \forall\, u, v \in \mathbf{V}, \\ F^\delta(\phi, t; v) &= \int_\Omega f\,\bar v\,\mathrm{d}x + \int_{\Gamma_a}(\Phi^\delta + i\alpha\, T^\delta)\,\bar v\,\mathrm{d}s + \int_{\Gamma_u}(\phi + i\alpha\, t)\,\bar v\,\mathrm{d}s \quad \forall\, v \in \mathbf{V}. \end{aligned}$$

About the variational problem (9), the following well-posedness holds.

Proposition 2.3: *Given $f \in H^{-1}(\Omega)$, $(\Phi^\delta, T^\delta) \in H^{-1/2}(\Gamma_a) \times H^{-1/2}(\Gamma_a)$, $(\phi, t) \in H^{-1/2}(\Gamma_u) \times H^{-1/2}(\Gamma_u)$, the problem* (9) *admits a unique solution $u \in \mathbf{V}$ which depends continuously on all data. Moreover,*

$$|||u^\delta|||_{1,\Omega} \le c\,(\|f\|_{-1,\Omega} + \|\Phi^\delta\|_{-1/2,\Gamma_a} + \alpha\|T^\delta\|_{-1/2,\Gamma_a} + \|\phi\|_{-1/2,\Gamma_u} + \alpha\|t\|_{-1/2,\Gamma_u}). \tag{10}$$

Proposition 2.3 can be proved in a similar way as that of [43, Proposition 2.2], and is omitted here.

Although $(\phi, t) \in H^{-1/2}(\Gamma_u) \times H^{-1/2}(\Gamma_u)$ suffices to guarantee the solvability of the forward complex BVP, in this paper, approximations to (ϕ, t) are searched in a more natural space $Q_{\Gamma_u} \times Q_{\Gamma_u}$. To the end, For any $(\psi, \tau) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$, denote by $u^\delta(\psi, \tau) = u_1^\delta(\psi, \tau) + i\,u_2^\delta(\psi, \tau) \in \mathbf{V}$ the solution of (9) with (ϕ, t) replaced by (ψ, τ). Define an objective functional

$$J_\varepsilon^\delta(\psi, \tau) = \frac{1}{2}\|u_2^\delta(\psi, \tau)\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\|\psi\|_{0,\Gamma_u}^2 + \frac{\varepsilon}{2}\|\tau\|_{0,\Gamma_u}^2,$$

and introduce the following Tikhonov regularization framework for Problem 2.2.

Problem 2.4: *Find $(\phi_\varepsilon^\delta, t_\varepsilon^\delta) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$ such that*

$$J_\varepsilon^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) = \inf_{(\psi,\tau)\in Q_{\Gamma_u}\times Q_{\Gamma_u}} J_\varepsilon^\delta(\psi, \tau).$$

We can verify that for any $(\phi, t), (\psi, \tau) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$,

$$\begin{aligned}
(J_\varepsilon^\delta)'(\phi, t)\,(\psi, \tau) &= (u_2^\delta(\phi, t), u_2^\delta(\psi, \tau) - u_2^\delta(0, 0))_{0,\Omega} + \varepsilon\,(\phi, \psi)_{0,\Gamma_u} + \varepsilon\,(t, \tau)_{0,\Gamma_u},\\
(J_\varepsilon^\delta)''(\phi, t)\,(\psi, \tau)^2 &= \|u_2^\delta(\psi, \tau) - u_2^\delta(0, 0)\|_{0,\Omega}^2 + \varepsilon\|\psi\|_{0,\Gamma_u}^2 + \varepsilon\|\tau\|_{0,\Gamma_u}^2.
\end{aligned}$$

Therefore, J_ε^δ is strictly convex for any $\varepsilon > 0$, and we have the following well-posedness result.

Proposition 2.5: *For any $\varepsilon > 0$, Problem 2.4 has a unique solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$ which depends continuously on all data. Moreover, $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ is characterized by*

$$\phi_\varepsilon^\delta = -\frac{1}{\varepsilon} w_{\varepsilon,2}^\delta|_{\Gamma_u}, \quad t_\varepsilon^\delta = -\frac{\alpha}{\varepsilon} w_{\varepsilon,1}^\delta|_{\Gamma_u}, \tag{11}$$

where $w_{\varepsilon,1}^\delta$ and $w_{\varepsilon,2}^\delta$ are the real and imaginary parts of the weak solution $w_\varepsilon^\delta \in \mathbf{V}$ of the adjoint BVP:

$$\begin{aligned}
-\nabla \cdot (\sigma\,\nabla w_\varepsilon^\delta) &= u_{\varepsilon,2}^\delta \quad \text{in } \Omega,\\
\sigma\,\partial_\nu w_\varepsilon^\delta + i\,\alpha\,w_\varepsilon^\delta &= 0 \quad \text{on } \Gamma,
\end{aligned} \tag{12}$$

and $u_{\varepsilon,2}^\delta$ is the imaginary part of the solution of Problem (9), *with (ϕ, t) being replaced by $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$.*

Proof: The well-posedness of Problem 2.4 follows from a standard result on convex minimization problems [46,47]. Moreover, the solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ is characterized by

$$(J_\varepsilon^\delta)'(\phi_\varepsilon^\delta, t_\varepsilon^\delta)\,(\psi, \tau) = 0 \quad \forall\,(\psi, \tau) \in Q_{\Gamma_u} \times Q_{\Gamma_u}. \tag{13}$$

With arguments similar to those in the proof of [42, Proposition 3.1], we have

$$(u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta), u_2^\delta(\psi, \tau) - u_2^\delta(0, 0))_{0,\Omega} = \alpha(w_1^\delta, \tau)_{0,\Gamma_u} + (w_2^\delta, \psi)_{0,\Gamma_u}.$$

Therefore,

$$(J_\varepsilon^\delta)'(\phi_\varepsilon^\delta, t_\varepsilon^\delta)\,(\psi, \tau) = (\alpha\,w_1^\delta + \varepsilon\,t_\varepsilon^\delta, \tau)_{0,\Gamma_u} + (w_2^\delta + \varepsilon\,\phi_\varepsilon^\delta, \psi)_{0,\Gamma_u}. \tag{14}$$

Substitute (14) into (13) and take $\psi = w_2^\delta|_{\Gamma_u} + \varepsilon\,\phi_\varepsilon^\delta$, $\tau = \alpha\,w_1^\delta|_{\Gamma_u} + \varepsilon\,t_\varepsilon^\delta$ to get (11). ■

3. Convergence and uniform boundedness

We first present a result on the limiting behavior of $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ as $\delta, \varepsilon \to 0$. For this purpose, assume the exact Cauchy data (ϕ, T) are compatible. Then Problem 1.2 admits a solution $(\phi^*, t^*) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$ ([48]) and the solution is unique ([49]). For our future theoretical analysis, we assume additionally that ϕ^* belongs to Q_{Γ_u}, and $(\Phi^\delta, T^\delta) \in Q_{\Gamma_a} \times H^{1/2}(\Gamma_a)$ satisfying

$$\|\Phi^\delta - \Phi\|_{0,\Gamma_a} \le \delta\, \|\Phi\|_{0,\Gamma_a}, \quad \|T^\delta - T\|_{1/2,\Gamma_a} \le \delta \|T\|_{1/2,\Gamma_a}.$$

with a known noise level δ.

Then the following result holds.

Proposition 3.1: *Fix $\alpha > 0$. Let $\varepsilon = \varepsilon(\delta)$ be chosen satisfying $\varepsilon \to 0$ and $\delta^2/\varepsilon \to 0$, as $\delta \to 0$. Denote by $(\phi_\varepsilon^\delta, t_\varepsilon^\delta) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$ the solution of Problem 2.4. Then the solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ converges to (ϕ^*, t^*) in $Q_{\Gamma_u} \times Q_{\Gamma_u}$ as $\delta \to 0$.*

The proof is similar to that of [43, Proposition 3.4] with slight modifications, and is hence omitted. For future use, we record a stability result about the forward problem (9).

Lemma 3.2: *For any $(\psi, \tau) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$, denote by $u^\delta(\psi,\tau) = u_1^\delta(\psi,\tau) + i\, u_2^\delta(\psi,\tau), u(\psi,\tau) = u_1(\psi,\tau) + i\, u_2(\psi,\tau) \in \mathbf{V}$ the unique solutions of the problem (9) for $\delta > 0$ and $\delta = 0$ respectively. Then there holds*

$$|||u^\delta(\psi,\tau) - u(\psi,\tau)|||_{1,\Omega} \le c\, \delta. \tag{15}$$

The proof of Lemma 3.2 is standard and is thus omitted.

We next give an estimate of the regularized solution with respect to the noise level δ. For this purpose, we make the following assumption.

