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A mixed discontinuous Galerkin method for an unsteady incompressible Darcy equation

Yanxia Qian[a], Fei Wang[b], Yongchao Zhang[a] and Weimin Han[c]

[a]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, People's Republic of China; [b]School of Mathematics and Statistics & State Key Laboratory of Multiphase Flow in Power Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi, People's Republic of China; [c]Department of Mathematics & Program in Applied Mathematical and Computational Sciences (AMCS), University of Iowa, Iowa City, IA, USA

ABSTRACT
We study a mixed discontinuous Galerkin (MDG) method for solving a time-dependent Darcy problem, which simulates an incompressible fluid such as water flowing in a rigid porous medium. The discretization of the unsteady Darcy problem relies on a backward Euler scheme for temporal variable and MDG method for spatial variables. Spatially semi-discrete and fully discrete schemes are analyzed. Existence and uniqueness of the numerical solutions are proved, and optimal order error estimates are derived for both the velocity and pressure variables. Finally, some test problems are provided to display the performance of the MDG method, and numerical results are reported to support the theoretical predictions.

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1. Introduction

We consider the following time-dependent Darcy problem [1], which models the unsteady flow of an incompressible fluid in a rigid porous medium:

$$\begin{aligned} &\dot{\boldsymbol{u}} + \alpha \boldsymbol{u} + \nabla p = \boldsymbol{f} && \text{in } \Omega \times [0,T], \\ &\operatorname{div} \boldsymbol{u} = 0 && \text{in } \Omega \times [0,T], \\ &\boldsymbol{u} = \boldsymbol{u}_0 && \text{in } \Omega \times \{t=0\}, \\ &p = g_D && \text{on } \partial\Omega \times [0,T], \end{aligned} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. The initial value $\boldsymbol{u}_0$ meets the divergence-free condition $\operatorname{div} \boldsymbol{u}_0 = 0$ in Ω. The unknowns are the velocity $\boldsymbol{u}$ and the pressure p, which are functions of the spatial variable $\mathbf{x}$ in Ω and temporal variable t in $[0,T]$ ($T > 0$ is a finite time), and we use $\dot{\boldsymbol{u}}$ for the time derivative of $\boldsymbol{u}$. Here, $\boldsymbol{f}$ is the density of body force, $\boldsymbol{u}_0$ is the initial value of the velocity, and g_D is the Dirichlet boundary value of the pressure. The parameter α is a positive constant representing the drag coefficient, which depends on the permeability of the porous medium and the fluid viscosity.

The classical Darcy model describes the flow of a compressible fluid through a porous media [2]. Many numerical methods have been developed to solve it. For a stationary flow, $\dot{\boldsymbol{u}} = \mathbf{0}$, the Darcy problem can be reduced to a Poisson equation for pressure and solved by the standard finite element methods. Then velocity field can be obtained by a post-processing step. However, there may be a

CONTACT Fei Wang feiwang.xjtu@xjtu.edu.cn

loss of accuracy for the numerical solution in velocity field and the local conservation may not be guaranteed. To approximate the velocity and pressure simultaneously, mixed finite element method [3] is a popular way to discretize the Darcy flow [4,5]. For the time dependent axisymmetric Darcy problem, finite element method is studied in [6]. In [7], the mortar finite element discretization is applied on the time dependent nonlinear Darcy's equations. In addition, the decoupled schemes [8] and partitioned methods [9,10] are also developed to solve the Stokes-Darcy and related problems.

Over the past several decades, discontinuous Galerkin (DG) methods have become very popular in the scientific computing and engineering communities, due to their flexibility in constructing feasible local shape function spaces. Compared with the standard finite element methods, DG methods enjoy the following good features: they (i) are locally (and globally) conservative; (ii) can be easily coupled with other methods like conforming or mixed finite element methods; (iii) are well-suited for *hp*-adaptivity. Therefore, DG methods have been applied to solve various partial differential equations, such as hyperbolic equations [11–14], Navier-Stokes equations [15,16], convection-diffusion equations [17], transport problems [18,19], variational inequalities [20–23] and much more. For more discussion about DG methods, we refer the reader to [24–26] and the references therein.

Mixed discontinuous Galerkin (MDG) methods have been developed for solving the Darcy flow problem. In [27], a local DG method [28] is coupled with the RT mixed method to solve the Darcy problem on two disjoint subdomain Ω_{LDG} and Ω_{RT}, respectively, with $\Omega = \Omega_{\mathrm{LDG}} \cup \Omega_{\mathrm{RT}}$. In [5], a stabilized mixed finite element method is studied for solving the Darcy flow problem. Then similar ideas about stabilization are applied to mixed DG methods in [29,30], so that there are no mesh-dependent parameters in the scheme. In [31], a reliable and efficient residual-type a posteriori error estimator is derived for an augmented DG formulation for the Darcy flow. To reduce the computational cost, a mixed dual-scale Galerkin method for the Darcy problem is proposed in [32]. These references are for the stationary Darcy flow. In [33], a mixed discontinuous Galerkin scheme is applied to solve a non-stationary Darcy problem, and the scheme is stabilized by penalty terms in both the primary and the flux unknowns. In [26], a unified framework is established on continuous and discontinuous finite element methods, and relationship of different finite element methods is discussed; furthermore, a new mixed DG scheme for elliptic problems is introduced with the P_{l+1}-P_l pairs ($l \geq 0$).

In this paper, we apply the mixed DG scheme to solve the time-dependent incompressible Darcy flow (1), which has a large number of applications, such as the simulation of water in underground rocks [1]. First, a spatially semi-discrete mixed discontinuous Galerkin scheme is introduced and its well-posedness is analyzed. Then, we give a prior error analysis, which shows that optimal convergence orders are reached for both velocity and pressure variables. Next, we consider a fully-discrete scheme with the backward Euler difference approximation for the temporal variable and an MDG discretization for the spatial variables. Optimal convergence orders for both temporal and spatial variables are proved.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, a spatially semi-discrete MDG scheme is introduced, the well-posedness and a priori error estimates are obtained. We show the stability of the fully-discrete scheme and prove optimal order error estimates for both velocity and pressure in Section 4. In Section 5, numerical examples are provided to confirm the theoretical findings and to illustrate the performance of the fully-discrete mixed DG scheme.

2. Preliminaries

In this section, we introduce the notation and recall some basic results.

For a nonnegative integer m and an open Lipschitz subdomain $D \subset \overline{\Omega}$, we denote the Sobolev space by $H^m(D) = W^{m,2}(D)$ with the norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. When $m = 0$, $H^0(D)$ coincides with the Lebesgue space $L^2(D)$, which is equipped with the usual L^2-inner product $(\cdot,\cdot)_D$ and L^2-norm $\|\cdot\|_{0,D}$. If D is chosen as Ω, we abbreviate them by the norms $\|\cdot\|_m$, $\|\cdot\|_0$, the semi-norm $|\cdot|_m$ and the inner product $(\cdot,\cdot)$, respectively. The space $H_0^m(D)$ denotes the closure in $H^m(D)$

of the set of the infinitely often differentiable functions with compact support in D. The same notation is used for the vector-valued counterparts such as $[L^2(D)]^d$ and $[H^m(D)]^d$.

To study the problem (1), we introduce the following function spaces

$$V = H(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^d : \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega)\}, \quad Q = L^2(\Omega).$$

The space V is equipped with the norm $\|\boldsymbol{v}\|_V = (\|\boldsymbol{v}\|_0^2 + \|\mathrm{div}\, \boldsymbol{v}\|_0^2)^{1/2}$. In addition, we need spaces of vector-valued functions such as $L^2(0, T; H^m(\Omega))$ and $C(0, T; H^m(\Omega))$ with the norms

$$\|\varphi\|_{L^2(0,T;H^m(\Omega))} = \left[\int_0^t \|\varphi(t)\|_m^2 \,\mathrm{d}t\right]^{1/2} \quad \text{and} \quad \|\varphi\|_{C(0,T;H^m(\Omega))} = \max_{0 \le t \le T} \|\varphi(t)\|_m.$$

Assume $\boldsymbol{f} \in L^2(0, T; [L^2(\Omega)]^d)$, $g_D \in L^2(0, T; H^{1/2}(\partial\Omega))$ and $\boldsymbol{u}_0 \in V$. The weak formulation of the problem (1) is to find $\boldsymbol{u} \in L^2(0, T; V)$ with $\dot{\boldsymbol{u}} \in L^2(0, T; V)$ and $p \in L^2(0, T; Q)$ such that for a.e. $t \in [0, T]$,

$$\begin{aligned} (\dot{\boldsymbol{u}}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}) - \langle \boldsymbol{v} \cdot \boldsymbol{n}, g_D\rangle_{\partial\Omega} \qquad \forall\, \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, q) &= 0 \quad \forall\, q \in Q, \end{aligned} \tag{2}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{3}$$

Here, the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\alpha \boldsymbol{u}, \boldsymbol{v}), \quad b(\boldsymbol{v}, q) = -(\mathrm{div}\, \boldsymbol{v}, q),$$

$\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, and $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$.

Obviously, $b(\cdot, \cdot)$ is continuous on $V \times Q$; its kernel

$$Z = \{\boldsymbol{v} \in V : \ b(\boldsymbol{v}, q) = 0 \;\forall\, q \in Q\}$$

is characterized as

$$Z = \{\boldsymbol{v} \in V : \ \mathrm{div}\, \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}.$$

Obviously,

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &\le \alpha \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \\ a(\boldsymbol{v}, \boldsymbol{v}) &\ge \alpha \|\boldsymbol{v}\|_0^2 = \alpha \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in Z. \end{aligned}$$

Moreover, the inf-sup condition holds: there exists a constant $\beta_1 > 0$ such that ([34])

$$\sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V} \ge \beta_1 \|q\|_0. \tag{4}$$

Next, the following lemma presents a trace inequality, the continuous and discrete Gronwall's inequalities.

Lemma 2.1 ([35]): *For any $\boldsymbol{v} \in V$, $\boldsymbol{v} \cdot \boldsymbol{n} \in H^{-1/2}(\partial\Omega)$ and there exists a positive constant c such that*

$$\|\boldsymbol{v} \cdot \boldsymbol{n}\|_{-1/2,\partial\Omega} \le c\, \|\boldsymbol{v}\|_V. \tag{5}$$

Lemma 2.2 (The continuous Gronwall lemma [36]): *Let G be a nonnegative function and let y, f, g be locally integrable non-negative functions on the interval* $[t_0, \infty)$. *Assume there exists a constant* $C_0 \geq 0$ *such that*

$$y(t) + G(t) \leq C_0 + \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t} g(\tau)\,y(\tau)\,\mathrm{d}\tau, \quad \forall\, t \in [t_0, \infty).$$

Then,

$$y(t) + G(t) \leq \left(C_0 + \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right) \exp\left(\int_{t_0}^{t} g(\tau)\,\mathrm{d}\tau\right), \quad \forall\, t \in [t_0, \infty).$$

Here, $\exp(\varphi)$ *denotes the exponential function* e^{φ}.

Lemma 2.3 (The discrete Gronwall lemma [37]): *Let* a_j, b_j, c_j, d_j, k *and* C_0, *for integers* $j \geq 0$, *be non-negative numbers such that*

$$a_n + k\sum_{j=0}^{n} b_j \leq k\sum_{j=0}^{n} d_j a_j + k\sum_{j=0}^{n} c_j + C_0 \quad \forall\, n \geq 0.$$

Then, if $k\,d_j < 1$ *for all j,*

$$a_n + k\sum_{j=0}^{n} b_j \leq \left(C_0 + k\sum_{j=0}^{n} c_j\right) \exp\left(k\sum_{j=0}^{n} \frac{d_j}{1 - k\,d_j}\right) \quad \forall\, n \geq 0.$$

Hereafter, C denotes a generic positive constant, independent of the mesh size h and time step k; its value may vary from case to case. For the solution of problem (2)–(3), we have the following lemma.

