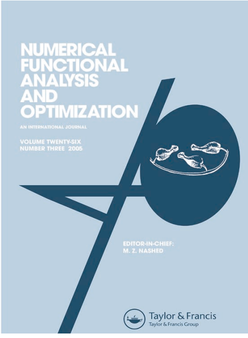

Numerical Functional Analysis and Optimization

ISSN: 0163-0563 (Print) 1532-2467 (Online) Journal homepage: http://www.tandfonline.com/loi/lnfa20

Discontinuous Galerkin Methods for Solving a Frictional Contact Problem with Normal Compliance

Wenqiang Xiao, Fei Wang & Weimin Han

NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
https://doi.org/10.1080/01630563.2018.1472609

Taylor & Francis
Taylor & Francis Group

Discontinuous Galerkin Methods for Solving a Frictional Contact Problem with Normal Compliance

Wenqiang Xiao[a], Fei Wang[a], and Weimin Han[a,b]

[a]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, China;
[b]Department of Mathematics, University of Iowa, Iowa City, Iowa, USA

ABSTRACT
Several discontinuous Galerkin (DG) methods are introduced for solving a frictional contact problem with normal compliance, which is modeled as a quasi-variational inequality. Consistency, boundedness, and stability are established for the DG methods. Two numerical examples are presented to illustrate the performance of the DG methods.

ARTICLE HISTORY
Received 17 August 2017
Accepted 30 April 2018

KEYWORDS
Contact problem; discontinuous Galerkin methods; normal compliance; quasi-variational inequality

AMS CLASSIFICATION
65N30; 49J40

1. Introduction

In industry and daily life, processes of frictional contact between deformable bodies or between a deformable body and a rigid foundation are very common. Much effort has been made in modeling, analysis, and numerical simulations of the frictional contact processes. The normal compliance contact condition was proposed in [1] in the study of dynamic contact problems, and it allows interpenetration of the body's surface into the foundation. Contact problems with a normal compliance condition have been studied in many articles, e.g. [2–9]. A frictional contact problem with normal compliance can be described by a quasi-variational inequality, and its existence and uniqueness are proved by using fixed-point arguments [10, 11]. Conforming finite element methods (FEMs) were studied for these problems, and a priori error estimates were derived in [6–8, 10]. Under a smallness assumption on the material coefficient, a priori error estimates were derived in [6]. The frictional contact problem with a reduced normal compliance law was studied in [8], and a Ceá-type error inequality was derived there. In [1], a priori error estimates of finite element

CONTACT Fei Wang feiwang.xjtu@xjtu.edu.cn School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, China.
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/lnfa.

approximation for a dynamic contact problem was established. A contact problem with the nonlinear elastic constitutive law was studied, and a priori error estimates was derived in [10]. For other more complicated contact problems with normal compliance, we refer to [12, 13] and the references therein.

Unlike the standard FEMs, discontinuous Galerkin (DG) methods do not require the ordinary inter-element smoothness in the discrete function spaces. In DG methods, a boundary value problem is discretized in an element-by-element fashion, and neighboring elements are joined together through the use of numerical traces. It is easier to construct DG methods that are locally conservative and that can capture non-smooth or oscillatory solutions effectively. Consequently, the DG methods have attracted the interest of many applied mathematicians and engineers, and have been applied to solve various differential equations in the past three decades. For a second-order boundary value problem, since no inter-element continuity is required in the discrete function spaces, DG methods allow general meshes with hanging nodes and elements of different shapes. In addition, locality of the discretization makes the DG methods ideally suited for parallel computing. The compact formulation can be applied near boundaries without special treatment, which increases the robustness of any boundary condition implementation (see [14–16] and the references therein).

In recent years, discontinuous Galerkin methods have been developed for solving a variety of variational inequalities, such as the gradient plasticity problem [17, 18], obstacle problems [19, 20], Signorini problem [21, 22], quasi-static contact problems [23], the plate contact problem [24–26], two membranes problem [27], and Stokes or Navier–Stokes flows with slip boundary condition [28, 29]. In this paper, we study DG methods for solving a quasi-variational inequality arising in frictional contact problems with normal compliance. We consider a process in which an elastic body, under the influence of a given body force and surface traction, comes into contact with a deformable foundation, and there exists penetration of the elastic body into the foundation. Therefore, the normal compliance condition is adopted and Coulomb's law of dry friction is used to describe the frictional phenomenon. Since the friction bound function depends on the unknown variable, the problem is highly nonlinear, and in the literature, no reference can be found on the study of DG methods for such a problem. We introduce four DG schemes to solve this problem and analyze properties of the methods. For numerical implementation, we use Uzawa algorithm to circumvent the difficulty from the non-differentiable term.

The paper is organized as follows: In Section 2, we introduce the frictional contact problem with normal compliance, and present its weak formulation as a quasi-variational inequality. In Section 3, we give the DG formulations, and show the consistency of the DG schemes, boundedness, and stability of the bilinear forms. Finally, in Section 4, we present results from numerical examples, paying particular attention to numerical convergence orders.

2. A frictional contact problem with normal compliance

Let $\Omega \subset R^d$ $(d=2,\ 3)$ be a bounded open connected domain. A Lipschitz boundary Γ of Ω consists of three non-overlapping parts Γ_D, Γ_F, and Γ_C, where displacement, force (surface traction), and contact boundary conditions will be specified, respectively. We use a vector-valued function $\boldsymbol{u} : \Omega \subset R^d \to R^d$ to denote the displacement, and use $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ and $\boldsymbol{\sigma}$ for the linearized strain tensor and the stress tensor, respectively, which belongs to S^d, the space of second order symmetric tensors on R^d. Let $\boldsymbol{\sigma}: \boldsymbol{\phi} = \sigma_{\mathrm{ij}}\phi_{\mathrm{ij}}$ be the inner product on space S^d, and the corresponding norm is $|\boldsymbol{\phi}| := (\phi : \phi)^{\frac{1}{2}}$. Here and below, the summation convention over a repeated index is adopted. For a vector $\boldsymbol{w}$, its normal component and tangential component on the boundary are $v_\nu = \boldsymbol{w} \cdot \boldsymbol{v}$, $\boldsymbol{w}_\tau = \boldsymbol{w} - \boldsymbol{w}_\nu \boldsymbol{v}$, where $\boldsymbol{v}$ is the unit outward normal vector on Γ. For a tensor-valued function $\boldsymbol{\sigma}$, the normal component and tangential component are $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{v}) \cdot \boldsymbol{v}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{v} - \sigma_\nu \boldsymbol{v}$, respectively. Due to the fact that $\boldsymbol{w}_\tau \cdot \boldsymbol{v} = 0$, $\boldsymbol{\sigma}_\tau \cdot \boldsymbol{v} = 0$, we have the decomposition formula

$$(\boldsymbol{\sigma}\boldsymbol{v}) \cdot \boldsymbol{w} = (\sigma_\nu \boldsymbol{v} + \boldsymbol{\sigma}_\tau) \cdot (\boldsymbol{w}_\nu \boldsymbol{v} + \boldsymbol{w}_\tau) = \sigma_\nu \boldsymbol{w}_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{w}_\tau. \tag{2.1}$$

$\mathrm{div}\boldsymbol{\sigma} = (\partial_{\mathrm{j}}\sigma_{\mathrm{ij}})_{1\le \mathrm{i} \le \mathrm{d}}$ is divergence of a tensor-valued function $\boldsymbol{\sigma}$. If vector $\boldsymbol{w}$ and tensor $\boldsymbol{\sigma}$ are continuously differentiable, then, we have the integration by a part formula:

$$\int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{w}) dx = \int_\Gamma (\boldsymbol{\sigma}\boldsymbol{v}) \cdot \boldsymbol{w} ds - \int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \boldsymbol{w} dx. \tag{2.2}$$

