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Discontinuous Galerkin Methods for a Stationary Navier–Stokes Problem with a Nonlinear Slip Boundary Condition of Friction Type

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Abstract In this work, several discontinuous Galerkin (DG) methods are introduced and analyzed to solve a variational inequality from the stationary Navier–Stokes equations with a nonlinear slip boundary condition of friction type. Existence, uniqueness and stability of numerical solutions are shown for the DG methods. Error estimates are derived for the velocity in a broken H^1-norm and for the pressure in an L^2-norm, with the optimal convergence order when linear elements for the velocity and piecewise constants for the pressure are used. Numerical results are reported to demonstrate the theoretically predicted convergence orders, as well as the capability in capturing the discontinuity, the ability in handling the shear layers, the capacity in dealing with the advection-dominated problem, and the application to the general polygonal mesh of the DG methods.

Keywords Discontinuous Galerkin methods · Navier–Stokes equations · Slip boundary condition · Variational inequality · Error analyses

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1 Introduction

The Navier–Stokes equations characterize a variety of flows and play an important role in many engineering applications. Let $\Omega \subset R^2$ be an open bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$. For the two-dimensional stationary incompressible flow problem in Ω, the momentum and continuity equations are

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, $\nu > 0$ denotes the kinematic viscosity, and $\mathbf{f}$ is a given external force density. Throughout this paper, the boldface symbols denote vector-valued quantities. We assume Γ consists of two components Γ_D and Γ_S: $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_S$, $\Gamma_D \cap \Gamma_S = \emptyset$ with both Γ_D and Γ_S non-empty. Over Γ_D, we specify the homogeneous Dirichlet boundary condition:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D. \tag{3}$$

For the boundary condition on Γ_S, we consider the normal direction and tangential direction separately. Let $\mathbf{n} = (\mathrm{n}_1, \mathrm{n}_2)^T$ be the unit outward normal on the boundary Γ_S, and let $\boldsymbol{\tau}$ be the unit tangential vector obtained by rotating $\mathbf{n}$ counterclockwise for an angle of $\frac{\pi}{2}$ radians. Then if $\mathbf{v}$ is a vector defined on the boundary, we write $\mathrm{v}_{\mathbf{n}} = \mathbf{v}\cdot\mathbf{n}$ for its normal component, and $\mathrm{v}_{\boldsymbol{\tau}} = \mathbf{v}\cdot\boldsymbol{\tau}$ for its tangential component. Denote by $\sigma_{\boldsymbol{\tau}}(\mathbf{u}) = \nu\frac{\partial \mathrm{u}_{\tau}}{\partial\mathbf{n}}$ the tangential component of stress vector defined on Γ_S. Over Γ_S, we specify a slip and non-leak boundary condition of friction type:

$$\mathrm{u}_{\mathbf{n}} = 0, \quad |\sigma_{\boldsymbol{\tau}}| \le g, \quad \sigma_{\boldsymbol{\tau}}\mathrm{u}_{\boldsymbol{\tau}} + g|\mathrm{u}_{\boldsymbol{\tau}}| = 0 \quad \text{on } \Gamma_S. \tag{4}$$

The function $g : \Gamma_S \to [0, \infty)$ is known as the threshold slip or barrier function. If $g \equiv 0$, then (4) reduces to the ordinary slip boundary condition: $\mathrm{u}_{\mathbf{n}} = 0$ and $\sigma_{\boldsymbol{\tau}} = 0$. The second and third relations in (4) are equivalent to the following implications:

$$|\sigma_{\boldsymbol{\tau}}| < g \Rightarrow \mathrm{u}_{\boldsymbol{\tau}} = 0, \quad \mathrm{u}_{\boldsymbol{\tau}} > 0 \Rightarrow \sigma_{\boldsymbol{\tau}} = -g, \quad \mathrm{u}_{\boldsymbol{\tau}} < 0 \Rightarrow \sigma_{\boldsymbol{\tau}} = g. \tag{5}$$

This friction type of boundary conditions was first introduced by Fujita [21] for applications in the blood flow in a vein of an arterial sclerosis patient, and flow through a canal with its bottom covered by sherbet of mud and pebbles.

The problem (1)–(4) is difficult to solve numerically because of the nonlinearity, the coupling between the velocity and the pressure, and the inequality form of the slip boundary condition. Well-posedness of the Stokes and Navier–Stokes equations with nonlinear slip boundary conditions has been discussed in several papers, e.g. [22–25,42,43,51,52]. Uzawa iterative algorithms were introduced in [37,44] for solving the inequality problem governed by Stokes equations, motivated by ideas presented in [32]. In addition, one can find analyses of finite element discretization for such variational inequalities in [2,18,35,36,38,41].

For discontinuous Galerkin (DG) methods of a second-order partial differential equation, discontinuous functions are used to approximate the unknown solution, and by adding some penalty terms, the approximate solutions between neighboring elements are connected. Relaxing the continuity of approximation functions across the finite element boundaries allows the DG methods to be easily implemented on highly unstructured meshes. The locality and flexibility also make the methods well suited for parallelization and applications of domain decomposition techniques. Due to these advantages, DG methods have been an active research area in recent years [3,5,8,10–13,19,29,45,49]. In particular, the methods have been used to solve variational inequality problems [15–17,33,34,55–58,60]. A DG formulation and algorithm of gradient plasticity of the second kind were developed and analyzed in [16,17]. A unified analysis is provided on DG methods for both the first and second kinds of elliptic variational inequality problems in [55], and DG methods for the obstacle and Signorini problems [33,34,56,58,60] and contact problems [57]. The interior penalty DG methods for the Stokes equations with a slip boundary condition were considered in [15]. To our knowledge, there has been no analysis of DG methods for the Navier–Stokes equations with such a nonlinear slip boundary condition of friction type. We note that the reference [15] is on a variational inequality for the Stokes equation, it provides sub-optimal order error estimates, and there is no numerical example. The reference [11,12,28] is on ordinary equality problems for the Stokes equations and for the Navier–Stokes equations, and it does not address the intrinsically more complicated inequality problems. In this paper, following the unified framework developed in [4,5,8,11,15,47,55], we present the interior penalty DG methods, local discontinuous Galerkin, discontinuous Galerkin of Brezzi and discontinuous Galerkin of Bassi for the problem (1)–(4), explore stability of the numerical schemes, prove existence and uniqueness of the solutions of the discrete problem, and derive error estimates under some solution regularity assumptions.

The outline of this paper is as follows. In Sect. 2, we bring in some notation and preliminary materials. In Sect. 3, we introduce four kinds of DG methods for the problem (1)–(4) and present some results needed later in the error analysis. In Sect. 4, we prove the stability, existence and uniqueness of the DG approximations. In Sect. 5, we derive error estimates for the numerical solutions in a broken H^1-norm for the velocity and the L^2-norm for the pressure. This paper ends with a section on numerical results, to illustrate the sharpness of the theoretical convergence orders and capability of the methods to capture the discontinuous velocity when slip phenomenon occurs; a wall-driven semi-circular cavity flow is also simulated on the ability of the methods in handling the boundary layers and on the effects by the nonlinear advection term; and finally, the interior penalty DG method is applied on general polygonal meshes. Throughout this paper, the letter C denotes a generic positive constant independent of the mesh size.

2 Variational Inequality

In this section, we introduce a variational inequality formulation for the problem (1)–(4).