(A1) There is a pair $(\alpha, z^*) \in \mathbb{R}^+ \times Q$ such that

$$\phi^* = \sigma\, \partial_\nu \tilde{w}_1^*|_{\Gamma_u}, \quad t^* = \alpha^2\, \tilde{w}_1^*|_{\Gamma_u}, \tag{16}$$

where $\mathbb{R}^+ := \{s \in \mathbb{R} \mid s > 0\}$ and $\tilde{w}_1^*$ is the real part of the weak solution $\tilde{w}^* = \tilde{w}_1^* + i\,\tilde{w}_2^* \in \mathbf{V}$ of the adjoint BVP:

$$\begin{aligned} -\nabla \cdot (\sigma\, \nabla \tilde{w}^*) &= z^* \quad \text{in } \Omega, \\ \sigma\, \partial_\nu \tilde{w}^* + i\,\alpha\, \tilde{w}^* &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{17}$$

Note that Assumption (**A1**) can be viewed as a kind of source condition about the exact solution (ϕ^*, t^*).

Theorem 3.3: *Let Assumption (A1) hold. Then the solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ of Problem 2.4 satisfies the following estimate*

$$\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u} + \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} \le c\,(\alpha\,\sqrt{\varepsilon} + \frac{\delta}{\sqrt{\varepsilon}}). \tag{18}$$

If we choose $\varepsilon = c\,\delta$, then

$$\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u} + \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} \le c\,(\alpha + 1)\sqrt{\delta}. \tag{19}$$

Proof: For any $(\psi, \tau) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$, denote $\tilde{u}(\psi, \tau) = u^\delta(\psi, \tau) - u^\delta(0,0) \in \mathbf{V}$. Then $\tilde{u}$ is bilinear and there holds

$$(\tilde{u}_2, z^*)_{0,\Omega} = (\psi, \tilde{w}_2)_{0,\Gamma_u} + (\alpha\, \tau, \tilde{w}_1)_{0,\Gamma_u}$$

or

$$(\tilde{u}_2(\psi, 0), z^*)_{0,\Omega} = (\psi, \tilde{w}_2)_{0,\Gamma_u}, \quad (\tilde{u}_2(0, \tau), z^*)_{0,\Omega} = (\alpha\, \tau, \tilde{w}_1)_{0,\Gamma_u}. \tag{20}$$

From the definitions of $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ and (ϕ^*, t^*), we have

$$\begin{aligned} J_\varepsilon^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) &= \frac{1}{2}\|u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta)\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\|\phi_\varepsilon^\delta\|_{0,\Gamma_u}^2 + \frac{\varepsilon}{2}\|t_\varepsilon^\delta\|_{0,\Gamma_u}^2 \\ &\le J_\varepsilon^\delta(\phi^*, t^*) = \frac{1}{2}\|u_2^\delta(\phi^*, t^*)\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\|\phi^*\|_{0,\Gamma_u}^2 + \frac{\varepsilon}{2}\|t^*\|_{0,\Gamma_u}^2 \end{aligned}$$

which implies

$$\begin{aligned} &\|u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta)\|_{0,\Omega}^2 + \varepsilon\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u}^2 + \varepsilon\|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u}^2 \\ &\quad \le \|u_2^\delta(\phi^*, t^*)\|_{0,\Omega}^2 - 2\,\varepsilon(\phi^*, \phi_\varepsilon^\delta - \phi^*)_{0,\Gamma_u} - 2\,\varepsilon(t^*, t_\varepsilon^\delta - t^*). \end{aligned} \tag{21}$$

Note that $u_2(\phi^*, t^*) = 0$ in Ω. Then from (15),

$$\|u_2^\delta(\phi^*, t^*)\|_{0,\Omega} = \|u_2^\delta(\phi^*, t^*) - u_2(\phi^*, t^*)\|_{0,\Omega} \le c\,\delta. \tag{22}$$

Moreover, from Assumption (A1) and by using *the equalities* of (20), we have

$$(\phi^*, \phi_\varepsilon^\delta - \phi^*)_{0,\Gamma_u} = \alpha\,(z^*, \tilde{u}_2(\phi_\varepsilon^\delta - \phi^*, 0))_{0,\Omega}, \tag{23}$$

$$(t^*, t_\varepsilon^\delta - t^*)_{0,\Gamma_u} = \alpha\,(z^*, \tilde{u}_2(0, t_\varepsilon^\delta - t^*))_{0,\Omega}. \tag{24}$$

Combine (21)–(24) to get

$$\begin{aligned} &\|u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta)\|_{0,\Omega}^2 + \varepsilon\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u}^2 + \varepsilon\|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u}^2 \\ &\quad \le c\,\delta^2 - 2\,\varepsilon\,\alpha\,(z^*, u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) - u_2^\delta(\phi^*, t^*))_{0,\Omega} \end{aligned}$$

or

$$\begin{aligned} &\|u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) + \varepsilon\,\alpha\, z^*\|_{0,\Omega}^2 + \varepsilon\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u}^2 + \varepsilon\|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u}^2 \\ &\quad \le c\,\delta^2 + 2\,\varepsilon\,\alpha\,(z^*, u_2^\delta(\phi^*, t^*))_{0,\Omega} + \varepsilon^2\,\alpha^2\,\|z^*\|_{0,\Omega}^2, \end{aligned} \tag{25}$$

where we use the fact that

$$\tilde{u}_2(\phi_\varepsilon^\delta - \phi^*, t_\varepsilon^\delta - t^*) = \tilde{u}_2(\phi_\varepsilon^\delta, t_\varepsilon^\delta) - \tilde{u}_2(\phi^*, t^*) = u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) - u_2^\delta(\phi^*, t^*).$$

Using Schwarz inequality and (15) again,

$$(z^*, u_2^\delta(\phi^*, t^*))_{0,\Omega} \le c\,\delta\,\|z^*\|_{0,\Omega}.$$

Therefore, (25) implies

$$\|u_2^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) + \varepsilon\,\alpha\, z^*\|_{0,\Omega}^2 + \varepsilon\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u}^2 + \varepsilon\|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u}^2 \le c\,\delta^2 + \varepsilon^2\,\alpha^2\,\|z^*\|_{0,\Omega}^2$$

which leads to (18).

The error estimate (19) follows directly from (18) when we set $\varepsilon = c\,\delta$. ■

Remark 3.1: For fixed $\alpha > 0$, e.g. $\alpha = 1$, we can also derive similar error estimates under the source condition, Assumption (*A1*). However, allowing a variable α could relax the restriction of the source condition.

Recall that the CCBM possesses many merits such as allowing weaker regularity of Dirichlet data when solving the forward problem, and transferring*the data fitted* from the boundary to the interior. With the introduction of the parameter α, we can further expect an improvement of the error estimate above and convenience of choosing the regularization parameter ε.

Lemma 3.4: *For any* $(\psi, \tau) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$, *denote by* $u^\delta(\psi, \tau) = u_1^\delta(\psi, \tau) + i\, u_2^\delta(\psi, \tau), u(\psi, \tau) = u_1(\psi, \tau) + i\, u_2(\psi, \tau) \in \mathbf{V}$ *the unique solutions of the problem* (9) *for* $\delta > 0$ *and* $\delta = 0$ *respectively. Then,*

$$\|u_2^\delta(\psi, \tau) - u_2(\psi, \tau)\|_{1,\Omega} \le c\, \alpha\, \delta. \tag{26}$$

Proof: Recall that in the weak sense, $u_2(\psi, \tau)$ satisfies (7), with t replaced by τ and $u_2^\delta(\psi, \tau) \in V$ satisfies

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u_2^\delta) &= 0 && \text{in } \Omega, \\ \sigma\, \partial_\nu u_2^\delta + \alpha\, u_1^\delta &= \alpha\, T^\delta && \text{on } \Gamma_a, \\ \sigma\, \partial_\nu u_2^\delta + \alpha\, u_1^\delta &= \alpha\, \tau && \text{on } \Gamma_u. \end{aligned} \tag{27}$$

Denote $\delta u_j = u_j^\delta(\psi, \tau) - u_j(\psi, \tau), j = 1,2$. Subtract (7) from (27) to give

$$\begin{aligned} -\nabla \cdot (\sigma \nabla \delta u_2) &= 0 && \text{in } \Omega, \\ \sigma\, \partial_\nu \delta u_2 + \alpha\, \delta u_1 &= \alpha\, (T^\delta - T) && \text{on } \Gamma_a, \\ \sigma\, \partial_\nu \delta u_2 + \alpha\, \delta u_1 &= 0 && \text{on } \Gamma_u \end{aligned} \tag{28}$$

whose weak form is

$$(\sigma \nabla \delta u_2, \nabla v)_{0,\Omega} = -\alpha(\delta u_1, v)_{0,\Gamma} + \alpha(T^\delta - T, v)_{0,\Gamma_a} \quad \forall\, v \in V.$$