Consider the following frictional contact problem with normal compliance,

$$\boldsymbol{\sigma} = \mathcal{C}\varepsilon(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.3}$$

$$-\mathrm{div}\ \boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{in } \Omega, \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{v} = \boldsymbol{f}_2 \quad \text{on } \Gamma_{\mathrm{F}}, \tag{2.6}$$

$$-\sigma_\nu = p_\nu(u_\nu - g_a) \quad \text{on } \Gamma_C, \tag{2.7}$$

$$|\boldsymbol{\sigma}_\tau| \le p_\tau(u_\nu - g_a) \quad \text{on } \Gamma_C, \tag{2.8}$$

$$\boldsymbol{\sigma}_\tau = -p_\tau(u_\nu - g_a)\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \quad \text{if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \text{ on } \Gamma_C. \tag{2.9}$$

Here, (2.3) represents the constitutive relation of the elastic material, (2.4) is the equilibrium equation, in which volume forces of density $\boldsymbol{f}_1$ acts in Ω. Boundary condition (2.5) means that the body is clamped on Γ_D, so

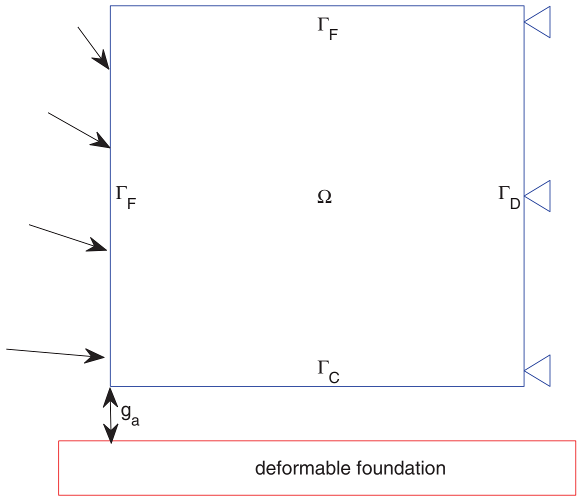

Figure 1. Setting of the problem for Example 4.1.

the displacement field vanishes there. Surface traction of density $\boldsymbol{f}_2$ acts on Γ_F in (2.6). (2.7) is the normal compliance condition, and it describes a reactive normal pressure that depends on the penetration of the elastic body in the deformable foundation. g_a is the initial gap between the body and foundation (Figure 1 in Section 4), u_ν is the normal displacement, and $u_\nu - g_a$, when it is positive, represents the penetration of the body in the foundation. The function p_ν is nonnegative with the property $p_\nu(t) = 0$ for $t \le 0$. The relations (2.8)–(2.9) form a version of Coulomb's law of dry friction, and are equivalent to

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \le p_\tau(u_\nu - g_a), \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu - g_a) \Rightarrow \boldsymbol{u}_\tau = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu - g_a) \Rightarrow \boldsymbol{\sigma}_\tau = -\kappa \boldsymbol{u}_\tau, \kappa \ge 0. \end{cases} \tag{2.10}$$

Here, $\boldsymbol{u}_\tau$ denotes the tangential displacement. p_τ is *friction bound* function, so it is nonnegative and $p_\tau(t) = 0$ for $t \le 0$. $\boldsymbol{\sigma}_\tau$ denotes the tangential force on Γ_C, the contact boundary. Obviously, the shear stress cannot exceed the maximal frictional resistance $p_\tau(u_\nu - g_a)$. When the strict inequality holds, it is *stick* state, i.e., the surface adheres to the foundation; and when the equality holds, a relative sliding happens, this is *slip* state; when $u_\nu < g_a$, there is no contact. Therefore, the contact surface Γ_C is divided into three zones: stick, slip, and separation.

We will make some assumptions on the data. The fourth-order elasticity tensor of the material $\mathcal{C} : \Omega \times S^d \to S^d$ is assumed to satisfy the following conditions:

$$\begin{cases} (a) & \mathcal{C}_{ijkl} \in L^\infty(\Omega), 1 \le i,j,k,l \le d; \\ (b) & (\mathcal{C}\boldsymbol{\sigma}) : \phi = \boldsymbol{\sigma} : (\mathcal{C}\phi) \quad \forall \boldsymbol{\sigma}, \phi \in \mathrm{S}^\mathrm{d}, \text{a.e. in } \Omega; \\ (c) & \exists \mathrm{C}_0 > 0 \text{ s.t. } \mathcal{C}\phi : \phi \ge \mathrm{C}_0|\phi|^2, \quad \forall \phi \in \mathrm{S}^\mathrm{d}, \text{a.e. in } \Omega. \end{cases} \tag{2.11}$$

For the elasticity tensor of a homogeneous and isotropic elastic material, $\mathcal{C}\varepsilon = \lambda(\mathrm{tr}\varepsilon)I + 2\mu\varepsilon$ with Lamé parameters $\lambda > 0$ and $\mu > 0$. The gap function has the properties

$$g_a \in L^2(\Gamma_C), \qquad g_a(\boldsymbol{x}) \geq 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \tag{2.12}$$

For the functions p_e $(e = \nu, \tau)$, we assume

$$\begin{cases} (a) & p_e : \Gamma_C \times R \to R_+; \\ (b) & \text{there exists } L_e > 0 \text{ such that } |p_e(\boldsymbol{x}, r_1) - p_e(\boldsymbol{x}, r_2)| \leq L_e |r_1 - r_2| \\ & \quad \forall r_1, r_2 \in R, \text{a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c) & \text{the mapping } \boldsymbol{x} \mapsto p_e(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_C, \text{ for any } r \in R; \\ (d) & p_e(\boldsymbol{x}, r) = 0 \text{ for all } r \leq 0, \quad \text{a.e. } \boldsymbol{x} \in \Gamma_C. \end{cases} \tag{2.13}$$

The condition (2.13)(b) means the functions to grow at most linearly. If there is separation between the body and the foundation, the condition (2.13)(d) implies that the normal and tangential components of the stress tensor vanish on the boundary Γ_C.

We define $V = \{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_D\}$. Then, by a standard procedure, we can derive the weak formulation of the problem (2.3)–(2.9): find $\boldsymbol{u} \in V$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in V, \tag{2.14}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V,$$
$$(\boldsymbol{f}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v} dx + \int_{\Gamma_F} \boldsymbol{f}_2 \cdot \boldsymbol{v} ds \quad \forall \boldsymbol{v} \in V,$$

and

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} p_\nu(u_\nu - g_a) v_\nu ds + \int_{\Gamma_C} p_\tau(u_\nu - g_a) |\boldsymbol{v}_\tau| ds \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{2.15}$$

Here, the functional j depends on u_ν, which makes this problem highly nonlinear. The existence of a unique solution for the quasi-variational inequality (2.14) is studied in several references, see, e.g. [11].

3. Discontinuous Galerkin methods

3.1. Notation

For simplicity, we only consider the case $d = 2$ in this paper; the three-dimensional case can be analyzed similarly. For a bounded domain

$D \subset R^2, H^m(D)$ is the standard Sobolev space, with the corresponding norm $||\cdot||_{m,D}$ and semi-norm $|\cdot|_{m,D}$. For a vector $\boldsymbol{u} = (u_1, u_2) \in [H^m(\Omega)]^2$, its norm and semi-norm are $||\boldsymbol{u}||_{m,D} := \left(\sum_{i=1}^{2} ||u_i||_{m,D}^2\right)^{1/2}$ and $|\boldsymbol{u}|_{m,D} := \left(\sum_{i=1}^{2} |u_i|_{m,D}^2\right)^{1/2}$. These definitions are readily extended for the norm and semi-norm of a matrix-valued function. To describe the numerical methods, we will need the following symbols:

$$
\begin{aligned}
&\{\mathcal{T}_h\}_h := \text{a family of regular triangulations of } \Omega,\\
&\mathcal{E}_h := \text{the set of all edges of } \mathcal{T}_h,\\
&\mathcal{E}_h^i := \mathcal{E}_h \backslash \Gamma,\\
&\mathcal{E}_h^0 := \mathcal{E}_h \backslash (\Gamma_F \cup \Gamma_C),\\
&K := \text{a triangle element} \in \mathcal{T}_h,\\
&h_K := \text{diam}(K),\\
&h := \max\{h_K : K \in \mathcal{T}_h\}.
\end{aligned}
$$

