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A coupled complex boundary method for an inverse conductivity problem with one measurement

Rongfang Gong[a], Xiaoliang Cheng[b] and Weimin Han[c]

[a]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, China; [b]Department of Mathematics, Zhejiang University, Hangzhou, China; [c]Department of Mathematics, University of Iowa, Iowa City, IA, USA

ABSTRACT

We recently proposed in [Cheng, XL et al. A novel coupled complex boundary method for inverse source problems *Inverse Problem* 2014 30 055002] a coupled complex boundary method (CCBM) for inverse source problems. In this paper, we apply the CCBM to inverse conductivity problems (ICPs) with one measurement. In the ICP, the diffusion coefficient q is to be determined from both Dirichlet and Neumann boundary data. With the CCBM, q is sought such that the imaginary part of the solution of a forward Robin boundary value problem vanishes in the problem domain. This brings in advantages on robustness and computation in reconstruction. Based on the complex forward problem, the Tikhonov regularization is used for a stable reconstruction. Some theoretical analysis is given on the optimization models. Several numerical examples are provided to show the feasibility and usefulness of the CCBM for the ICP. It is illustrated that as long as all the subdomains share some portion of the boundary, our CCBM-based Tikhonov regularization method can reconstruct the diffusion parameters stably and effectively.

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1. Introduction

Parameter identification in elliptic boundary value problems arises in many applications such as groundwater management, crack identifications, modeling of car wind-shields, image processing, and so on [1–5]. Extensive literature exists for theoretical and numerical investigation of the parameter identification problem. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a boundary $\Gamma := \partial\Omega$. Then a commonly studied parameter identification problem is to determine the diffusion parameter q in the inverse conductivity problem (ICP),[6,7] also known as electrical impedance tomography [8,9] in medical imaging:

Problem 1.1: Given f in Ω, g_2, and g_1 on Γ, find q in Ω satisfying equation

$$-\nabla \cdot (q\nabla u) = f \text{ in } \Omega, \tag{1}$$

the Dirichlet boundary condition

$$u = g_2 \text{ on } \Gamma, \tag{2}$$

and the Neumann boundary condition

$$q\frac{\partial u}{\partial \nu} = g_1 \text{ on } \Gamma. \tag{3}$$

CONTACT Rongfang Gong grf_math@nuaa.edu.cn

It is well known that reconstruction of a spatially dependent parameter in partial differential equations is nonlinear and ill-posed.[10,11] In general, there are infinitely many solutions. Nevertheless, there are many results about the unique identifiability of solutions given the complete knowledge of the Dirichlet to Neumann mapping $Q_q : g_2 \to g_1$, see [12–16] etc. and references therein. In [12], it is proved that a smooth $q \in C^\infty(\overline{\Omega})$ can be uniquely determined given the complete knowledge of the mapping Q_q for $d \geq 3$. For $d = 2$, this uniqueness is shown to hold when $q \in W^{2,p}(\Omega)$ with $p > 1$ [14] or when $q \in L^\infty(\Omega)$ [16]. Moreover, certain a priori information allows conditional stability, see e.g. [17–19].

Instead of discussing the problem of determining q from the mapping Q_q, in this work, we consider the problem of computing q from a single pair of g_1 and g_2. With a single-boundary current measurement, the identifiability and uniqueness are not guaranteed for a general spatially dependent q. One frequently used technique is to assume some a priori information about the unknown parameter q, including its form, value on the boundary and so on. For instance, the unique identifiability is extensively discussed for piecewise constant parameters of the form [20,21].

$$q = 1 + k\chi_D,$$

where χ_D is the characteristic function of an unknown subdomain $D \subset \Omega$ and k is an unknown constant. We refer to [22–27] for more results on the unique identifiability for Problem 1.1 with a single measurement.

In this paper, we consider numerically computing the diffusion coefficient $q \in B := L^\infty(\Omega)$ in Problem 1.1 with the Tikhonov regularization. Using the Tikhonov technique, one associates either the Dirichlet boundary condition or the Neumann boundary condition with the differential Equation (1) to form a forward BVP, and uses the remaining boundary condition for data fitting. However, applying the Tikhonov framework directly here may cause some numerical problems. For example, if we take (1) and (3) to form the forward BVP, we will need an additional condition, such as $\int_\Omega u \, dx = 0$, to guarantee the solution uniqueness. If we take (1) and (2) to form the forward BVP, there is a complication due to the need of computing certain norm of $q\frac{\partial u}{\partial \nu} - g_1$ in the Tikhonov objective function with the regular conforming finite element methods, because for the numerical solution u^h, $u^h|_\Gamma$ is a piecewise polynomial and $q\frac{\partial u^h}{\partial \nu}$ has jumps at boundary nodes. In [28], a coupled complex boundary method (CCBM) was proposed to solve an inverse source problem, where both Dirichlet and Neumann boundary conditions are used simultaneously in the forward BVP and thus the problems mentioned above are avoided. In this paper, the CCBM is applied to solve Problem 1.1. The idea of the CCMB is to couple the Neumann data and the Dirichlet data in a Robin boundary condition in such a way that the Neumann data and the Dirichlet data are the real part and imaginary part of the Robin boundary condition, respectively. As a result, the data needed to fit are transferred from the boundary Γ to the domain Ω.

The paper is organized as follows. The CCBM for Problem 1.1 is proposed and studied in Section 2. An output least-square framework with the Tikhonov regularization is introduced in Section 3, where some theoretical results are provided about the new regularization framework. Several numerical examples are presented in Section 4 to demonstrate the usefulness of the proposed method.

2. A reformulation through a CCBM

We first introduce notations for function spaces and sets. Assume the boundary Γ is Lipschitz continuous. Let $V := H^1(\Omega)$ be the standard real Sobolev space with the inner product $(\cdot,\cdot)_{1,\Omega}$ and the norm $\|\cdot\|_{1,\Omega}$. Denote by $V_\Gamma := H^{1/2}(\Gamma)$ the trace space of V. Let $Q := L^2(\Omega)$ and $Q_\Gamma := L^2(\Gamma)$. Denoted by $\mathbf{V}$, $\mathbf{V}_\Gamma$, $\mathbf{Q}$, and $\mathbf{Q}_\Gamma$ the complex versions of V, V_Γ, Q, and Q_Γ, respectively. Define the inner product $((\cdot,\cdot))_{1,\Omega}$ and the norm $|||\cdot|||_{1,\Omega}$ in $\mathbf{V}$ as follows: $\forall\, u, v \in \mathbf{V}$, $((u,v))_{1,\Omega} = (u,\bar{v})_{1,\Omega}$, $|||v|||_{1,\Omega}^2 = ((v,v))_{1,\Omega}$. Set $A = \{q \in B \mid q_{\min} \leq q \leq q_{\max}\}$ with $q_{\min}, q_{\max} \in L^\infty(\Omega)$ and

$q_{\max} > q_{\min} > 0$, and A^0 the interior of A. Moreover, assume $g_1, g_2 \in H^{-1/2}(\Gamma)$, the dual of V_Γ, and denote by c a constant with possibly a different value at a different place.

Consider the complex boundary value problem

$$\begin{cases} -\nabla \cdot (q\nabla u) = f & \text{in } \Omega, \\ q\frac{\partial u}{\partial \nu} + i\,u = g := g_1 + i\,g_2 & \text{on } \Gamma, \end{cases} \tag{4}$$

where $i = \sqrt{-1}$ is the imaginary unit. We will assume $f \in Q$. For a solution of (4), $u = u_1 + i\,u_2$, the real-valued functions u_1, u_2 satisfy

$$\begin{cases} -\nabla \cdot (q\nabla u_1) = f & \text{in } \Omega, \\ q\frac{\partial u_1}{\partial \nu} - u_2 = g_1 & \text{on } \Gamma, \end{cases} \tag{5}$$

and

$$\begin{cases} -\nabla \cdot (q\nabla u_2) = 0 & \text{in } \Omega, \\ q\frac{\partial u_2}{\partial \nu} + u_1 = g_2 & \text{on } \Gamma. \end{cases} \tag{6}$$

If $u_2 \equiv 0$ in Ω, then $u_2 \equiv 0$, $\frac{\partial u_2}{\partial n} \equiv 0$ on Γ. From (5) and (6), (u_1, q) satisfy the original problem (1)–(3). Conversely, if (u, q) satisfy (1)–(3), then obviously, they satisfy (4).