Taking $v = \delta\, u_2$, we obtain

$$|\delta u_2|_{1,\Omega}^2 \le c\alpha\, (\|\delta u_1\|_{1/2,\Gamma} + \delta)\|\delta u_2\|_{-1/2,\Gamma} \le c\, \alpha\, \delta \|\delta u_2\|_{-1/2,\Gamma}. \tag{29}$$

where we use, due to the estimate (15),

$$\|\delta u_1\|_{1/2,\Gamma} \le \|\delta u_1\|_{1,\Omega} \le c\, \delta.$$

Next we estimate the $\|\delta u_2\|_{-1/2,\Gamma}$. To the end, for any $\lambda \in H^{1/2}(\Gamma)$, define $p_\lambda \in V$ the weak solution of the BVP:

$$\begin{aligned} -\nabla \cdot (\sigma\, \nabla p_\lambda) &= 0 && \text{in } \Omega, \\ \sigma\, \partial_\nu p_\lambda &= \lambda && \text{on } \Gamma, \end{aligned}$$

satisfying $\int_\omega p_\lambda \mathrm{d}x = 0$. Multiply the first equation of (28) with p_λ, integrate over Ω, and take integration by part to produce

$$\begin{aligned} 0 &= \int_\Gamma (\delta u_2 \sigma\, \partial_\nu p_\lambda - p_\lambda \sigma\, \partial_\nu \delta u_2) \mathrm{d}s \\ &= \int_\Gamma \lambda \delta u_2 \mathrm{d}s + \alpha \int_\Gamma p_\lambda \delta u_1 \mathrm{d}s - \alpha \int_{\Gamma_a} p_\lambda (T^\delta - T) \mathrm{d}s \end{aligned}$$

which gives

$$\begin{aligned}\int_\Gamma \lambda\delta u_2 \mathrm{d}s &= -\alpha\int_\Gamma p_\lambda \delta u_1 \mathrm{d}s + \alpha\int_{\Gamma_a} p_\lambda (T^\delta - T)\mathrm{d}s\\ &\le c\,\alpha\|p_\lambda\|_{0,\Gamma}\|\delta u_1\|_{0,\Gamma} + c\,\alpha\|p_\lambda\|_{0,\Gamma}\|T^\delta - T\|_{0,\Gamma}\\ &\le c\,\alpha\,\delta\|p_\lambda\|_{1,\Omega} \le c\,\alpha\,\delta\|\lambda\|_{-1/2,\Gamma} \le c\,\alpha\,\delta\|\lambda\|_{1/2,\Gamma},\end{aligned}$$

where we use again the estimate, due to the estimate (15),

$$\|\delta u_1\|_{0,\Gamma} \le \|\delta u_1\|_{1,\Omega} \le c\,|||\delta u|||_{1,\Omega} \le c\,\delta.$$

Hence,

$$\|\delta u_2\|_{-1/2,\Gamma} = \sup \frac{\int_\Gamma \lambda\delta u_2 \mathrm{d}s}{\|\lambda\|_{1/2,\Gamma}} \le c\,\alpha\,\delta. \tag{30}$$

Substitute (30) back into (29) to obtain

$$|\delta u_2|_{1,\Omega} \le c\,\alpha\,\delta. \tag{31}$$

We finally estimate the $\|\delta u_2\|_{0,\Omega}$. For the purpose, let $w = w_1 + i\,w_2 \in \mathbf{V}$ be the weak solution of the BVP:

$$\begin{aligned} -\nabla\cdot(\sigma\,\nabla w) &= \delta u_2 && \text{in } \Omega,\\ \sigma\,\partial_\nu w + i\,\alpha w &= 0 && \text{on } \Gamma.\end{aligned}$$

Then the real part w_1 satisfies

$$\begin{aligned} -\nabla\cdot(\sigma\,\nabla w_1) &= \delta u_2 && \text{in } \Omega,\\ \sigma\,\partial_\nu w_1 &= \alpha w_2 && \text{on } \Gamma.\end{aligned}$$

Multiply the first equation of the BVP above with δu_2, integrate over Ω, and take integration by part again to give

$$\begin{aligned}\|\delta u_2\|_{0,\Omega}^2 &= \int_\Omega (-\nabla\cdot(\sigma\nabla w_1)\delta u_2)\mathrm{d}x\\ &= \int_\Gamma [(\sigma\nabla\delta u_2)w_1 - (\sigma\nabla w_1)\delta u_2]\mathrm{d}s\\ &= \alpha\int_{\Gamma_a}(T^\delta - T)w_1\mathrm{d}s - \alpha\int_\Gamma \delta u_1 w_1 \mathrm{d}s - \alpha\int_\Gamma \delta u_2 w_2 \mathrm{d}s\\ &= c\,\alpha\,\delta(\|w_1\|_{-1/2,\Gamma} + \|w_2\|_{-1/2,\Gamma}) \le c\,\alpha\,\delta|||w|||_{1,\Omega} \le c\,\alpha\,\delta\|\delta u_2\|_{0,\Omega}\end{aligned}$$

which leads to

$$\|\delta u_2\|_{0,\Omega} \le c\,\alpha\,\delta. \tag{32}$$

By combining (31) and (32), we arrive at (26). The proof is completed. ■

Now we are in a position to give an improvement of the error estimate.

Theorem 3.5: *Let Assumptions (A1) hold. Then for the solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ of Problem 2.4, the following estimate holds:*

$$\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u} + \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} \le c\,\alpha\,(\sqrt{\varepsilon} + \frac{\delta}{\sqrt{\varepsilon}}). \tag{33}$$

If we choose $\varepsilon = c\,\delta$*, then*

$$\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u} + \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} \le c\,\alpha\sqrt{\delta}. \tag{34}$$

If we choose $\varepsilon = c\,\delta^2$*, then*

$$\|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u} + \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} \le c\,\alpha. \tag{35}$$

The proof is similar to that of Theorem 3.3, with (22) replaced by

$$\|u_2^\delta(\phi^*, t^*)\|_{0,\Omega} = \|u_2^\delta(\phi^*, t^*) - u_2(\phi^*, t^*)\|_{0,\Omega} \le c\,\alpha\,\delta.$$

Remark 3.2: In the case α is small, the estimates (33) and (34) are better than the ones (18) and (19). The estimate (35) indicates that when α is quite small, a small regularization parameter ε can also lead to a reasonable solution.

Note that although an approximation of (ϕ^*, t^*) is sought in the space $Q_{\Gamma_u} \times Q_{\Gamma_u}$, from the optimality equalities (11), we actually have $(\phi_\varepsilon^\delta, t_\varepsilon^\delta) \in H^{1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$. We finally give a uniform boundedness result of the solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ with respect to small values of the regularization parameter, which is the motivation of introducing a parameter α in CCBM.

Theorem 3.6: *Let* $\alpha = \mathcal{O}(\sqrt{\varepsilon})$*. Then for any fixed* $\delta \ge 0$*, both* ϕ_ε^δ *and* t_ε^δ *are uniformly bounded in* $H^{1/2}(\Gamma_u)$ *and thus in* Q_{Γ_u} *as well, with respect to small* $\varepsilon > 0$*.*

Proof: Recall that $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ is the optimal solution of Problem 2.4, and (ϕ^*, t^*) is the unique solution of Problem 1.2 corresponding to noise-free data. Then, there holds

$$\begin{aligned} J_\varepsilon^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) &\le J_\varepsilon^\delta(\phi^*, t^*) \\ &= \frac{1}{2}\|u_2^\delta(\phi^*, t^*)\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\|\phi^*\|_{0,\Gamma_u}^2 + \frac{\varepsilon}{2}\|t^*\|_{0,\Gamma_u}^2 \\ &\le c\,\alpha^2\,\delta^2 + \frac{\varepsilon}{2}\|\phi^*\|_{0,\Gamma_u}^2 + \frac{\varepsilon}{2}\|t^*\|_{0,\Gamma_u}^2, \end{aligned}$$

which gives

$$\|\phi_\varepsilon^\delta\|_{0,\Gamma_u}^2 + \|t_\varepsilon^\delta\|_{0,\Gamma_u}^2 \le c\frac{\alpha^2\delta^2}{\varepsilon} + \|\phi^*\|_{0,\Gamma_u}^2 + \|t^*\|_{0,\Gamma_u}^2 \le c\,\delta^2 + \|\phi^*\|_{0,\Gamma_u}^2 + \|t^*\|_{0,\Gamma_u}^2, \tag{36}$$

where we use the estimate (26). Therefore both ϕ_ε^δ and t_ε^δ are uniformly bounded in Q_{Γ_u} with respect to ε.