Then, we introduce finite element spaces:

$$
\begin{aligned}
V_h &= \{\boldsymbol{v}_h \in [L^2(\Omega)]^2 : v_{hi}|_K \in P_1(K)\,\forall K \in \mathcal{T}_h, i = 1, 2\},\\
Q_h &= \{\phi_h \in [L^2(\Omega)]_s^{2\times 2} : \phi_{hij}|_K \in P_l(K)\,\forall K \in \mathcal{T}_h, i, j = 1, 2\}, \qquad l = 0 \text{ or } 1.
\end{aligned}
$$

Here, $P_l(K)$ is the space of all polynomials in K with the total degree no more than $l \geq 0$. On any element, $K \in \mathcal{T}_h$, $\varepsilon_h(\boldsymbol{v})$, and $\text{div}_h\phi$ are defined by the relations $\varepsilon_h(\boldsymbol{v}) = \varepsilon(\boldsymbol{v})$ and $\text{div}_h\phi = \text{div}\phi$ for any vector-valued function $\boldsymbol{v}$ and matrix-valued function ϕ.

Let $e \in \mathcal{E}_h^i$ be an interior edge shared by two neighboring elements K^+ and K^-, and $\boldsymbol{n}^\pm = \boldsymbol{n}|_{\partial K^\pm}$ is the unit outward normal vector on $\partial K^\pm$. Then, for a vector-valued function $\boldsymbol{v}$ and a matrix-valued function ϕ, averages $\{\cdot\}$, and jumps $[\cdot]$, $[\![\cdot]\!]$ across the edge e are defined as follows:

$$
\begin{aligned}
&\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \qquad [\![\boldsymbol{v}]\!] = \frac{1}{2}(\boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{n}^+ \otimes \boldsymbol{v}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^- + \boldsymbol{n}^- \otimes \boldsymbol{v}^-),\\
&\{\phi\} = \frac{1}{2}(\phi^+ + \phi^-), \qquad [\phi] = \phi^+\boldsymbol{n}^+ + \phi^-\boldsymbol{n}^-,
\end{aligned}
$$

where

$$
\boldsymbol{v}^\pm = \boldsymbol{v}|_{\partial K^\pm}, \quad \phi^\pm = \phi|_{\partial K^\pm}.
$$

If e lies on the boundary Γ, we define

$$
\begin{aligned}
&\{\boldsymbol{v}\} = \boldsymbol{v}, \qquad [\![\boldsymbol{v}]\!] = \frac{1}{2}(\boldsymbol{v} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{v}),\\
&\{\phi\} = \phi, \qquad [\phi] = \phi\boldsymbol{n}.
\end{aligned}
$$

Here, $\boldsymbol{u} \otimes \boldsymbol{v}$ is a matrix with $u_i v_j$ as its (i, j)-th element.

For a vector-valued function $\boldsymbol{v}$ and a matrix-valued function ϕ, by direct calculation, we can verify the following identity:

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\phi\boldsymbol{n}_K)\cdot\boldsymbol{v}ds=\sum_{e\in\mathcal{E}_h^i}\int_e[\phi]\cdot\{\boldsymbol{v}\}ds+\sum_{e\in\mathcal{E}_h}\int_e\{\phi\}:[\![\boldsymbol{v}]\!]ds. \tag{3.1}$$

To present the DG schemes, we introduce two lifting operators $\boldsymbol{r}_0:\left(L^2(\mathcal{E}_h^0)\right)_s^{2\times2}\to Q_h$ and $\boldsymbol{r}_e:\left(L^2(e)\right)_s^{2\times2}\to Q_h$, defined by

$$\int_\Omega\boldsymbol{r}_0(\phi):\xi dx=-\int_{\mathcal{E}_\mathrm{h}^0}\phi:\{\xi\}ds\quad\forall\xi\in\mathrm{Q_h}, \tag{3.2}$$

$$\int_\Omega\boldsymbol{r}_e(\phi):\xi dx=-\int_\mathrm{e}\phi:\{\xi\}ds\quad\forall\xi\in\mathrm{Q_h}. \tag{3.3}$$

The two lifting operators are related by the equality

$$\boldsymbol{r}_0(\phi)=\sum_{e\in\mathcal{E}_h^0}\boldsymbol{r}_e(\phi|_e)\quad\forall\phi\in\left(L^2(\mathcal{E}_h^0)\right)_s^{2\times2}.$$

Thus,

$$||\boldsymbol{r}_0(\phi)||^2=||\sum_{e\in\mathcal{E}_h^0}\boldsymbol{r}_e(\phi|_e)||^2\le3\sum_{e\in\mathcal{E}_h^0}||\boldsymbol{r}_e(\phi|_e)||^2. \tag{3.4}$$

3.2. DG formulations

In this subsection, we present some DG formulations for solving the quasi-variational inequality (2.14). On any element $K\in\mathcal{T}_h$, using integration by part formula, we multiply (2.3) and (2.4) by $\mathcal{C}^{-1}\phi$ and $\boldsymbol{v}$, respectively, integrate over an element $K\in\mathcal{T}_h$, and perform integration by parts to get:

$$\int_K\mathcal{C}^{-1}\boldsymbol{\sigma}:\phi dx=-\int_\mathrm{K}\boldsymbol{u}\cdot\mathrm{div}\phi dx+\int_{\partial\mathrm{K}}\boldsymbol{u}\cdot(\phi\boldsymbol{n}_K)ds,$$
$$\int_K\boldsymbol{f}_1\cdot\boldsymbol{v}dx=\int_K\boldsymbol{\sigma}:\varepsilon(\boldsymbol{v})dx-\int_{\partial K}(\boldsymbol{\sigma}\boldsymbol{n}_K)\cdot\boldsymbol{v}ds.$$

A subscript h is added on $\boldsymbol{\sigma},\phi,\boldsymbol{u},\boldsymbol{v}$, div and ε in these equations. Then, we sum these two equalities over all the elements $K\in\mathcal{T}_h$, and use numerical traces $\widehat{\boldsymbol{u}}_h$ and $\widehat{\boldsymbol{\sigma}}_h$ to approximate $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ over element edges:

$$\int_\Omega\mathcal{C}^{-1}\boldsymbol{\sigma}_h:\phi_h dx=-\int_\Omega\boldsymbol{u}_h\cdot\mathrm{div}_h\phi_h dx+\sum_{K\in T_h}\int_{\partial K}\widehat{\boldsymbol{u}}_h\cdot(\phi_h\boldsymbol{n}_K)ds,$$

$$\int_\Omega\boldsymbol{f}_1\cdot\boldsymbol{v}_h dx=\int_\Omega\boldsymbol{\sigma}_h:\varepsilon_h(\boldsymbol{v}_h)dx-\sum_{K\in T_h}\int_{\partial K}(\widehat{\boldsymbol{\sigma}}_h\boldsymbol{n}_K)\cdot\boldsymbol{v}_h ds$$

for all $(\phi_h, \boldsymbol{v}_h) \in Q_h \times V_h$. to choose the numerical fluxes $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{\boldsymbol{u}}_h$, we should guarantee the stability and consistency of the scheme.

To derive a formulation, which does not rely on $\boldsymbol{\sigma}_h$ explicitly, following an argument similar to the one used in [21, 23], we have:

$$\begin{aligned}\int_\Omega \mathcal{C}\varepsilon_h(\boldsymbol{u}_h) : \varepsilon_h(\boldsymbol{v}_h)dx + \int_{\mathcal{E}_h^i} \{\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h\} \cdot [\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)]ds \\ + \int_{\mathcal{E}_h} [\![\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h]\!] : \{\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)\}ds \\ - \int_{\mathcal{E}_h^i} [\widehat{\boldsymbol{\sigma}}_h] \cdot \{\boldsymbol{v}_h\}ds - \int_{\mathcal{E}_h} [\![\boldsymbol{v}_h]\!] : \{\widehat{\boldsymbol{\sigma}}_h\}ds = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v}_h dx.\end{aligned} \tag{3.5}$$

Then, we can obtain DG schemes from (3.5) through proper choices of numerical fluxes.