First, we introduce some notation. For a given integer m, we shall use the standard Sobolev space $H^m(\Omega)$ [1]

$$H^m(\Omega)=\{\mathrm{v}\in L^2(\Omega):\partial^k\mathrm{v}\in L^2(\Omega)\ \forall\, k:|k|\le m\},$$

where $k=(k_1,k_2)$, k_1 and k_2 being nonnegative integers, $|k|=k_1+k_2$, and

$$\partial^k\mathrm{v}=\frac{\partial^{|k|}\mathrm{v}}{\partial x^{k_1}\partial y^{k_2}}.$$

It is a Hilbert space with the norm and the corresponding seminorm:

$$\|\mathrm{v}\|_{m,\Omega}=\left[\sum_{0\le|k|\le m}\int_\Omega|\partial^k\mathrm{v}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\right]^{1/2},\quad |\mathrm{v}|_{m,\Omega}=\left[\sum_{|k|=m}\int_\Omega|\partial^k\mathrm{v}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\right]^{1/2}.$$

For functions vanishing on the boundary $\partial\Omega$, we use

$$H_0^1(\Omega)=\left\{\mathrm{v}\in H^1(\Omega):\mathrm{v}|_{\partial\Omega}=0\right\}.$$

We shall also need the following space of functions with zero mean value:

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \right\}.$$

Define $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^2$ and

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\Gamma_D} = \mathbf{0},\ \mathrm{v}_{\mathbf{n}}|_{\Gamma_S} = 0\}, \quad Q = L_0^2(\Omega), \quad \mathbf{Y} = [L^2(\Omega)]^2,$$

and $\mathbf{V}$ is equipped with the norm $\|\nabla(\cdot)\|_{0,\Omega}$. For simplicity, we drop Ω in the notation for norms in the rest of this paper. The scalar product and norm in Q are the usual $L^2(\Omega)$ inner product and the corresponding norm $\|\cdot\|_0$. In addition, define

$$\mathbf{V}_{\mathrm{div}} = \{\mathbf{u} \in \mathbf{V} :\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}.$$

We use the notation

$$(p, q) := \int_\Omega p(\mathbf{x}) q(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall\, p, q \in L^2(\Omega),$$

$$(\mathbf{v}, \mathbf{w}) := \int_\Omega \sum_{i=1}^{2} \mathrm{v}_i(\mathbf{x}) \mathrm{w}_i(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{Y},$$

$$(\nabla \mathbf{v}, \nabla \mathbf{w}) := \int_\Omega \sum_{i=1}^{2} \left(\frac{\partial \mathrm{v}_i}{\partial x} \frac{\partial \mathrm{w}_i}{\partial x} + \frac{\partial \mathrm{v}_i}{\partial y} \frac{\partial \mathrm{w}_i}{\partial y} \right) \mathrm{d}\mathbf{x} \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega),$$

and define

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad d(\mathbf{v}, p) = -(\nabla \cdot \mathbf{v}, p), \quad c(\mathbf{u}; \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w})$$

for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, and $p \in Q$.

The following inequality will be used repeatedly [26,54]:

$$|c(\mathbf{u}; \mathbf{v}, \mathbf{w})| \le N \|\nabla \mathbf{u}\|_0 \|\nabla \mathbf{v}\|_0 \|\nabla \mathbf{w}\|_0 \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V},$$

where

$$N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}} \frac{c(\mathbf{u}; \mathbf{v}, \mathbf{w})}{\|\nabla \mathbf{u}\|_0 \|\nabla \mathbf{v}\|_0 \|\nabla \mathbf{w}\|_0} < \infty.$$

Following [20,22,32], the variational inequality formulation of the problem (1)–(4) is to find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} &\nu a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v} - \mathbf{u}) + d(\mathbf{v} - \mathbf{u}, p) + j(\mathrm{v}_{\boldsymbol{\tau}}) - j(\mathrm{u}_{\boldsymbol{\tau}}) \ge (\mathbf{f}, \mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{V}, \\ &d(\mathbf{u}, q) = 0 \quad \forall\, q \in Q, \end{aligned} \tag{6}$$

where

$$j(\eta) = \int_{\Gamma_S} g\,|\eta|\,\mathrm{d}s, \quad \eta \in L^2(\Gamma_S).$$

Obviously, j is a continuous functional defined on $L^2(\Gamma_S)$. It is known that there exists a positive constant $\beta > 0$ such that [51]

$$\beta \|q\|_0 \le \sup_{\mathbf{v} \in \mathbf{V}} \frac{d(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|_0} \quad \forall\, q \in Q.$$

Thus, the variational inequality (6) is equivalent to finding $\mathbf{u} \in \mathbf{V}_{\text{div}}$ such that

$$\nu a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}-\mathbf{u}) + j(\mathrm{v}_{\boldsymbol{\tau}}) - j(\mathrm{u}_{\boldsymbol{\tau}}) \geq (\mathbf{f}, \mathbf{v}-\mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}. \tag{7}$$

Existence and uniqueness of a solution to the problem (7) is guaranteed if [43]

$$\mathbf{f} \in \mathbf{Y}, \quad g \in L^2(\Gamma_S), \quad 4\kappa N(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) < \nu^2,$$

where $\kappa > 0$ is a constant found in the inequality

$$|(\mathbf{f}, \mathbf{v}) - j(\mathrm{v}_{\boldsymbol{\tau}})| \leq \kappa(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S})\|\nabla \mathbf{v}\|_0 \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}.$$

Moreover, the solution can be bounded as follows,

$$\|\nabla \mathbf{u}\|_0 \leq \frac{2\kappa}{\nu}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}).$$

3 Discontinuous Galerkin Methods

3.1 Notation

To simplify the exposition, we assume Ω is an open bounded polygon. Let $\{\mathscr{T}_h\}$ be a family of locally quasi-uniform partitions of the domain Ω into triangles, i.e., it is regular and satisfies the inverse assumption [9], h being the mesh size. For $\mathscr{T}_h = \{K\}$, let $e = \partial K_i \cap \partial K_j$ $(i \neq j)$ be the common boundary between two elements K_i and K_j in $\mathscr{T}_h$. The diameters of K and e are denoted by h_K and h_e. Let $\mathscr{E}_h$ and $\mathscr{E}_h^I$ be the union of all the edges of the subdivision $\mathscr{T}_h$ and the set of interior edges, respectively. Besides, we denote by $\mathscr{E}_h^S$ the set of all edges lying on $\overline{\Gamma}_S$ and $\mathscr{E}_h^* = \mathscr{E}_h^I \cup \mathscr{E}_h^S$.

For vectors $\mathbf{v}$ and $\mathbf{n}$, let $\mathbf{v} \otimes \mathbf{n}$ denote the matrix whose (i, j)th component is $\mathrm{v}_i \mathrm{n}_j$. For two matrix-valued variables $\mathbb{A}$ and $\mathbb{B}$, we define $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^2 \mathbb{A}_{ij}\mathbb{B}_{ij}$. Let $e = \partial K_1 \cap \partial K_2$, and $\mathbf{n}_1$ and $\mathbf{n}_2$ be the unit normal vectors on e pointing to the exterior of K_1 and K_2, respectively. We define the average $\{\cdot\}$ and jump $[\cdot]$ on e for a scalar q, a vector $\mathbf{v}$, and a matrix $\mathbb{A}$, respectively, by

$$\{q\} = \frac{1}{2}(q|_{\partial K_1} + q|_{\partial K_2}), \quad [q] = q|_{\partial K_1}\mathbf{n}_1 + q|_{\partial K_2}\mathbf{n}_2,$$
$$\{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}|_{\partial K_1} + \mathbf{v}|_{\partial K_2}), \quad [\mathbf{v}] = \mathbf{v}|_{\partial K_1} \cdot \mathbf{n}_1 + \mathbf{v}|_{\partial K_2} \cdot \mathbf{n}_2,$$
$$\{\mathbb{A}\} = \frac{1}{2}(\mathbb{A}|_{\partial K_1} + \mathbb{A}|_{\partial K_2}), \quad [\mathbb{A}] = \mathbb{A}|_{\partial K_1}\mathbf{n}_1 + \mathbb{A}|_{\partial K_2}\mathbf{n}_2.$$