Lemma 2.4: *Assume* $\boldsymbol{f} \in L^2(0, T; [L^2(\Omega)]^d), g_D \in L^2(0, T; H^{1/2}(\partial\Omega))$ *and* $\boldsymbol{u}_0 \in V$. *Suppose that* $(\boldsymbol{u}, p)$ *satisfies (2)–(3). Then, for any* $t \in [0, T]$,

$$\begin{aligned} &\|\boldsymbol{u}\|_{C(0,T;[L^2(\Omega)]^d)} + \|\dot{\boldsymbol{u}}\|_{L^2(0,T;[L^2(\Omega)]^d)} + \|\boldsymbol{u}\|_{L^2(0,T;V)} \\ &\quad \leq C\left[\|\boldsymbol{u}_0\|_0 + \|\boldsymbol{f}\|_{L^2(0,T;[L^2(\Omega)]^d)} + \|g_D\|_{L^2(0,T;H^{1/2}(\partial\Omega))}\right]. \end{aligned} \tag{6}$$

Proof: Taking $\boldsymbol{v} = \boldsymbol{u}$ and $q = p$ in (2), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_0^2 + \alpha\|\boldsymbol{u}\|_V^2 \leq \|\boldsymbol{f}\|_0\|\boldsymbol{u}\|_V + \|g_D\|_{1/2,\partial\Omega}\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{-1/2,\partial\Omega}.$$

Apply (5),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{u}\|_V^2 \leq \frac{1}{\alpha}\|\boldsymbol{f}\|_0^2 + \frac{c^2}{\alpha}\|g_D\|_{1/2,\partial\Omega}^2. \tag{7}$$

Integrate (7) from 0 to t and multiply by 2 to yield

$$\|\boldsymbol{u}(\cdot,t)\|_0^2 + \alpha\int_0^t \|\boldsymbol{u}(\cdot,\tau)\|_V^2\,\mathrm{d}\tau \leq \|\boldsymbol{u}(\cdot,0)\|_0^2 + \frac{2}{\alpha}\int_0^t \left(\|\boldsymbol{f}(\cdot,\tau)\|_0^2 + c^2\|g_D(\cdot,\tau)\|_{1/2,\partial\Omega}^2\right)\mathrm{d}\tau.$$

Similarly, taking $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and $q = p$ in (2), we can derive that

$$\int_0^t \|\dot{\boldsymbol{u}}(\cdot,t)\|_0^2\,\mathrm{d}t + \|\boldsymbol{u}(\cdot,t)\|_0^2 \leq C\|\boldsymbol{u}(\cdot,0)\|_0^2 + C\int_0^t \|\boldsymbol{f}(\cdot,\tau)\|_0^2\,\mathrm{d}\tau + C\int_0^t \|g_D(\cdot,\tau)\|_{1/2,\partial\Omega}^2\,\mathrm{d}\tau.$$

With the initial value condition (3), we obtain (6) from the above two inequalities. ■

3. Spatially semi-discrete MDG scheme

3.1. Notation and mixed DG scheme

For brevity, we only consider the case $d = 2$; the results can be similarly extended to the case of $d = 3$. Let $\{\mathcal{T}_h\}_h$ be a regular family of quasi-uniform triangulations of $\overline{\Omega}$. The notation $\mathcal{E}_h$ stands for the union of the boundaries of all $K \in \mathcal{T}_h$, $\mathcal{E}_h^i \subset \mathcal{E}_h$ is the set of interior edges and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ is the set of boundary edges. Let $|K|$ be the area of K, h_K be the diameter of the element $K \in \mathcal{T}_h$, and $h = \max\{h_K : K \in \mathcal{T}_h\}$. Similarly, denote the length of edge e by h_e for any $e \in \mathcal{E}_h$.

Let e be an edge shared by two neighboring elements K_1 and K_2, with their outward unit normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ on e. For a scalar function q, let $q_i = q|_{\partial K_i}$, define the jump $[\![q]\!]$ and average $\{q\}$ by

$$[\![q]\!] = q_1\boldsymbol{n}_1 + q_2\boldsymbol{n}_2 \quad \text{and} \quad \{q\} = \frac{1}{2}(q_1 + q_2) \quad \forall\, e \in \mathcal{E}_h^i.$$

For a vector function $\boldsymbol{v}$, define $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ analogously and set

$$[\boldsymbol{v}] = \boldsymbol{v}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{n}_2 \quad \text{and} \quad \{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2) \quad \forall\, e \in \mathcal{E}_h^i.$$

On the boundary, define

$$[\![q]\!] = q\boldsymbol{n} \quad \text{and} \quad \{\boldsymbol{v}\} = \boldsymbol{v} \quad \forall\, e \in \mathcal{E}_h^\partial.$$

Let $l \geq 0$ be an integer. We define the discontinuous finite element spaces V_h and Q_h by

$$V_h = \{\boldsymbol{v}_h \in [L^2(\Omega)]^2 : \boldsymbol{v}_h \in [\mathcal{P}_{l+1}(K)]^2 \quad \forall\, K \in \mathcal{T}_h\},$$
$$Q_h = \{q_h \in L^2(\Omega) : q_h \in \mathcal{P}_l(K) \quad \forall\, K \in \mathcal{T}_h\}.$$

where $\mathcal{P}_l(K)$ denotes the set of all polynomials in K with the total degree no more than l.

For $V(h) = V_h + [H^1(\Omega)]^2$, define the norm by

$$\|\boldsymbol{v}\|_h^2 = \|\boldsymbol{v}\|_0^2 + \|\mathrm{div_h}\boldsymbol{v}\|_0^2 + \sum_{e\in\mathcal{E}_h^i} h_\mathrm{e}^{-1}\|[\boldsymbol{v}]\|_\mathrm{e}^2.$$

Here, $\mathrm{div_h}$ is the differential operator div defined piecewise on each element $K \in \mathcal{T}_h$, and $\|[\boldsymbol{v}]\|_e$ denotes the $L^2(e)$-norm of $[\boldsymbol{v}]$.

We introduce an interpolation operator $\Pi_h : [H^1(\Omega)]^2 \to V_h$ satisfying [38]

$$\int_K q_h \mathrm{div}\,(\Pi_h\boldsymbol{v} - \boldsymbol{v})\,\mathrm{d}\mathbf{x} = 0 \quad \forall\, q_h \in \mathcal{P}_l(K),$$
$$\int_e q_h(\Pi_h\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_e\,\mathrm{d}s = 0 \quad \forall\, q_h \in \mathcal{P}_l(e),$$
$$\Pi_h\boldsymbol{v}|_e \cdot \boldsymbol{n}_e \in \mathcal{P}_l(e). \tag{8}$$

and

$$\|\Pi_h\boldsymbol{v} - \boldsymbol{v}\|_{0,K} + h_K|\Pi_h\boldsymbol{v} - \boldsymbol{v}|_{1,K} \leq Ch^{l+2}|\boldsymbol{v}|_{l+2,K} \quad \forall\, \boldsymbol{v} \in [H^{l+2}(\Omega)]^2. \tag{9}$$

Let $P_h : Q \to Q_h$ denote the L^2-orthogonal projection defined by

$$\int_K (P_h\varphi - \varphi)q_h\,\mathrm{d}\mathbf{x} = 0 \quad \forall\, q_h \in Q_h,\ K \in \mathcal{T}_h. \tag{10}$$

Assume $\varphi \in H^{l+1}(\Omega)$. Then

$$\|P_h\varphi - \varphi\|_{0,K} + h_K|P_h\varphi - \varphi|_{1,K} \leq Ch^{l+1}|\varphi|_{l+1,K} \quad \forall\, K \in \mathcal{T}_h. \tag{11}$$

Moreover, together with the fact that div $\Pi_h \boldsymbol{v} \in \mathcal{P}_l(K)$, the definitions of Π_h and P_h imply that

$$\operatorname{div} \Pi_h \boldsymbol{v} = P_h(\operatorname{div} \boldsymbol{v}). \tag{12}$$

Following the construction of the bilinear forms for the MDG scheme in [26], we define

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_{\mathcal{T}_h} \alpha \boldsymbol{u}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\mathbf{x} + \int_{\mathcal{E}_h^i} \eta_e h_e^{-1} [\boldsymbol{u}_h][\boldsymbol{v}_h] \,\mathrm{d}s, \tag{13}$$

$$b_h(\boldsymbol{v}_h, p_h) = -\int_{\mathcal{T}_h} p_h \mathrm{div_h} \boldsymbol{v}_h \,\mathrm{d}\mathbf{x} + \int_{\mathcal{E}_h^i} \{p_h\}[\boldsymbol{v}_h] \,\mathrm{d}s, \tag{14}$$

where η_e is the penalty parameter.

A spatially semi-discrete MDG approximation for the time-dependent Darcy problem (1) is to find $(\boldsymbol{u}_h(t), p_h(t)) \in V_h \times Q_h$ for all $t \in [0, T]$ such that

$$\begin{aligned} &(\dot{\boldsymbol{u}}_h, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, g_D \rangle_{\mathcal{E}_h^\partial} \quad \forall\, \boldsymbol{v}_h \in V_h, \\ &b_h(\boldsymbol{u}_h, q_h) = 0 \quad \forall\, q_h \in Q_h, \end{aligned} \tag{15}$$

with the initial condition

$$\boldsymbol{u}_h(0) = \Pi_h \boldsymbol{u}_0. \tag{16}$$

This mixed DG scheme can be regarded as a special case of the LDG method. However, the finite element pairs $\mathcal{P}_{l+1}$-$\mathcal{P}_l$ for the DG spaces V_h and Q_h are different from that in the usual LDG method [28,39].

3.2. Well-posednesss of the semi-discrete scheme

In order to establish the well-posedness of the semi-discrete scheme, we define the subspace Z_h of V_h by

$$Z_h = \{\boldsymbol{v}_h \in V_h : \ b_h(\boldsymbol{v}_h, q_h) = 0 \quad \forall\, q_h \in Q_h\}.$$

To characterize the kernel Z_h, for any $q_h \in Q_h$, we define the lifting operator (cf. [24,40]) $r_e : L^2(e) \to Q_h$ by

$$\int_\Omega r_e(\varphi) q_h \,\mathrm{d}\mathbf{x} = -\int_e \varphi\{q_h\} \,\mathrm{d}s \quad \forall\, e \in \mathcal{E}_h. \tag{17}$$

Let

$$r(\varphi) = \sum_{e \in \mathcal{E}_h^i} r_e(\varphi).$$

Then obviously

$$\|r(\varphi)\|_0^2 = \|\sum_{e \in \mathcal{E}_h^i} r_e(\varphi)\|_0^2 \le 3 \sum_{e \in \mathcal{E}_h^i} \|r_e(\varphi)\|_0^2.$$

Lemma 3.1: *For any $\boldsymbol{v}_h \in Z_h$, we have $\mathrm{div}_h \boldsymbol{v}_h + r([\boldsymbol{v}_h]) = 0$.*

Proof: If $\boldsymbol{v}_h \in Z_h$, then for any $q_h \in Q_h$,

$$\begin{aligned} 0 = b_h(\boldsymbol{v}_h, q_h) &= -\int_{\mathcal{T}_h} q_h \mathrm{div_h} \boldsymbol{v}_h \, \mathrm{d}\mathbf{x} + \int_{\mathcal{E}_h^i} \{q_h\}[\boldsymbol{v}_h] \, \mathrm{d}s \\ &= -\int_{\Omega} q_h \mathrm{div_h} \boldsymbol{v}_h \, \mathrm{d}\mathbf{x} - \int_{\Omega} q_h r([\boldsymbol{v}_h]) \, \mathrm{d}\mathbf{x} \\ &= -\int_{\Omega} q_h (\mathrm{div_h} \boldsymbol{v}_h + r([\boldsymbol{v}_h])) \, \mathrm{d}\mathbf{x}. \end{aligned}$$

By taking $q_h = \mathrm{div_h} \boldsymbol{v}_h + r([\boldsymbol{v}_h]) \in Q_h$, we conclude that $\mathrm{div}_h \boldsymbol{v}_h + r([\boldsymbol{v}_h]) = 0$. ■

Lemma 3.2 ([40]): *There exists a positive constant C_1, independent of h, such that*

$$\|r_e(\varphi)\|_0 \leq C_1 h_e^{-1/2} \|\varphi\|_e.$$

Coercivity of $a_h(\boldsymbol{v}_h, \boldsymbol{v}_h)$ on Z_h is considered next.