The above consideration implies that Problem 1.1 is equivalent to the following problem.

Problem 2.1: Find $q \in A$ such that

$$u_2 = 0 \text{ in } \Omega,$$

where u_2 is the imaging part of the solution $u = u_1 + i\,u_2$ of the BVP

$$\begin{cases} -\nabla \cdot (q\nabla u) = f & \text{in } \Omega \\ q\frac{\partial u}{\partial \nu} + i\,u = g & \text{on } \Gamma. \end{cases} \tag{7}$$

Before discussing Problem 2.1, we consider the well-posedness of the BVP (7) for a given $q \in A$. For any $u, v \in \mathbf{V}$, define

$$a(q; u, v) = \int_\Omega q\nabla u \cdot \nabla \bar{v} \mathrm{d}x + i \int_\Gamma u\,\bar{v} \mathrm{d}s, \quad m(v) = \int_\Omega f\,\bar{v} \mathrm{d}x + \int_\Gamma g\,\bar{v} \mathrm{d}s.$$

Then the weak form of the BVP (7) is:

$$\text{Find } u \in \mathbf{V} \text{ such that } a(q; u, v) = m(v) \quad \forall\, v \in \mathbf{V}. \tag{8}$$

It is not difficult to verify that for any $q \in A$ and $u, v \in \mathbf{V}$, we have

$$\operatorname{Re} a(q; v, v) \geq \alpha |||v|||_{1,\Omega}^2, \tag{9}$$

$$|a(q; u, v)| \leq \beta\, |||u|||_{1,\Omega} |||v|||_{1,\Omega}, \tag{10}$$

$$|m(v)| \leq c\,\gamma |||v|||_{1,\Omega}. \tag{11}$$

where $\gamma = \|f\|_{0,\Omega} + \|g_1\|_{-1/2,\Gamma} + \|g_2\|_{-1/2,\Gamma}$, α, and β are constants which are independent of q, u and v but may depend on $\Omega, q_{\min}$ and $q_{\max}$. Then for given $q \in A, f \in Q, g_1, g_2 \in H^{-1/2}(\Gamma)$, by the complex version of Lax–Milgram Lemma,[29, p.368–369] Problem (8) admits a unique solution $u \in \mathbf{V}$ which depends continuously on all data. Moreover,

$$|||u|||_{1,\Omega} \leq \frac{c\,\gamma}{\alpha}. \tag{12}$$

Since for each $q \in A$, there is a unique solution $u \in \mathbf{V}$, we define a mapping F from A to $\mathbf{V}$: $\forall\, q \in A$, $F(q) = u$, the solution of the problem (8). Regarding the mapping F, it is easy to verify that

$$|||F(q_1) - F(q_2)|||_{1,\Omega} \le \frac{c\,\gamma}{\alpha^2}\,\|q_1 - q_2\|_{\infty,\Omega} \quad \forall\, q_1, q_2 \in A. \tag{13}$$

Furthermore, we have the following differentiation of the operator F.

Proposition 2.2: *For each $q \in A^0$, F is differentiable at q, and $\delta u = F'(q)\,\delta q$ is the unique solution to the variational equation*

$$a(q; \delta u, v) = -(\delta q\,\nabla u, \nabla \bar{v})_{0,\Omega} \quad \forall\, v \in \boldsymbol{V} \tag{14}$$

with $u = F(q)$. Moreover,

$$\|F'(q)\|_{B\to \mathbf{V}} \le \frac{c\,\gamma}{\alpha^2}.$$

Proof: For any $q \in A^0$ and $\delta q \in B$ with $q + \delta q \in A$, $F(q + \delta q) \in \mathbf{V}$ is well defined. Set $\delta w = F(q + \delta q) - F(q)$. Then by the definition of F, we have

$$a(q + \delta q; \delta w, v) = -(\delta q\,\nabla u, \nabla \bar{v})_{0,\Omega} \quad \forall\, v \in \mathbf{V}. \tag{15}$$

By applying the complex version of Lax–Milgram Lemma again, we know that the variational problem (14) has a unique solution $\delta u \in \mathbf{V}$.

Combine (14) and (15) to get

$$a(q; \delta w - \delta u, v) = -(\delta q\,\nabla \delta w, \nabla \bar{v})_{0,\Omega} \quad \forall\, v \in \mathbf{V}.$$

Take $v = \delta w - \delta u$, and use ellipticity (9) of $a(q; \cdot, \cdot)$, Cauchy–Schwarz inequality to get

$$|||\delta w - \delta u|||_{1,\Omega} \le \frac{1}{\alpha}\|\delta q\|_{\infty,\Omega}|||\delta w|||_{1,\Omega}$$

Applying (13), we obtain

$$|||\delta w - \delta u|||_{1,\Omega} \le \frac{c\,\gamma}{\alpha^3}\|\delta q\|^2_{\infty,\Omega},$$

which gives

$$\begin{aligned}\frac{|||F(q + \delta q) - F(q) - \delta u|||_{1,\Omega}}{\|\delta q\|_{\infty,\Omega}} &= \frac{|||\delta w - \delta u|||_{1,\Omega}}{\|\delta q\|_{\infty,\Omega}} \\ &\le \frac{c\,\gamma}{\alpha^3}\|\delta q\|_{\infty,\Omega} = O(\|\delta q\|_{\infty,\Omega}).\end{aligned} \tag{16}$$

Thus, F is differentiable at q and $F'(q)\delta q = \delta u$.

Set $v = \delta u$ in (14), and use (9) as well as Cauchy–Schwarz inequality again to obtain

$$\alpha|||\delta u|||^2_{1,\Omega} \le \|\delta q\|_{\infty,\Omega}|||u|||_{0,\Omega}|||\delta u|||_{0,\Omega}.$$

Apply (12) to give

$$|||\delta u|||_{1,\Omega} \le \frac{c\,\gamma}{\alpha^2}\|\delta q\|_{\infty,\Omega}, \tag{17}$$

which shows that $F'(q)$ is uniformly bounded. □

For $q \in A^b := A\backslash A^0$, denote by $\tilde{B} := \tilde{B}(q)$ the largest subset of B such that $\forall\, \delta q \in \tilde{B}$ and sufficiently small $t > 0$, $q + t\,\delta q \in A$. Then the map F is also differentiable at each $q \in A^b$, and $\|F'(q)\|_{\tilde{B}\to \mathbf{V}}$ is bounded.

We proceed to compute the second derivative of F. Let $\delta^2 w = F'(q+\delta q_1)\delta q_2 - F'(q)\delta q_2$. By definition (14), we have

$$a(q+\delta q_1; F'(q+\delta q_1)\delta q_2, v) = -(\delta q_2 \nabla F(q+\delta q_1), \nabla \bar{v})_{0,\Omega} \quad \forall v \in \mathbf{V}$$

and

$$\begin{aligned} a(q+\delta q_1; F'(q)\delta q_2, v) &= a(q; F'(q)\delta q_2, v) + (\delta q_1 \nabla (F'(q)\delta q_2), \nabla \bar{v})_{0,\Omega} \\ &= -(\delta q_2 \nabla F(q), \nabla \bar{v})_{0,\Omega} + (\delta q_1 \nabla (F'(q)\delta q_2), \nabla \bar{v})_{0,\Omega}. \end{aligned}$$

Then

$$\begin{aligned} a(q+\delta q_1; \delta^2 w, v) = &-(\delta q_2 \nabla (F(q+\delta q_1) - F(q) - F'(q)\delta q_1), \nabla \bar{v})_{0,\Omega} \\ &- (\delta q_1 \nabla (F'(q)\delta q_2), \nabla \bar{v})_{0,\Omega} \\ &- (\delta q_2 \nabla (F'(q)\delta q_1), \nabla \bar{v})_{0,\Omega}, \end{aligned} \tag{18}$$

from which we have the following result.