Note that $u_\varepsilon^\delta = u^\delta(\phi_\varepsilon^\delta, t_\varepsilon^\delta) = u_{\varepsilon,1}^\delta + i\,u_{\varepsilon,2}^\delta \in \mathbf{V}$ is the solution of Problem (9), with (ϕ, t) being replaced by $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$, and $w_\varepsilon^\delta = w_{\varepsilon,1}^\delta + i\,w_{\varepsilon,2}^\delta \in \mathbf{V}$ is the weak solutions of the adjoint problem (12). Then due to (10), both u_ε^δ and w_ε^δ are bounded uniformly in $\mathbf{V}$ with respect to ε, that is,

$$|||u_\varepsilon^\delta|||_{1,\Omega} \le c, \quad |||w_\varepsilon^\delta|||_{1,\Omega} \le c\|u_{\varepsilon,2}^\delta\|_{0,\Omega} \le c.$$

Following the similar arguments as those for δu_2 in the proof of Lemma 3.4, we get

$$\|u^\delta_{\varepsilon,2}\|_{1,\Omega} \le c\alpha, \quad |||w^\delta_\varepsilon||| \le c\|u^\delta_{\varepsilon,2}\|_{0,\Omega} \le c\alpha.$$

Using the similar arguments as those for δu_2 in the proof of Lemma 3.4 again, we further obtain

$$\|w^\delta_{\varepsilon,2}\|_{1,\Omega} \le c\alpha^2.$$

Therefore, we have

$$\|\phi^\delta_\varepsilon\|_{\frac{1}{2},\Gamma_u} = \| - \frac{1}{\varepsilon} w^\delta_{\varepsilon,2}\|_{\frac{1}{2},\Gamma_u} \le c\frac{1}{\varepsilon}\|w^\delta_{\varepsilon,2}\|_{1,\Omega} \le c\frac{\alpha^2}{\varepsilon} = \mathcal{O}(1),$$
$$\|t^\delta_\varepsilon\|_{\frac{1}{2},\Gamma_u} = \| - \frac{\alpha}{\varepsilon} w^\delta_{\varepsilon,1}\|_{\frac{1}{2},\Gamma_u} \le c\|\frac{\alpha}{\varepsilon} w^\delta_{\varepsilon,1}\|_{1,\Omega} \le c\frac{\alpha}{\varepsilon}\|w^\delta_{\varepsilon,1}\|_{1,\Omega} \le c\frac{\alpha^2}{\varepsilon} = \mathcal{O}(1)$$

and the proof is completed. ■

Remark 3.3: Conventionally, instead of having the estimate (36), by using (15), and noticing $u_2(\phi^*, t^*) = 0$, we have

$$\begin{aligned} J^\delta_\varepsilon(\phi^\delta_\varepsilon, t^\delta_\varepsilon) &\le J^\delta_\varepsilon(\phi^*, t^*) \\ &= \frac{1}{2}\|u^\delta_2(\phi^*, t^*)\|^2_{0,\Omega} + \frac{\varepsilon}{2}\|\phi^*\|^2_{0,\Gamma_u} + \frac{\varepsilon}{2}\|t^*\|^2_{0,\Gamma_u} \\ &\le c\delta^2 + \frac{\varepsilon}{2}\|\phi^*\|^2_{0,\Gamma_u} + \frac{\varepsilon}{2}\|t^*\|^2_{0,\Gamma_u}, \end{aligned}$$

which gives

$$\|\phi^\delta_\varepsilon\|^2_{0,\Gamma_u} + \|t^\delta_\varepsilon\|^2_{0,\Gamma_u} \le c\frac{\delta^2}{\varepsilon} + \|\phi^*\|^2_{0,\Gamma_u} + \|t^*\|^2_{0,\Gamma_u}.$$

This also lead to the uniform boundedness of ϕ^δ_ε and t^δ_ε in Q_{Γ_u} under the assumption that $\delta^2/\varepsilon \to 0$, as $\varepsilon \to 0$, which indicates the regularization parameter ε should be chosen not too small. However, for the uniform boundedness in Theorem 3.6, no such assumption is made and thus a reasonable solution could be obtained for any small value of ε for a fixed δ. This is a strong property because the smaller the parameter ε is, the better the approximation to the original problem is. In addition, Theorem 3.6 also provides a guidance on how to choose α properly; see the numerical experiments reported in Section 6.

4. Recover the Robin coefficient from Cauchy data

This section is devoted to a stable computation of the Robin coefficient with the reconstructed Cauchy data $(\phi^\delta_\varepsilon, t^\delta_\varepsilon)$ in Section 2 through the Newton law (4).

Suppose that instead of knowing g, we only have the noisy Robin data $g^\delta \in Q_{\Gamma_u}$ satisfying

$$\|g^\delta - g\|_{0,\Gamma_u} \le \delta\,\|g\|_{0,\Gamma_u}.$$

Moreover, for later use, assume the exact Dirichlet data $t^* \in L^\infty(\Gamma_u)$ and define

$$\beta^\delta(x) = \begin{cases} \eta\delta, & 0 \le t^\delta_\varepsilon(x) \le \delta, \\ -\eta\delta, & -\delta \le t^\delta_\varepsilon(x) \le 0, \qquad x \in \Gamma_u, \\ 0, & |t^\delta_\varepsilon(x)| > \delta, \end{cases} \tag{37}$$

where η (≤ 1) is a small positive constant. It is easy to verify that $\beta^\delta \in L^\infty(\Gamma_u)$ and

$$\left|\frac{t_\varepsilon^\delta}{t_\varepsilon^\delta + \beta^\delta}\right| \leq 1. \tag{38}$$

Then instead of using (4) directly, an approximation to γ^* is produced by

$$\gamma_\varepsilon^\delta = \frac{g^\delta - \phi_\varepsilon^\delta}{t_\varepsilon^\delta + \beta^\delta}. \tag{39}$$

Compared with (4), which is equivalent to solving $t_\varepsilon^\delta \gamma = g^\delta - \phi_\varepsilon^\delta$ for γ, the formula (39), which is equivalent to solving $(t_\varepsilon^\delta + \beta^\delta)\gamma_\varepsilon^\delta = g^\delta - \phi_\varepsilon^\delta$ for $\gamma_\varepsilon^\delta$, is a least square output Tikhonov regularization with parameter β^δ.

About $\gamma_\varepsilon^\delta$, we have the following convergence result:

Theorem 4.1: *Fix $\alpha > 0$. Let $\varepsilon = \varepsilon(\delta)$ be chosen satisfying $\varepsilon \to 0$ and $\delta^2/\varepsilon \to 0$, as $\delta \to 0$. Denote by $\gamma_\varepsilon^\delta$ the approximate Robin coefficient computed through* (39), *where $(\phi_\varepsilon^\delta, t_\varepsilon^\delta) \in Q_{\Gamma_u} \times Q_{\Gamma_u}$ is the solution of Problem* 2.4. *Then the solution sequence $\{\gamma_\varepsilon^\delta\}_{\delta>0}$ converges to γ^* in the following sense as $\delta \to 0$,*

$$\lim_{n\to\infty} \|t_\varepsilon^\delta t^*(\gamma_\varepsilon^\delta - \gamma^*)\|_{0,1,\Gamma_u} = 0. \tag{40}$$

Proof: Recall that

$$\gamma^* = \frac{g - \phi^*}{t^*}.$$

Then from (37) to (39) and Proposition 3.1,

$$\begin{aligned}
\|t_\varepsilon^\delta t^*(\gamma_\varepsilon^\delta - \gamma^*)\|_{0,1,\Gamma_u} &= \int_{\Gamma_u} |t_\varepsilon^\delta t^*(\gamma_\varepsilon^\delta - \gamma^*)| \mathrm{d}s \\
&= \int_{\Gamma_u} |t_\varepsilon^\delta t^*| |\frac{g^\delta - \phi_\varepsilon^\delta}{t_\varepsilon^\delta + \beta^\delta} - \frac{g - \phi^*}{t^*}| \mathrm{d}s \\
&\leq |(g^\delta - g + \phi^* - \phi_\varepsilon^\delta, t^*)_{0,\Gamma_u}| \\
&\quad + |(t_\varepsilon^\delta - t^*, \phi^* - g)_{0,\Gamma_u}| + |(\phi^* - g, \beta^\delta)_{0,\Gamma_u}| \\
&\leq c\,(\delta + \|\phi_\varepsilon^\delta - \phi^*\|_{0,\Gamma_u} + \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} + \eta\delta|\Gamma^\delta|) \\
&\to 0
\end{aligned}$$

as $\delta \to 0$. The proof is completed. ■

Remark 4.1: Note that if $t^*(x_0) = 0$ or $t_\varepsilon^\delta(x_0) = 0$ at a point $x_0 \in \Gamma_u$, we have no convergence $\gamma_\varepsilon^\delta(x_0) \to \gamma^*(x_0)$. This is not surprising because in Robin problem (1)–(2), if $t^*(x_0) = 0$, we can not get any information about $\gamma^*(x_0)$. Numerical experiments of Section 6 also show that the accuracy of $\gamma_\varepsilon^\delta$ gets worse near the points where t^* vanishes.