In order to be consistent with the boundary condition, for all the DG schemes in this paper, we always make the following choices:

$$\begin{cases} \widehat{\boldsymbol{u}}_h = \{\boldsymbol{u}_h\} \quad \text{on } \mathcal{E}_h \backslash \Gamma_D, \\ \widehat{\boldsymbol{u}}_h = \boldsymbol{0} \quad \text{on } \Gamma_D, \\ \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_\mathrm{F}, \end{cases} \tag{3.6}$$

and on Γ_C,

$$\begin{cases} -\hat{\sigma}_{h\nu} = p_\nu(\hat{u}_{h\nu} - g_a), \\ |\widehat{\boldsymbol{\sigma}}_{h\tau}| \le p_\tau(\hat{u}_{h\nu} - g_a), \\ \hat{\sigma}_{h\tau} = -p_\tau(\hat{u}_{h\nu} - g_a)\dfrac{\widehat{\boldsymbol{u}}_{h\tau}}{|\widehat{\boldsymbol{u}}_{h\tau}|} \quad \text{if } \widehat{\boldsymbol{u}}_{h\tau} \neq \boldsymbol{0}. \end{cases} \tag{3.7}$$

Note that $\widehat{\boldsymbol{u}}_h = \boldsymbol{u}_h$ on $\Gamma_C \cup \Gamma_F$.

To develop an interior penalty (IP) formulation ([30]), we choose $\widehat{\boldsymbol{\sigma}}_h = \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\} - \eta_e h_e^{-1}[\![\boldsymbol{u}_h]\!]$ on $\mathcal{E}_h^0$, where η_e is a bounded, positive, piecewise constant function on $\mathcal{E}_h^0$. Then we obtain from (3.5) that

$$B_{1,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v}_h dx + \int_{\Gamma_F} \boldsymbol{f}_2 \cdot \boldsymbol{v}_h ds + \int_{\Gamma_C} \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} \cdot \boldsymbol{v}_h ds, \tag{3.8}$$

where

$$\begin{aligned}B_{1,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega \mathcal{C}\varepsilon_h(\boldsymbol{u}_h) : \varepsilon_h(\boldsymbol{v}_h)dx - \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_h]\!] : \{\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)\}ds \\ - \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h]\!] : \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\}ds + \int_{\mathcal{E}_h^0} \eta_e h_e^{-1} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]ds.\end{aligned} \tag{3.9}$$

Let $\boldsymbol{v}_h = \boldsymbol{w}_h - \boldsymbol{u}_h$ with $\boldsymbol{w}_h \in V_h$. Using (2.1), we have

$$\int_{\Gamma_C} (\widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu}) \cdot (\boldsymbol{w}_h - \boldsymbol{u}_h)ds = \int_{\Gamma_C} \widehat{\sigma}_{h\nu}(\mathrm{w}_{h\nu} - \mathrm{u}_{h\nu})ds + \int_{\Gamma_C} \widehat{\boldsymbol{\sigma}}_{h\tau} \cdot (\boldsymbol{w}_{h\tau} - \boldsymbol{u}_{h\tau})ds.$$

Table 1. Choices of $\hat{\boldsymbol{\sigma}}_h$ on $\mathcal{E}_h^0$.

Methods	Numerical flux $\hat{\boldsymbol{\sigma}}_h$ on $\mathcal{E}_h^0$
IP [30]	$\hat{\boldsymbol{\sigma}}_h = \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\} - \eta_e h_e^{-1}[\![\boldsymbol{u}_h]\!]$
Bassi et al. [31]	$\hat{\boldsymbol{\sigma}}_h = \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\} + \eta_e\{\mathcal{C}\boldsymbol{r}_e([\![\boldsymbol{u}_h]\!])\}$
Brezzi et al. [32]	$\hat{\boldsymbol{\sigma}}_h = \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\} + \{\mathcal{C}\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\} + \eta_e\{\mathcal{C}\boldsymbol{r}_e([\![\boldsymbol{u}_h]\!])\}$
LDG [33]	$\hat{\boldsymbol{\sigma}}_h = \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\} + \{\mathcal{C}\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\} - \eta_e h_e^{-1}[\![\boldsymbol{u}_h]\!]$

By (3.7), we have on Γ_C,

$$\begin{aligned}\hat{\sigma}_{h\nu}(w_{h\nu} - u_{h\nu}) &= -p_\nu(\hat{u}_{h\nu} - g_a)(w_{h\nu} - u_{h\nu})\\ &= -p_\nu(u_{h\nu} - g_a)w_{h\nu} + p_\nu(u_{h\nu} - g_a)u_{h\nu},\end{aligned}$$

and

$$\begin{aligned}\hat{\sigma}_{h\tau}\cdot(\boldsymbol{w}_{h\tau} - \boldsymbol{u}_{h\tau}) &= \hat{\sigma}_{h\tau}\cdot\boldsymbol{w}_{h\tau} - \hat{\sigma}_{h\tau}\cdot\boldsymbol{u}_{h\tau}\\ &\geq -|\hat{\sigma}_{h\tau}||\boldsymbol{w}_{h\tau}| - \hat{\sigma}_{h\tau}\cdot\boldsymbol{u}_{h\tau}\\ &\geq -p_\tau(u_{h\nu} - g_a)|\boldsymbol{w}_{h\tau}| + p_\tau(u_{h\nu} - g_a)|\boldsymbol{u}_{h\tau}|.\end{aligned}$$

Therefore, we obtain the IP scheme from (3.8),

$$B_{1,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + j(\boldsymbol{u}_h, \boldsymbol{w}_h) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq (\boldsymbol{f}, \boldsymbol{w}_h - \boldsymbol{u}_h) \quad \forall \boldsymbol{w}_h \in V_h. \tag{3.10}$$

With the help of the lift operator $\boldsymbol{r}_0$, we can rewrite $B_{1,h}^{(1)}$ as

$$\begin{aligned}B_{2,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{v}_h) :=& \int_\Omega \mathcal{C}\varepsilon_h(\boldsymbol{u}_h) : (\varepsilon_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!]))dx\\ &+ \int_\Omega \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]) : \mathcal{C}\varepsilon_h(\boldsymbol{v}_h)dx + \int_{\mathcal{E}_h^0} \eta_e h_e^{-1}[\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]ds.\end{aligned} \tag{3.11}$$

Note that (3.9) and (3.11) are equivalent on V_h.

Similarly, we can introduce three other DG methods. We list the choices of $\widehat{\boldsymbol{\sigma}_h}$ for these methods in Table 1. Furthermore, we list the bilinear forms of the DG methods in Table 2 with $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{E}_h^0}$, and

$$\begin{aligned}g &= \int_\Omega \mathcal{C}\varepsilon_h(\boldsymbol{u}_h) : \varepsilon_h(\boldsymbol{v}_h)dx,\\ \beta^j &= \int_{\mathcal{E}_h^0} \eta_e h_e^{-1}[\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]ds,\\ \beta^r &= \sum_{e\in\mathcal{E}_h^0}\int_\Omega \eta_e \mathcal{C}\boldsymbol{r}_e([\![\boldsymbol{u}_h]\!]) : \boldsymbol{r}_e([\![\boldsymbol{v}_h]\!])dx.\end{aligned}$$

Let B_h represent one of the four bilinear forms $B_{1,h}^{(i)}$, $1 \leq i \leq 4$. The corresponding DG formulation is to find $\boldsymbol{u}_h \in V_h$ such that

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + j(\boldsymbol{u}_h, \boldsymbol{v}_h) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h) \quad \forall \boldsymbol{v}_h \in V_h. \tag{3.12}$$

Table 2. DG formulations.