Proposition 2.3: *For each $q \in A^0$, F is twice-differentiable at q, and $\delta^2 u = F''(q)\,(\delta q_1, \delta q_2)$ is the unique solution to the variational equation*

$$\begin{aligned} a(q; \delta^2 u, v) = &-(\delta q_1 \nabla (F'(q)\delta q_2), \nabla \bar{v})_{0,\Omega} \\ &- (\delta q_2 \nabla (F'(q)\delta q_1), \nabla \bar{v})_{0,\Omega} \quad \forall v \in \boldsymbol{V}, \end{aligned} \tag{19}$$

with $u = F(q)$. Moreover,

$$\|F''(q)\|_{B\times B\to \mathbf{V}} \le \frac{c\gamma}{\alpha^3}. \tag{20}$$

Proof: The variational problem (19) is well posed and thus $\delta^2 u$ is well defined. We combine (18) and (19) to get

$$\begin{aligned} a(q; \delta^2 w - \delta^2 u, v) = &-(\delta q_1 \nabla \delta^2 w, \nabla \bar{v})_{0,\Omega} \\ &- (\delta q_2 \nabla (F(q+\delta q_1) - F(q) - F'(q)\delta q_1), \nabla \bar{v})_{0,\Omega}. \end{aligned}$$

Choosing $v = \delta^2 w - \delta^2 u$ and using Cauchy–Schwarz inequality, we have

$$\begin{aligned} \alpha|||\delta^2 w - \delta^2 u|||_{1,\Omega} \le &\|\delta q_1\|_{\infty,\Omega}|||\delta^2 w|||_{1,\Omega} \\ &+ \|\delta q_2\|_{\infty,\Omega}|||F(q+\delta q_1) - F(q) - F'(q)\delta q_1|||_{1,\Omega}. \end{aligned}$$

Similarly, choose $v = \delta^2 w$ in (18) and use Cauchy–Schwarz inequality again to give

$$\begin{aligned} \alpha|||\delta^2 w|||_{1,\Omega} \le &\|\delta q_2\|_{\infty,\Omega}|||F(q+\delta q_1) - F(q) - F'(q)\delta q_1|||_{1,\Omega} \\ &+ \|\delta q_1\|_{\infty,\Omega}|||F'(q)\delta q_2|||_{1,\Omega} + \|\delta q_2\|_{\infty,\Omega}|||F'(q)\delta q_1|||_{1,\Omega}. \end{aligned}$$

Combining the two relations above and using estimates (16)–(17), we obtain

$$\begin{aligned} \alpha|||\delta^2 w - \delta^2 u|||_{1,\Omega} &\le \frac{c\gamma}{\alpha^3}(3 + \frac{1}{\alpha}\|\delta q_1\|_{\infty,\Omega})\|\delta q_1\|^2_{\infty,\Omega}\|\delta q_2\|_{\infty,\Omega} \\ &\le \frac{c\gamma}{\alpha^3}\|\delta q_1\|^2_{\infty,\Omega}\|\delta q_2\|_{\infty,\Omega} \end{aligned}$$

for $\|\delta q_1\|_{\infty,\Omega}$ small enough. Therefore,

$$|||\delta^2 w - \delta^2 u|||_{1,\Omega} \le \frac{c\,\gamma}{\alpha^4}\|\delta q_1\|^2_{\infty,\Omega}\|\delta q_2\|_{\infty,\Omega},$$

which shows that F is twice-differentiable at q and $\delta^2 u = F''(q)\,(\delta q_1, \delta q_2)$.

Choosing $v = \delta^2 u$ in (19) and using (9), Cauchy–Schwarz inequality as well as (17), we arrive at

$$|||\delta^2 u|||_{1,\Omega} \le \frac{c\,\gamma}{\alpha^3}\|\delta q_1\|_{\infty,\Omega}\|\delta q_2\|_{\infty,\Omega},$$

which gives (20).

3. Tikhonov regularization

Based on the complex model, Problem 2.1, we give a Tikhonov regularization framework for the ICP. Assume available values of the data g_1 and g_2 contain noise:

$$g_i^\delta = g_i + \text{noise}, \quad i = 1, 2.$$

The natural space for g_i^δ and g_i is Q_Γ. We assume

$$\|g_i^\delta - g_i\|_{0,\Gamma} \le \delta, \quad i = 1, 2,$$

with noise level δ. Then the forward complex variational problem (8) is modified to

$$\text{Find } u^\delta \in \mathbf{V} \text{ such that } a(q; u^\delta, v) = m^\delta(v) \quad \forall\, v \in \mathbf{V} \tag{21}$$

with

$$m^\delta(v) = \int_\Omega f\,\bar{v}\mathrm{d}x + \int_\Gamma g^\delta\,\bar{v}\mathrm{d}s \text{ and } g^\delta := g_1^\delta + i\,g_2^\delta.$$

The form $m^\delta(\cdot)$ is continuous and for each $q \in A$, the problem (21) has a unique solution u^δ. Similar to the mapping F, we define $F^\delta : A \to \mathbf{V}$ through $F^\delta(q) = u^\delta$ for $q \in A$. Then F^δ has properties similar to those of the map F. Moreover, it is easy to verify that for each $q \in A$,

$$|||F^\delta(q) - F(q)|||_{1,\Omega} \le c\,\delta. \tag{22}$$

In the following, we compute a stable approximation to $q^\dagger$ which is an exact solution of Problem 1.1 or 2.1. For any $q \in A$, denoted by $u^\delta = F^\delta(q) \in \mathbf{V} = u_1^\delta(q) + i\,u_2^\delta(q)$ the unique solution of the variational problem (21). We define a Tikhonov regularization objective functional

$$J_\varepsilon^\delta(q) = \frac{1}{2}\|u_2^\delta(q)\|^2_{0,\Omega} + \frac{\varepsilon}{2}\|q\|^2_{0,\Omega}, \quad \varepsilon > 0,$$

and introduce the following optimization problem:

Problem 3.1: Find $q_\varepsilon^\delta \in Q_{ad}$, an admissible subset of A, such that

$$J_\varepsilon^\delta(q_\varepsilon^\delta) = \inf_{q \in Q_{ad}} J_\varepsilon^\delta(q).$$

In the following, we assume Q_{ad} is a finite dimensional closed convex subset of A. For example, Q_{ad} can be the subset of A of piecewise constant functions or piecewise polynomials of some degree, corresponding to certain partition of $\overline{\Omega}$. Note that in practice, the coefficient function q is often reconstructed as a piecewise constant.

We first address the solution existence of Problem 3.1.

Proposition 3.2: *Problem 3.1 has a solution.*

Proof: Let m be the infimum value in Problem 3.1 and $\{q_n\}_{n\geq 1} \subset Q_{ad}$ be a mininizing sequence. Denote $u_n = F^\delta(q_n)$. From the estimate (12), $\{u_n\}_{n\geq 1}$ is bounded uniformly in $\mathbf{V}$. Then there is a subsequence of the sequence $\{n\}$, still denoted by $\{n\}$, and some elements $q_\infty \in Q_{ad}, u_\infty \in \mathbf{V}$ such that as $n \to \infty$,

$$q_n \to q_\infty \text{ in } B, \qquad u_n \rightharpoonup u_\infty \text{ in } \mathbf{V}$$
$$u_n \to u_\infty \text{ in } \mathbf{Q}, \qquad u_n \to u_\infty \text{ in } \mathbf{Q}_\Gamma.$$

With arguments similar to those in the proofs of [30, Theorem 3.2] or those in the proofs of [31, Lemma 2.1], we conclude that

$$J_\varepsilon^\delta(q_\infty) \leq m,$$

which shows that $q_\infty \in Q_{ad}$ is a solution of Problem 3.1. The proof is completed. □

It is well known that the solution uniqueness of Problem 3.1 is not guaranteed since the regularized functional is in general non-convex.