We also define a matrix-valued jump $[\![\cdot]\!]$ for a vector $\mathbf{v}$ by $[\![\mathbf{v}]\!] = \mathbf{v}|_{\partial K_1} \otimes \mathbf{n}_1 + \mathbf{v}|_{\partial K_2} \otimes \mathbf{n}_2$ on e. If e is a part of the boundary $\partial\Omega$, the above definitions are modified as follows:

$$\{q\} = q, \ \{\mathbf{v}\} = \mathbf{v}, \quad \{\mathbb{A}\} = \mathbb{A} \quad \text{and} \quad [q] = q, \ [\mathbf{v}] = \mathbf{v}, \quad [\mathbb{A}] = \mathbb{A}, \ [\![\mathbf{v}]\!] = \mathbf{v} \otimes \mathbf{n}.$$

By a straightforward computation, we know that

$$\sum_{K \in \mathscr{T}_h} \int_{\partial K} q\mathbf{v} \cdot \mathbf{n}\, \mathrm{d}s = \int_{\mathscr{E}_h^I} [q] \cdot \{\mathbf{v}\}\, \mathrm{d}s + \int_{\mathscr{E}_h} \{q\}[\mathbf{v}]\, \mathrm{d}s,$$
$$\sum_{K \in \mathscr{T}_h} \int_{\partial K} \mathbf{v} \cdot \mathbb{A}\mathbf{n}\, \mathrm{d}s = \int_{\mathscr{E}_h^I} [\mathbb{A}] \cdot \{\mathbf{v}\}\, \mathrm{d}s + \int_{\mathscr{E}_h} \{\mathbb{A}\} : [\![\mathbf{v}]\!]\, \mathrm{d}s.$$

Let $k \geq 1$ be an integer. Define the finite element space $\mathbf{V}_h$ for the velocity by

$$\mathbf{V}_h = \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathbf{v}|_K \in [P_k(K)]^2 \ \ \forall K \in \mathscr{T}_h\},$$

and the finite element space Q_h for the pressure by

$$Q_h = \{q \in L_0^2(\Omega) : q|_K \in P_{k-1}(K) \ \ \forall K \in \mathscr{T}_h\},$$

where $P_k(K)$ denotes the space of polynomials of a degree at most k over the set K.

3.2 DG Methods for the Variational Inequality

We define the usual interior penalty forms for Navier–Stokes problem. The forms $a_h(\cdot,\cdot)$, $d_h(\cdot,\cdot)$, $j(\cdot)$ and $\mathcal{F}(\cdot)$ correspond to the DG discretization of the viscous term, divergence term, friction term and the right term of the Eq. (1), respectively,

$$a_h^*(\mathbf{v},\mathbf{w}) = \sum_{K\in\mathscr{T}_h}\int_K \nabla\mathbf{v} : \nabla\mathbf{w}\,\mathrm{d}\mathbf{x} - \int_{\mathscr{E}_h^*}\{\nabla\mathbf{v}\} : [\![\mathbf{w}]\!]\,\mathrm{d}s,$$

$$\alpha(\mathbf{v},\mathbf{w}) = \sum_{e\in\mathscr{E}_h^*}\gamma h_e^{-1}\int_e [\![\mathbf{v}]\!] : [\![\mathbf{w}]\!]\,\mathrm{d}s,$$

$$d_h(\mathbf{v},q) = -\sum_{K\in\mathscr{T}_h}\int_K q\,\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\mathscr{E}_h^*}\{q\}[\mathbf{v}]\,\mathrm{d}s,$$

$$j(\mathrm{v}_{\boldsymbol{\tau}}) = \int_{\mathscr{E}_h^S} g\,|\mathrm{v}_{\boldsymbol{\tau}}|\,\mathrm{d}s, \qquad \mathcal{F}(\mathbf{v}) = \sum_{K\in\mathscr{T}_h}\int_K \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},$$

where $\gamma > 0$ is a parameter to be specified later.

Let us briefly sketch the derivation of the consistency term in $a_h(\cdot,\cdot)$ and the cause of the inequality. Since $\nabla\mathbf{u}$ is continuous on the elements, $[\nabla\mathbf{u}] = \mathbf{0}$ on the interior edges. For an arbitrary $\mathbf{v} \in \mathbf{V}_h$, multiplying (1) by $\mathbf{v}-\mathbf{u}$, integrating on an element K, performing an integration by parts, and summing over all elements, we see that

$$\int_\Omega -\nu\Delta\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}\mathbf{x} = \sum_{K\in\mathscr{T}_h}\int_K \nu\nabla\mathbf{u} : \nabla(\mathbf{v}-\mathbf{u})\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathscr{T}_h}\int_{\partial K}\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}s. \tag{8}$$

We rewrite the edge integral term in (8):

$$\begin{aligned} -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}s &= -\int_{\mathscr{E}_h}\nu\{\nabla\mathbf{u}\} : [\![\mathbf{v}-\mathbf{u}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_h^I}\nu[\nabla\mathbf{u}]\cdot\{\mathbf{v}-\mathbf{u}\}\,\mathrm{d}s \\ &= -\int_{\mathscr{E}_h^*}\nu\{\nabla\mathbf{u}\} : [\![\mathbf{v}-\mathbf{u}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_h^S}\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}s. \end{aligned} \tag{9}$$

For the second term in (9), by the definition of $\sigma_{\boldsymbol{\tau}}$, and with the boundary conditions (4), there holds

$$\begin{aligned} -\int_{\mathscr{E}_h^S}\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}s &= -\int_{\mathscr{E}_h^S}\nu\left(\frac{(\partial\mathrm{u}_{\mathbf{n}})}{\partial\mathbf{n}}\mathbf{n} + \frac{\partial(\mathrm{u}_{\boldsymbol{\tau}})}{\partial\mathbf{n}}\boldsymbol{\tau}\right)\cdot((\mathrm{v}_{\mathbf{n}}-\mathrm{u}_{\mathbf{n}})\mathbf{n} + (\mathrm{v}_{\boldsymbol{\tau}}-\mathrm{u}_{\boldsymbol{\tau}})\boldsymbol{\tau})\,\mathrm{d}s \\ &= -\int_{\mathscr{E}_h^S}\nu\frac{\partial\mathrm{u}_{\mathbf{n}}}{\partial\mathbf{n}}\cdot(\mathrm{v}_{\mathbf{n}}-\mathrm{u}_{\mathbf{n}})\,\mathrm{d}s - \int_{\mathscr{E}_h^S}\nu\frac{\partial\mathrm{u}_{\boldsymbol{\tau}}}{\partial\mathbf{n}}\cdot(\mathrm{v}_{\boldsymbol{\tau}}-\mathrm{u}_{\boldsymbol{\tau}})\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathscr{E}_h^S} \nu \frac{\partial \mathrm{u}_{\boldsymbol{\tau}}}{\partial \mathbf{n}} \cdot (\mathrm{v}_{\boldsymbol{\tau}} - \mathrm{u}_{\boldsymbol{\tau}})\, \mathrm{d}s = \int_{\mathscr{E}_h^S} \sigma_{\boldsymbol{\tau}} (\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{\boldsymbol{\tau}})\, \mathrm{d}s \\
&\le \int_{\mathscr{E}_h^S} (g|\mathrm{v}_{\boldsymbol{\tau}}| - g|\mathrm{u}_{\boldsymbol{\tau}}|)\, \mathrm{d}s. \qquad (10)
\end{aligned}
$$

Using (9), (10) in (8), we have

$$
\begin{aligned}
\int_\Omega -\nu \Delta \mathbf{u} \cdot (\mathbf{v} - \mathbf{u})\, \mathrm{d}\mathbf{x} \le \sum_{K \in \mathscr{T}_h} \int_K \nu \nabla \mathbf{u} : \nabla(\mathbf{v} - \mathbf{u})\, \mathrm{d}\mathbf{x} - \int_{\mathscr{E}_h^*} \nu \{\nabla \mathbf{u}\} : [\![\mathbf{v} - \mathbf{u}]\!]\, \mathrm{d}s \\
+ \int_{\mathscr{E}_h^S} (g|\mathrm{v}_{\boldsymbol{\tau}}| - g|\mathrm{u}_{\boldsymbol{\tau}}|)\, \mathrm{d}s.
\end{aligned}
$$