Methods	Bilinear forms
IP [30]	$B_{1,h}^{(1)} = g - \langle [\![\boldsymbol{u}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)\}\rangle - \langle [\![\boldsymbol{v}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\}\rangle + \beta^j$
	$B_{2,h}^{(1)} = g + (\mathcal{C}\varepsilon_h(\boldsymbol{u}_h), \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + (\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]), \mathcal{C}\varepsilon_h(\boldsymbol{v}_h)) + \beta^j$
Bassi et al. [31]	$B_{1,h}^{(2)} = g - \langle [\![\boldsymbol{u}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)\}\rangle - \langle [\![\boldsymbol{v}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\}\rangle + \beta^r$
	$B_{2,h}^{(2)} = g + (\mathcal{C}\varepsilon_h(\boldsymbol{u}_h), \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + (\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]), \mathcal{C}\varepsilon_h(\boldsymbol{v}_h)) + \beta^r$
Brezzi et al. [32]	$B_{1,h}^{(3)} = g - \langle [\![\boldsymbol{u}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)\}\rangle - \langle [\![\boldsymbol{v}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\}\rangle + (\mathcal{C}\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]), \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + \beta^r$
	$B_{2,h}^{(3)} = g + (\mathcal{C}\varepsilon_h(\boldsymbol{u}_h), \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + (\mathcal{C}\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]), \varepsilon_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + \beta^r$
LDG [33]	$B_{1,h}^{(4)} = g - \langle [\![\boldsymbol{u}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{v}_h)\}\rangle - \langle [\![\boldsymbol{v}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{u}_h)\}\rangle + (\mathcal{C}\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]), \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + \beta^j$
	$B_{2,h}^{(4)} = g + (\mathcal{C}\varepsilon_h(\boldsymbol{u}_h), \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + (\mathcal{C}\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]), \varepsilon_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + \beta^j$

3.3. Consistency, boundedness, and stability

Here, we present some properties of the four DG methods introduced previously. Recall that $\boldsymbol{u}$ is the solution of (2.14).

Lemma 3.1 (Consistency). *Assume* $\boldsymbol{u} \in [H^2(\Omega)]^2$. *Then for* $B_h(\boldsymbol{w}, \boldsymbol{v}) = B_{1,h}^{(i)}(\boldsymbol{w}, \boldsymbol{v})$ *with* $i = 1, \cdots, 4$, *we have:*

$$B_h(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v}_h) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}) \quad \forall \boldsymbol{v}_h \in V_h. \tag{3.13}$$

Proof. Since $\boldsymbol{u} \in [H^2(\Omega)]^2$, on any interior edge e, $[\![\boldsymbol{u}]\!] = \boldsymbol{0}$, $\{\boldsymbol{u}\} = \boldsymbol{u}$, $\{\varepsilon(\boldsymbol{u})\} = \varepsilon(\boldsymbol{u})$, $[\boldsymbol{\sigma}] = \boldsymbol{0}$, $\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}$. Then, for any $\boldsymbol{v}_h \in V_h$,

$$\begin{aligned} B_h(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) &= \int_\Omega \mathcal{C}\varepsilon(\boldsymbol{u}) : \varepsilon_h(\boldsymbol{v}_h - \boldsymbol{u}) dx - \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h - \boldsymbol{u}]\!] : \mathcal{C}\varepsilon(\boldsymbol{u}) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \varepsilon_h(\boldsymbol{v}_h - \boldsymbol{u}) dx - \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h - \boldsymbol{u}]\!] : \boldsymbol{\sigma} ds \end{aligned}$$

Using integration by parts and (3.1), we get,

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \varepsilon_h(\boldsymbol{v}_h - \boldsymbol{u}) dx &= -\sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}\boldsymbol{\sigma} \cdot (\boldsymbol{v}_h - \boldsymbol{u}) dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\boldsymbol{\sigma}\boldsymbol{n}_K) \cdot (\boldsymbol{v}_h - \boldsymbol{u}) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{f}_1 \cdot (\boldsymbol{v}_h - \boldsymbol{u}) dx + \int_{\mathcal{E}_h} \boldsymbol{\sigma} : [\![\boldsymbol{v}_h - \boldsymbol{u}]\!] ds. \end{aligned}$$

Therefore,

$$\begin{aligned} B_h(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) &= \int_\Omega \boldsymbol{f}_1 \cdot (\boldsymbol{v}_h - \boldsymbol{u}) dx + \int_{\Gamma_F} \boldsymbol{f}_2 \cdot (\boldsymbol{v}_h - \boldsymbol{u}) ds + \int_{\Gamma_C} (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot (\boldsymbol{v}_h - \boldsymbol{u}) ds \\ &= (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}) + \int_{\Gamma_C} (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot (\boldsymbol{v}_h - \boldsymbol{u}) ds. \end{aligned} \tag{3.14}$$

Here,

$$(\boldsymbol{\sigma\nu})\cdot(\boldsymbol{v}_h-\boldsymbol{u})=\sigma_\nu(v_{h\nu}-\mathrm{u}_\nu)+\boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_{h\tau}-\boldsymbol{u}_\tau).$$

Using the boundary conditions (2.7)–(2.9) and (2.13), we have:

$$\begin{aligned}\sigma_\nu(v_{h\nu}-u_\nu)&=-p_\nu(u_\nu-g_a)(v_{h\nu}-u_\nu)\\&=-p_\nu(u_\nu-g_a)v_{h\nu}+p_\nu(u_\nu-g_a)u_\nu,\\ \boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_{h\tau}-\boldsymbol{u}_\tau)&=\boldsymbol{\sigma}_\tau\cdot\boldsymbol{v}_{h\tau}-\boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}_\tau\\&\geq-|\boldsymbol{\sigma}_\tau||\boldsymbol{v}_{h\tau}|-\boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}_\tau\\&\geq-p_\tau(u_\nu-g_a)|\boldsymbol{v}_{h\tau}|+p_\tau(u_\nu-g_a)|\boldsymbol{u}_\tau|.\end{aligned}$$

Applying these relations in (3.14) together with the definition (2.15) of the functional j, we observe that (3.13) holds. □

Denote $V(h)=V_h+V\cap[H^2(\Omega)]^2$. Recall that $\mathcal{C}$ is bounded, symmetric, and positive definite. We define seminorms and norms for $\boldsymbol{v}\in V(h)$ by the following relations:

$$|\boldsymbol{v}|^2_{a,K}:=\int_K\mathcal{C}\varepsilon(\boldsymbol{v}):\varepsilon(\boldsymbol{v})dx,\qquad |\boldsymbol{v}|^2_{a,h}:=\sum_{K\in\mathcal{T}_h}|\boldsymbol{v}|^2_{a,K},\qquad |\boldsymbol{v}|^2_*:=\sum_{e\in\mathcal{E}_h^0}h_e^{-1}||[\![\boldsymbol{v}]\!]||^2_{0,e},$$

where

$$||[\![\boldsymbol{v}]\!]||^2_{0,e}=\int_e[\![\boldsymbol{v}]\!]:[\![\boldsymbol{v}]\!]ds.$$

Then define norms by

$$|||\boldsymbol{v}|||^2_*:=|\boldsymbol{v}|^2_{a,h}+|\boldsymbol{v}|^2_*,\qquad |||\boldsymbol{v}|||^2:=|||\boldsymbol{v}|||^2_*+\sum_{K\in\mathcal{T}_h}h_K^2|\boldsymbol{v}|^2_{2,K}. \tag{3.15}$$

By Korn's inequality on the discontinuous finite element space, the norm $|||\cdot|||_*$ is equivalent to the usual DG-norm $(||\cdot||^2_{1,h}+|\cdot|^2_*)^{\frac{1}{2}}$ ([34]). Set the norm $||\cdot||^2_{0,h}:=\sum_{K\in\mathcal{T}_h}||\cdot||^2_{0,K}$.