Next we present a continuous dependence result.

Proposition 3.3: *For a convergent sequence of the boundary data, the solution sequence of Problem 3.1 contains a subsequence converging to a solution of Problem 3.1 with the true measurement.*

Proof: Let $\{g^n\}_{n\geq 1} \subset \mathbf{Q}_\Gamma$ with $g^n \to g^\infty$ as $n \to \infty$. For each n, denote by q^n a solution of Problem 3.1, with g^δ replaced by g^n. Set $u^n = F^n(q^n)$, where the map $F^n : A \to \mathbf{V}$ is defined as F^δ, with g^δ replaced by g^n. Then with arguments similar to those in the proof of Proposition 3.2 and those in the proof [30, Theorem 3.3], we conclude that there is a subsequence of $\{q^n\}_{n\geq 1}$, still denoted by $\{q^n\}_{n\geq 1}$, converging to $q^\infty \in Q_{ad}$, and

$$J_\varepsilon^\infty(q^\infty) \leq J_\varepsilon^\infty(\tilde{q}) \quad \forall\, \tilde{q} \in A,$$

where $J_\varepsilon^\infty(q)$ is defined similar to $J_\varepsilon^\delta(q)$, with g^δ replaced by g^∞. Therefore, $q^\infty \in A$ is a solution of Problem 3.1 corresponding to data g^∞, and the proof is completed. □

Next we discuss a relation between Problem 2.1 and its regularization, Problem 3.1. Recall that F is the forward operator corresponding to the exact data g. We introduce the following assumption on the attainability of the measurement data.

(A1) There is $q^\dagger \in A$ such that $u_2^\dagger = 0$ in Ω, where $u_2^\dagger \in V$ is the imaginary part of $u^\dagger = F(q^\dagger) \in \mathbf{V}$.

Proposition 3.4: *For a sequence of noise levels $\{\delta_n\}_{n\geq 1}$ which converges to 0 in $\mathbb{R}$ as $n \to \infty$, let $\varepsilon_n = \varepsilon(\delta_n)$ be chosen with $\varepsilon_n \to 0$ as $n \to \infty$, and $q_{\varepsilon_n}^{\delta_n} \in Q_{ad}$ be a solution of Problem 3.1 with g^δ and ε replaced by g^{δ_n} and ε_n, respectively. Then under the assumption (A1), the solution sequence $\{q_{\varepsilon_n}^{\delta_n}\}_{n\geq 1}$ has a subsequence converging to a solution of Problem 2.1 for the exact data g.*

Proof: We use the abbreviations $q^n = q_{\varepsilon_n}^{\delta_n}$ and $g^n = g^{\delta_n}$. Let $F^n : A \to \mathbf{V}$ be the forward mapping defined as F^δ for data g^n, and set $u^n = F^n(q^n)$. Then $\{u^n\}_{n\geq 1}$ is bounded uniformly. Arguments similar to those used in the proof of Proposition 3.2 show that there is a subsequence $\{n'\}$ of $\{n\}$, functions $q^\infty \in Q_{ad}, u^\infty = F(q^\infty)$ such that

$$q^{n'} \to q^\infty \text{ in } B, \qquad u^{n'} \rightharpoonup u^\infty \text{ in } \mathbf{V},$$
$$u^{n'} \to u^\infty \text{ in } \mathbf{Q}, \qquad u^{n'} \to u^\infty \text{ in } \mathbf{Q}_\Gamma.$$

For the imaginary part u_2^∞ of u^∞, we have

$$\frac{1}{2}\|u_2^\infty\|_{0,\Omega}^2 \le \liminf_{n'\to\infty} J_{\varepsilon_{n'}}^{\delta_{n'}}(q^{n'}) \le \liminf_{n'\to\infty} J_{\varepsilon_{n'}}^{\delta_{n'}}(q^\dagger) = \liminf_{n'\to\infty} \frac{\varepsilon_{n'}}{2}\|q^\dagger\|_{0,\Omega} = 0,$$

implying $u_2^\infty = 0$ in Ω. Thus, q^∞ is a solution of Problem 2.1 for the exact data g. □

The convergence speed of q_ε^δ to $q^\dagger$ can be arbitrarily slow, as is shown in [32]. If q is searched in a smoother set with inner product, $\{q \in V \mid q_{\min} \le q \le q_{\max}\}$ for instance, a convergence rate $\sqrt{\delta}$ can be expected under so-called source conditions about $q^\dagger - \bar{q}$, where $\bar{q} \in A$ is an a priori guess of $q^\dagger$. We refer to [21,33,34] and references therein about convergence rate issues of inverse diffusion parameters.

4. Numerical results

In this section, some numerical results are reported to show the feasibility and effectivity of the proposed CCBM-based Tikhonov regularization method for the ICP.

With a problem domain Ω, a source function f, a Neumann data g_1, and a prescribed true diffusion parameter $q^\dagger$, using the standard linear finite element method, we solve the forward BVP

$$-\nabla\cdot(q^\dagger\nabla u) = f \text{ in } \Omega, \quad q^\dagger\frac{\partial u}{\partial \nu} = g_1 \text{ on } \Gamma, \quad \int_\Omega u\,\mathrm{d}x = 0 \tag{23}$$

to get the measurement $g_2 = u|_\Gamma$. Specifically, applying the linear finite element method to weak form

$$\int_\Omega q^\dagger\nabla u\cdot\nabla v\mathrm{d}x = \int_\Omega f\,v\mathrm{d}x + \int_\Gamma g_1\,v\,\mathrm{d}s \quad \forall\, v\in V,$$

we obtain a reduced algebraic system

$$K\,u = b \tag{24}$$

We need to impose the condition $\int_\Omega u\,\mathrm{d}x = 0$ for a unique solution u. Note that the finite element discretization of $\int_\Omega u\,\mathrm{d}x = 0$ is

$$e\,M\,u = 0,$$

where M is the symmetrical mass matrix from the discretization of $\int_\Omega \phi\,\varphi\,\mathrm{d}x, \phi,\varphi \in V$ and e is a row vector with all components being one.

With the QR-decomposition of K, (24) reduces to

$$R\,u = Q^T b, \tag{25}$$

where Q is an orthogonal matrix, Q^T is the transpose of Q, and R is an upper triangular matrix. Then all elements in the last row of R are zeros. We replace the last row of R by the row vector $e\,M$ and replace the last element of $Q^T b$ by zero to get a well-conditioned system

$$\tilde{R}\,u = \tilde{b}. \tag{26}$$

Solve (26) to obtain the forward solution u; then the Dirichlet data g_2 is extracted. Uniformly distributed noises with level δ are added to both g_1 and g_2 to get g_1^δ and g_2^δ:

$$g_k^\delta(x) = [1 + \delta\cdot(2\,\mathrm{rand}(x) - 1)]\,g_k(x), \quad x\in\Gamma, \quad k = 1,2,$$

where $\mathrm{rand}(x)$ returns a pseudo-random value drawn from a uniform distribution on $[0,1]$.

Given $\Omega, f, g_1^\delta, g_2^\delta$, and Q_{ad}, we solve Problem 3.1 to obtain approximations of $q^\dagger$. Due to Proposition 2.3, J_ε^δ is twice differentiable, a sequential quadratic programming (SQP) method (see

Table 1. Results for $\delta = 0$ in Example 1.