We then use the following variant of Lesaint–Raviart upwinding scheme [40] introduced in [29] to discretize the nonlinear convection term in (8). The superscript int (resp. ext) refers to the trace of the function on a side of K coming from the interior of e (resp. coming from the exterior of e on that side). When the side of K belongs to $\partial\Omega$, we take the exterior trace to be zero. Define

$$
\begin{aligned}
c_h(\mathbf{z}_h; \mathbf{u}_h; \mathbf{v}_h, \mathbf{w}_h) = &\sum_{K \in \mathscr{T}_h} \int_K \left((\mathbf{u}_h \cdot \nabla \mathbf{v}_h) \cdot \mathbf{w}_h + \frac{1}{2} \operatorname{div} \mathbf{u}_h\, \mathbf{v}_h \cdot \mathbf{w}_h \right) \mathrm{d}\mathbf{x} \\
&- \frac{1}{2} \int_{\mathscr{E}_h^*} [\mathbf{u}_h] \{\mathbf{v}_h \cdot \mathbf{w}_h\}\, \mathrm{d}s \\
&+ \sum_{K \in \mathscr{T}_h} \int_{\partial K_-^{\mathbf{z}_h} \setminus \mathscr{E}_h^S} |\{\mathbf{u}_h\} \cdot \mathbf{n}| \left(\mathbf{v}_h^{\text{int}} - \mathbf{v}_h^{\text{ext}} \right) \cdot \mathbf{w}_h^{\text{int}}\, \mathrm{d}s \quad \forall\, \mathbf{z}_h, \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h,
\end{aligned}
$$

and

$$
c_h^{NL}(\mathbf{z}_h; \mathbf{u}_h; \mathbf{v}_h, \mathbf{w}_h) = \sum_{K \in \mathscr{T}_h} \int_{\partial K_-^{\mathbf{z}_h} \setminus \mathscr{E}_h^S} |\{\mathbf{u}_h \cdot \mathbf{n}\}| \left(\mathbf{v}_h^{\text{int}} - \mathbf{v}_h^{\text{ext}} \right) \cdot \mathbf{w}_h^{\text{int}}\, \mathrm{d}s,
$$

where

$$
\partial K_-^{\mathbf{z}_h} = \{\mathbf{x} \in \partial K : \mathbf{z}_h(\mathbf{x}) \cdot \mathbf{n} < 0\},
$$

and the superscript $\mathbf{z}_h$ indicates the dependence of $\partial K_-^{\mathbf{z}_h}$ on $\mathbf{z}_h$.

Now, we study DG methods for the Navier–Stokes equations with a nonlinear slip boundary conditions of friction type. In [4] several DG methods are discussed for the elliptic problem and are extended to the elliptic variational inequalities in [55], for the sake of simplicity, we select the following DG methods for the considered problem here. Let $\underline{\mathcal{L}} : \mathbf{V}_h \to \underline{\Sigma}_h = \{\mathbb{v} \in [L^2(\Omega)]^{2\times 2} : \mathbb{v}|_K \in [P_k(K)]^{2\times 2} \ \forall K \in \mathscr{T}_h\}$ and $\underline{r_e} : \mathbf{V}_h \to \underline{\Sigma}_h$ be the two lifting operators defined by

$$
\begin{aligned}
&\int_\Omega \underline{\mathcal{L}}(\mathbf{v}) : \mathbb{w}_h\, \mathrm{d}\mathbf{x} = \int_{\mathscr{E}_h^*} [\![\mathbf{v}]\!] : \{\mathbb{w}_h\}\, \mathrm{d}\mathbf{s}, \\
&\int_\Omega \underline{r_e}(\mathbf{v}) \cdot \mathbb{w}_h\, \mathrm{d}\mathbf{x} = \int_e [\![\mathbf{v}]\!] : \{\mathbb{w}_h\}\, \mathrm{d}\mathbf{x} \ \ \forall\, \mathbf{v} \in \mathbf{V}_h, \quad \mathbb{w}_h \in \underline{\Sigma}_h.
\end{aligned}
$$

The bilinear form, trilinear form and friction term $d_h(\cdot,\cdot)$, $c_h(\cdot;\cdot,\cdot)$ and $j(\cdot)$ will be the same as the definitions above, we present the choice of $a_h(\cdot,\cdot)$ for various DG methods.

1. IPG method [3,19,50,59]

$$
a_h^{\text{IP}}(\mathbf{v}, \mathbf{w}) = a_h^*(\mathbf{v}, \mathbf{w}) + \epsilon \int_{\mathscr{E}_h^*} \{\nabla \mathbf{w}\} : [\![\mathbf{v}]\!]\, \mathrm{d}s + \alpha(\mathbf{v}, \mathbf{w}) \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{V}_h,
$$

Remark 1

(i) When $\epsilon = 0$, $a_h(\cdot,\cdot)$ is the incomplete interior penalty DG(IIPG) scheme. When $\epsilon = -1$ and 1, $a_h(\cdot,\cdot)$ becomes the symmetric interior penalty DG (SIPG) scheme and non-symmetric interior penalty DG (NIPG) scheme, respectively.
(ii) $\alpha(\mathbf{v},\mathbf{w})$ is the penalty term, and the selection of γ will affect the stability of the discrete scheme.

2. LDG method [10,11,13]

$$a_h^{\mathrm{LDG}}(\mathbf{v},\mathbf{w}) = a_h^*(\mathbf{v},\mathbf{w}) - \int_{\mathscr{E}_h^*} \{\nabla \mathbf{w}\} : [\![\mathbf{v}]\!]\,\mathrm{d}s + (\underline{\mathcal{L}}(\mathbf{v}), \underline{\mathcal{L}}(\mathbf{w}))_\Omega + \alpha(\mathbf{v},\mathbf{w}) \quad \forall\, \mathbf{v},\mathbf{w} \in \mathbf{V}_h.$$

3. Brezzi et al. [8]

$$a_h^{\mathrm{Br}}(\mathbf{v},\mathbf{w}) = a_h^*(\mathbf{v},\mathbf{w}) - \int_{\mathscr{E}_h^*} \{\nabla \mathbf{w}\} : [\![\mathbf{v}]\!]\,\mathrm{d}s + (\underline{\mathcal{L}}(\mathbf{v}), \underline{\mathcal{L}}(\mathbf{w}))_\Omega + \alpha^*(\mathbf{v},\mathbf{w}) \quad \forall\, \mathbf{v},\mathbf{w} \in \mathbf{V}_h,$$

and

$$\alpha^*(\mathbf{v},\mathbf{w}) = \sum_{e\in\mathscr{E}_h^*} \int_\Omega \gamma h_e^{-1} \underline{r_e}(\mathbf{v}) \cdot \underline{r_e}(\mathbf{w})\,\mathrm{d}\mathbf{x} \quad \forall\, \mathbf{v},\mathbf{w} \in \mathbf{V}_h.$$

4. Bassi et al. [5]

$$a_h^{\mathrm{Ba}}(\mathbf{v},\mathbf{w}) = a_h^*(\mathbf{v},\mathbf{w}) - \int_{\mathscr{E}_h^*} \{\nabla \mathbf{w}\} : [\![\mathbf{v}]\!]\,ds + \alpha^*(\mathbf{v},\mathbf{w}) \quad \forall\, \mathbf{v},\mathbf{w} \in \mathbf{V}_h.$$

These four methods can all be expressed as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\nu a_h(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + d_h(\mathbf{v}_h - \mathbf{u}_h, p_h) + c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathrm{v}_{h\boldsymbol{\tau}}) - j(\mathrm{u}_{h\boldsymbol{\tau}}) \\ \geq \mathcal{F}(\mathbf{v}_h - \mathbf{u}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \tag{11}$$

$$d_h(\mathbf{u}_h, q_h) = 0 \quad \forall\, q_h \in Q_h, \tag{12}$$

where $a_h(\cdot,\cdot)$ stands for any one of a_h^{IP}, a_h^{LDG}, a_h^{Br} and a_h^{Ba}.