For the lifting operator $\boldsymbol{r}_e$, we have the following lemma.

Lemma 3.2 ([21]). For any $\boldsymbol{v}\in V(h)$ and $e\in\mathcal{E}_h^0$,

$$C_1h_e^{-1}||[\![\boldsymbol{v}]\!]||^2_{0,e}\leq||\boldsymbol{r}_e([\![\boldsymbol{v}]\!])||^2_{0,h}\leq C_2h_e^{-1}||[\![\boldsymbol{v}]\!]||^2_{0,e}. \tag{3.16}$$

So, from (3.16) and (3.4), we have

$$||\boldsymbol{r}_0([\![\boldsymbol{v}]\!])||^2_{0,h}=||\sum_{e\in\mathcal{E}_h^0}\boldsymbol{r}_e([\![\boldsymbol{v}]\!])||^2_{0,h}\leq3C_2\sum_{e\in\mathcal{E}_h^0}h_e^{-1}||[\![\boldsymbol{v}]\!]||^2_{0,e}=3C_2|\boldsymbol{v}|^2_*.$$

Then similar to [21, 23], we have the following results concerning the boundedness and stability of B_h.

Lemma 3.3 (Boundedness). *There is a constant $C_b>0$ such that for $1 \le i \le 4$, $B_h = B_{1,h}^{(i)}$ satisfies*

$$B_h(\boldsymbol{u}, \boldsymbol{v}) \le C_b|||\boldsymbol{u}||||||\boldsymbol{v}||| \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V(h). \tag{3.17}$$

Lemma 3.4 (Stability). *Let $\eta_0 := \min_{e\in\mathcal{E}_h^0}\eta_e$ be large enough. Then there is a constant $\theta \in (0,1)$ such that for $1 \le i \le 4, B_h = B_{1,h}^{(i)}$ and $B_{2,h}^{(i)}$ satisfy*

$$B_h(\boldsymbol{v}, \boldsymbol{v}) \ge \theta|||\boldsymbol{v}|||^2 \quad \forall \boldsymbol{v} \in V_h. \tag{3.18}$$

Proof. Note that $|||\boldsymbol{v}||| = |||\boldsymbol{v}|||_*$ for $\boldsymbol{v} \in V_h$. Since $B_{1,h}^{(i)}$ and $B_{2,h}^{(i)}$ coincide on V_h, once the stability for $B_{2,h}^{(i)}$ on V_h, the stability of $B_{1,h}^{(i)}$ on V_h follows. Following [21, 23], we apply the Cauchy–Schwarz inequality and Lemma 3.2 to get

$$B_{2,h}^{(1)}(\boldsymbol{v}, \boldsymbol{v}) \ge (1-\epsilon)|\boldsymbol{v}|_{a,h}^2 + \left(\eta_0 - \frac{3C_2||\mathcal{C}||_{L^\infty(\Omega)}}{\epsilon}\right)|\boldsymbol{v}|_*^2,$$

$$B_{2,h}^{(2)}(\boldsymbol{v}, \boldsymbol{v}) \ge (1-\epsilon)|\boldsymbol{v}|_{a,h}^2 + \left(\eta_0 C_0 C_1 - \frac{3C_2||\mathcal{C}||_{L^\infty(\Omega)}}{\epsilon}\right)|\boldsymbol{v}|_*^2,$$

$$B_{2,h}^{(3)}(\boldsymbol{v}, \boldsymbol{v}) \ge (1-\epsilon)|\boldsymbol{v}|_{a,h}^2 + \left(\eta_0 C_0 C_1 + 3C_2||\mathcal{C}||_{L^\infty(\Omega)}\left(1 - \frac{1}{\epsilon}\right)\right)|\boldsymbol{v}|_*^2,$$

$$B_{2,h}^{(4)}(\boldsymbol{v}, \boldsymbol{v}) \ge (1-\epsilon)|\boldsymbol{v}|_{a,h}^2 + \left(\eta_0 + 3C_2||\mathcal{C}||_{L^\infty(\Omega)}\left(1 - \frac{1}{\epsilon}\right)\right)|\boldsymbol{v}|_*^2.$$

Here, C_0 is the constant in (2.11)(c). Let η_0 be large enough. We can then take $\theta = 1-\epsilon$ and (3.18) holds.

4. Numerical examples

For the numerical examples, we consider functions p_ν, p_τ of the form

$$p_\nu(t) = k_\nu(t)_+^{m_\nu}, \quad p_\tau(t) = k_\tau(t)_+^{m_\tau}, \tag{4.1}$$

where $k_\nu, m_\nu, k_\tau, m_\tau$ are the material interface parameters and all non-negative. Here, $(t)_+$ is the positive part of function t, i.e. $(t)_+ = t$ for $t \ge 0$, and $(t)_+ = 0$ for $t \ge 0$. In [3, 4], it is proved that the (2.14) has a locally unique solution, for $1 \le m_\nu, m_\tau < \infty$ if $d=2$, and $1 \le m_\nu, m_\tau < 4$ if $d=3$ (in the case $3 \le m_\nu, m_\tau < 4$ and $d=3$, (2.14) is replaced by a weaker formulation). For simplicity, we consider a reduced normal compliance law, i.e. $m_\tau = 0$ in (4.1) [8]. Then the functional $j(\boldsymbol{u}, \boldsymbol{v})$ can be written as:

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} k_\nu(u_\nu - g_a)_+^{m_\nu} v_\nu ds + \int_{\Gamma_C} k_\tau|\boldsymbol{v}_\tau| ds = j_\nu(\boldsymbol{u}, \boldsymbol{v}) + j_\tau(\boldsymbol{u}, \boldsymbol{v}). \tag{4.2}$$

To implement the DG method (3.12), following the idea in [35–37], we use Uzawa iteration by introducing a Lagrange multiplier $\boldsymbol{\lambda}_h$. Then the DG scheme is equivalent to the system

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + j_\nu(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} k_\tau \boldsymbol{\lambda}_{h\tau} \cdot \boldsymbol{v}_{h\tau} ds = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h, \tag{4.3}$$

$$|\boldsymbol{\lambda}_{h\tau}| \leq 1, \quad \boldsymbol{\lambda}_{h\tau} \cdot \boldsymbol{u}_{h\tau} = |\boldsymbol{u}_{h\tau}| \text{ a.e. on } \Gamma_C, \boldsymbol{\lambda}_{h\tau} \in (L^\infty(\Gamma_C))^d. \tag{4.4}$$

The following is the Uzawa iteration algorithm:

Step 1. Choose $\boldsymbol{\lambda}_h^0 = \mathbf{0}$, and find $\boldsymbol{u}_h^0$, solution of the problem

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h. \tag{4.5}$$

Step 2. For $n = 1, 2, \ldots$, update the Lagrangian multiplier

$$\boldsymbol{\lambda}_h^n = P(\boldsymbol{\lambda}_h^{n-1} + \rho k_\tau \boldsymbol{u}_h^{n-1}), \tag{4.6}$$

and find $\boldsymbol{u}_h^n \in V_h$, solution of the problem

$$B_h(\boldsymbol{u}_h^n, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - j_\nu(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - \int_{\Gamma_C} k_\tau \boldsymbol{\lambda}_{h\tau}^n \cdot \boldsymbol{v}_{h\tau} ds, \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h. \tag{4.7}$$

Here, ρ is a positive constant and P is a projection operator defined as:

$$P(\mu) = \sup(-1, \inf(1, \mu)) \quad \forall \mu \in L^\infty(\Gamma_C). \tag{4.8}$$

Step 3. If $||\boldsymbol{u}^n - \boldsymbol{u}^{n-1}|| < \epsilon$, a specified error tolerance, stop; otherwise, go to Step 2.