Remark 4.2: If $|t^*|$ has positive lower bound, we can prove that (40) reduces to

$$\lim_{n\to\infty} \|\gamma_{\varepsilon_n}^{\delta_n} - \gamma^*\|_{0,1,\Gamma_u} = 0.$$

For instance, suppose there is a positive constant c_0 such that

$$|t^*| \geq c_0 \text{ a.e. on } \Gamma_u.$$

We modify the definition of β^δ to

$$\beta^\delta(x) = \begin{cases} \eta c_0, & 0 \leq t_\varepsilon^\delta(x) \leq \eta c_0, \\ -\eta c_0, & -\eta c_0 \leq t_\varepsilon^\delta(x) \leq 0, \\ 0, & |t_\varepsilon^\delta(x)| > c_0/2, \end{cases} \quad x \in \Gamma_u.$$

Let $\Gamma^\delta := \{x \in \Gamma_u | \, \beta^\delta(x) \neq 0\}$. Then

$$\begin{aligned} \|t_\varepsilon^\delta - t^*\|_{0,\Gamma_u} &\geq \|t_\varepsilon^\delta - t^*\|_{0,\Gamma^\delta} \geq \|t^*\|_{0,\Gamma^\delta} - \|t_\varepsilon^\delta\|_{0,\Gamma^\delta} \\ &\geq c_0|\Gamma^\delta|^{1/2} - \eta c_0 |\Gamma^\delta|^{1/2} = (1-\eta)\, c_0 |\Gamma^\delta|^{1/2} \geq 0, \end{aligned}$$

which leads to $|\Gamma^\delta| \to 0$ as $\delta \to 0$. Therefore, with similar arguments as those in the proof of Theorem 4.1, there holds

$$\|\gamma_{\varepsilon_n}^{\delta_n} - \gamma^*\|_{0,1,\Gamma_u} \leq c\,(\delta_n + \|\phi_{\varepsilon_n}^{\delta_n} - \phi^*\|_{0,\Gamma_u} + \|t_{\varepsilon_n}^{\delta_n} - t^*\|_{0,\Gamma_u} + \eta c_0 |\Gamma^\delta|) \to 0.$$

5. A computational scheme for the regularized solution

By Proposition 2.5, the optimal regularized Cauchy solution $(\phi_\varepsilon^\delta, t_\varepsilon^\delta)$ satisfies the system of (11), (12) and

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u_\varepsilon^\delta) &= f && \text{in } \Omega, \\ \sigma\, \partial_\nu u_\varepsilon^\delta + i\,\alpha\, u_\varepsilon^\delta &= \Phi^\delta + i\,\alpha\, T^\delta && \text{on } \Gamma_a, \\ \sigma\, \partial_\nu u_\varepsilon^\delta + i\,\alpha\, u_\varepsilon^\delta &= \phi_\varepsilon^\delta + i\,\alpha\, t_\varepsilon^\delta && \text{on } \Gamma_u. \end{aligned} \tag{41}$$

Recall that $u_\varepsilon^\delta = u_{\varepsilon,1}^\delta + i\, u_{\varepsilon,2}^\delta$ and $w_\varepsilon^\delta = w_{\varepsilon,1}^\delta + i\, w_{\varepsilon,2}^\delta$. Then we introduce the following solver for Problem 2.4:

Algorithm 5.1: Given problem domain Ω, functions $f, \Phi^\delta, T^\delta, g^\delta$ and set parameters $\varepsilon, \alpha, \eta$.

1. Solve

$$\begin{aligned} &(\sigma\, \nabla u_{\varepsilon,1}^\delta, \nabla v)_{0,\Omega} - \alpha\,(u_{\varepsilon,2}^\delta, v)_{0,\Gamma} + \frac{1}{\varepsilon}(w_{\varepsilon,2}^\delta, v)_{0,\Gamma_u} \\ &\quad = (f, v)_{0,\Omega} + (\Phi^\delta, v)_{0,\Gamma_a}\ \forall\, v \in V, \\ &(\sigma\, \nabla u_{\varepsilon,2}^\delta, \nabla v)_{0,\Omega} + \alpha\,(u_{\varepsilon,1}^\delta, v)_{0,\Gamma} + \frac{\alpha^2}{\varepsilon}\,(w_{\varepsilon,1}^\delta, v)_{0,\Gamma_u} = \alpha\,(T^\delta, v)_{0,\Gamma_a}\ \forall\, v \in V, \\ &\quad -(u_{\varepsilon,2}^\delta, v)_{0,\Omega} + (\sigma\, \nabla w_{\varepsilon,1}^\delta, \nabla v)_{0,\Omega} - \alpha\,(w_{\varepsilon,2}^\delta, v)_{0,\Gamma} = 0\ \forall\, v \in V, \\ &(\sigma\, \nabla w_{\varepsilon,2}^\delta, \nabla v)_{0,\Omega} + \alpha\,(w_{\varepsilon,1}^\delta, v)_{0,\Gamma} = 0\ \forall\, v \in V. \end{aligned} \tag{42}$$

2. Compute

$$\phi_\varepsilon^\delta = -\frac{1}{\varepsilon} w_{\varepsilon,2}^\delta|_{\Gamma_u}, \quad t_\varepsilon^\delta = -\frac{\alpha}{\varepsilon} w_{\varepsilon,1}^\delta|_{\Gamma_u}. \tag{43}$$

3. Compute β^δ by (37) and

$$\gamma_\varepsilon^\delta = \frac{g^\delta - \phi_\varepsilon^\delta}{t_\varepsilon^\delta + \beta^\delta}. \tag{44}$$

For actual reconstruction, (42)–(44) need to be solved numerically. Standard conforming linear finite element methods are applied to solve (42). Specifically, let $\{\mathcal{T}_h\}_h$ be a regular family of finite element partitions of $\bar{\Omega}$ and define the linear finite element spaces

$$V^h = \{v \in C(\overline{\Omega}) \mid v \text{ is linear in } K \,\forall\, K \in \mathcal{T}_h\},$$

where $h > 0$ is the meshsize. Then a finite element discretization of Algorithm 5.1 reads:

Algorithm 5.2: Given problem domain Ω, functions $f, \Phi^\delta, T^\delta, g^\delta$ and set parameters $\varepsilon, \alpha, \eta$.
1. Solve

$$\begin{aligned}
&(\sigma \nabla u^{\delta,h}_{\varepsilon,1}, \nabla v)_{0,\Omega} - \alpha\, (u^{\delta,h}_{\varepsilon,2}, v)_{0,\Gamma} + \frac{1}{\varepsilon}(w^{\delta,h}_{\varepsilon,2}, v)_{0,\Gamma_u}\\
&\quad = (f, v)_{0,\Omega} + (\Phi^\delta, v)_{0,\Gamma_a} \,\forall\, v \in V, \qquad (45)\\
&(\sigma \nabla u^{\delta,h}_{\varepsilon,2}, \nabla v)_{0,\Omega} + \alpha\, (u^{\delta,h}_{\varepsilon,1}, v)_{0,\Gamma} + \frac{\alpha^2}{\varepsilon}\,(w^{\delta,h}_{\varepsilon,1}, v)_{0,\Gamma_u} = \alpha\, (T^\delta, v)_{0,\Gamma_a} \,\forall\, v \in V,\\
&\quad - (u^{\delta,h}_{\varepsilon,2}, v)_{0,\Omega} + (\sigma \nabla w^{\delta,h}_{\varepsilon,1}, \nabla v)_{0,\Omega} - \alpha\, (w^{\delta,h}_{\varepsilon,2}, v)_{0,\Gamma} = 0 \,\forall\, v \in V,\\
&(\sigma \nabla w^{\delta,h}_{\varepsilon,2}, \nabla v)_{0,\Omega} + \alpha\, (w^{\delta,h}_{\varepsilon,1}, v)_{0,\Gamma} = 0 \,\forall\, v \in V. \qquad (46)
\end{aligned}$$

2. Compute

$$\phi^{\delta,h}_\varepsilon = -\frac{1}{\varepsilon} w^{\delta,h}_{\varepsilon,2}|_{\Gamma_u}, \quad t^{\delta,h}_\varepsilon = -\frac{\alpha}{\varepsilon} w^{\delta,h}_{\varepsilon,1}|_{\Gamma_u}. \qquad (47)$$

3. Compute $\beta^{\delta,h}$ by (37) with t^δ_ε being replaced by $t^{\delta,h}_\varepsilon$, and

$$\gamma^{\delta,h}_\varepsilon = \frac{g^\delta - \phi^{\delta,h}_\varepsilon}{t^{\delta,h}_\varepsilon + \beta^{\delta,h}}. \qquad (48)$$

6. Numerical results

In this section, we present some numerical results to illustrate the feasibility and effectiveness of the parameter-dependent CCBM-based Tikhonov regularization for solving the Cauchy problem and the inverse Robin coefficient problem. Denote by (ϕ^*, t^*, γ^*) the true Neumann and Dirichlet data as well as the true Robin coefficient on Γ_u, and by $(\phi^{\delta,h}_\varepsilon, t^{\delta,h}_\varepsilon, \gamma^{\delta,h}_\varepsilon)$ its approximation computed from (45)–(47). Note that (45) reduces to a linear system $A\,x = b$, which can be solved by the biconjugate gradient method. To better investigate the uniform boundedness of the solutions of the Cauchy problem, we define the L^2-norm relative errors for the solutions $\phi^{\delta,h}_\varepsilon$ and $t^{\delta,h}_\varepsilon$:

$$\mathrm{E}_\phi = \frac{\|\phi^{\delta,h}_\varepsilon - \phi^*\|_{0,\Gamma_u}}{\|\phi^*\|_{0,\Gamma_u}}, \quad \mathrm{E}_t = \frac{\|t^{\delta,h}_\varepsilon - t^*\|_{0,\Gamma_u}}{\|t^*\|_{0,\Gamma_u}}.$$

In the following examples, let $\Omega \subset \mathbb{R}^2$ be a ring with inner radius $r_1 = 0.6$ and external radius $r_2 = 1$. The exact Cauchy data (Φ, T) on the external circle Γ_a is computed from a true state u^* given in advance: $\Phi = \sigma\,\partial_\nu u^*|_{\Gamma_a}, T = u^*|_{\Gamma_a}$. The true Cauchy solution (ϕ^*, t^*) on the inner circle Γ_u is $(\sigma\,\partial_\nu u^*|_{\Gamma_u}, u^*|_{\Gamma_u})$. For a true γ^*, the Robin data g on Γ_u is computed through $g = \phi^* + \gamma^* t^*$. Then

Table 1. E_ϕ vs. ε and δ (Example 6.1).