It is easy to check that the solution of (6) satisfies the following consistency condition:

$$\nu a_h(\mathbf{u}, \mathbf{v} - \mathbf{u}) + d_h(\mathbf{v} - \mathbf{u}, p) + c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathrm{v}_{\boldsymbol{\tau}}) - j(\mathrm{u}_{\boldsymbol{\tau}}) \geq \mathcal{F}(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{V}(h), \tag{13}$$

$$d_h(\mathbf{u}, q) = 0 \quad \forall\, q \in Q_h. \tag{14}$$

4 Stability, Existence and Uniqueness

In this section, we consider the well-posedness of the DG methods (11)–(12).

Let e be an edge of $K \in \mathscr{T}_h$. There exists a constant C that depends only on the lower bound of the minimum angle of K such that for any function $\varphi \in H^1(K)$,

$$h_K \|\varphi\|_{0,e}^2 \leq C \left(\|\varphi\|_{0,K}^2 + h_K^2 |\nabla\varphi|_{0,K}^2 \right), \quad h_K \|\frac{\partial\varphi}{\partial \mathbf{n}}\|_{0,e}^2 \leq C \left(|\varphi|_{1,K}^2 + h_K^2 |\nabla\varphi|_{1,K}^2 \right). \tag{15}$$

In particular, for any $\mathbf{v} \in \mathbf{V}_h$, the following inequalities are valid [7,9,39]:

$$h_e \|\nabla \mathbf{v}|_K\|_{0,e}^2 \leq C \left(\|\nabla\mathbf{v}\|_{0,K}^2 + h_K^2 \|\Delta\mathbf{v}\|_{0,K}^2 \right), \quad h_K^2 \|\Delta\mathbf{v}\|_{0,K}^2 \leq C \|\nabla\mathbf{v}\|_{0,K}^2. \tag{16}$$

Let $\mathbf{V}(h) = \mathbf{V}_h + \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $Q(h) = Q_h + Q \cap H^1(\Omega)$, we define the broken Sobolev norms on $\mathbf{V}(h)$ and norm on $Q(h)$ as follows:

$$|\mathbf{v}|_{*,e}^2 = \sum_{e \in \mathscr{E}_h^*} \frac{1}{h_e} \|[\![\mathbf{v}]\!]\|_{0,e}^2, \quad |||\mathbf{v}|||_1^2 = |\mathbf{v}|_{1,h}^2 + |\mathbf{v}|_{*,e}, \quad |||\mathbf{v}|||^2 = |||\mathbf{v}\,|||_1^2 + \sum_{K \in \mathscr{T}_h} h_K^2 |\mathbf{v}|_{2,K}^2,$$
$$\|q\|_*^2 = \|q\|_0^2 + \sum_{K \in \mathscr{T}_h} h_K^2 |q|_{1,K}^2,$$

where $|\cdot|_{1,h}^2 = \sum_K |\cdot|_{1,K}^2$. In fact, $|||\cdot|||$ and $|||\cdot|||_1$ are equivalent by the standard inverse inequality for $\mathbf{v} \in \mathbf{V}_h$ [6,7,9]. Let us recall some properties of $a_h(\cdot,\cdot)$, $d_h(\cdot,\cdot)$ and $c_h(\cdot;\cdot;\cdot,\cdot)$ before presenting the well-posedness result of problem (11)–(12).

Note that

$$a_h^*(\mathbf{v},\mathbf{v}) = |\mathbf{v}|_{1,h}^2 - \int_{\mathscr{E}_h^*} \{\nabla \mathbf{v}\} : [\![\mathbf{v}]\!]\,\mathrm{d}s, \quad \alpha(\mathbf{v},\mathbf{v}) = \gamma |\mathbf{v}|_{*,e}^2. \tag{17}$$

By inequalities (15) and (16), we find that (cf. [11,39] for more information)

$$\left|(1-\epsilon)\int_{\mathscr{E}_h^*} \{\nabla \mathbf{v}\} : [\![\mathbf{v}]\!]\,\mathrm{d}s\right| = \left|(1-\epsilon)\int_{\Omega} \underline{\mathcal{L}}(\mathbf{v}) : \{\nabla \mathbf{v}\}\,\mathrm{d}s\right| \le \varepsilon |\mathbf{v}|_{1,h}^2 + \frac{(1-\epsilon)}{4\varepsilon}\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2. \tag{18}$$

For the global lifting operator $\underline{\mathcal{L}}$, we observe that it can be extended to operator $\underline{\mathcal{L}} : \mathbf{V}(h) \to \underline{\Sigma}_h$, and from [4,53], there holds

$$\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 \le C_{\text{lift}} |\mathbf{v}|_{*,e}^2. \tag{19}$$

Besides, as in [4,5,53], using the definition of the local lifting operator $\underline{r_e}$, the Cauchy-Schwarz inequality, the trace inequality and the inverse inequality, we get for all $\mathbf{v} \in \mathbf{V}_h$,

$$C_{\text{Br}}^1 \cdot \|\underline{r_e}(\mathbf{v})\|_0^2 \le |\mathbf{v}|_{*,e}^2 \le (C_{\text{Br}})^{-1} \cdot \|\underline{r_e}(\mathbf{v})\|_0^2. \tag{20}$$

From (17) and (18), we obtain

$$\begin{aligned} a_h^{\text{IP}}(\mathbf{v},\mathbf{v}) &\ge |\mathbf{v}|_{1,h}^2 - \varepsilon|\mathbf{v}|_{1,h}^2 - \frac{(1-\epsilon)}{4\varepsilon}\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \gamma|\mathbf{v}|_{*,e}^2 \\ &\ge (1-\varepsilon)|\mathbf{v}|_{1,h}^2 + \left(\gamma + \frac{(1-\epsilon)}{4\varepsilon} C_{\text{lift}}\right)|\mathbf{v}|_{*,e}^2 \ge \gamma_0\, |||\,\mathbf{v}|||_1, \end{aligned}$$

with $\gamma_0 = \min\{1-\varepsilon, \gamma + \frac{(1-\epsilon)}{4\varepsilon} C_{\text{lift}}\}$.

Combining (19) and with the definition of $a_h^{\text{LDG}}(\cdot,\cdot)$, we obtain

$$\begin{aligned} a_h^{\text{LDG}}(\mathbf{v},\mathbf{v}) &\ge |\mathbf{v}|_{1,h}^2 - \varepsilon|\mathbf{v}|_{1,h}^2 - \frac{1}{2\varepsilon}\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \gamma|\mathbf{v}|_{*,e}^2 \\ &\ge (1-\varepsilon)|\mathbf{v}|_{1,h}^2 + \left(1 - \frac{1}{2\varepsilon}\right)\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \gamma|\mathbf{v}|_{*,e}^2 \\ &\ge (1-\varepsilon)|\mathbf{v}|_{1,h}^2 + \left(\gamma + \left(\frac{1}{2\varepsilon} - 1\right) C_{\text{lift}}\right)|\mathbf{v}|_{*,e}^2 \\ &\ge \gamma_0\, |||\,\mathbf{v}|||_1, \quad \gamma_0 = \min\left\{1-\varepsilon, \gamma + \left(\frac{1}{2\varepsilon} - 1\right) C_{\text{lift}}\right\} \text{ with } 1 > \varepsilon \ge \frac{1}{2}. \end{aligned}$$

Then in view of $a_h^{\mathrm{Br}}(\cdot,\cdot)$ and using (20), one can see that

$$\begin{aligned} a_h^{\mathrm{Br}}(\mathbf{v},\mathbf{v}) &\geq |\mathbf{v}|_{1,h}^2 - \varepsilon|\mathbf{v}|_{1,h}^2 - \frac{1}{2\varepsilon}\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \|\underline{r_e}(\mathbf{v})\|_0^2 \\ &\geq (1-\varepsilon)|\mathbf{v}|_{1,h}^2 + \left(C_{\mathrm{Br}} + \left(\frac{1}{2\varepsilon} - 1\right)C_{\mathrm{lift}}\right)|\mathbf{v}|_{*,e}^2 \\ &\geq \gamma_0 \,|||\, \mathbf{v} |||_1, \quad \gamma_0 = \min\left\{1-\varepsilon, C_{\mathrm{Br}} + \left(\frac{1}{2\varepsilon} - 1\right)C_{\mathrm{lift}}\right\} \text{ with } 1 > \varepsilon \geq \frac{1}{2}. \end{aligned}$$