Lemma 3.3: *Assume $\eta_0 := \min_{e \in \mathcal{E}_h^i} \eta_e > 3C_1$. Then there exists a constant $C_2 > 0$ such that*

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_2 \|\boldsymbol{v}_h\|_h^2 \quad \forall\, \boldsymbol{v}_h \in Z_h. \tag{18}$$

Proof: By using Lemmas 3.1 and 3.2, for any $\boldsymbol{v}_h \in Z_h$, one finds

$$\|\mathrm{div}_h \boldsymbol{v}_h\|_0^2 = \|r([\boldsymbol{v}_h])\|_0^2 \leq 3 \sum_{e \in \mathcal{E}_h^i} \|r_e([\boldsymbol{v}_h])\|_0^2 \leq 3C_1 \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[\boldsymbol{v}_h]\|_e^2.$$

In addition,

$$\begin{aligned} &\int_{\mathcal{T}_h} \alpha \boldsymbol{v}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\mathbf{x} \geq \alpha \|\boldsymbol{v}_h\|_0^2, \\ &\int_{\mathcal{E}_h^i} \frac{\eta_e}{h_e} [\boldsymbol{v}_h][\boldsymbol{v}_h] \, \mathrm{d}s \geq \eta_0 \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[\boldsymbol{v}_h]\|_e^2. \end{aligned}$$

Therefore,

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq \alpha \|\boldsymbol{v}_h\|_0^2 + (\eta_0 - 3C_1) \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[\boldsymbol{v}_h]\|_e^2 + \|\mathrm{div}_h \boldsymbol{v}_h\|_0^2.$$

Thus, (18) holds with $C_2 = \min\{\alpha, 1, \eta_0 - 3C_1\}$. ■

From Cauchy-Schwarz inequality and inverse inequality, we can derive the following continuity of the bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$.

Lemma 3.4 ([26]): *For the bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, we have*

$$\begin{array}{ll} a_h(\boldsymbol{u}, \boldsymbol{v}) \leq C\|\boldsymbol{u}\|_h \|\boldsymbol{v}\|_h & \forall\, \boldsymbol{u}, \boldsymbol{v} \in V(h), \\ b_h(\boldsymbol{v}, p_h) \leq C\|p_h\|_0 \|\boldsymbol{v}\|_h & \forall\, \boldsymbol{v} \in V(h) \quad \forall\, p_h \in Q_h, \\ b_h(\boldsymbol{v}, p) \leq C\left(\|p\|_0 + h\,|p|_1\right) \|\boldsymbol{v}\|_h & \forall\, \boldsymbol{v} \in V(h) \quad \forall\, p \in H^1(\Omega). \end{array}$$

We then turn to the discrete inf-sup condition of the bilinear form $b_h(\cdot,\cdot)$.

Lemma 3.5: *There exists a constant $\beta_2 > 0$ such that*

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{b_h(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_h} \geq \beta_2 \|p_h\|_0 \quad \forall\, p_h \in Q_h. \tag{19}$$

Proof: Following [41,42], we construct a Fortin operator to prove (19). Let G be a convex and bounded domain containing $\overline{\Omega}$. Given $\boldsymbol{v} \in V$, let

$$f_v = \begin{cases} \operatorname{div} \boldsymbol{v} & \text{in } \Omega, \\ 0 & \text{in } G \backslash \Omega. \end{cases} \tag{20}$$

Since $f_v \in L^2(G)$ and G is convex, the unique weak solution $z \in H_0^1(G)$ of the boundary value problem

$$\Delta z = f_v \quad \text{in } G, \qquad z = 0 \quad \text{on } \partial G$$

has the regularity $z \in H^2(G)$, and there exists a constant $C > 0$ such that

$$\|z\|_{2,G} \leq C\, \|f_v\|_{0,G} = C\, \|\operatorname{div} \boldsymbol{v}\|_0 \leq C\, \|\boldsymbol{v}\|_V. \tag{21}$$

Furthermore, we see that $\operatorname{div} \boldsymbol{v} = \operatorname{div}\nabla z$ in Ω. Recalling the operator Π_h defined in (8), we construct a Fortin operator $\pi_h \colon V \to V_h$ by

$$\pi_h \boldsymbol{v} = \Pi_h(\nabla z). \tag{22}$$

By (12), we deduce that

$$\operatorname{div} \pi_h \boldsymbol{v} = \operatorname{div} \Pi_h(\nabla z) = P_h(\operatorname{div}\nabla z) = P_h(\operatorname{div}\boldsymbol{v}) \quad \text{in } \Omega \tag{23}$$

and

$$\begin{aligned} b_h(\pi_h \boldsymbol{v}, q_h) &= -\int_{\mathcal{T}_h} q_h \operatorname{div}(\pi_h \boldsymbol{v})\, d\mathbf{x} + \int_{\mathcal{E}_h^i} \{q_h\}[\pi_h \boldsymbol{v}]\, ds \\ &= -\int_{\mathcal{T}_h} q_h P_h(\operatorname{div}\boldsymbol{v})\, d\mathbf{x} + \int_{\mathcal{E}_h^i} \{q_h\}[\Pi_h \nabla z]\, ds \\ &= -\int_{\mathcal{T}_h} q_h \operatorname{div}\boldsymbol{v}\, d\mathbf{x} + \int_{\mathcal{E}_h^i} \{q_h\}[\nabla z]\, ds \\ &= b(\boldsymbol{v}, q_h) \end{aligned} \tag{24}$$

for any $\boldsymbol{v} \in V$ and any $q_h \in Q_h$.

Moreover, by (9), (21), (22) and (23), we derive

$$\|\pi_h \boldsymbol{v}\|_h \leq \left(\|\Pi_h(\nabla z)\|_0^2 + \|\operatorname{div} \boldsymbol{v}\|_0^2 + \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[\Pi_h(\nabla z)]\|_e^2 \right)^{1/2} \leq C\, \|\boldsymbol{v}\|_V \tag{25}$$

based on the relations

$$\begin{aligned} \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[\Pi_h(\nabla z)]\|_e^2 &= \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|[\nabla z - \Pi_h(\nabla z)]\|_e^2 \\ &\leq C \sum_{K \in \mathcal{T}_h} h_K^{-1} (h_K^{-1} \|\nabla z - \Pi_h(\nabla z)\|_{0,K}^2 + h_K \|\nabla z - \Pi_h(\nabla z)\|_{1,K}^2) \end{aligned}$$

$$\leq C \sum_{K\in\mathcal{T}_h} h_K^{-1}(h_K^{-1}h_K^2\|\nabla z\|_{1,K}^2 + h_K\|\nabla z\|_{1,K}^2)$$
$$\leq C \sum_{K\in\mathcal{T}_h} \|\nabla z\|_{1,K}^2 \leq C\|z\|_2^2 \leq C\|\boldsymbol{v}\|_V^2,$$

and

$$\|\Pi_h(\nabla z)\|_0^2 \leq \|\nabla z\|_0^2 + \|\nabla z - \Pi_h(\nabla z)\|_0^2$$
$$\leq \|\nabla z\|_0^2 + Ch^2\|\nabla z\|_1^2 \leq C\|z\|_2^2 \leq C\|\boldsymbol{v}\|_V^2.$$

The identity (24) and inequality (25) show that π_h is a Fortin operator. Then the inf-sup condition (19) can be proved by a standard argument [41,42]. ■

Lemma 3.6: *Assume* $\boldsymbol{f} \in L^2(0,T;[L^2(\Omega)]^2)$, $g_D \in L^2(0,T;H^{1/2}(\partial\Omega))$ *and* $\boldsymbol{u}_0 \in [L^2(\Omega)]^2$. *Then (15) has a unique solution.*

Proof: We first prove the existence and uniqueness of $\boldsymbol{u}_h$. Observe that $\boldsymbol{u}_h \in Z_h$ is determined by

$$(\dot{\boldsymbol{u}}_h, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, g_D\rangle_{\mathcal{E}_h^\partial} \quad \forall\, \boldsymbol{v}_h \in Z_h,$$
$$\boldsymbol{u}_h(0) = \Pi_h\boldsymbol{u}_0.$$

This is an initial value problem for a linear system of ODEs with continuous coefficients and source terms. Hence it has a unique solution $\boldsymbol{u}_h \in Z_h$.

Once $\boldsymbol{u}_h \in Z_h$ is determined, the existence and uniqueness of p_h can be obtained by using the discrete inf-sup condition (19). ■

Remark 3.7: In the definition (13) of the bilinear form $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$, if we replace the penalty term $\int_{\mathcal{E}_h^i} \frac{\eta_e}{h_e}[\boldsymbol{u}_h][\boldsymbol{v}_h]ds$ by $\sum_{e\in\mathcal{E}_h^i}\int_\Omega \eta_e r_e([\boldsymbol{u}_h])r_e([\boldsymbol{v}_h])d\mathbf{x}$, we can obtain another mixed DG scheme, which is the dual form of the method of Brezzi et al. [43]. The well-posedness of the corresponding scheme can be proved similarly.

3.3. Error estimates for the semi-discrete scheme

We begin the error estimation with a consistency result for the semi-discrete MDG scheme.