ε	(q_1, q_2)	L2Err	$\|u^{\delta}_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.1404, 0.2256)	8.4679e−1	7.3179e−1	5
10^{-1}	(0.4607, 0.7816)	4.7243e−1	3.4600e−1	6
10^{-2}	(0.7611, 1.3130)	1.1984e−1	7.0491e−2	10
10^{-3}	(0.6664, 1.4968)	3.0969e−2	9.0364e−3	6
10^{-4}	(0.6722, 1.5171)	3.0783e−2	1.0111e−3	6
10^{-5}	(0.6728, 1.5192)	3.1084e−2	1.8717e−4	6
10^{-6}	(0.6729, 1.5194)	3.1117e−2	1.0480e−4	6
10^{-7}	(0.6729, 1.5195)	3.1121e−2	9.6583e−5	6
10^{-8}	(0.6729, 1.5195)	3.1121e−2	9.5762e−5	6
10^{-9}	(0.6729, 1.5195)	3.1121e−2	9.5680e−5	6
10^{-10}	(0.6729, 1.5195)	3.1121e−2	9.5672e−5	6

Table 2. Results for $\delta = 5\%$ in Example 1.

ε	(q_1, q_2)	L2Err	$\|u^{\delta}_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.1399, 0.2246)	8.4749e−1	7.3035e−1	5
10^{-1}	(0.4598, 0.7798)	4.7361e−1	3.4602e−1	6
10^{-2}	(0.7607, 1.3123)	1.2026e−1	7.0685e−2	10
10^{-3}	(0.6669, 1.4967)	3.0785e−2	9.2093e−3	6
10^{-4}	(0.6728, 1.5172)	3.0610e−2	1.8960e−3	6
10^{-5}	(0.6734, 1.5193)	3.0918e−2	1.6119e−3	6
10^{-6}	(0.6735, 1.5195)	3.0952e−2	1.6042e−3	6
10^{-7}	(0.6735, 1.5196)	3.0955e−2	1.6037e−3	6
10^{-8}	(0.6735, 1.5196)	3.0956e−2	1.6037e−3	6
10^{-9}	(0.6735, 1.5196)	3.0956e−2	1.6037e−3	6
10^{-10}	(0.6735, 1.5196)	3.0956e−2	1.6037e−3	6

[35] and references therein for instance) is used to solve the discrete version of the optimization problem. At the continuous level, applying the SQP method to Problem 3.1 reads:

Step 1 Give an initial guess $q^0 \in A$, a tolerance error $\epsilon > 0$, and set $k = 0$.
Step 2 Solve a quadratic programming subproblem

$$d^k = \arg \min_{(d+q^k)\in A} J^{\delta}_{\varepsilon}(q^k) + J^{\delta'}_{\varepsilon}(q^k)\, d + J^{\delta''}_{\varepsilon}(q^k)\, d^2. \tag{27}$$

Step 3 Solve a linear search problem

$$\alpha^k = \arg \min_{(d^k+\alpha\, q^k)\in A} \varphi(\alpha) := J^{\delta}_{\varepsilon}(d^k + \alpha\, q^k). \tag{28}$$

Step 4 Set $q^{k+1} = q^k + \alpha^k d^k$. If $\|q^{k+1} - q^k\|_{0,\Omega} \le \epsilon$, stop; otherwise, set $k = k+1$ and go to Step 2.

In the following experiments, we set $\epsilon = 10^{-6}$.

Example 1: In this example, let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}, f = 1$ in Ω and $g_1 = -0.5$ on Γ. Assume $q^{\dagger}$ needed to be recovered is piecewise constant in Ω, i.e.

$$q^{\dagger} = \begin{cases} 0.75 & \text{in } \Omega_1 := \{(x, y) \in \Omega \mid x^2 + y^2 < 0.25\}, \\ 1.5 & \text{on } \Omega_2 := \overline{\Omega}/\Omega_1. \end{cases}$$

Table 3. Results for $\delta = 10\%$ in Example 1.

ε	(q_1, q_2)	L2Err	$\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.1393, 0.2236)	8.4815e−1	7.2893e−1	5
10^{-1}	(0.4589, 0.7781)	4.7478e−1	3.4604e−1	6
10^{-2}	(0.7604, 1.3117)	1.2067e−1	7.0910e−2	10
10^{-3}	(0.6674, 1.4968)	3.0608e−2	9.6468e−3	6
10^{-4}	(0.6733, 1.5174)	3.0456e−2	3.3566e−3	6
10^{-5}	(0.6739, 1.5195)	3.0773e−2	3.2034e−3	6
10^{-6}	(0.6740, 1.5197)	3.0808e−2	3.1995e−3	6
10^{-7}	(0.6740, 1.5197)	3.0811e−2	3.1992e−3	6
10^{-8}	(0.6740, 1.5197)	3.0812e−2	3.1992e−3	6
10^{-9}	(0.6740, 1.5197)	3.0812e−2	3.1992e−3	6
10^{-10}	(0.6740, 1.5197)	3.0812e−2	3.1992e−3	6

We rewrite $q^\dagger = 0.75\chi_{\Omega_1} + 1.5\chi_{\Omega_2}$, where for a set K, χ_K is the characteristic function of K, i.e. its value is 1 in K and 0 outside. Sixty-eight nodes on Γ are used to sample the data g_2, obtained from the numerical solution of (23) and is polluted by random noise.

For the inverse problem, assume the admissible set

$$Q_{ad} = \{q = q_1\,\chi_{\Omega_1} + q_2\,\chi_{\Omega_2} \mid 0.1\,q^\dagger \le q \le 10\,q^\dagger\}.$$

We then choose the initial guess $q^0 = 0.1$ and apply the SQP method to solve Problem 3.1. The results for different noise level δ and regularization parameter ε are given in Tables 1–3. Specifically, Table 1 contains results for $\delta = 0$, Table 2 for $\delta = 5\%$, and Table 3 for $\delta = 10\%$. To better assess the accuracy of approximate solutions, we define the L^2-norm relative error in an approximate solution q^δ_ε:

$$\text{L2Err} := \frac{\|q^\delta_\varepsilon - q^\dagger\|_{0,\Omega}}{\|q^\dagger\|_{0,\Omega}},$$

and list its value in the third column of each table. Recall that in Problem 2.1, q is searched so that the imaginary part of the solution of the BVP (7) vanishes in Ω. So we show in the fourth column of each table the L^2-norm of the imaginary parts $u^\delta_{\varepsilon,2}$ of the solution u^δ_ε of (21), with q replaced by q^δ_ε. In addition, the teration numbers denoted by 'Iternum' needed in SQP methods are listed in the fifth columns.

We can see from Tables 1–3 that piecewise constant diffusion parameters can be recovered well. In comparison, the reconstruction of q is less accurate in Ω_1 than in Ω_2. We attribute this to the fact that $\overline{\Omega_1}$ does not interesect the boundary Γ, and therefore, neither the Dirichlet data nor the Neumann data apply directly on the boundary of Ω_1 (cf. the numerical results in the next two examples). Tables 1–3 show that the reconstruction is stable and the regularized solutions are not sensitive to the smallness of the regularization parameters, i.e. satisfactory solutions can be obtained for rather small regularization parameters.

Example 2: In this example, let $\Omega = (-1,1)^2, f(x,y) = x+y+2, g_1(x,y) = x+y-1$. Assume the true $q^\dagger = j$ in Ω_j, $1 \le j \le 4$, with $\Omega_1 = [-1,0)^2$, $\Omega_2 = [0,1] \times [-1,0)$, $\Omega_3 = [-1,0) \times [0,1]$ and $\Omega_4 = [0,1]^2$. Then $q^\dagger = \chi_{\Omega_1} + 2\,\chi_{\Omega_2} + 3\,\chi_{\Omega_3} + 4\,\chi_{\Omega_4}$. Eighty nodes on Γ are used to sample the data g^δ_2. Assume the admissible set

$$Q_{ad} = \left\{ q = \sum_{j=1}^{4} q_j\,\chi_{\Omega_j} \mid 0.1\,q^\dagger \le q \le 10\,q^\dagger \right\}.$$

Table 4. Results for $\delta = 0$ in Example 2.