From (20), it is easy to obtain that

$$\begin{aligned} a_h^{\mathrm{Ba}}(\mathbf{v},\mathbf{v}) &\geq |\mathbf{v}|_{1,h}^2 - \varepsilon|\mathbf{v}|_{1,h}^2 - \frac{1}{2\varepsilon}\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 + \|\underline{r_e}(\mathbf{v})\|_0^2 \\ &\geq (1-\varepsilon)|\mathbf{v}|_{1,h}^2 + \left(C_{\mathrm{Br}} + \frac{C_{\mathrm{lift}}}{2\varepsilon}\right)|\mathbf{v}|_{*,e}^2 \\ &\geq \gamma_0 \,|||\, \mathbf{v} |||_1, \quad \gamma_0 = \min\left\{1-\varepsilon, C_{\mathrm{Br}} + \frac{C_{\mathrm{lift}}}{2\varepsilon}\right\}. \end{aligned}$$

The above analysis leads to a coercivity result on the bilinear form $a_h(\cdot,\cdot)$.

Lemma 1 (Stability) *There exists a constant γ_0 independent of h such that*

$$a_h^{\star}(\mathbf{v},\mathbf{v}) \geq \gamma_0 \,|||\, \mathbf{v} \,|||_1^2 \quad \forall \mathbf{v} \in \mathbf{V}_h,$$

where $\star =$ IP, LDG, Br, Ba *and*

$$\begin{cases} if \star = \mathrm{IP},\ \gamma_0 = \min\left\{1-\varepsilon, \gamma + \dfrac{(1-\epsilon)}{4\varepsilon}C_{\mathrm{lift}}\right\}\} \quad \epsilon = 0 \text{ or } -1, \\ \qquad = \min\{1,\gamma\},\ \ \epsilon = 1, \quad \text{for any } 0 < \varepsilon < 1, \\ if \star = \mathrm{LDG},\ \gamma_0 = \min\left\{1-\varepsilon, \gamma + \left(\dfrac{1}{2\varepsilon} - 1\right)C_{\mathrm{lift}}\right\}, \quad \text{for any } \dfrac{1}{2} \leq \varepsilon < 1, \\ if \star = \mathrm{Br},\ \gamma_0 = \min\left\{1-\varepsilon, C_{\mathrm{Br}} + \left(\dfrac{1}{2\varepsilon} - 1\right)C_{\mathrm{lift}}\right\}, \quad \text{for any } \dfrac{1}{2} \leq \varepsilon < 1, \\ if \star = \mathrm{Ba},\ \gamma_0 = \min\left\{1-\varepsilon, C_{\mathrm{Br}} + \dfrac{C_{\mathrm{lift}}}{2\varepsilon}\right\}, \quad \text{for any } 0 < \varepsilon < 1. \end{cases}$$

Regarding boundedness of the bilinear forms $a_h(\cdot,\cdot)$ and $d_h(\cdot,\cdot)$, we have the following result.

Lemma 2 (Boundedness) [4,39,53] *There exists a constant C independent of h such that for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}(h)$, $q \in Q(h)$,*

$$\begin{aligned} |a_h(\mathbf{w},\mathbf{v})| &\leq C \,|||\, \mathbf{w} \,|||\ \ |||\mathbf{v}|||, \\ d_h(\mathbf{v},q) &\leq C \,|||\, \mathbf{v} \,|||\ \ \|q\|_*. \end{aligned} \tag{21}$$

Moreover, for all $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$,

$$d_h(\mathbf{v},q) \leq C \,|||\, \mathbf{v} \,|||_1\ \ \|q\|_0. \tag{22}$$

Lemma 3 [27–29] *There exist constants C_0 and C_1 independent of h such that*

$$c_h(\mathbf{v}_h; \mathbf{v}_h; \mathbf{w}_h, \mathbf{w}_h) \geq 0 \quad \forall\, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h,$$
$$|c_h(\mathbf{z}_h; \mathbf{u}_h; \mathbf{v}_h, \mathbf{w}_h)| \leq C_0 \,|||\, \mathbf{u}_h \,|||_1 \,|||\mathbf{v}_h\, |||_1 \,|||\mathbf{w}_h\, |||_1 \quad \forall\, \mathbf{z}_h, \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h,$$
$$|c_h^{NL}(\mathbf{z}_h; \mathbf{w}_h; \mathbf{z}_h, \mathbf{v}_h) - c_h^{NL}(\mathbf{w}_h; \mathbf{w}_h; \mathbf{z}_h, \mathbf{v}_h)| \leq C_1 \|\mathbf{z}_h - \mathbf{w}_h\|_1 \|\mathbf{z}_h\|_1 \|\mathbf{v}_h\|_1 \quad \forall\, \mathbf{z}_h, \mathbf{w}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

We recall properties of the Raviart–Thomas interpolation operator Π which is useful in the following argument. Define a subspace $\mathring{\mathbf{V}}_h$ of $\mathbf{V}_h$:

$$\mathring{\mathbf{V}}_h = \{\mathbf{v}_h \in \mathbf{V}_h : [\mathbf{v}_h]|_e = 0 \ \forall e \in \mathscr{E}_h^*, \ \mathbf{v}_h|_e = \mathbf{0} \ \forall e \in \mathscr{E}_h^S\}.$$

Lemma 4 [28,47,48] *The Raviart–Thomas interpolation operator $\Pi \in \mathscr{L}([H^1(\Omega)]^2; \mathring{\mathbf{V}}_h)$ satisfies: for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$,*

$$\int_K q\, \nabla \cdot (\Pi \mathbf{v} - \mathbf{v})\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall\, K \in \mathscr{T}_h, \forall\, q \in P_{k-1}(K),$$
$$\int_e q\, (\Pi \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}\, \mathrm{d}s = 0 \quad \forall\, e \in \mathscr{E}_h, \forall\, q \in P_{k-1}(e),$$
$$\Pi \mathbf{v}|_e \cdot \mathbf{n} \in P_{k-1}(e) \quad \forall\, e \in \mathscr{E}_h,$$
$$\|\Pi \mathbf{v} - \mathbf{v}\|_{0,K} + h_K \|\nabla(\Pi \mathbf{v} - \mathbf{v})\|_{0,K} \leq C h_K \|\mathbf{v}\|_{0,K} \quad \forall\, K \in \mathscr{T}_h,$$
$$|||\, \Pi \mathbf{v}|||_1 \leq C \|\nabla \mathbf{v}\|_0,$$

with a constant C independent of h_K.

Lemma 5 [47,48] *There exists a constant $\beta > 0$, independent of h, such that*

$$\beta \leq \inf_{q \in Q_h} \sup_{\mathbf{v} \in \mathring{\mathbf{V}}_h} \frac{d_h(\mathbf{v}, q)}{|||\mathbf{v}\,|||_1 \cdot \|q\|_0}. \tag{23}$$

We now present an existence and uniqueness result for the DG methods (11)–(12).

Theorem 1 *Let $\mathbf{f} \in \mathbf{Y}$ and $g \in L^2(\Gamma_S)$ be given with*

$$\frac{2\kappa(2C_0 + C_1)}{\nu^2 \gamma_0^2} (\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) < 1. \tag{24}$$

Then, the problem (11)–(12) admits a unique solution $(\mathbf{u}_h, p_h) \in \widetilde{\mathcal{S}}$, where

$$\widetilde{\mathcal{S}} = \Bigg\{ (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h : \ |||\mathbf{v}_h|||_1 \leq \frac{2\kappa}{\nu\gamma_0} \left(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S} \right),$$
$$\|q_h\|_0 \leq \frac{1}{\beta} \left(\|\boldsymbol{f}\|_0 + \left(\kappa + \frac{2C\kappa}{\gamma_0} \right) (\|\boldsymbol{f}\|_0 + \|g\|_{0,\Gamma_S}) \right) \Bigg\}.$$

Proof The proof of this theorem is divided into three steps.