ε	$\delta = 1\%$	$\delta = 5\%$	$\delta = 10\%$	$\delta = 20\%$
10^{-1}	9.9983e−1	9.9886e−1	9.9766e−1	9.9287e−1
10^{-2}	9.8479e−1	9.3087e−1	8.9213e−1	7.9003e−1
10^{-3}	7.0079e−1	3.4163e−1	2.2541e−1	1.5121e−1
10^{-4}	6.8557e−2	6.9651e−2	8.8492e−2	1.1695e−1
10^{-5}	3.2806e−2	6.6010e−2	8.8818e−2	1.1614e−1
10^{-6}	3.1947e−2	6.5865e−2	8.8926e−2	1.1607e−1
10^{-7}	3.1880e−2	6.5853e−2	8.8936e−2	1.1606e−1
10^{-8}	3.1874e−2	6.5852e−2	8.8938e−2	1.1606e−1
10^{-9}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1
10^{-10}	3.1873e−2	6.5852e−2	8.8938e−2	1.1606e−1
10^{-11}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1
10^{-12}	3.1873e−2	6.5851e−2	8.8939e−2	1.1606e−1
10^{-13}	3.1873e−2	6.5852e−2	8.8939e−2	1.1606e−1
10^{-14}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1
10^{-15}	3.1873e−2	6.5852e−2	8.8938e−2	1.1606e−1
10^{-16}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1
10^{-17}	3.1873e−2	6.5852e−2	8.8939e−2	1.1606e−1
10^{-18}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1
10^{-19}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1
10^{-20}	3.1873e−2	6.5851e−2	8.8938e−2	1.1606e−1

for a noise level, a uniformly distributed random noise is added to (Φ, T, g) to get $(\Phi^\delta, T^\delta, g^\delta)$:

$$\begin{aligned}
\Phi^\delta(x) &= [1 + \delta \cdot (2\,\mathrm{rand}(x) - 1)]\,\Phi(x), \quad x \in \Gamma_a,\\
T^\delta(x) &= [1 + \delta \cdot (2\,\mathrm{rand}(x) - 1)]\,T(x), \quad x \in \Gamma_a,\\
g^\delta(x) &= [1 + \delta \cdot (2\,\mathrm{rand}(x) - 1)]\,g(x), \quad x \in \Gamma_u,
\end{aligned}$$

where $\mathrm{rand}(x)$ returns a pseudo-random value drawn from a uniform distribution on $[0,1]$. All experiments are implemented on a finite element mesh with 384 nodes, 648 elements and mesh-size $h = 0.1289$. Moreover, as indicated by Theorem 3.6, in the following examples, we choose $\alpha = \mathcal{O}(\sqrt{\varepsilon}) = C_\alpha\sqrt{\varepsilon}$, where $C_\alpha > 0$ is a constant for one reconstruction. In addition, for simplicity of the statements, let $\sigma \equiv 1$ in Ω in all experiments.

Example 6.1: We first consider an analytical example where $|t^*| > c_0 > 0$ for some constant c_0 ([15,43]). Let $u^*(x_1,x_2) = e^{x_1}\cos(x_2)$. Then $f(x_1,x_2) = 0$, $T(x_1,x_2) = e^{x_1}\cos(x_2)$, $\Phi(x_1,x_2) = e^{x_1}(x_1\cos(x_2) - x_2\sin(x_2))$, $\phi^*(x_1,x_2) = \frac{5}{3}e^{x_1}(x_2\sin(x_2) - x_1\cos(x_2))$ and $t^*(x_1,x_2) = e^{x_1}\cos(x_2)$.

The system (45) is solved and then the formulas (46) are applied to compute approximate solutions $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$ of (ϕ^*, t^*) from the boundary data (Φ^δ, T^δ). The errors in $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$ are listed in Tables 1 and 2. We observe that the results are quite satisfactory. In particular, Tables 1 and 2 show that solutions are uniformly bounded with respect to small values of the regularization parameter ε which matches the declaration of Theorem 3.6. Therefore, in the case α is small, a reasonably good approximate solution can be reconstructed for a relative small value of ε. In this example, $C_\alpha = 420, 160, 110, 60$ for $\delta = 1\%, 5\%, 10\%$ and 20%, respectively. Numerical experiments indicate that the higher the noise level, the smaller the suggested value of C_α. However, we observe that although the value of C_α affects the solution accuracy, $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$ is less sensitive to C_α. A quite large range of values of C_α can produce satisfactory approximations to (ϕ^*, t^*). Moreover, the numerical results also confirm the limiting behavior of the regularized solutions with respect to the noise level δ: the smaller δ is, the better the approximate solution $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$ is, which demonstrates the stability of the proposed Tikhonov regularization reconstruction framework.

For the reconstruction of the Robin coefficient, with $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$, the formula (47) is used to obtain $\gamma_\varepsilon^{\delta,h}$. Three different real coefficients γ^* are tested: $\gamma^* \equiv 1$, $\gamma^* = \frac{5}{3}x_1$ and $\gamma^* = \mathrm{sgn}(x_2)e^{x_1}$. Since

Table 2. E_t vs. ε and δ (Example 6.1).

ε	$\delta = 1\%$	$\delta = 5\%$	$\delta = 10\%$	$\delta = 20\%$
10^{-1}	9.9926e−1	9.9498e−1	9.8967e−1	9.6914e−1
10^{-2}	9.3259e−1	7.0578e−1	5.8221e−1	4.5965e−1
10^{-3}	1.8416e−1	1.0135e−1	1.0274e−1	1.7641e−1
10^{-4}	1.6579e−2	3.0213e−2	5.2008e−2	1.4637e−1
10^{-5}	6.4708e−3	2.5280e−2	4.7862e−2	1.4340e−1
10^{-6}	5.9135e−3	2.4828e−2	4.7463e−2	1.4310e−1
10^{-7}	5.8617e−3	2.4784e−2	4.7423e−2	1.4307e−1
10^{-8}	5.8572e−3	2.4779e−2	4.7415e−2	1.4307e−1
10^{-9}	5.8576e−3	2.4779e−2	4.7418e−2	1.4307e−1
10^{-10}	5.8543e−3	2.4779e−2	4.7417e−2	1.4307e−1
10^{-11}	5.8544e−3	2.4779e−2	4.7419e−2	1.4307e−1
10^{-12}	5.8557e−3	2.4780e−2	4.7419e−2	1.4307e−1
10^{-13}	5.8565e−3	2.4779e−2	4.7418e−2	1.4307e−1
10^{-14}	5.8565e−3	2.4778e−2	4.7417e−2	1.4307e−1
10^{-15}	5.8564e−3	2.4779e−2	4.7419e−2	1.4307e−1
10^{-16}	5.8558e−3	2.4778e−2	4.7418e−2	1.4307e−1
10^{-17}	5.8559e−3	2.4779e−2	4.7418e−2	1.4307e−1
10^{-18}	5.8557e−3	2.4779e−2	4.7419e−2	1.4307e−1
10^{-19}	5.8564e−3	2.4779e−2	4.7417e−2	1.4307e−1
10^{-20}	5.8551e−3	2.4780e−2	4.7416e−2	1.4307e−1

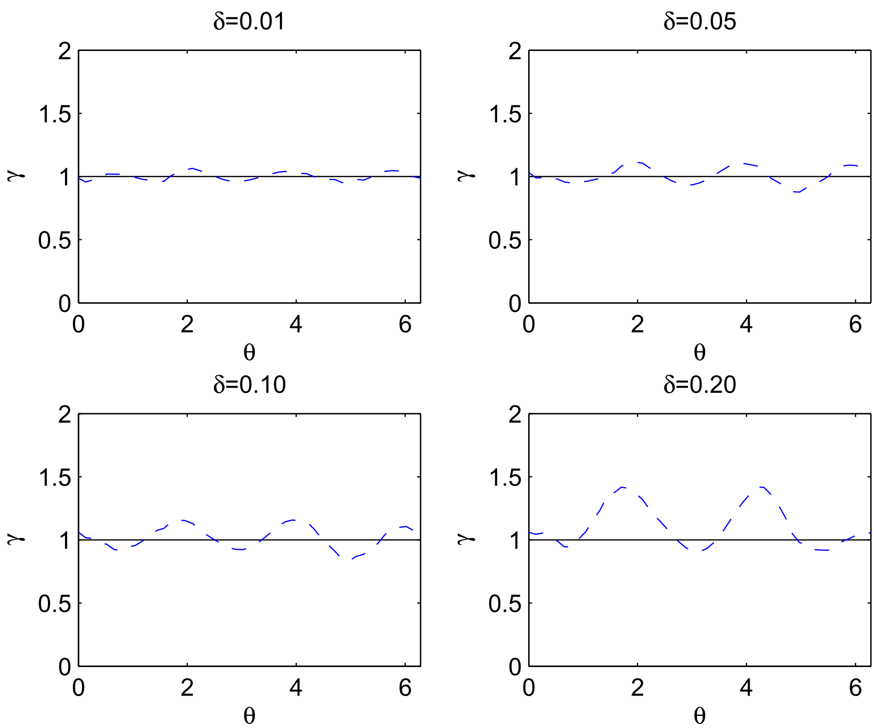

Figure 1. Reconstructed Robin coefficients for different δ when $\gamma^* \equiv 1$ (Example 6.1).