ε	(q_1, q_2, q_3, q_4)	L2Err	$\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.8777, 0.6152, 0.6033, 0.5091)	8.1369e−1	1.1429	10
10^{-1}	(1.0558, 1.0611, 1.1103, 1.1494)	6.4760e−1	5.0219e−1	11
10^{-2}	(1.0457, 1.4670, 1.6742, 2.1988)	4.1984e−1	1.9295e−1	14
10^{-3}	(1.0214, 1.7635, 2.3791, 3.2099)	1.8852e−1	5.5051e−2	18
10^{-4}	(1.0083, 1.9395, 2.8027, 3.8319)	4.8613e−2	1.0687e−2	20
10^{-5}	(1.0074, 1.9594, 2.9152, 3.9834)	1.7476e−2	3.6707e−3	20
10^{-6}	(1.0073, 1.9612, 2.9296, 4.0030)	1.4748e−2	3.2667e−3	20
10^{-7}	(1.0073, 1.9614, 2.9310, 4.0050)	1.4517e−2	3.2402e−3	20
10^{-8}	(1.0073, 1.9615, 2.9312, 4.0052)	1.4494e−2	3.2377e−3	20
10^{-9}	(1.0073, 1.9615, 2.9312, 4.0052)	1.4492e−2	3.2374e−3	20
10^{-10}	(1.0073, 1.9615, 2.9312, 4.0052)	1.4491e−2	3.2374e−3	20

Table 5. Results for $\delta = 5\%$ in Example 2.

ε	(q_1, q_2, q_3, q_4)	L2Err	$\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.8773, 0.6147, 0.6045, 0.5092)	8.1360e-1	1.1433	9
10^{-1}	(1.0548, 1.0583, 1.1138, 1.1499)	6.4731e−1	5.0287e−1	11
10^{-2}	(1.0448, 1.4597, 1.6814, 2.2024)	4.1889e−1	1.9351e−1	14
10^{-3}	(1.0207, 1.7531, 2.3881, 3.2220)	1.8629e−1	5.5291e−2	19
10^{-4}	(1.0103, 1.9133, 2.8213, 3.8621)	4.4188e−2	1.0664e−2	20
10^{-5}	(1.0090, 1.9397, 2.9251, 4.0242)	1.8183e−2	4.2824e−3	21
10^{-6}	(1.0090, 1.9417, 2.9391, 4.0438)	1.7423e−2	4.1105e−3	21
10^{-7}	(1.0089, 1.9419, 2.9405, 4.0458)	1.7403e−2	4.1054e−3	21
10^{-8}	(1.0089, 1.9419, 2.9407, 4.0460)	1.7402e−2	4.1051e−3	21
10^{-9}	(1.0089, 1.9419, 2.9407, 4.0460)	1.7402e−2	4.1050e−3	21
10^{-10}	(1.0089, 1.9419, 2.9407, 4.0460)	1.7402e−2	4.1050e−3	21

Table 6. Results for $\delta = 10\%$ in Example 2.

ε	(q_1, q_2, q_3, q_4)	L2Err	$\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.8770, 0.6141, 0.6057, 0.5092)	8.1352e−1	1.1437	10
10^{-1}	(1.0538, 1.0555, 1.1174, 1.1505)	6.4702e−1	5.0356e−1	11
10^{-2}	(1.0438, 1.4503, 1.6900, 2.2042)	4.1814e−1	1.9432e−1	15
10^{-3}	(1.0211, 1.7361, 2.3962, 3.2368)	1.8413e−1	5.5642e−2	20
10^{-4}	(1.0106, 1.9000, 2.8198, 3.8896)	4.2725e−2	1.1091e−2	21
10^{-5}	(1.0080, 1.9323, 2.9131, 4.0290)	2.0844e−2	5.3996e−3	22
10^{-6}	(1.0078, 1.9348, 2.9260, 4.0473)	2.0026e−2	5.0301e−3	22
10^{-7}	(1.0078, 1.9350, 2.9273, 4.0493)	2.0002e−2	5.0009e−3	22
10^{-8}	(1.0078, 1.9351, 2.9274, 4.0495)	2.0001e−2	4.9982e−3	22
10^{-9}	(1.0078, 1.9351, 2.9274, 4.0495)	2.0000e−2	4.9978e−3	22
10^{-10}	(1.0078, 1.9351, 2.9274, 4.0495)	2.0000e−2	4.9979e−3	22

Again we choose initial guess $q^0 = 0.1$ and apply the SQP method to solve Problem 3.1. The results for $\delta = 0, 5\%, 10\%$ and different ε are reported in Tables 4–6. The corresponding 'L2Err', $\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$ and 'Iternum' are shown in the 3rd–5th columns, respectively.

We can see from Tables 4–6 that the approximations are very well, and the reconstruction is stable. Again, the regularized solution is not sensitive to the small regularization parameter.

Example 3: In the third example, we consider a 3D problem. Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid -1 < x, y < 1, 0 < z < 1\}$, $f(x, y, z) = x + y + z$, $g_1(x, y, z) = x + y + y - 5/8$. Again, assume the true $q^\dagger = j$ in Ω_j, $1 \le j \le 4$, where

Table 7. Results for $\delta = 0$ in Example 3.

ε	(q_1, q_2, q_3, q_4)	L2Err	$\|u^{\delta}_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.4194, 0.5002, 0.5028, 0.4267)	8.4834e−1	1.4798	7
10^{-1}	(0.9848, 1.2857, 1.3441, 1.4299)	5.7323e−1	6.2070e−1	9
10^{-2}	(1.0444, 1.9055, 2.1679, 2.5055)	3.1287e−1	1.9974e−1	13
10^{-3}	(0.9944, 2.0939, 2.8858, 3.3247)	1.2621e−1	4.7854e−2	18
10^{-4}	(0.9778, 2.0719, 3.2265, 3.6564)	7.6374e−2	1.5564e−2	20
10^{-5}	(0.9755, 2.0694, 3.2802, 3.7064)	7.5315e−2	1.5050e−2	20
10^{-6}	(0.9753, 2.0691, 3.2860, 3.7117)	7.5345e−2	1.5119e−2	20
10^{-7}	(0.9752, 2.0691, 3.2867, 3.7123)	7.5350e−2	1.5127e−2	20
10^{-8}	(0.9752, 2.0691, 3.2867, 3.7123)	7.5350e−2	1.5128e−2	20
10^{-9}	(0.9752, 2.0691, 3.2867, 3.7123)	7.5350e−2	1.5128e−2	20
10^{-10}	(0.9752, 2.0691, 3.2867, 3.7123)	7.5350e−2	1.5128e−2	20

Table 8. Results for $\delta = 5\%$ in Example 3.

ε	(q_1, q_2, q_3, q_4)	L2Err	$\|u^{\delta}_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.4178, 0.4981, 0.5033, 0.4255)	8.4863e−1	1.4786	7
10^{-1}	(0.9873, 1.2843, 1.3430, 1.4282)	5.7365e−1	6.1896e−1	9
10^{-2}	(1.0494, 1.9049, 2.1646, 2.4959)	3.1474e−1	1.9758e−1	13
10^{-3}	(0.9972, 2.0986, 2.8904, 3.2983)	1.3091e−1	4.4905e−2	18
10^{-4}	(0.9816, 2.0721, 3.2170, 3.5947)	8.5024e−2	1.3592e−2	21
10^{-5}	(0.9801, 2.0634, 3.2698, 3.6387)	8.3217e−2	1.2736e−2	21
10^{-6}	(0.9800, 2.0630, 3.2740, 3.6420)	8.3185e−2	1.2759e−2	21
10^{-7}	(0.9799, 2.0630, 3.2744, 3.6424)	8.3182e−2	1.2762e−2	21
10^{-8}	(0.9799, 2.0630, 3.2744, 3.6424)	8.3182e−2	1.2762e−2	21
10^{-9}	(0.9799, 2.0630, 3.2745, 3.6424)	8.3182e−2	1.2763e−2	21
10^{-10}	(0.9799, 2.0630, 3.2744, 3.6424)	8.3182e−2	1.2762e−2	21

Table 9. Results for $\delta = 10\%$ in Example 3.