First, we show the existence of $\mathbf{u}_h$. Define $\mathbf{V}_{h\sigma} = \{\mathbf{v}_h \in \mathbf{V}_h, d_h(\mathbf{v}_h, q_h) = 0, \ \forall\, q_h \in Q_h\}$. We then give the equivalent problem to (11)–(12): find $\mathbf{u}_h \in \mathbf{V}_{h\sigma}$ such that

$$\nu a_h(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathrm{v}_{h\boldsymbol{\tau}}) - j(\mathrm{u}_{h\tau}) \geq \mathcal{F}(\mathbf{v}_h - \mathbf{u}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_{h\sigma}. \tag{25}$$

Given $\mathbf{u}_h \in \mathbf{V}_{h\sigma} = \{\mathbf{v}_h \in \mathbf{V}_h, d_h(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h\}$, consider the following variational inequality:

$$\text{Find } \mathbf{w}_h \in \mathbf{V}_{h\sigma} \text{ such that}$$
$$\nu a_h(\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) + j(\mathrm{v}_{h\boldsymbol{\tau}}) - j(\mathrm{w}_{h\tau}) \geq \mathcal{F}(\mathbf{v}_h - \mathbf{w}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{w}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_{h\sigma}. \tag{26}$$

Coercivity of $a_h(\cdot)$, continuities of $j(\cdot)$ and $\mathcal{F}(\cdot)$, along with Lemma 3 imply that the problem (26) admits a unique solution $\mathbf{w}_h \in \mathbf{V}_{h\sigma}$. Thus problem (26) defines a map $\mathcal{G}_h : \mathbf{V}_{h\sigma} \to \mathbf{V}_{h\sigma}$, and $\mathbf{u}_h$ is the solution of problem (25) is equivalent to the existence of a fixed point of the map $\mathcal{G}_h$ defined by

$$\mathcal{G}_h(\mathbf{u}_h) = \mathbf{u}_h.$$

Now we show $\mathcal{G}_h$ is continuous. Setting $\mathbf{v}_h = \mathbf{0}$ and $\mathbf{v}_h = 2\mathbf{w}_h$ in (26), respectively, we obtain

$$\nu a_h(\mathbf{w}_h, \mathbf{w}_h) + j(\mathrm{w}_{h\boldsymbol{\tau}}) = \mathcal{F}(\mathbf{w}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{w}_h).$$

Then

$$\begin{aligned}\nu\gamma_0 \,|||\, \mathbf{w}_h |||_1^2 &\leq \mathcal{F}(\mathbf{w}_h) - j(\mathrm{w}_{h\boldsymbol{\tau}}) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{w}_h)\\ &\leq \kappa(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) \,|||\, \mathbf{w}_h \,|||_1 + C_0 \,|||\, \mathbf{u}_h \,|||_1^2 \,|||\mathbf{w}_h \,|||_1 .\end{aligned}$$

Define a sphere $\mathcal{S}$ in $\mathbf{V}_{h\sigma}$:

$$\mathcal{S} = \left\{\mathbf{u}_h \in \mathbf{V}_{h\sigma} : |||\mathbf{u}_h|||_1 \leq \frac{2\kappa}{\nu\gamma_0}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S})\right\}.$$

Thus we have

$$\begin{aligned}|||\mathbf{w}_h|||_1 &\leq \frac{\kappa}{\nu\gamma_0}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) + \frac{C_0}{\nu\gamma_0} \,|||\, \mathbf{u}_h |||_1^2\\ &\leq \frac{\kappa}{\nu\gamma_0}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) + \frac{C_0}{\nu\gamma_0} \cdot \frac{4\kappa^2(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S})^2}{\nu^2\gamma_0^2}\\ &\leq \frac{2\kappa}{\nu\gamma_0}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}).\end{aligned}$$

Then, we demonstrate that $\mathcal{G}_h$ is a continuous map. Given $\mathbf{u}_{1h}, \mathbf{u}_{2h} \in \mathbf{V}_{h\sigma}$, $\mathbf{w}_{1h} = \mathcal{G}_h(\mathbf{u}_{1h})$ and $\mathbf{w}_{2h} = \mathcal{G}_h(\mathbf{u}_{2h})$ satisfy

$$\nu a_h(\mathbf{w}_{1h}, \mathbf{v}_h - \mathbf{w}_{1h}) + j(\mathrm{v}_{h\boldsymbol{\tau}}) - j(\mathrm{w}_{1h\boldsymbol{\tau}}) \geq \mathcal{F}(\mathbf{v}_h - \mathbf{w}_{1h}) - c_h(\mathbf{u}_{1h}; \mathbf{u}_{1h}; \mathbf{u}_{1h}, \mathbf{v}_h - \mathbf{w}_{1h}) \tag{27}$$

and

$$\nu a_h(\mathbf{w}_{2h}, \mathbf{v}_h - \mathbf{w}_{2h}) + j(\mathrm{v}_{h\boldsymbol{\tau}}) - j(\mathrm{w}_{2h\boldsymbol{\tau}}) \geq \mathcal{F}(\mathbf{v}_h - \mathbf{w}_{2h}) - c_h(\mathbf{u}_{2h}; \mathbf{u}_{2h}; \mathbf{u}_{2h}, \mathbf{v}_h - \mathbf{w}_{2h}). \tag{28}$$

Choosing $\mathbf{v}_h = \mathbf{w}_{2h}$ in (27) and $\mathbf{v}_h = \mathbf{w}_{1h}$ in (28) and adding the two resulting inequalities, we have

$$\nu a_h(\mathbf{w}_{1h} - \mathbf{w}_{2h}, \mathbf{w}_{1h} - \mathbf{w}_{2h}) \leq c_h(\mathbf{u}_{1h}; \mathbf{u}_{1h}; \mathbf{u}_{1h}, \mathbf{w}_{2h} - \mathbf{w}_{1h}) - c_h(\mathbf{u}_{2h}; \mathbf{u}_{2h}; \mathbf{u}_{2h}, \mathbf{w}_{2h} - \mathbf{w}_{1h}). \tag{29}$$

Since

$$
\begin{aligned}
& c_h(\mathbf{u}_{1h};\mathbf{u}_{1h};\mathbf{u}_{1h},\mathbf{w}_{2h}-\mathbf{w}_{1h}) - c_h(\mathbf{u}_{2h};\mathbf{u}_{2h};\mathbf{u}_{2h},\mathbf{w}_{2h}-\mathbf{w}_{1h}) \\
&\quad = c_h(\mathbf{u}_{2h};\mathbf{u}_{2h};\mathbf{u}_{1h}-\mathbf{u}_{2h},\mathbf{w}_{2h}-\mathbf{w}_{1h}) + c_h(\mathbf{u}_{1h};\mathbf{u}_{1h}-\mathbf{u}_{2h};\mathbf{u}_{1h},\mathbf{w}_{2h}-\mathbf{w}_{1h}) \\
&\qquad + c_h^{NL}(\mathbf{u}_{1h};\mathbf{u}_{2h};\mathbf{u}_{1h},\mathbf{w}_{2h}-\mathbf{w}_{1h}) - c_h^{NL}(\mathbf{u}_{2h};\mathbf{u}_{2h};\mathbf{u}_{1h},\mathbf{w}_{2h}-\mathbf{w}_{1h}),
\end{aligned}
$$

we apply Lemmas 1 and 3 in (29) to obtain

$$
\begin{aligned}
\nu\gamma_0 \,|||\, \mathbf{w}_{1h}-\mathbf{w}_{2h}|||_1^2 &\le C_0 \,|||\, \mathbf{u}_{1h}-\mathbf{u}_{2h}\,|||_1 \;\; |||\mathbf{w}_{1h}-\mathbf{w}_{2h}\,|||_1 \,(|||\mathbf{u}_{1h}\,|||_1 + |||\,\mathbf{u}_{2h}|||_1) \\
&\quad + C_1 \,|||\, \mathbf{u}_{1h}\,|||_1 \;\; |||\mathbf{u}_{1h}-\mathbf{u}_{2h}\,|||_1 \;\; |||\mathbf{w}_{1h}-\mathbf{w}_{2h}|||_1 \\
&\le \frac{2\kappa(2C_0+C_1)}{\nu\gamma_0}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) \,|||\, \mathbf{u}_{1h}-\mathbf{u}_{2h}\,|||_1 \;\; |||\mathbf{w}_{1h}-\mathbf{w}_{2h}|||_1
\end{aligned}
$$

by (24), which implies

$$
|||\mathbf{w}_{1h}-\mathbf{w}_{2h}|||_1 < |||\mathbf{u}_{1h}-\mathbf{u}_{2h}\,|||_1 \,.
$$