Now, we present numerical results on two-dimensional problems solved by the IP method. In all the examples, the domain is a square, and uniform triangulations of the domain are used. We divide the unit interval into h^{-1} equal sub-intervals and start with $h = 1/4$, which is decreased by half subsequently. We set the error tolerance $\epsilon = 10^{-8}$. We adopt the numerical solution on the mesh $h = \frac{1}{128}$ as the "exact" solution $\boldsymbol{u}_*$ for computing the errors of the numerical solutions on coarser meshes. Let E be Young's modulus and s be the Poisson ratio of the material, the Lamé coefficients are

$$\lambda = \frac{Es}{(1+s)(1-2s)}, \qquad \mu = \frac{E}{2(1+s)}.$$

The penalty parameter η is chosen to be 30μ for two examples.

Example 4.1. The physical setting is shown in Figure 1. The domain $\Omega = (0, 1) \times (0.05, 1.05)$ is the cross-section of a three-dimensional linearized elastic body and plane stress condition is assumed. On $\Gamma_D = \{1\} \times (0.05, 1.05)$, the body is clamped. $\Gamma_F = (\{0\} \times (0.05, 1.05)) \cup ((0, 1) \times \{1.05\})$. Oblique tractions act on the part $\{0\} \times (0.05, 1.05)$ and the part $(0, 1) \times \{1.05\}$ is traction free. The contact part of the boundary is $\Gamma_C = (0, 1) \times \{0.05\}$.

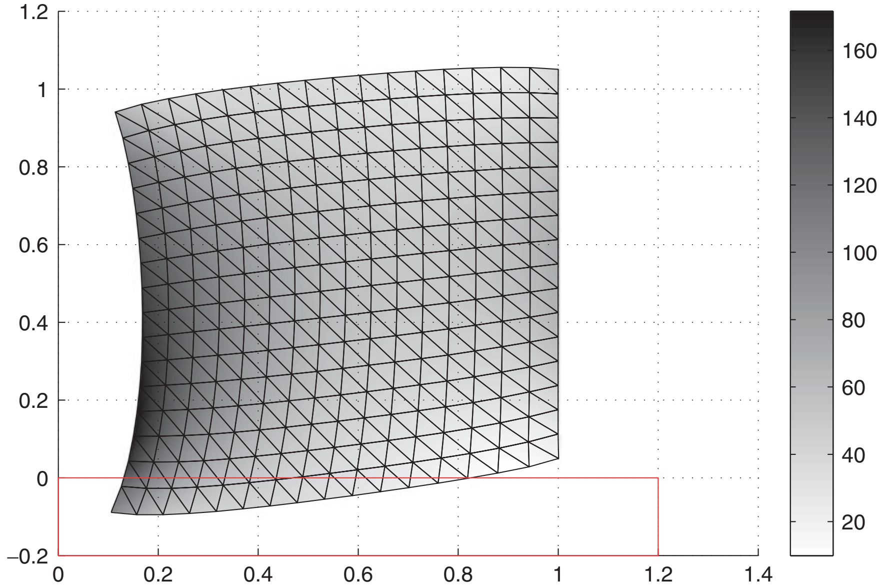

Figure 2. Example 4.1. Deformed configuration with $h = 1/16$.

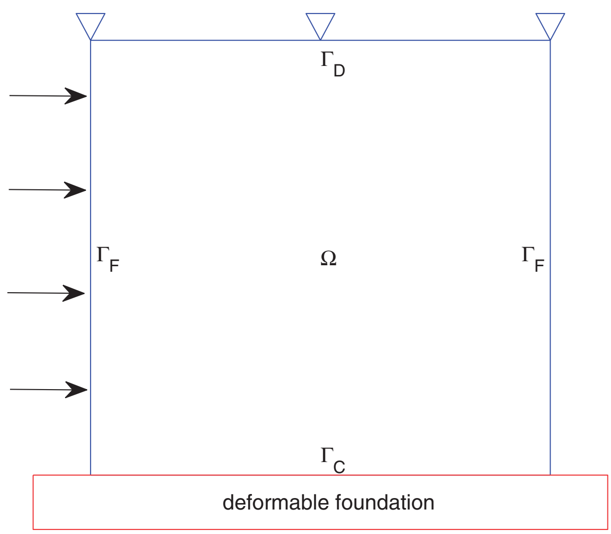

Figure 3. Setting of the problem for Example 4.2.

We use the following data (the unit daN/mm^2 stands for "decaNewtons per square millimeter"):

$$E = 2000\text{daN}/\text{mm}^2, \qquad s = 0.4, \qquad \boldsymbol{f}_1 = (0,0)\text{daN}/\text{mm}^2,$$
$$\boldsymbol{f}_2 = (200(5 - x_2), -190)\text{daN}/\text{mm}^2, k_\tau = 450, k_\nu = 1, m_\nu = 1, g_a = 0.05\text{mm}.$$

Figure 2 shows the deformed mesh with $h = 1/16$, and the red rectangle stands for foundation below the elastic body. Initially, there is no contact

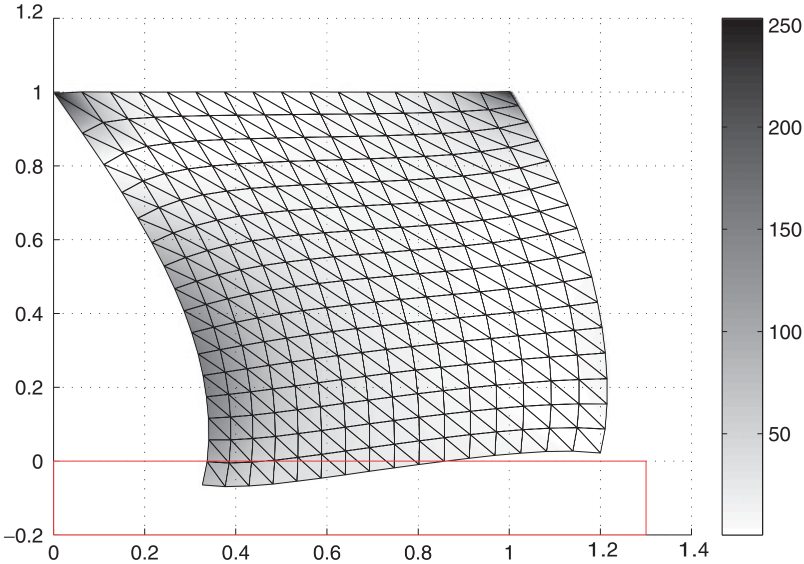

Figure 4. Example 4.2. Deformed configuration with $h = 1/16$.

Table 3 Numerical solution error $|\boldsymbol{u}_* - \boldsymbol{u}_h|_{1,h}$.

h	Example 4.1.	Order	Example 4.2.	Order
1/4	6.1438e−002	–	1.0134e−001	–
1/8	3.0672e−002	1.0022	5.9611e−002	0.7656
1/16	1.5566e−002	0.9785	3.3589e−002	0.8276
1/32	7.4369e−003	1.0656	1.8337e−002	0.8733

since the gap function $g_a = 0.05\text{mm}$, but we see that penetration of the elastic body into foundation occurs after surface traction acts on the boundary Γ_F.

Example 4.2. The physical setting is shown in Figure 3, where $\Omega = (0,1) \times (0,1)$. On $\Gamma_D = (0,1) \times \{1\}$ the body is clamped. $\Gamma_F = (\{0\} \times (0,1)) \cup (\{1\} \times (0,1))$. Horizontal tractions act on the part $\{0\} \times (0,1)$ and the part $\{1\} \times (0,1)$ is traction free. The contact part of the boundary is $\Gamma_C = (0,1) \times \{0\}$. We use the following data:

$$E = 2500\text{daN/mm}^2, \qquad s = 0.2, \qquad \boldsymbol{f}_1 = (0,0)\text{daN/mm}^2,$$
$$\boldsymbol{f}_2 = (880,0)\text{daN/mm}^2, k_\tau = 250, k_\nu = 1, m_\nu = 1, g_a = 0\text{mm}.$$

Figure 4 shows the deformed mesh with $h = 1/16$, and penetration occurs at some places.