ε	(q_1, q_2, q_3, q_4)	L2Err	$\|u^{\delta}_{\varepsilon,2}\|_{0,\Omega}$	Iternum
1	(0.4528, 0.5152, 0.6000, 0.8000)	7.8537e−1	1.3388	6
10^{-1}	(0.9898, 1.2832, 1.3422, 1.4265)	5.7402e−1	6.1713e−1	8
10^{-2}	(1.0578, 1.9473, 2.1533, 2.5388)	3.0866e−1	1.8576e−1	7
10^{-3}	(1.0033, 2.0906, 2.8545, 3.2854)	1.3417e−1	4.4603e−2	18
10^{-4}	(0.9879, 2.0682, 3.1678, 3.5653)	8.6007e−2	1.3114e−2	20
10^{-5}	(0.9858, 2.0620, 3.2216, 3.6091)	8.2857e−2	1.1838e−2	20
10^{-6}	(0.9855, 2.0611, 3.2276, 3.6139)	8.2629e−2	1.1839e−2	20
10^{-7}	(0.9855, 2.0611, 3.2282, 3.6144)	8.2608e−2	1.1841e−2	20
10^{-8}	(0.9855, 2.0611, 3.2282, 3.6144)	8.2606e−2	1.1841e−2	20
10^{-9}	(0.9855, 2.0611, 3.2282, 3.6144)	8.2605e−2	1.1841e−2	20
10^{-10}	(0.9855, 2.0611, 3.2282, 3.6144)	8.2605e−2	1.1841e−2	20

$$\begin{aligned}
\Omega_1 &:= \{(x,y,z) \in \Omega \mid -1 \le x, y < 0, 0 \le z << 1\},\\
\Omega_2 &:= \{(x,y,z) \in \Omega \mid 0 \le x, z \le 1, -1 \le y < 0\},\\
\Omega_3 &:= \{(x,y,z) \in \Omega \mid -1 \le x < 0, 0 \le y, z \le 1\},
\end{aligned}$$

and $\Omega_4 := [0,1]^3$. Then $q^\dagger = \chi_{\Omega_1} + 2\,\chi_{\Omega_2} + 3\,\chi_{\Omega_3} + 4\,\chi_{\Omega_4}$. Six hundred and fifty nodes on Γ are used to sample the data g_2^δ. Assume the admissible set has the same form as the one in Example 2.

Again we choose initial guess $q^0 = 0.1$ and apply the SQP method described above to solve Problem 3.1. The results for $\delta = 0, 5\%, 10\%$ and different ε are reported in Tables 7–9. The

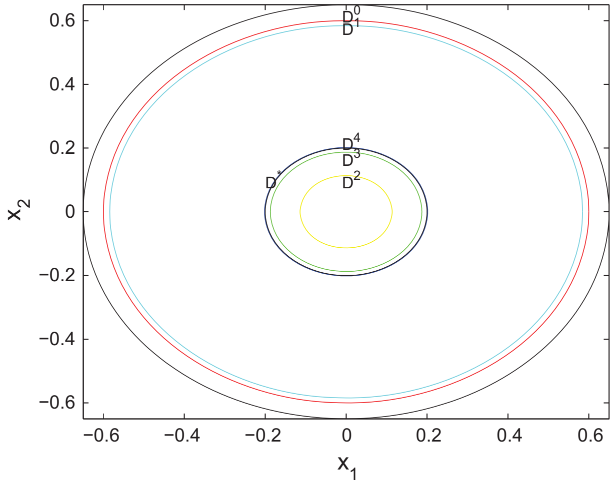

Figure 1. Iterations of D for $k = 10, \delta = 0$ and $\varepsilon = 10^{-6}$.

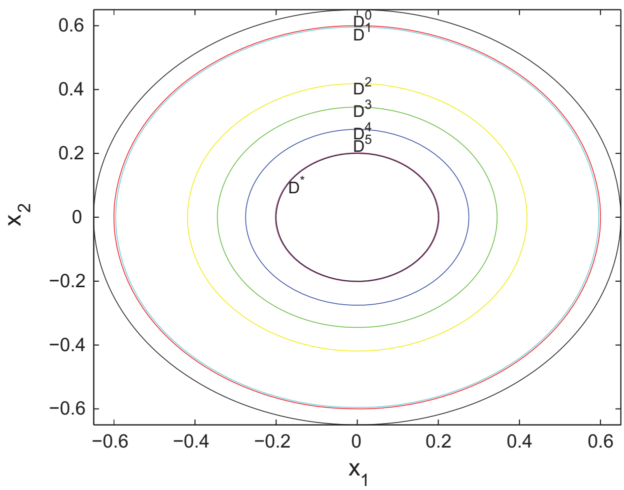

Figure 2. Iterations of D for $k = 0.8, \delta = 0$ and $\varepsilon = 10^{-6}$.

Table 10. Results for $k = 10$ and $\delta = 0\%$ in Example 4.

ε	r^h	L2Err	$\Vert u^{\delta}_{\varepsilon,2}\Vert_{0,\Omega}$	Iternum
10^{-2}	0	8.0822e−1	4.4693e−4	1
10^{-3}	0	8.0822e−1	4.4693e−4	2
10^{-4}	0.0001	8.0822e−1	4.4693e−4	2
10^{-5}	0.1932	2.0912e−1	3.2243e−6	5
10^{-6}	0.2016	1.2883e−1	3.3393e−8	4
10^{-7}	0.2025	1.2880e−1	6.1881e−12	3
10^{-8}	0.2025	1.2880e−1	7.7768e−12	3
10^{-9}	0.2025	1.2880e−1	7.5570e−12	3
10^{-10}	0.2025	1.2880e−1	7.5257e−12	3

Table 11. Results for $k = 10$ and $\delta = 5\%$ in Example 4.

ε	r^h	L2Err	$\lVert u^{\delta}_{\varepsilon,2} \rVert_{0,\Omega}$	Iternum
10^{-2}	0	8.0822e−1	4.4744e−4	1
10^{-3}	0	8.0822e−1	4.4744e−4	2
10^{-4}	0.0001	8.0822e−1	4.4744e−4	2
10^{-5}	0.1938	2.0027e−1	2.9877e−6	5
10^{-6}	0.2016	1.0163e−1	1.7351e−7	4
10^{-7}	0.2026	1.3200e−1	1.3866e−7	3
10^{-8}	0.2026	1.3166e−1	1.3855e−7	3
10^{-9}	0.2026	1.3164e−1	1.3854e−7	3
10^{-10}	0.2026	1.3164e−1	1.3854e−7	3

Table 12. Results for $k = 10$ and $\delta = 10\%$ in Example 4.

ε	r^h	L2Err	$\lVert u^{\delta}_{\varepsilon,2} \rVert_{0,\Omega}$	Iternum
10^{-2}	0	8.0822e−1	4.4824e−4	1
10^{-3}	0	8.0822e−2	4.4824e−4	2
10^{-4}	0.0001	8.0822e−2	4.4824e−4	2
10^{-5}	0.1937	2.0048e−2	3.4131e−6	5
10^{-6}	0.2015	1.0062e−2	5.9041e−8	4
10^{-7}	0.2028	1.3547e−2	5.5513e−12	3
10^{-8}	0.2028	1.3488e−2	5.5469e−12	3
10^{-9}	0.2028	1.3485e−2	5.5467e−12	3
10^{-10}	0.2028	1.3487e−2	5.5468e−12	3

Table 13. Results for $k = 0.8$ and $\delta = 0\%$ in Example 4.