$\phi_\varepsilon^{\delta,h}$ and $t_\varepsilon^{\delta,h}$ are uniformly bounded when ε is small, we fix $\varepsilon = 10^{-20}$. The approximation $\gamma_\varepsilon^{\delta,h}$ to γ^* for different noise level δ is shown in Figures 1–3, where θ is the angular variable of the point (x_1, x_2) on Γ_u, ranging from 0 to 2π. The black and solid line represents γ^* while the blue and dashed one represents $\gamma_\varepsilon^{\delta,h}$. We conclude from them that when t^* stays away from 0, the reconstruction through

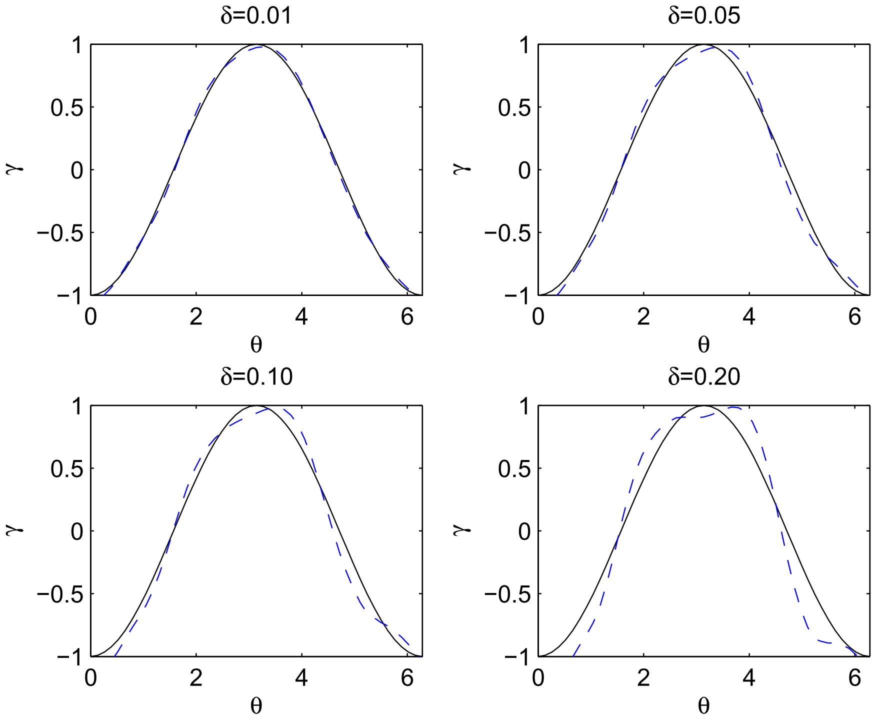

Figure 2. Reconstructed Robin coefficients for different δ when $\gamma^* = \frac{5}{3}x_1$ (Example 6.1).

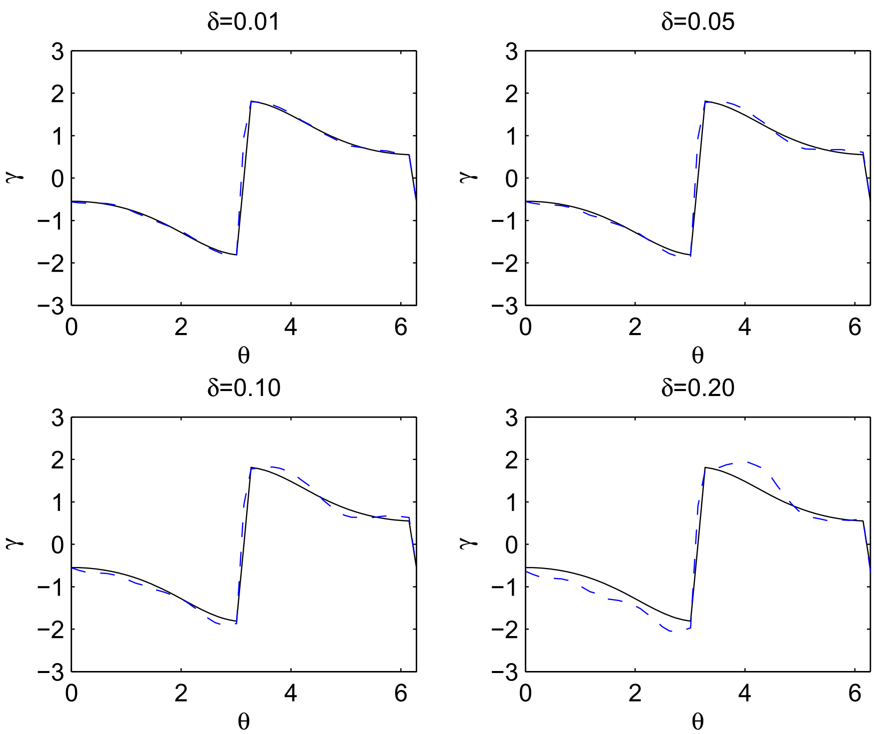

Figure 3. Reconstructed Robin coefficients for different δ,when $\gamma^* = \mathrm{sgn}(x_2)\mathrm{e}^{x_1}$ (Example 6.1).

Table 3. E_ϕ vs. ε and δ (Example 6.2).

ε	$\delta = 1\%$	$\delta = 5\%$	$\delta = 10\%$	$\delta = 20\%$
10^{-1}	9.9957e−1	9.9794e−1	9.9480e−1	9.8806e−1
10^{-2}	9.6411e−1	8.8497e−1	8.0436e−1	6.8553e−1
10^{-3}	4.8436e−1	1.9295e−1	1.0485e−1	8.6116e−2
10^{-4}	4.2611e−2	5.7040e−2	7.2771e−2	8.7114e−2
10^{-5}	3.2753e−2	5.3309e−2	7.1537e−2	8.6760e−2
10^{-6}	3.2377e−2	5.2920e−2	7.1397e−2	8.6714e−2
10^{-7}	3.2346e−2	5.2881e−2	7.1383e−2	8.6709e−2
10^{-8}	3.2343e−2	5.2877e−2	7.1382e−2	8.6709e−2
10^{-9}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-10}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-11}	3.2342e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-12}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-13}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-14}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-15}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-16}	3.2342e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-17}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-18}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-19}	3.2343e−2	5.2876e−2	7.1381e−2	8.6709e−2
10^{-20}	3.2342e−2	5.2876e−2	7.1381e−2	8.6709e−2

Table 4. E_t vs. ε and δ (Example 6.2).

ε	$\delta = 1\%$	$\delta = 5\%$	$\delta = 10\%$	$\delta = 20\%$
10^{-1}	9.9793e−1	9.9002e−1	9.7529e−1	9.4547e−1
10^{-2}	8.3114e−1	5.2623e−1	3.5515e−1	2.9346e−1
10^{-3}	5.4581e−2	2.6169e−2	5.9081e−2	1.3040e−1
10^{-4}	1.4638e−2	3.4408e−2	6.7455e−2	1.3381e−1
10^{-5}	1.1988e−2	3.3874e−2	6.7303e−2	1.3391e−1
10^{-6}	1.1676e−2	3.3779e−2	6.7274e−2	1.3392e−1
10^{-7}	1.1644e−2	3.3769e−2	6.7270e−2	1.3392e−1
10^{-8}	1.1640e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-9}	1.1640e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-10}	1.1640e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-11}	1.1641e−2	3.3768e−2	6.7270e−2	1.3392e−1
10^{-12}	1.1639e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-13}	1.1641e−2	3.3768e−2	6.7270e−2	1.3392e−1
10^{-14}	1.1640e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-15}	1.1640e−2	3.3768e−2	6.7270e−2	1.3392e−1
10^{-16}	1.1641e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-17}	1.1640e−2	3.3768e−2	6.7270e−2	1.3392e−1
10^{-18}	1.1641e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-19}	1.1640e−2	3.3768e−2	6.7269e−2	1.3392e−1
10^{-20}	1.1639e−2	3.3768e−2	6.7270e−2	1.3392e−1

Newton law (47) for the Robin coefficient is stable and satisfactory. In the three experiments, $\beta^\delta \equiv 0$ since $|t^*| > 0.5$.