The numerical errors $|\boldsymbol{u}_* - \boldsymbol{u}_h|_{1,h}$ and numerical convergence orders are reported in Table 3. We observe that the numerical convergence orders are near 1, an expected result since linear elements are used in the numerical methods.

Funding

The work of the first and second authors was partially supported by the National Natural Science Foundation of China [Grant No. 11771350]. The work of the third author was partially supported by NSF under Grant DMS-1521684.

References

[1] Martins, J. T., Oden, J. T. (1987). Existence and uniqueness results for dynamics contact problems with nonlinear normal and friction interface laws. *Nonlinear Anal.* 11(3):407–428.

[2] Oden, J. T., Martins, J. A. C. (1985). Models and computational methods for dynamic fiction phenomena. *Comput. Methods Appl. Mech. Eng.* 52(1–3):527–634.

[3] Klarbring, A., Mikelić, A., Shillor, M. (1988). Frictional contact problems with normal compliance. *Int. J. Eng. Sci.* 26(8):811–832.

[4] Klarbring, A., Mikelić, A., Shillor, M. (1989). On friction problems with normal compliance *Nonlinear Anal.* 13(8):935–955.

[5] Andersson, L. E. (1991). A quasistatic frictional problem with normal compliance. *Nonlinear Anal.* 16(4):347–369.

[6] Lee, C. Y., Oden, J. T. (1993). A priori error estimation of hp-finite element approximations of frictional contact problems with normal compliance. *Int. J. Eng. Sci.* 31(6):927–952.

[7] Lee, C. Y., Oden, J. T. (1993). Theory and approximation of quasistatic frictional contact problems. *Comput Methods Appl. Mech Eng.* 106(3):407–429.

[8] Han, W. (1996). On the numerical approximation of a frictional contact problem with normal compliance. *Numer. Func. Anal. Opt.* 17(3–4):307–321.

[9] Rochdi, M., Shillor, M., Sofonea, M. (1998). Quasistatic viscoelastic contact with normal compliance and friction. *J. Elasticity* 51(2):105–126.

[10] Han, W., Sofonea, M. (1999). Analysis and numerical approximation of an elastic frictional contact problem with normal compliance. *Appl. Math. (Warsaw)* 26(4):415–435.

[11] Sofonea, M., Matei, A. (2012). *Mathematical Models in Contact Mechanics*. London: Cambridge University Press.

[12] Han, W., Shillor, M., Sofonea, M. (2001). Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage. *J. Comput. Appl. Math.* 137(2):377–398.

[13] Amassad, A., Fabre, C., Sofonea, M. (2004). A quasistatic viscoplastic contact problem with normal compliance and friction. *IMA J. Appl. Math.* 69(5):463–482.

[14] Cockburn, B., Karniadakis, G. E., Shu, C.-W. (2000). Discontinuous Galerkin methods: Theory, computation and applications. *Lecture Notes in Comput. Sci. Engrg.* New York: Springer-Verlag, 11:1119–1148.

[15] Arnold, D. N., Brezzi, F., Cockburn, B., Marini, L. D. (2002). Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* 39(5):1749–1779.

[16] Wihler, T. P. (2006). Locking-free adaptive discontinuous Galerkin FEM for linear elasticity problems. *Math. Comp.* 75(255):1087–1102.

[17] Djoko, J. K., Ebobisse, F., Mcbride, A. T., Reddy, B. D. (2007). A discontinuous Galerkin formulation for classical and gradient plasticity – part 1: Formulation and analysis. *Comp. Methods Appl. Mech. Eng.* 196(37–40):3881–3897.

[18] Djoko, J. K., Ebobisse, F., Mcbride, A. T., Reddy, B. D. (2007). A discontinuous Galerkin formulation for classical and gradient plasticity – part 2: Algorithms and numerical analysis. *Comp. Methods Appl. Mech. Eng.* 197(1–4):1–21.

[19] Wang, F., Han, W., Cheng, X. (2010). Discontinuous Galerkin methods for solving elliptic variational inequalities. *SIAM J. Numer. Anal.* 48(2):708–733.

[20] Wang, F. (2013). Discontinuous Galerkin methods for solving double obstacle problem. *Numer. Methods Partial Differ. Eq.* 29(2):706–720.

[21] Wang, F., Han, W., Cheng, X. (2011). Discontinuous Galerkin methods for solving signorini problem. *IMA J. Numer. Anal.* 31(4):1754–1772.

[22] Zeng, Y., Chen, J., Wang, F. (2015). Error estimates of the weakly over-penalized symmetric interior penalty method for two variational inequalities. *Comput. Math. Appl.* 69(8):760–770.

[23] Wang, F., Han, W., Cheng, X. (2014). Discontinuous Galerkin methods for solving a quasistatic contact problem. *Numer. Math.* 126(4):771–800.

[24] Brenner, S. C., Sung, L., Zhang, H., Zhang, Y. (2012). A quadratic C^0 interior penalty method for the displacement obstacle problem of clamped kirchhoff plates. *SIAM J. Numer. Anal.* 50(6):3329–3350.

[25] Wang, F., Han, W., Huang, J., Zhang, T. (2015). 33:199–222. Discontinuous Galerkin methods for an elliptic variational inequality of fourth-order. In: Han, W., Migorski, S., Sofonea, M., eds. *Advances in Variational and Hemivariational Inequalities with Applications*. Switzerland: Springer International Publishing.

[26] Wang, F., Zhang, T., Han, W. (2019). C^0 discontinuous Galerkin methods for a Kirchhoff plate contact problem. *J. Comput. Math.* 37:1–17.

[27] Wang, F. (2013). Discontinuous Galerkin methods for two membranes problem. *Num. Func. Anal. Opt.* 34(2):220–235.

[28] Djoko, J. K. (2013). Discontinuous Galerkin finite element discretization for steady stokes flows with threshold slip boundary condition. *Quaestiones Mathematicae* 36(4):501–516.

[29] Jing, F., Han, W., Yan, W., Wang, F. (2018). Discontinuous Galerkin methods for a stationary Navier–Stokes problem with a nonlinear slip boundary condition of friction type. *J. Sci. Comput.* 76(2):888–892.

[30] Douglas, J., Dupont, T. (1976). Interior penalty procedures for elliptic and parabolic Galerkin methods. *Lect. Notes Phys.* 58:207–216. Berlin: Springer-Verlag.

[31] Bassi, F., Rebay, S., Mariotti, G., Pedinotti, S., Savini, M. (1997). A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows. In: Decuypere, R., Dibelius, G., eds. *Proceedings of 2nd European Conference on Turbomachinery, Fluid Dynamics and Thermodynamics*. Antwerpen, Belgium: Technologisch Instituut, pp. 99–108.

[32] Brezzi, F., Manzini, G., Marini, D., Pietra, P., Russo, A. (1999). Discontinuous finite elements for diffusion problems. In: *Atti Convegno in Onore Di F. Brioschi (Milan, 1997)*. Milan, Italy: Istituto Lombardo, Accademia di Scienze e Lettere, pp. 197–217.

[33] Cockburn, B., Shu, C.-W. (1998). The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.* 35(6):2440–2463.

[34] Chen, Y., Huang, J., Huang, X., Xu, Y. (2010). On the local discontinuous Galerkin method for linear elasticity. *Math. Probl. Eng.* 2010:242–256.

[35] Porwal, K. (2017). Discontinuous Galerkin methods for a contact problem with tresca friction arising in linear elasticity. *Appl. Numer. Math.* 112:182–202.

[36] Glowinski, R. (1984). *Numerical Methods for Nonlinear Variational Problems*. New York: Springer-Verlag.

[37] Glowinski, R., Lions, J.-L., Trémolières, R. (1981). *Numerical Analysis of Variational Inequalities*. North-Holland, Amsterdam: Elsevier.