Lemma 3.8: *For the solution of (2)–(3), assume that* $(\boldsymbol{u}(t), p(t)) \in V \times H^1(\Omega)$ *for a.e.* $t \in [0,T]$. *Then for a.e.* $t \in [0,T]$,

$$\begin{aligned}(\dot{\boldsymbol{u}}(t), \boldsymbol{v}_h) + a_h(\boldsymbol{u}(t), \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p(t)) &= (\boldsymbol{f}(t), \boldsymbol{v}_h) - \langle \boldsymbol{v}_h\cdot\boldsymbol{n}, g_D(t)\rangle_{\mathcal{E}_h^\partial} \quad \forall\, \boldsymbol{v}_h \in V_h,\\ b_h(\boldsymbol{u}(t), q_h) &= 0 \quad \forall\, q_h \in Q_h.\end{aligned} \tag{26}$$

Proof: For a fixed $t > 0$ with $(\boldsymbol{u}, p) = (\boldsymbol{u}(t), p(t)) \in V \times H^1(\Omega)$, we have $[\boldsymbol{u}] = 0$ and $[\![p]\!] = 0$ on each edge $e \in \mathcal{E}_h^i$. Thus,

$$\begin{aligned}(\dot{\boldsymbol{u}}, \boldsymbol{v}_h) + a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p) &= (\dot{\boldsymbol{u}}, \boldsymbol{v}_h) + \alpha(\boldsymbol{u}, \boldsymbol{v}_h)\\ &\quad + \int_{\mathcal{E}_h^i}\frac{\eta_e}{h_e}[\boldsymbol{u}][\boldsymbol{v}_h]\,ds - (p, \mathrm{div}_h\boldsymbol{v}_h) + \int_{\mathcal{E}_h^i}\{p\}[\boldsymbol{v}_h]\,ds\\ &= (\dot{\boldsymbol{u}}, \boldsymbol{v}_h) + \alpha(\boldsymbol{u}, \boldsymbol{v}_h) + (\nabla p, \boldsymbol{v}_h) - \int_{\mathcal{E}_h}\{\boldsymbol{v}_h\}[\![p]\!]\,ds\end{aligned}$$

$$= (\dot{\boldsymbol{u}}, \boldsymbol{v}_h) + \alpha(\boldsymbol{u}, \boldsymbol{v}_h) + (\nabla p, \boldsymbol{v}_h) - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, g_D \rangle_{\mathcal{E}_h^\partial}$$
$$= (\boldsymbol{f}, \boldsymbol{v}_h) - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, g_D \rangle_{\mathcal{E}_h^\partial}$$

and

$$b_h(\boldsymbol{u}, q_h) = -(\mathrm{div}_h \boldsymbol{u}, q_h) + \int_{\mathcal{E}_h^i} \{q_h\}[\boldsymbol{u}]\,\mathrm{d}s = (\mathrm{div}\,\boldsymbol{u}, q_h) = 0.$$

This completes the proof of the lemma. ■

Now we derive a priori error estimates for the numerical solution of the spatially semi-discrete MDG scheme.

Theorem 3.9: *Let* $(\boldsymbol{u}, p)$ *and* $(\boldsymbol{u}_h, p_h)$ *be the solutions of* (2)–(3) *and* (15)–(16), *respectively. Assume that* $\boldsymbol{u} \in C(0, T; [H^{l+1}(\Omega)]^2) \cap L^2(0, T; [H^{l+2}(\Omega)]^2)$, $\dot{\boldsymbol{u}} \in L^2(0, T; [H^{l+1}(\Omega)]^2)$, *and* $p \in L^2(0, T; H^{l+1}(\Omega))$. *Then,*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_0^2 + \int_0^t \|\boldsymbol{u}(\tau) - \boldsymbol{u}_h(\tau)\|_h^2\,\mathrm{d}\tau$$
$$\leq C h^{2(l+1)} \|\boldsymbol{u}\|^2_{C(0,T;H^{l+1}(\Omega)^2)} + C h^{2(l+1)} \int_0^t (|\dot{\boldsymbol{u}}|^2_{l+1} + |\boldsymbol{u}|^2_{l+2} + |p|^2_{l+1})\,\mathrm{d}\tau. \tag{27}$$

Proof: Subtracting (15) from (26), we have, for a.e. $t \in [0, T]$,

$$\begin{aligned} &(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}_h, \boldsymbol{v}_h) + a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p - p_h) = 0 \quad \forall\, \boldsymbol{v}_h \in V_h, \\ &b_h(\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0 \quad \forall\, q_h \in Q_h. \end{aligned} \tag{28}$$

Decompose the errors as

$$\mathbf{e} = \boldsymbol{u} - \boldsymbol{u}_h = \mathbf{e}_1 + \mathbf{e}_2 \quad \text{with} \quad \mathbf{e}_1 = \boldsymbol{u} - \Pi_h \boldsymbol{u} \quad \text{and} \quad \mathbf{e}_2 = \Pi_h \boldsymbol{u} - \boldsymbol{u}_h,$$
$$\zeta = p - p_h = \zeta_1 + \zeta_2 \quad \text{with} \quad \zeta_1 = p - P_h p \quad \text{and} \quad \zeta_2 = P_h p - p_h.$$

Then, we rewrite (28) as

$$\begin{aligned} (\dot{\mathbf{e}}_2, \boldsymbol{v}_h) + a_h(\mathbf{e}_2, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \zeta_2) &= -(\dot{\mathbf{e}}_1, \boldsymbol{v}_h) - a_h(\mathbf{e}_1, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, \zeta_1), \\ b_h(\mathbf{e}_2, q_h) &= -b_h(\mathbf{e}_1, q_h), \end{aligned} \tag{29}$$

for all $\boldsymbol{v}_h \in V_h$, $q_h \in Q_h$. According to the definition of Π_h, one finds

$$b_h(\mathbf{e}_1, q_h) = 0 \quad \forall\, q_h \in Q_h.$$

Let $\boldsymbol{v}_h = \mathbf{e}_2$ and $q_h = \zeta_2$ in (29). Then

$$\begin{aligned} (\dot{\mathbf{e}}_2, \mathbf{e}_2) + a_h(\mathbf{e}_2, \mathbf{e}_2) + b_h(\mathbf{e}_2, \zeta_2) &= -(\dot{\mathbf{e}}_1, \mathbf{e}_2) - a_h(\mathbf{e}_1, \mathbf{e}_2) - b_h(\mathbf{e}_2, \zeta_1), \\ b_h(\mathbf{e}_2, \zeta_2) &= 0. \end{aligned} \tag{30}$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_2\|_0^2 + C_2\|\mathbf{e}_2\|_h^2 \leq -(\dot{\mathbf{e}}_1, \mathbf{e}_2) - a_h(\mathbf{e}_1, \mathbf{e}_2) - b_h(\mathbf{e}_2, \zeta_1). \tag{31}$$

By using the Cauchy-Schwarz inequality, we estimate the right terms as follows:

$$-(\dot{\mathbf{e}}_1, \mathbf{e}_2) \leq \|\dot{\mathbf{e}}_1\|_0 \|\mathbf{e}_2\|_0 \leq \frac{1}{2}\|\mathbf{e}_2\|_0^2 + \frac{1}{2}\|\dot{\mathbf{e}}_1\|_0^2,$$

$$-a_h(\mathbf{e}_1,\mathbf{e}_2) \le \|\mathbf{e}_1\|_h\|\mathbf{e}_2\|_h \le \frac{C_2}{4}\|\mathbf{e}_2\|_h^2 + \frac{1}{C_2}\|\mathbf{e}_1\|_h^2,$$
$$-b_h(\mathbf{e}_2,\zeta_1) \le \|\zeta_1\|_0\|\mathbf{e}_2\|_h \le \frac{C_2}{4}\|\mathbf{e}_2\|_h^2 + \frac{1}{C_2}\|\zeta_1\|_0^2.$$

Combining above inequalities with (31), we have for a.e. $t \in [0,T]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_2\|_0^2 + \frac{C_2}{2}\|\mathbf{e}_2\|_h^2 \le \frac{1}{2}\|\mathbf{e}_2\|_0^2 + \frac{1}{2}\|\dot{\mathbf{e}}_1\|_0^2 + \frac{1}{C_2}\|\mathbf{e}_1\|_h^2 + \frac{1}{C_2}\|\zeta_1\|_0^2. \tag{32}$$

Integrating (32) with respect to t from 0 to t, applying the continuous Gronwall's lemma, and noting that $\mathbf{e}_2(0) = \mathbf{0}$, we obtain

$$\|\mathbf{e}_2(t)\|_0^2 + C_2\int_0^t \|\mathbf{e}_2\|_h^2\,\mathrm{d}\tau \le C\int_0^t (\|\dot{\mathbf{e}}_1\|_0^2 + \|\mathbf{e}_1\|_h^2 + \|\zeta_1\|_0^2)\,\mathrm{d}\tau.$$

By the definitions of norm $\|\cdot\|_h$, the operators Π_h and P_h, we obtain

$$\|\dot{\mathbf{e}}_1\|_0 \le Ch^{l+1}|\dot{\boldsymbol{u}}|_{l+1},$$
$$\|\zeta_1\|_0 \le Ch^{l+1}|p|_{l+1}$$

and

$$\|\mathbf{e}_1\|_h^2 = \|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|_0^2 + \|\mathrm{div_h}(\boldsymbol{u}-\Pi_h\boldsymbol{u})\|_0^2 + \sum_{e\in\mathcal{E}_h^i}\frac{1}{h_e}\|[\boldsymbol{u}-\Pi_h\boldsymbol{u}]\|_e^2 \le h^{2(l+1)}|\boldsymbol{u}|_{l+2,\Omega}^2.$$

From the above inequalities, we have

$$\|\mathbf{e}_2(t)\|_0^2 + C_2\int_0^t \|\mathbf{e}_2\|_h^2\,\mathrm{d}\tau \le Ch^{2(l+1)}\int_0^t \left(|\dot{\boldsymbol{u}}|_{l+1}^2 + |\boldsymbol{u}|_{l+2}^2 + |p|_{l+1}^2\right)\mathrm{d}\tau. \tag{33}$$

From (33), the triangle inequality for norms, and the bounds on $\mathbf{e}_1$, we obtain (27). ■

Under higher regularity assumptions, we can obtain an error bound in the $\|\cdot\|_h$-norm.

Theorem 3.10: *Let* $(\boldsymbol{u},p)$ *and* $(\boldsymbol{u}_h,p_h)$ *be the solutions of* (2)–(3) *and* (15)–(16), *respectively. Assume* $\boldsymbol{u} \in C(0,T;[H^{l+2}(\Omega)]^2)$, $\dot{\boldsymbol{u}} \in L^2(0,T;[H^{l+2}(\Omega)]^2)$, $p \in C(0,T;H^{l+1}(\Omega))$, *and* $\dot{p} \in L^2(0,T;H^{l+1}(\Omega))$. *Then*

$$\int_0^t \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}_h\|_0^2\,\mathrm{d}\tau + \|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_h^2$$
$$\le Ch^{2(l+1)}\left(\|\boldsymbol{u}\|_{C(0,T;H^{l+2}(\Omega)^2)}^2 + \|p\|_{C(0,T;H^{l+1}(\Omega))}^2\right) + Ch^{2(l+1)}\int_0^t \left(|\dot{\boldsymbol{u}}|_{l+2}^2 + |\dot{p}|_{l+1}^2\right)\mathrm{d}\tau.$$

Proof: Taking $\boldsymbol{v}_h = \dot{\mathbf{e}}_2$ and $q_h = \zeta_2$ in (29), we have

$$(\dot{\mathbf{e}}_2,\dot{\mathbf{e}}_2) + a_h(\mathbf{e}_2,\dot{\mathbf{e}}_2) + b_h(\dot{\mathbf{e}}_2,\zeta_2)$$
$$= -(\dot{\mathbf{e}}_1,\dot{\mathbf{e}}_2) - a_h(\mathbf{e}_1,\dot{\mathbf{e}}_2) - b_h(\dot{\mathbf{e}}_2,\zeta_1),$$
$$b_h(\mathbf{e}_2,\zeta_2) = 0. \tag{34}$$

For all $q \in Q_h$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} b_h(\mathbf{e}_2, q) = b_h(\dot{\mathbf{e}}_2, q). \tag{35}$$

Hence,

$$(\dot{\mathbf{e}}_2, \dot{\mathbf{e}}_2) + a_h(\mathbf{e}_2, \dot{\mathbf{e}}_2) = -(\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2) - a_h(\mathbf{e}_1, \dot{\mathbf{e}}_2) - b_h(\dot{\mathbf{e}}_2, \zeta_1). \tag{36}$$