Example 6.2: In the second example, we consider a problem where t^* vanishes at some boundary points. Specifically, let $u^*(x_1, x_2) = \sin(x_1 + x_2)$. Then $f(x_1, x_2) = -2\sin(x_1 + x_2)$, $T(x_1, x_2) = \sin(x_1 + x_2)$, $\Phi(x_1, x_2) = (x_1 + x_2)\cos(x_1 + x_2)$, $\Phi^*(x_1, x_2) = -\frac{5}{3}(x_1 + x_2)\cos(x + x_2)$ and $t^*(x_1, x_2) = \sin(x_1 + x_2)$. Note that $t^*(\pm(\sqrt{2}/2), \mp(\sqrt{2}/2)) = 0$

Again, (45) and (46) are used to compute approximate solutions $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$. The errors in $\phi_\varepsilon^{\delta,h}$ and $t_\varepsilon^{\delta,h}$ are given in Tables 3 and 4, which show again the convergence, the stability and the uniformness of the solutions of Problem 2.4. In this example, choose $C_\alpha = 290, 130, 80, 50$ for $\delta = 1\%, 5\%, 10\%$ and 20% respectively. We can also see that the bigger the noise level is, the smaller the suggested value

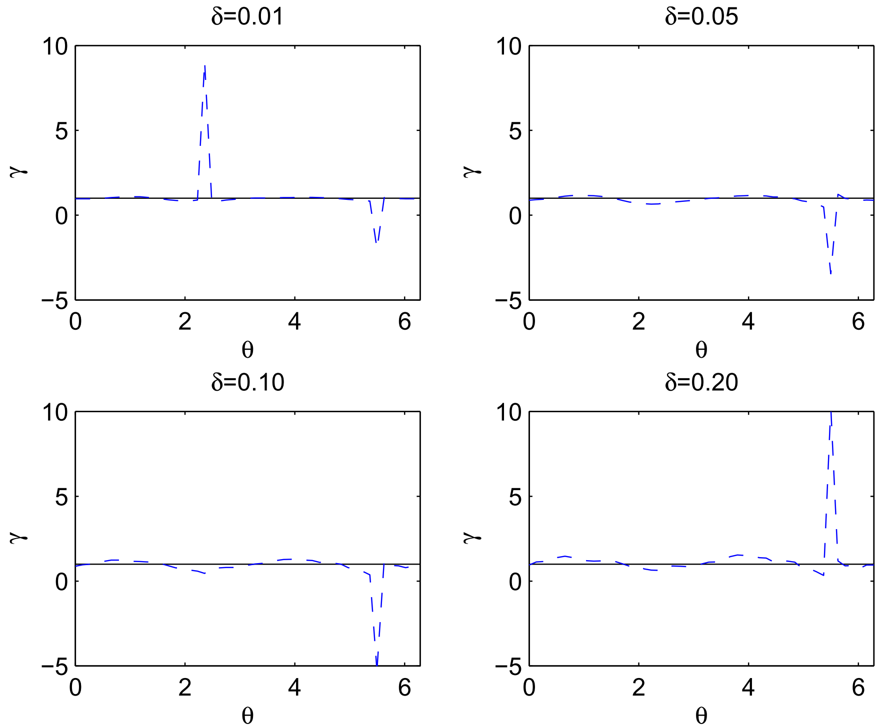

Figure 4. Reconstructed Robin coefficients for different δ when $\gamma^* \equiv 1$ (Example 6.2).

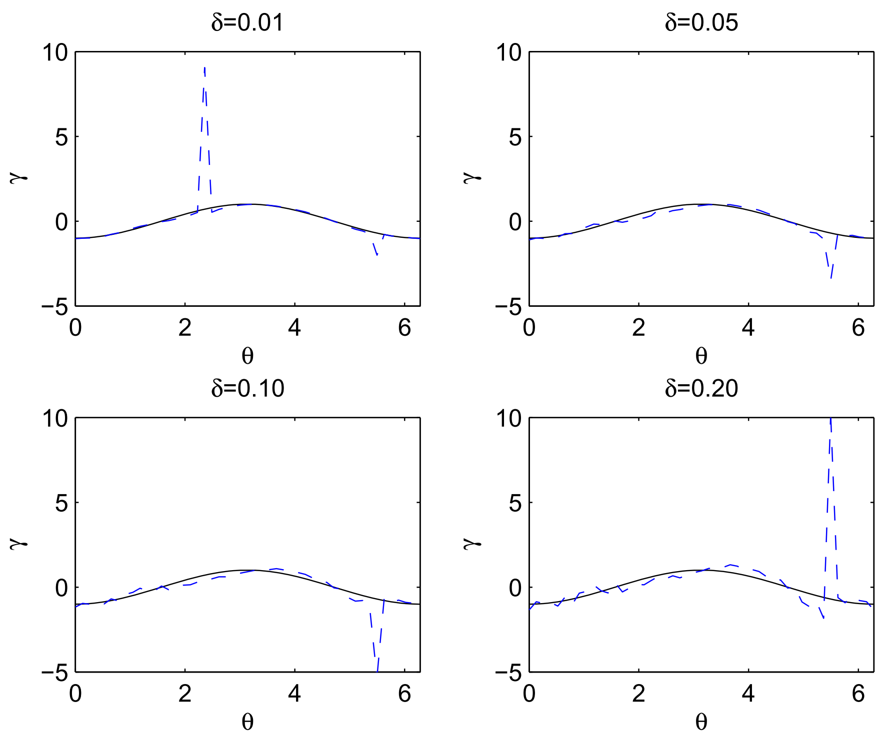

Figure 5. Reconstructed Robin coefficients for different δ when $\gamma^* = \frac{5}{3}x_1$ (Example 6.2).

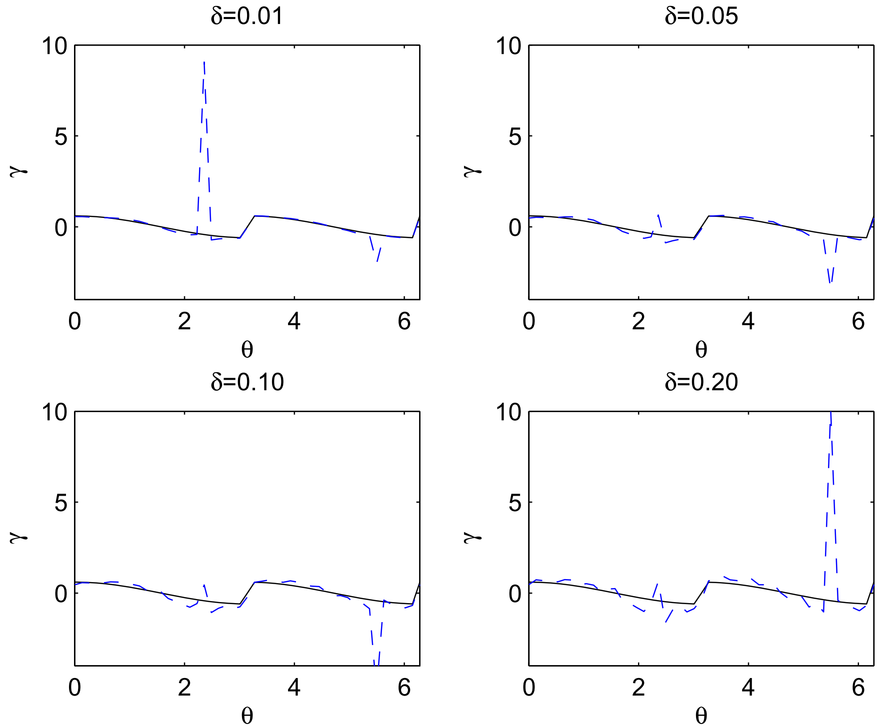

Figure 6. Reconstructed Robin coefficients for different δ when $\gamma^* = \mathrm{sgn}(x_2)e^{x_1}$ (Example 6.2).

of C_α is. Note again the accuracy in $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$ depends weakly on the value of C_α and a large range of values of C_α can produce satisfactory approximations to (ϕ^*, t^*).

Like Example 6.1, with $(\phi_\varepsilon^{\delta,h}, t_\varepsilon^{\delta,h})$, approximations $\gamma_\varepsilon^{\delta,h}$ to three different γ^* $(1, \frac{5}{3}x, \mathrm{sign}(x_2)e^{x_1})$ are computed through the formula (47). Again, set $\varepsilon = 10^{-20}$. Moreover, for β^δ in (37), choose $\eta = 0.01$. The approximate $\gamma_\varepsilon^{\delta,h}$ to three different γ^* for $\delta = 1\%, 5\%, 10\%$ and 20% are plotted in Figures 4–6, from which we conclude that the reconstruction is stable and reasonable. Nevertheless, these figures also show that the accuracy in $\gamma_\varepsilon^{\delta,h}$ gets worse near the zero points of t^*, which is in accordance with the theoretical observation in Remark 4.1.

7. Conclusions

In this paper, a parameter-dependent CCBM-based Tikhonov regularization framework is presented for solving the reduced Cauchy problem coming from an inverse Robin problem. Compared with the existing work, the contributions of this paper are two aspects. On one hand, the nonlinear Robin inverse problem is transferred to a linear Cauchy one. As a result, when applying the Tikhonov regularization, the problem is further reduced to a strictly convex optimal one which can be solved through the optimality equations, and thus no iteration is needed. On the other hand, with the introduction of a positive parameter α, we don't need to choose the regularization parameter ε. As shown by theoretical analysis and numerical experiments, when set $\alpha = \mathcal{O}(\sqrt{\varepsilon}) = C_\alpha\sqrt{\varepsilon}$ for some constant C_α, the solutions are constant with respect to small ε. The solution accuracy is far less sensitive to C_α than ε.

Disclosure statement

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