ε	r^h	L2Err	$\lVert u^{\delta}_{\varepsilon,2} \rVert_{0,\Omega}$	Iternum
10^{-2}	0.7213	1.3961e−1	1.3407e−3	3
10^{-3}	0.3067	4.6842e−2	1.9124e−5	4
10^{-4}	0.2122	1.4289e−2	7.5294e−7	5
10^{-5}	0.2029	6.8465e−3	3.5216e−7	5
10^{-6}	0.2019	5.5564e−3	3.1985e−7	5
10^{-7}	0.2018	5.4189e−3	3.1689e−7	5
10^{-8}	0.2018	5.3946e−3	3.1637e−7	5
10^{-9}	0.2018	5.4199e−3	3.1691e−7	5
10^{-10}	0.2018	5.4041e−3	3.1658e−7	5

Table 14. Results for $k = 0.8$ and $\delta = 5\%$ in Example 4.

ε	r^h	L2Err	$\lVert u^{\delta}_{\varepsilon,2} \rVert_{0,\Omega}$	Iternum
10^{-2}	0.7200	1.3934e−1	1.3305e−3	2
10^{-3}	0.3066	4.6826e−2	1.9503e−5	4
10^{-4}	0.2121	1.4238e−2	1.1541e−6	5
10^{-5}	0.2028	6.8176e−3	7.5694e−7	5
10^{-6}	0.1894	1.2948e−2	4.5423e−7	6
10^{-7}	0.2097	1.2669e−2	1.0322e−6	5
10^{-8}	0.1892	1.3066e−2	4.5191e−7	6
10^{-9}	0.2016	5.1639e−3	7.1728e−7	5
10^{-10}	0.2018	5.4292e−3	7.2276e−7	5

Table 15. Results for $k = 0.8$ and $\delta = 10\%$ in Example 4.

ε	r^h	L2Err	$\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$	Iternum
10^{-2}	0.7224	1.3984e−1	1.506e−3	3
10^{-3}	0.3059	4.6635e−2	2.0442e−5	4
10^{-4}	0.2114	1.3817e−2	2.3360e−6	5
10^{-5}	0.2030	6.9604e−3	1.9779e−6	5
10^{-6}	0.2014	4.7602e−3	1.9267e−6	5
10^{-7}	0.1990	4.0329e−3	1.8570e−6	5
10^{-8}	0.2007	3.3257e−3	1.9050e−6	5
10^{-9}	0.2012	4.3293e−3	1.9192e−6	5
10^{-10}	0.2014	4.8325e−3	1.9280e−6	5

corresponding “L2Err”, $\|u^\delta_{\varepsilon,2}\|_{0,\Omega}$ and “Iternum" are shown in the third-fifth columns of Tables 7–9, respectively. Similar to Example 2, the approximations are very well, the reconstruction is stable, and the regularized solution is not sensitive to the small ε.

Example 4: In the last example, we consider an example where the distribution domain of the searched conductivity is unknown. In the literature, most of the numerical experiments are implemented for the case where the coefficient q has the form

$$q = 1 + (k-1)\,\chi_D \quad \text{in } \Omega \tag{29}$$

with a known k and an unknown D. As a result, the ICP reduces to that of recovering a subset D in $\overline{\Omega}$ from the given Cauchy data g_1^δ and g_2^δ on Γ. In the case of a single boundary measurement, the uniqueness is not guaranteed for a general D. Under some condition, the question of the uniqueness is answered when D is a disk,[27] a polygon,[24] a ball,[27,36] and a convex cylinder.[23] We consider a model problem similar to the one in [37]. Specifically, let $k = 10, f = 0$, Ω be the same as Example 1, and the true D^* be a disk centered at $x^* = (0,0)$ with radus r^*. Note that since D is unknown and will move during the iteration, standard finite element methods do not work. We apply the layer potential technique (see [20,27] for instance) to the complex reconstruction framework considered in this paper. For this purpose, we modify the condition $\int_\Omega u\,\mathrm{d}x = 0$ to $\int_\Gamma u\,\mathrm{d}x = 0$, and let $\Gamma(x) := \frac{1}{2\pi}\ln|x|$ the fundamental solution of Δ in $\mathbb{R}^2$.

The Cauchy data g_1 and g_2 are computed from

$$g_1(x) = x_1 + \int_{\partial D^*} \nabla_x\Gamma(x-y)\cdot\nu_x\,\varphi_{D^*}(y)\mathrm{d}s_y, \quad x \in \Gamma,$$

$$g_2(x) = x_1 + \int_{\partial D^*} \Gamma(x-y)\,\varphi_{D^*}(y)\mathrm{d}s_y, \quad x \in \Gamma,$$

where $x = (x_1, x_2)$ and

$$\varphi_{D^*}(x) = \frac{2\,(k-1)}{k+1}\,\frac{x_1}{r^*}.$$

Two hundred nodes on Γ are used for the sampled data g_1 and g_2. Again uniformly distributed noises with a level δ are added to g_1 and g_2 to generate g_1^δ and g_2^δ. Denote $\tilde{Q}_{ad}$ for the set of all disk D contained in Ω. Then Problem 3.1 amounts to solving

$$D^\delta_\varepsilon = \arg\min_{D\in\tilde{Q}_{ad}} \frac{1}{2}\|u_2(D)\|^2_{0,\Omega} + \frac{\varepsilon}{2}\,|k^2-1|\,|D|, \tag{30}$$

where $|D|$ is the measure of D, and $u_2(D) \in V$ is the imaginary part of the solution $u \in \mathbf{V}$ of the problem (21) with q being characterized by (29). Without going into details, we only mention that the solution of the forward problem (7), with g replaced by $g^\delta := g_1^\delta + i\,g_2^\delta$, can be expressed by a sum of two single-layer potentials on Ω and D:

$$u(x) = \int_{\Gamma} \Gamma(x-y)\,\varphi(y)\mathrm{d}\sigma(y) + \int_{\partial D} \Gamma(x-y)\,\psi(y)\mathrm{d}\sigma(y), \quad x \in \Omega.$$

The complex functions φ on Γ and ψ on ∂D are determined from the boundary condition of (7) and the continuous condition $\frac{\partial u^e}{\partial \nu}|_{\partial D} = k\frac{\partial u^i}{\partial \nu}|_{\partial D}$, where $u^e = u|_{\Omega/\bar{D}}$, $u^i = u|_D$.

Then choosing the initial guess $D^0 := \{x \in \mathbb{R}^2 \mid |x| \le 0.6\}$, we apply the SQP method to the optimization problem (30). The results corresponding to $k = 10 > 1$ and $\delta = 0, 5\%, 10\%$ are listed in Tables 10–12. The experiment is repeated for $k = 0.8 < 1$ and the results are reported in Tables 13–15. We plot in Figure 1 the evolution of the approximate D for $k = 10$, $\delta = 0$ and $\varepsilon = 10^{-6}$, where D^1, D^2, D^3, and D^4 are the approximations of D^* at iterations 1–4. The approximations of D^* are plotted in Figure 2 for $k = 0.8$, $\delta = 0$, and $\varepsilon = 10^{-6}$, with D^1, D^2, D^3, D^4, and D^5 corresponding to iterations 1–5 respectively. The same kind of conclusions on the performance of the proposed method as in the previous examples can be drawn from Tables 10–15 and Figures 1 and 2.

5. Concluding remarks

We conclude from Examples 1–3 that as long as all the subdomains share some portion of the boundary, with the CCBM-based Tikhonov regularization method presented in this work, the piecewise constant diffusion parameters can be constructed stably and effectively. As is shown in Example 4, our method also works well in the case where the distributions of the positions of the piecewise constant diffusion parameters are not known exactly. All numerical experiments indicate that the iteration converges even when the initial guess is not close to the solution. Moreover, the results are satisfactory for rather small regularization parameters. In comparison, in standard regularization methods for the inverse parameter problems, choosing a proper regularization parameter is crucial. Our new method allows the use of very small regularization parameters based solely on the consideration of the solution accuracy, rather than on a balance between the solution accuracy and the solution stability.

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