So,

$$\|\dot{\mathbf{e}}_2\|_0^2 + \frac{C_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_2\|_h^2 \le -(\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2) - a_h(\mathbf{e}_1, \dot{\mathbf{e}}_2) - b_h(\dot{\mathbf{e}}_2, \zeta_1). \tag{37}$$

The first term on the right hand side of (37) can be bounded as

$$-(\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2) \le \|\dot{\mathbf{e}}_1\|_0\|\dot{\mathbf{e}}_2\|_0 \le \frac{1}{2}\|\dot{\mathbf{e}}_2\|_0^2 + \frac{1}{2}\|\dot{\mathbf{e}}_1\|_0^2.$$

Then,

$$\frac{\|\dot{\mathbf{e}}_2\|_0^2}{2} + \frac{C_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_2\|_h^2 \le \frac{1}{2}\|\dot{\mathbf{e}}_1\|_0^2 - a_h(\mathbf{e}_1, \dot{\mathbf{e}}_2) - b_h(\dot{\mathbf{e}}_2, \zeta_1). \tag{38}$$

Integrating (38) from 0 to t, due to the fact that $\mathbf{e}_2(0) = \mathbf{0}$, we get

$$\int_0^t \|\dot{\mathbf{e}}_2\|_0^2 \,\mathrm{d}\tau + C_2\|\mathbf{e}_2(t)\|_h^2 \le \int_0^t \|\dot{\mathbf{e}}_1\|_0^2 \,\mathrm{d}\tau - 2\int_0^t a_h(\mathbf{e}_1, \dot{\mathbf{e}}_2)\,\mathrm{d}\tau - 2\int_0^t b_h(\dot{\mathbf{e}}_2, \zeta_1)\,\mathrm{d}\tau. \tag{39}$$

For the last two terms of (39), by integrating by parts and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
-2\int_0^t a_h(\mathbf{e}_1, \dot{\mathbf{e}}_2)\,\mathrm{d}\tau &= -2a_h(\mathbf{e}_1(t), \mathbf{e}_2(t)) + 2\int_0^t a_h(\dot{\mathbf{e}}_1, \mathbf{e}_2)\,\mathrm{d}\tau \\
&\le \frac{C_2}{4}\|\mathbf{e}_2(t)\|_h^2 + \frac{4}{C_2}\|\mathbf{e}_1(t)\|_h^2 + \int_0^t \|\dot{\mathbf{e}}_1\|_h^2\,\mathrm{d}\tau + \int_0^t \|\mathbf{e}_2\|_h^2\,\mathrm{d}\tau, \\
-2\int_0^t b_h(\dot{\mathbf{e}}_2, \zeta_1)\,\mathrm{d}\tau &= -2b_h(\mathbf{e}_2(t), \zeta_1(t)) + 2\int_0^t b_h(\mathbf{e}_2, \dot{\zeta}_1)\,\mathrm{d}\tau \\
&\le \frac{C_2}{4}\|\mathbf{e}_2(t)\|_h^2 + \frac{4}{C_2}\|\zeta_1(t)\|_0^2 + \int_0^t \|\dot{\zeta}_1\|_0^2\,\mathrm{d}\tau + \int_0^t \|\mathbf{e}_2\|_h^2\,\mathrm{d}\tau.
\end{aligned}$$

Combining the above inequalities with (39), we obtain

$$\int_0^t \|\dot{\mathbf{e}}_2\|_0^2\,\mathrm{d}\tau + \frac{C_2}{2}\|\mathbf{e}_2(t)\|_h^2 \le \int_0^t \|\dot{\mathbf{e}}_1\|_0^2\,\mathrm{d}\tau + 2\int_0^t \|\mathbf{e}_2\|_h^2\,\mathrm{d}\tau + \frac{4}{C_2}\|\mathbf{e}_1(t)\|_h^2 + \int_0^t \|\dot{\mathbf{e}}_1\|_h^2\,\mathrm{d}\tau + \frac{4}{C_2}\|\zeta_1(t)\|_0^2 + \int_0^t \|\dot{\zeta}_1\|_0^2\,\mathrm{d}\tau. \tag{40}$$

Then apply the continuous Gronwall's lemma to obtain

$$\int_0^t \|\dot{\mathbf{e}}_2\|_0^2 \,\mathrm{d}\tau + \frac{C_2}{2}\|\mathbf{e}_2(t)\|_h^2 \le C\left(\|\mathbf{e}_1(t)\|_h^2 + \|\zeta_1(t)\|_0^2\right) + C\int_0^t \left(\|\dot{\mathbf{e}}_1\|_0^2 + \|\dot{\mathbf{e}}_1\|_h^2 + \|\dot{\zeta}_1\|_0^2\right) \mathrm{d}\tau.$$

By the error estimates (9) and (11), we get

$$\int_0^t \|\dot{\mathbf{e}}_2\|_0^2 \,\mathrm{d}\tau + \frac{C_2}{2}\|\mathbf{e}_2(t)\|_h^2 \le Ch^{2(l+1)}\|\boldsymbol{u}\|^2_{C(0,T;H^{l+2}(\Omega)^2)} + Ch^{2(l+1)}\|p\|^2_{C(0,T;H^{l+1}(\Omega))} + Ch^{2(l+1)}\int_0^t \left(|\dot{\boldsymbol{u}}|_{l+2}^2 + |\dot{p}|_{l+1}^2\right) \mathrm{d}\tau. \tag{41}$$

Then the proof of Theorem 3.10 is completed from (41), the triangle inequality for norms and the bounds on $\mathbf{e}_1$. ■

Finally, we bound the error $p - p_h$.

Theorem 3.11: *Under the assumptions of Theorem 3.10, we have*

$$\int_0^t \|p - p_h\|_0^2 \,\mathrm{d}\tau \le Ch^{2(l+1)}\left(\|\boldsymbol{u}\|^2_{C(0,T;H^{l+2}(\Omega)^2)} + \|p\|^2_{C(0,T;H^{l+1}(\Omega))} + \int_0^t |\dot{\boldsymbol{u}}|_{l+2}^2 \,\mathrm{d}\tau + \int_0^t |\dot{p}|_{l+1}^2 \,\mathrm{d}\tau\right).$$

Proof: From (29), we have

$$b_h(\boldsymbol{v}_h, \zeta_2) = -(\dot{\mathbf{e}}, \boldsymbol{v}_h) - a_h(\mathbf{e}, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, \zeta_1) \quad \forall\, \boldsymbol{v}_h \in V_h. \tag{42}$$

The inf-sup condition (19) guarantees that

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{b_h(\boldsymbol{v}_h, \zeta_2)}{\|\boldsymbol{v}_h\|_h} \ge \beta_2 \|\zeta_2\|_0. \tag{43}$$

Applying the Cauchy-Schwarz inequality and Lemma 3.4, we obtain

$$\begin{aligned} -(\dot{\mathbf{e}}, \boldsymbol{v}_h) &\le \|\dot{\mathbf{e}}\|_0 \|\boldsymbol{v}_h\|_0 \le \|\dot{\mathbf{e}}\|_0 \|\boldsymbol{v}_h\|_h, \\ -a_h(\mathbf{e}, \boldsymbol{v}_h) &\le \|\mathbf{e}\|_h \|\boldsymbol{v}_h\|_h, \\ -b_h(\boldsymbol{v}_h, \zeta_1) &\le \|\zeta_1\|_0 \|\boldsymbol{v}_h\|_h, \end{aligned}$$

then,

$$\|\zeta_2\|_0 \le C\left(\|\dot{\mathbf{e}}\|_0 + \|\mathbf{e}\|_h + \|\zeta_1\|_0\right).$$

Integrating above formulas with respect to t from 0 to t, we arrive at

$$\begin{aligned} \int_0^t \|\zeta_2\|_0^2 \,\mathrm{d}\tau &\le C\int_0^t \left(\|\dot{\mathbf{e}}\|_0^2 + \|\mathbf{e}\|_h^2 + \|\zeta_1\|_0^2\right) \mathrm{d}\tau \\ &\le Ch^{2(l+1)}\left(\|\boldsymbol{u}\|^2_{C(0,T;H^{l+2}(\Omega)^2)} + \|p\|^2_{C(0,T;H^{l+1}(\Omega))}\right) \\ &\quad + Ch^{2(l+1)}\int_0^t \left(|\dot{\boldsymbol{u}}|_{l+2}^2 + |\dot{p}|_{l+1}^2\right) \mathrm{d}\tau, \end{aligned}$$

where we applied error bounds from Theorem 3.10 and (11). By this error bound, the triangle inequality for norms, and error bounds on ζ_1, we get the stated error bound for the pressure variable approximation. ■

4. Fully-discrete MDG scheme

In this section, we study a fully-discrete scheme which is constructed with the MDG discretization for the spatial variable and backward Euler difference approximation for the temporal variable. Introduce a partition of the time interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \le n \le N$ (N an integer), such that $0 = t_0 < t_1 < \cdots < t_N = T$. In order to simplify the notation, we consider only evenly spaced nodes $t_n = nk$ ($n = 0, 1, \ldots, N$) with the time step $k = T/N$. For a function $\varphi(\cdot, t)$ continuous in t, we write $\varphi^n(\cdot) = \varphi(\cdot, t_n)$ and set

$$\delta_k \varphi^n = \frac{\varphi^n - \varphi^{n-1}}{k}.$$

For convenience, in this section we assume

$$\boldsymbol{f} \in C(0, T; L^2(\Omega)^2), \quad g_D \in C(0, T; H^{1/2}(\partial\Omega)). \tag{44}$$

Then

$$\boldsymbol{f}^n(\cdot) = \boldsymbol{f}(\cdot, t_n) \in L^2(\Omega)^2, \quad g_D^n(\cdot) = g_D(\cdot, t_n) \in H^{1/2}(\partial\Omega).$$

We comment that without the condition (44), we can take $\boldsymbol{f}^n$ and g_D^n to be averages of $\boldsymbol{f}$ and g_D over the time interval $[t_{n-1}, t_n]$.

A fully discrete approximation of (2)–(3) is to find $(\boldsymbol{u}_h^n, p_h^n) \in V_h \times Q_h$ such that for $1 \le n \le N$,

$$\begin{aligned} (\delta_k \boldsymbol{u}_h^n, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h^n) &= (\boldsymbol{f}^n, \boldsymbol{v}_h) - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, g_D^n \rangle_{\mathcal{E}_h^\partial} \quad \forall\, \boldsymbol{v}_h \in V_h, \\ b_h(\boldsymbol{u}_h^n, q_h) &= 0 \quad \forall\, q_h \in Q_h, \end{aligned} \tag{45}$$

and

$$\boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}_0. \tag{46}$$

Thanks to the properties of $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ listed in Lemmas 3.4 and 3.5, we can show that the fully discrete solution exists and is unique.

4.1. Stability of the fully-discrete scheme

We show that the numerical solution $\{(\boldsymbol{u}_h^n, p_h^n)\}_{n=1}^N$ of the fully discrete scheme (45)–(46) is bounded.

Theorem 4.1: *There exists positive constants k_0 and C such that for $k \le k_0$ and $1 \le n \le N$,*

$$\|\boldsymbol{u}_h^n\|_0^2 + \sum_{j=1}^n \|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\|_0^2 + k\sum_{j=1}^n \|\boldsymbol{u}_h^j\|_h^2 \le C\,\|\boldsymbol{u}_h^0\|_0^2 + C\,k\sum_{j=1}^n \left(\|\boldsymbol{f}^j\|_0^2 + \|g_D^j\|_{1/2,\partial\Omega}^2\right). \tag{47}$$

Proof: Taking $\boldsymbol{v}_h = 2k\boldsymbol{u}_h^n \in V_h$ and $q_h = p_h^n$ in (45) and applying the following relation

$$(a - b, 2a) = a^2 - b^2 + |a - b|^2, \tag{48}$$

we obtain

$$\begin{aligned} &\|\boldsymbol{u}_h^n\|_0^2 - \|\boldsymbol{u}_h^{n-1}\|_0^2 + \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_0^2 + 2C_2k\|\boldsymbol{u}_h^n\|_h^2 \\ &\quad\le 2k(\boldsymbol{f}^n, \boldsymbol{u}_h^n) + 2k\langle \boldsymbol{u}_h^n \cdot \boldsymbol{n}, g_D^n\rangle_{\mathcal{E}_h^\partial} \\ &\quad\le k\|\boldsymbol{f}^n\|_0^2 + k\|\boldsymbol{u}_h^n\|_0^2 + \frac{kc^2}{C_2}\|g_D^n\|_{1/2,\partial\Omega}^2 + C_2k\|\boldsymbol{u}_h^n\|_h^2. \end{aligned} \tag{49}$$

Change n to j in (49) and make a summation for $j = 1, \cdots, n$,

$$\|\boldsymbol{u}_h^n\|_0^2 + \sum_{j=1}^{n} \|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\|_0^2 + C_2 k \sum_{j=1}^{n} \|\boldsymbol{u}_h^j\|_h^2 \le \|\boldsymbol{u}_h^0\|_0^2 + k \sum_{j=1}^{n} \left(\|\boldsymbol{u}_h^j\|_0^2 + \|\boldsymbol{f}^j\|_0^2 + \frac{c^2}{C_2} \|g_D^j\|_{1/2,\partial\Omega}^2 \right).$$

Then, we finish the proof of (47) by applying the discrete Gronwall lemma. ■

4.2. Error estimates of the fully-discrete scheme

In the following, we will assume the condition $k \le k_0$ from Theorem 4.1 is true. We provide error estimates for the fully-discrete MDG scheme in this subsection.

Theorem 4.2: *Let $\{(\boldsymbol{u}_h^n, p_h^n)\}_{n=1}^N$ be the numerical solutions of (45)–(46). Under the assumptions of Theorem 3.9, $\ddot{\boldsymbol{u}} \in L^2(0,T;[L^2(\Omega)]^2)$ and $p \in C(0,T;L^2(\Omega))$, we have a constant $C > 0$ such that for $1 \le n \le N$,*

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0^2 + \sum_{j=1}^{n} \|(\boldsymbol{u}^j - \boldsymbol{u}_h^j) - (\boldsymbol{u}^{j-1} - \boldsymbol{u}_h^{j-1})\|_0^2 + k \sum_{j=1}^{n} \|\boldsymbol{u}^j - \boldsymbol{u}_h^j\|_h^2$$
$$\le C h^{2(l+1)} \|\boldsymbol{u}\|_{C(0,T;H^{l+1}(\Omega)^2)}^2 + C k^2 \int_0^{t_n} \|\ddot{\boldsymbol{u}}\|_0^2 \,\mathrm{d}t + C h^{2(l+1)} \int_0^{t_n} (|\dot{\boldsymbol{u}}|_{l+1}^2 + |\boldsymbol{u}|_{l+2}^2 + |p|_{l+1}^2) \,\mathrm{d}t.$$

Proof: Under the stated regularity assumptions, (26) holds for all $t \in [0,T]$. Subtracting (45) from (26) at the time $t = t_n$, we have

$$\begin{aligned} (\delta_k \boldsymbol{u}^n - \delta_k \boldsymbol{u}_h^n, \boldsymbol{v}_h) + a_h(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p^n - p_h^n) &= (R^n, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in V_h, \\ b_h(\boldsymbol{u}^n - \boldsymbol{u}_h^n, q_h) &= 0 \quad \forall\, q_h \in Q_h, \end{aligned} \tag{50}$$

where

$$R^n = \delta_k \boldsymbol{u}^n - \dot{\boldsymbol{u}}(\cdot, t_n).$$

To bound R^n, we use the Taylor's expansion

$$\boldsymbol{u}(\cdot, t_n) - \boldsymbol{u}(\cdot, t_{n-1}) = k \dot{\boldsymbol{u}}(\cdot, t_n) - \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \ddot{\boldsymbol{u}}(\cdot, t) \,\mathrm{d}t.$$

As a result,

$$R^n = -\frac{1}{k} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \ddot{\boldsymbol{u}}(\cdot, t) \,\mathrm{d}t.$$

By using Cauchy-Schwarz inequality, we have

$$\|R^n\|_0^2 \le \frac{k}{3} \int_{t_{n-1}}^{t_n} \|\ddot{\boldsymbol{u}}\|_0^2 \,\mathrm{d}t. \tag{51}$$

Write the errors $\mathbf{e}^n = \boldsymbol{u}^n - \boldsymbol{u}_h^n$ and $\zeta^n = p^n - p_h^n$ as

$$\begin{aligned} \mathbf{e}^n &= \mathbf{e}_1^n + \mathbf{e}_2^n \quad \text{with} \quad \mathbf{e}_1^n = \boldsymbol{u}^n - \Pi_h \boldsymbol{u}^n \quad \text{and} \quad \mathbf{e}_2^n = \Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n, \\ \zeta^n &= \zeta_1^n + \zeta_2^n \quad \text{with} \quad \zeta_1^n = p^n - P_h p^n \quad \text{and} \quad \zeta_2^n = P_h p^n - p_h^n. \end{aligned}$$

Then, from (50), for any $\boldsymbol{v}_h \in V_h$ and any $q_h \in Q_h$,

$$\begin{aligned}(\delta_k \mathbf{e}_2^n, \boldsymbol{v}_h) + a_h(\mathbf{e}_2^n, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \zeta_2^n) &= (R^n, \boldsymbol{v}_h) - (\delta_k \mathbf{e}_1^n, \boldsymbol{v}_h) - a_h(\mathbf{e}_1^n, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, \zeta_1^n), \\ b_h(\mathbf{e}_2^n, q_h) &= -b_h(\mathbf{e}_1^n, q_h).\end{aligned} \tag{52}$$

According to the definition of Π_h, it is easily to get

$$b_h(\mathbf{e}_1^n, q_h) = 0 \quad \forall\, q_h \in Q_h.$$

Therefore,

$$b_h(\mathbf{e}_2^n, q_h) = 0 \quad \forall\, q_h \in Q_h. \tag{53}$$

Taking $\boldsymbol{v}_h = \mathbf{e}_2^n$ in the first relation of (52), we have

$$(\delta_k \mathbf{e}_2^n, \mathbf{e}_2^n) + a_h(\mathbf{e}_2^n, \mathbf{e}_2^n) = (R^n, \mathbf{e}_2^n) - (\delta_k \mathbf{e}_1^n, \mathbf{e}_2^n) - a_h(\mathbf{e}_1^n, \mathbf{e}_2^n) - b_h(\mathbf{e}_2^n, \zeta_1^n). \tag{54}$$

Apply (48) and Lemma 3.3 to yield

$$\begin{aligned}&\frac{1}{2k}(\|\mathbf{e}_2^n\|_0^2 - \|\mathbf{e}_2^{n-1}\|_0^2 + \|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0^2) + C_2\|\mathbf{e}_2^n\|_h^2 \\ &\quad \le (R^n, \mathbf{e}_2^n) - (\delta_k \mathbf{e}_1^n, \mathbf{e}_2^n) - a_h(\mathbf{e}_1^n, \mathbf{e}_2^n) - b_h(\mathbf{e}_2^n, \zeta_1^n).\end{aligned} \tag{55}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}(R^n, \mathbf{e}_2^n) &\le \|R^n\|_0\|\mathbf{e}_2^n\|_0 \le \frac{1}{2}\|\mathbf{e}_2^n\|_0^2 + \frac{1}{2}\|R^n\|_0^2, \\ -(\delta_k \mathbf{e}_1^n, \mathbf{e}_2^n) &\le \|\delta_k \mathbf{e}_1^n\|_0\|\mathbf{e}_2^n\|_0 \le \frac{1}{2}\|\mathbf{e}_2^n\|_0^2 + \frac{1}{2}\|\delta_k \mathbf{e}_1^n\|_0^2, \\ -a_h(\mathbf{e}_1^n, \mathbf{e}_2^n) &\le \|\mathbf{e}_1^n\|_h\|\mathbf{e}_2^n\|_h \le \frac{C_2}{4}\|\mathbf{e}_2^n\|_h^2 + \frac{1}{C_2}\|\mathbf{e}_1^n\|_h^2, \\ -b_h(\mathbf{e}_2^n, \zeta_1^n) &\le \|\zeta_1^n\|_0\|\mathbf{e}_2^n\|_h \le \frac{C_2}{4}\|\mathbf{e}_2^n\|_h^2 + \frac{1}{C_2}\|\zeta_1^n\|_0^2.\end{aligned}$$

Apply these inequalities in (55),

$$\begin{aligned}&\frac{1}{2k}(\|\mathbf{e}_2^n\|_0^2 - \|\mathbf{e}_2^{n-1}\|_0^2 + \|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0^2) + \frac{C_2}{2}\|\mathbf{e}_2^n\|_h^2 \\ &\quad \le \frac{1}{2}\|R^n\|_0^2 + \|\mathbf{e}_2^n\|_0^2 + \frac{1}{2}\|\delta_k \mathbf{e}_1^n\|_0^2 + \frac{1}{C_2}\|\mathbf{e}_1^n\|_h^2 + \frac{1}{C_2}\|\zeta_1^n\|_0^2.\end{aligned} \tag{56}$$

Changing n to j, multiplying $2k$ on both sides of the inequality (56), making a summation for $j = 1, \cdots, n$, and using the fact that $\mathbf{e}_2^0 = \mathbf{0}$, we obtain

$$\begin{aligned}&\|\mathbf{e}_2^n\|_0^2 + \sum_{j=1}^{n}\|\mathbf{e}_2^j - \mathbf{e}_2^{j-1}\|_0^2 + C_2 k\sum_{j=1}^{n}\|\mathbf{e}_2^j\|_h^2 \\ &\quad \le k\sum_{j=1}^{n}\left(\|R^j\|_0^2 + 2\|\mathbf{e}_2^j\|_0^2 + \|\delta_k \mathbf{e}_1^j\|_0^2 + \frac{2}{C_2}\|\mathbf{e}_1^j\|_h^2 + \frac{2}{C_2}\|\zeta_1^j\|_0^2\right).\end{aligned} \tag{57}$$

By (9) and (11),

$$\|\delta_k \mathbf{e}_1^j\|_0^2 = \frac{1}{k^2}\|\int_{t_{j-1}}^{t_j}(I-\Pi_h)\dot{\boldsymbol{u}}\,\mathrm{d}t\|_0^2 \le \frac{1}{k}\int_{t_{j-1}}^{t_j} Ch^{2(l+1)}|\dot{\boldsymbol{u}}|_{l+1}^2\,\mathrm{d}t, \tag{58}$$

and

$$\|\mathbf{e}_1^j\|_h^2 = \|(I-\Pi_h)\boldsymbol{u}^j\|_h^2 \le Ch^{2(l+1)}|\boldsymbol{u}^j|_{l+2}^2,$$
$$\|\zeta_1^j\|_0^2 = \|(I-P_h)p^j\|_0^2 \le Ch^{2(l+1)}|p^j|_{l+1}^2.$$

By applying the discrete Gronwall lemma to (57), we get

$$\|\mathbf{e}_2^n\|_0^2 + \sum_{j=1}^n \|\mathbf{e}_2^j - \mathbf{e}_2^{j-1}\|_0^2 + C_2 k\sum_{j=1}^n \|\mathbf{e}_2^j\|_h^2$$
$$\le Ck^2\int_0^{t_n}\|\ddot{\boldsymbol{u}}\|_0^2\,\mathrm{d}t + Ch^{2(l+1)}\int_0^{t_n}(|\dot{\boldsymbol{u}}|_{l+1}^2 + |\boldsymbol{u}|_{l+2}^2 + |p|_{l+1}^2)\,\mathrm{d}t. \tag{59}$$

Combining this inequality with the error bounds on $\mathbf{e}_1^n$, we complete the proof. ■

Theorem 4.3: *Under the conditions of Theorem* 3.10 *and* $\ddot{\boldsymbol{u}} \in L^2(0,T;[L^2(\Omega)]^2)$, *we have a constant* $C>0$ *such that for* $1 \le n \le N$,

$$\frac{1}{k}\sum_{j=1}^n \|(\boldsymbol{u}^j - \boldsymbol{u}_h^j) - (\boldsymbol{u}^{j-1} - \boldsymbol{u}_h^{j-1})\|_0^2 + \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_h^2 + \sum_{j=1}^n \|(\boldsymbol{u}^j - \boldsymbol{u}_h^j) - (\boldsymbol{u}^{j-1} - \boldsymbol{u}_h^{j-1})\|_h^2$$
$$\le Ch^{2(l+1)}\|\boldsymbol{u}\|_{C(0,T;H^{l+2}(\Omega)^2)}^2 + Ch^{2(l+1)}\|p\|_{C(0,T;H^{l+1}(\Omega))}^2$$
$$+ Ck^2\int_0^{t_n}\|\ddot{\boldsymbol{u}}\|_0^2\,\mathrm{d}t + Ch^{2(l+1)}\int_0^{t_n}(|\dot{\boldsymbol{u}}|_{l+2}^2 + |\dot{\boldsymbol{u}}|_{l+1}^2 + |\dot{p}|_{l+1}^2)\,\mathrm{d}t.$$

Proof: Taking $\boldsymbol{v}_h = 2(\mathbf{e}_2^n - \mathbf{e}_2^{n-1})$ and $q_h = \zeta_2^n$ in (52), by (53), we get $b_h(\mathbf{e}_2^{n-1}, \zeta_2^n) = 0$, $b_h(\mathbf{e}_2^n, \zeta_2^n) = 0$ and

$$\frac{2}{k}\|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0^2 + a_h(\mathbf{e}_2^n, \mathbf{e}_2^n) - a_h(\mathbf{e}_2^{n-1}, \mathbf{e}_2^{n-1}) + a_h(\mathbf{e}_2^n - \mathbf{e}_2^{n-1}, \mathbf{e}_2^n - \mathbf{e}_2^{n-1})$$
$$= 2(R^n, \mathbf{e}_2^n - \mathbf{e}_2^{n-1}) - 2(\delta_k\mathbf{e}_1^n, \mathbf{e}_2^n - \mathbf{e}_2^{n-1})$$
$$- 2a_h(\mathbf{e}_1^n, \mathbf{e}_2^n - \mathbf{e}_2^{n-1}) - 2b_h(\mathbf{e}_2^n - \mathbf{e}_2^{n-1}, \zeta_1^n). \tag{60}$$

Applying the Cauchy-Schwarz inequality,

$$2(R^n, \mathbf{e}_2^n - \mathbf{e}_2^{n-1}) \le 2\|R^n\|_0\|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0 \le 2k\|R^n\|_0^2 + \frac{1}{2k}\|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0^2,$$
$$-2(\delta_k\mathbf{e}_1^n, \mathbf{e}_2^n - \mathbf{e}_2^{n-1}) \le 2\|\delta_k\mathbf{e}_1^n\|_0\|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0 \le 2k\|\delta_k\mathbf{e}_1^n\|_0^2 + \frac{1}{2k}\|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0^2.$$

Combining above inequalities with (60), we get

$$\frac{1}{k}\|\mathbf{e}_2^n - \mathbf{e}_2^{n-1}\|_0^2 + a_h(\mathbf{e}_2^n, \mathbf{e}_2^n) - a_h(\mathbf{e}_2^{n-1}, \mathbf{e}_2^{n-1}) + a_h(\mathbf{e}_2^n - \mathbf{e}_2^{n-1}, \mathbf{e}_2^n - \mathbf{e}_2^{n-1})$$
$$\le 2k\|R^n\|_0^2 + 2k\|\delta_k\mathbf{e}_1^n\|_0^2 - 2a_h(\mathbf{e}_1^n, \mathbf{e}_2^n - \mathbf{e}_2^{n-1}) - 2b_h(\mathbf{e}_2^n - \mathbf{e}_2^{n-1}, \zeta_1^n). \tag{61}$$

Changing n to j in (61) and making a summation for $j = 1, \cdots, n$, by Lemma 3.4, we obtain

$$\begin{aligned}
&\frac{1}{k}\sum_{j=1}^{n}\|\mathbf{e}_2^j - \mathbf{e}_2^{j-1}\|_0^2 + C_2\|\mathbf{e}_2^n\|_h^2 + C_2\sum_{j=1}^{n}\|\mathbf{e}_2^j - \mathbf{e}_2^{j-1}\|_h^2 \\
&\quad \le 2k\sum_{j=1}^{n}\|R^j\|_0^2 + 2k\sum_{j=1}^{n}\|\delta_k\mathbf{e}_1^j\|_0^2 \\
&\qquad - 2\sum_{j=1}^{n}a_h(\mathbf{e}_1^j, \mathbf{e}_2^j - \mathbf{e}_2^{j-1}) - 2\sum_{j=1}^{n}b_h(\mathbf{e}_2^j - \mathbf{e}_2^{j-1}, \zeta_1^j).
\end{aligned}$$

For the last two terms, we have

$$\begin{aligned}
-2\sum_{j=1}^{n}a_h(\mathbf{e}_1^j, \mathbf{e}_2^j - \mathbf{e}_2^{j-1}) &= -2\sum_{j=1}^{n}a_h(\mathbf{e}_1^j, \mathbf{e}_2^j) + 2\sum_{j=0}^{n-1}a_h(\mathbf{e}_1^{j+1}, \mathbf{e}_2^j) \\
&= -2a_h(\mathbf{e}_1^n, \mathbf{e}_2^n) + 2a_h(\mathbf{e}_1^1, \mathbf{e}_2^0) + 2\sum_{j=1}^{n-1}a_h(\mathbf{e}_1^{j+1} - \mathbf{e}_1^j, \mathbf{e}_2^j) \\
&\le \frac{C_2}{4}\|\mathbf{e}_2^n\|_h^2 + \frac{4}{C_2}\|\mathbf{e}_1^n\|_h^2 + k\sum_{j=1}^{n-1}\|\mathbf{e}_2^j\|_h^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|\mathbf{e}_1^{j+1} - \mathbf{e}_1^j\|_h^2,
\end{aligned}$$

and

$$\begin{aligned}
-2\sum_{j=1}^{n}b_h(\mathbf{e}_2^j - \mathbf{e}_2^{j-1}, \zeta_1^j) &= -2\sum_{j=1}^{n}b_h(\mathbf{e}_2^j, \zeta_1^j) + 2\sum_{j=0}^{n-1}b_h(\mathbf{e}_2^j, \zeta_1^{j+1}) \\
&= -2b_h(\mathbf{e}_2^n, \zeta_1^n) + 2b_h(\mathbf{e}_2^0, \zeta_1^1) + 2\sum_{j=1}^{n-1}b_h(\mathbf{e}_2^j, \zeta_1^{j+1} - \zeta_1^j) \\
&\le \frac{C_2}{4}\|\mathbf{e}_2^n\|_h^2 + \frac{4}{C_2}\|\zeta_1^n\|_0^2 + k\sum_{j=1}^{n-1}\|\mathbf{e}_2^j\|_h^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|\zeta_1^{j+1} - \zeta_1^j\|_0^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{k}\sum_{j=1}^{n}\|\mathbf{e}_2^j - \mathbf{e}_2^{j-1}\|_0^2 + \frac{C_2}{2}\|\mathbf{e}_2^n\|_h^2 + C_2\sum_{j=1}^{n}\|\mathbf{e}_2^j - \mathbf{e}_2^{j-1}\|_h^2 \\
&\quad \le 2k\sum_{j=1}^{n}\|\mathbf{e}_2^j\|_h^2 + 2k\sum_{j=1}^{n}(\|R^j\|_0^2 + \|\delta_k\mathbf{e}_1^j\|_0^2) + \frac{4}{C_2}\|\mathbf{e}_1^n\|_h^2 \\
&\qquad + \frac{1}{k}\sum_{j=1}^{n-1}\|\mathbf{e}_1^{j+1} - \mathbf{e}_1^j\|_h^2 + \frac{4}{C_2}\|\zeta_1^n\|_0^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|\zeta_1^{j+1} - \zeta_1^j\|_0^2.
\end{aligned} \tag{62}$$

Similar to (58), we obtain

$$\|\mathbf{e}_1^{j+1} - \mathbf{e}_1^j\|_h^2 = \|\int_{t_j}^{t_{j+1}}(I - \Pi_h)\dot{\boldsymbol{u}}\,\mathrm{d}t\|_h^2 \le k\int_{t_j}^{t_{j+1}}Ch^{2(l+1)}|\dot{\boldsymbol{u}}|_{l+2}^2\,\mathrm{d}t,$$

$$\|\zeta_1^{j+1}-\zeta_1^j\|_0^2=\|\int_{t_j}^{t_{j+1}}(I-P_h)\dot{p}\,\mathrm{d}t\|_0^2\le k\int_{t_j}^{t_{j+1}}Ch^{2(l+1)}|\dot{p}|_{l+1}^2\,\mathrm{d}t.$$

Combine above inequalities, (51) and (62), then apply the discrete Gronwall lemma to get

$$\begin{aligned}&\frac{1}{k}\sum_{j=1}^{n}\|\mathbf{e}_2^j-\mathbf{e}_2^{j-1}\|_0^2+\frac{C_2}{2}\|\mathbf{e}_2^n\|_h^2+C_2\sum_{j=1}^{n}\|\mathbf{e}_2^j-\mathbf{e}_2^{j-1}\|_h^2\\&\le Ch^{2(l+1)}(\|\boldsymbol{u}\|^2_{C(0,T;H^{l+2}(\Omega)^2)}+\|p\|^2_{C(0,T;H^{l+1}(\Omega))})+Ck^2\int_0^{t_n}\|\ddot{\boldsymbol{u}}\|_0^2\,\mathrm{d}t\\&\quad+Ch^{2(l+1)}\int_0^{t_n}(|\dot{\boldsymbol{u}}|_{l+2}^2+|\dot{\boldsymbol{u}}|_{l+1}^2+|\dot{p}|_{l+1}^2)\,\mathrm{d}t.\end{aligned}\tag{63}$$

Then, we finish the proof of Theorem 4.3 by triangle inequalities and (9). ■

Finally, we derive an error estimate for the pressure variable p.

Theorem 4.4: *Under the conditions of Theorem* 4.3, *we have*

$$\begin{aligned}k\sum_{j=1}^{n}\|p^j-p_h^j\|_0^2\le Ck^2\int_0^{t_n}\|\ddot{\boldsymbol{u}}\|_0^2dt+Ch^{2(l+1)}(\|\boldsymbol{u}\|^2_{C(0,T;H^{l+2}(\Omega)^2)}+\|p\|^2_{C(0,T;H^{l+1}(\Omega))})\\+Ch^{2(l+1)}\int_0^{t_n}\left(|\dot{\boldsymbol{u}}|_{l+2}^2+|\dot{\boldsymbol{u}}|_{l+1}^2+|\dot{p}|_{l+1}^2\right)\mathrm{d}t.\end{aligned}$$

Proof: Rewrite (52) as

$$b_h(\boldsymbol{v}_h,\zeta_2^n)=(R^n,\boldsymbol{v}_h)-(\delta_k\mathbf{e}^n,\boldsymbol{v}_h)-a_h(\mathbf{e}^n,\boldsymbol{v}_h)-b_h(\boldsymbol{v}_h,\zeta_1^n).$$

The inf-sup condition (19) guarantees that

$$\sup_{\boldsymbol{v}_h\in V_h}\frac{b_h(\boldsymbol{v}_h,\zeta_2^n)}{\|\boldsymbol{v}_h\|_h}\ge\beta_2\|\zeta_2^n\|_0,\tag{64}$$

which leads to

$$\|p^n-p_h^n\|_0^2\le C\left(\|R^n\|_0^2+\|\delta_k\mathbf{e}^n\|_0^2+\|\mathbf{e}^n\|_h^2+\|\zeta_1^n\|_0^2\right).\tag{65}$$

We change the index n to j in (65), and sum the inequalities for $1\le j\le n$,

$$k\sum_{j=1}^{n}\|p^j-p_h^j\|_0^2\le Ck\sum_{j=1}^{n}(\|R^j\|_0^2+\|\delta_k\mathbf{e}^j\|_0^2+\|\mathbf{e}^j\|_h^2+\|\zeta_1^j\|_0^2).$$

By making use of Theorem 4.3, we have

$$\begin{aligned}k\sum_{j=1}^{n}\|p^j-p_h^j\|_0^2\le Ck^2\int_0^{t_n}\|\ddot{\boldsymbol{u}}\|_0^2\,\mathrm{d}t+Ch^{2(l+1)}(\|\boldsymbol{u}\|^2_{C(0,T;H^{l+2}(\Omega)^2)}+\|p\|^2_{C(0,T;H^{l+1}(\Omega))})\\+Ch^{2(l+1)}\int_0^{t_n}\left(|\dot{\boldsymbol{u}}|_{l+2}^2+|\dot{\boldsymbol{u}}|_{l+1}^2+|\dot{p}|_{l+1}^2\right)\mathrm{d}t.\end{aligned}$$

Thus, the proof is completed. ■

5. Numerical tests

In this section, we present computer simulation results on several two-dimensional examples to illustrate numerical evidence of the theoretical error estimates. In all the examples, we solve the time-dependent Darcy problem (1) with $\alpha = 1$ over the spatial domain $\Omega = (0,1) \times (0,1)$. In the numerical simulations, the penalty parameter η_e is taken to be 0.1.

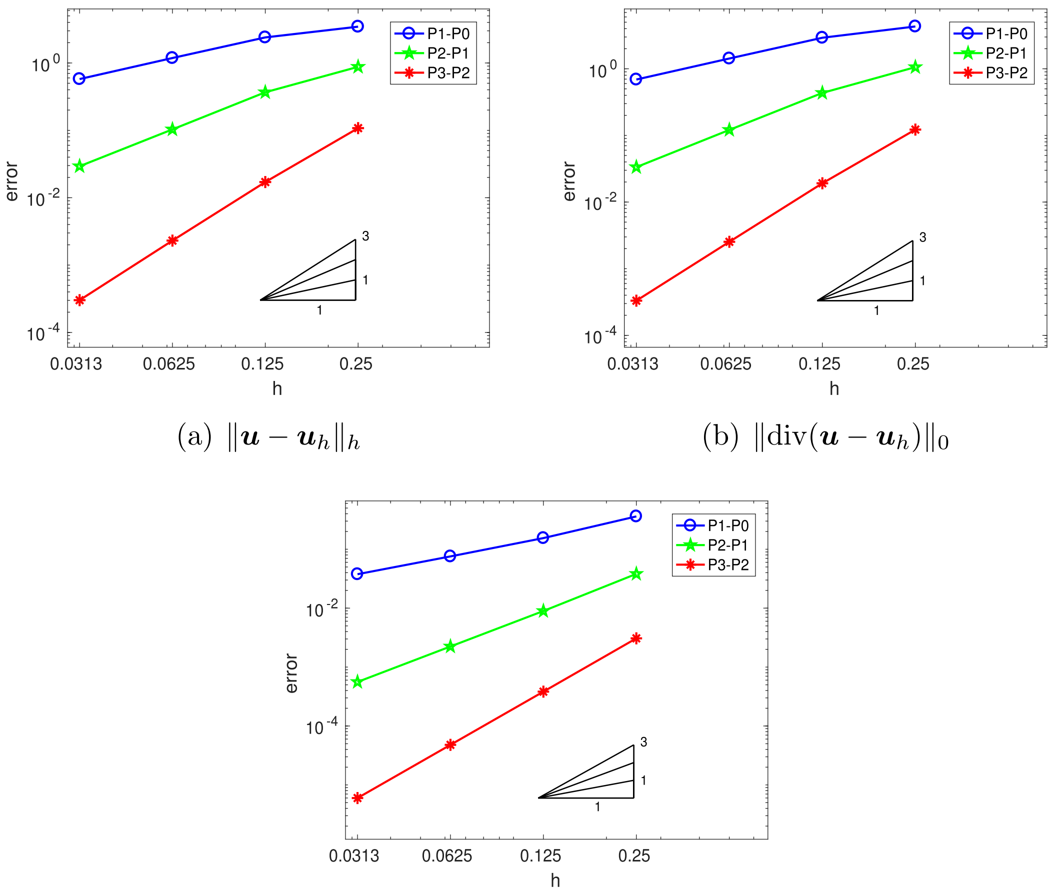
Example 5.1: The problem data are so chosen that the true solution is

$$u_1(x,y,t) = (x^2(y-1)^2 + y)\cos(t),$$

$$u_2(x,y,t) = \left(-\frac{2}{3}x(y-1)^3 + 2 - \pi\,\sin(\pi x)\right)\cos(t),$$

$$p(x,y,t) = (2 - \pi\,\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\cos(t).$$

In this example, we compute the numerical solutions for the element pairs $\mathcal{P}_1$-$\mathcal{P}_0$, $\mathcal{P}_2$-$\mathcal{P}_1$ and $\mathcal{P}_3$-$\mathcal{P}_2$ on the uniform triangular meshes. Fix a small time-step $k = 2^{-10}$. We present the numerical results of $\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$, $\|\mathrm{div}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0$ and $\|p - p_h\|_0$ at the final time $T = 0.1$ in Figure 1. The numerical results confirm the optimal convergence orders in $\|\cdot\|_h$ for $\boldsymbol{u}$ norm and L^2-norm for p, which are proved in Section 4.



(a) $\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$ (b) $\|\mathrm{div}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0$ (c) $\|p - p_h\|_0$

Figure 1. Numerical convergence orders for Example 5.1 at time $T = 0.1$.

Example 5.2: For this example, the problem data are so chosen that the true solution is

$$u_1(x,y,t) = x^2 y e^{-t},$$
$$u_2(x,y,t) = -xy^2 e^{-t},$$
$$p(x,y,t) = (xy - 1/4)e^{-t}.$$

The DG finite element pair $\mathcal{P}_2$-$\mathcal{P}_1$ is employed in the numerical scheme.

In Table 1, we list the numerical errors at time $T = 1$ and convergence rates with respect to h, as k is fixed to be 2^{-10}. In Table 2, with $h = 2^{-6}$ fixed, we provide numerical errors and numerical convergence orders with respect to k. The theoretically predicted optimal convergence orders in Section 4 are observed with respect to both h and k.

Example 5.3: In this last example, the problem data are chosen so that the true solution is

$$u_1(x,y,t) = \sin(xt)\ \sin(yt),$$
$$u_2(x,y,t) = \cos(xt)\ \cos(yt),$$
$$p(x,y,t) = \sin((x-y)t).$$

We consider numerical solutions at the time $T = 1$. In Table 3, we list numerical results with the use of the mixed DG finite element pair $\mathcal{P}_1$-$\mathcal{P}_0$ and a fixed small step-size $k = 2^{-10}$. The linear convergence order in h is evident. To explore the numerical convergence orders with respect to the time step-size k,

Table 1. Numerical convergence orders for h in Example 5.2 with the time step $k = 2^{-10}$.

$1/h$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$	Order	$\|\text{div}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0$	Order	$\|p - p_h\|_0$	Order
4	4.91089e−02	–	3.99189e−02	–	1.01481e−03	–
8	1.28926e−02	1.9294	1.06358e−02	1.9081	2.53548e−04	2.0009
16	3.28341e−03	1.9733	2.71952e−03	1.9675	6.33814e−05	2.0001
32	8.37682e−04	1.9707	6.95844e−04	1.9665	1.58452e−05	2.0000

Table 2. Numerical convergence orders for k in Example 5.2 with the mesh $h = 2^{-6}$.

$1/k$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$	Order	$\|\text{div}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0$	Order	$\|p - p_h\|_0$	Order
4	8.26619e−02	–	3.93362e−02	–	2.30372e−02	–
8	4.26435e−02	0.9549	2.06164e−02	0.9321	1.07996e−02	1.0930
16	2.16882e−02	0.9754	1.05676e−02	0.9641	5.23113e−03	1.0458
32	1.09424e−02	0.9870	5.35382e−03	0.9810	2.57470e−03	1.0227

Table 3. Numerical convergence orders for h in Example 5.3 with the finite element pair $\mathcal{P}_1$-$\mathcal{P}_0$ and the time-step $k = 2^{-10}$.

$1/h$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$	Order	$\|\text{div}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0$	Order	$\|p - p_h\|_0$	Order
4	1.98248e+00	–	1.63041e+00	–	9.49870e−02	–
8	1.00114e+00	0.9857	8.43981e−01	0.9500	4.74494e−02	1.0013
16	4.89519e−01	1.0322	4.18319e−01	1.0126	2.36720e−02	1.0032
32	2.40941e−01	1.0227	2.07382e−01	1.0123	1.18207e−02	1.0019

Table 4. Numerical convergence orders for k in Example 5.3 with the finite element pair $\mathcal{P}_2$-$\mathcal{P}_1$ and the mesh-size $h = 1/64$.

$1/k$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$	Order	$\|\text{div}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0$	Order	$\|p - p_h\|_0$	Order
4	2.46117e−01	–	8.85582e−02	–	8.19958e−02	–
8	1.27881e−01	0.9445	4.71559e−02	0.9092	3.96573e−02	1.0480
16	6.52589e−02	0.9706	2.43650e−02	0.9526	1.94753e−02	1.0259
32	3.29857e−02	0.9843	1.24162e−02	0.9726	9.64719e−03	1.0135

we need numerical solutions with sufficient accuracy in h. For this purpose, we employ the DG pair $\mathcal{P}_2$-$\mathcal{P}_1$ with the finite element mesh-size $h = 1/64$. The numerical results are reported in Table 4, which clearly displays the first order convergence with respect to the time step-size k.

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