So the map $\mathcal{G}_h : \mathbf{V}_{h\sigma} \to \mathbf{V}_{h\sigma}$ is a contraction. By the Brouwer fixed point theorem, $\mathcal{G}_h$ has a fixed point. Hence the discrete problem (25) admits a solution $\mathbf{u}_h \in \mathbf{V}_{h\sigma}$.

Next, we show the existence of p_h by the inf-sup condition.

For all $\mathbf{v}_h \in \overset{\circ}{\mathbf{V}}_h$, notice that the integral term on slip boundary is not included in this subspace, then the similar technique can be applied as the Stokes equations with Dirichlet boundary condition on $\partial\Omega$. Define the polar set of $\overset{\circ}{\mathbf{V}}_h$ as

$$
\overset{\circ}{\mathbf{V}}{}_h^{*} = \left\{ \phi \in (\overset{\circ}{\mathbf{V}}_h)' : \phi(\mathbf{v}) = 0 \;\; \forall\, \mathbf{v}_h \in \overset{\circ}{\mathbf{V}}_h \right\}.
$$

We define the map $B' : Q_h \to \overset{\circ}{\mathbf{V}}{}_h^{*}$ by

$$
B'q_h(\mathbf{v}_h) = d_h(\mathbf{v}_h, q_h) \quad \forall(\mathbf{v}_h, q_h) \in \overset{\circ}{\mathbf{V}}_h \times Q_h,
$$

and, using the solution $\mathbf{u}_h$ found in the preceding step, then we can define a map ϕ in $\overset{\circ}{\mathbf{V}}{}_h^{*}$ by the following equation

$$
\phi(\mathbf{v}_h) = \mathcal{F}(\mathbf{v}_h) - \nu a_h(\mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_h;\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h).
$$

From Lemma 5 and B' is the isomorphism from Q_h to $\overset{\circ}{\mathbf{V}}{}_h^{*}$, there exists a $p_h \in Q_h$ such that [26,47,54]

$$
B' p_h = \phi,
$$

equivalently,

$$
d_h(\mathbf{v}_h, p_h) = \mathcal{F}(\mathbf{v}_h) - \nu a_h(\mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_h;\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \overset{\circ}{\mathbf{V}}_h.
$$

Moreover,

$$
\begin{aligned}
\beta\|p_h\|_0 &\le \sup_{\mathbf{v}_h\in\overset{\circ}{\mathbf{V}}_h} \frac{d_h(\mathbf{v}_h,q_h)}{|||\mathbf{v}_h|||_1} \le \sup_{\mathbf{v}_h\in\overset{\circ}{\mathbf{V}}_h} \frac{\mathcal{F}(\mathbf{v}_h) - \nu a_h(\mathbf{u}_h,\mathbf{v}_h) - c(\mathbf{u}_h;\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)}{|||\mathbf{v}_h|||_1} \\
&\le \|\mathbf{f}\|_0 + C\nu|||\mathbf{u}_h|||_1 + C_0|||\mathbf{u}_h|||_1^2 \\
&\le \|\mathbf{f}\|_0 + \left(\kappa + \frac{2C\kappa}{\gamma_0}\right)(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}).
\end{aligned}
$$

Thus the pair $(\mathbf{u}_h, p_h)$ is the solution to (11)–(12).

Finally, we show that the solution pair $(\mathbf{u}_h, p_h)$ is unique. Suppose that $(\mathbf{u}_{1h}, p_{1h})$ and $(\mathbf{u}_{2h}, p_{2h})$ are two solutions of the problem (11)–(12). We similarly derive the following inequality

$$\nu\gamma_0 \,|||\, \mathbf{u}_{1h} - \mathbf{u}_{2h} |||_1^2 \le \frac{2\kappa(2C_0 + C_1)}{\nu\gamma_0}(\|\mathbf{f}\|_0 + \|g\|_{0,\Gamma_S}) \,|||\, \mathbf{u}_{1h} - \mathbf{u}_{2h} \,|||_1^2 .$$

Recalling the assumption (24), we see that $\mathbf{u}_{1h} = \mathbf{u}_{2h}$. We use $\mathbf{u}_h$ for $\mathbf{u}_{1h}$ and $\mathbf{u}_{2h}$, for all $\mathbf{w}_h \in \mathring{\mathbf{V}}_h, q_h \in Q_h$. Replacing $\mathbf{v}_h$ by $\mathbf{v}_h \pm \mathbf{w}_h$ in (11), we have

$$\nu a_h(\mathbf{u}_h, \mathbf{v}_h) + d_h(\mathbf{v}_h, p_{1h}) = \mathcal{F}(\mathbf{v}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h).$$

and

$$\nu a_h(\mathbf{u}_h, \mathbf{v}_h) + d_h(\mathbf{v}_h, p_{2h}) = \mathcal{F}(\mathbf{v}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h).$$

Thus,

$$d_h(\mathbf{v}_h, p_{1h} - p_{2h}) = 0.$$

From Lemma 5, there holds

$$\beta\|p_{1h} - p_{2h}\|_0 \le \sup_{\mathbf{v}_h \in \mathring{\mathbf{V}}_h} \frac{d(\mathbf{v}_h, p_{1h} - p_{2h})}{|||\mathbf{v}_h|||_1} = 0,$$

implying that $\|p_{1h} - p_{2h}\|_0 = 0$, and so $p_{1h} = p_{2h}$. □

5 Error Estimates

In this section, our task is to bound the errors $|||\mathbf{u} - \mathbf{u}_h|||_1$ and $\|p - p_h\|_0$. By the triangle inequality,

$$|||\mathbf{u} - \mathbf{u}_h \,|||_1 + \|p - p_h\|_0 \le |||\mathbf{u}_h - \mathbf{v}_h \,|||_1 + \|p_h - q_h\|_0 + |||\mathbf{u} - \mathbf{v}_h \,|||_1 + \|p - q_h\|_0,$$

where we choose the Crouzeix–Raviart type interpolation $\mathbf{v}_h$ of the velocity $\mathbf{u}$ and classical interpolation q_h of the pressure p [7,14,26,30,47], which satisfies

$$d_h(\mathbf{v}_h - \mathbf{u}, \psi) = 0 \quad \forall\, \mathbf{u} \in \mathbf{V}(h),\ \psi \in Q_h. \tag{30}$$

Then we have the following error estimate:

Theorem 2 *Assume* (24) *and* $g \in L^\infty(\Gamma_S)$. *If* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *are the solutions of the problems* (6) *and* (11)–(12)*, respectively, then*

$$\|p - p_h\|_0 + |||\mathbf{u} - \mathbf{u}_h|||_1 \le C\Big((|||\mathbf{u} - \mathbf{v}_h \,|||_1 + \|\mathbf{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}}\|_{0,\Gamma_S}^{1/2}) + \|p - q_h\|_0\Big), \tag{31}$$

where $C = C(\nu, \gamma_0, \mathbf{f}, g, \Omega)$.

Proof First, we bound $|||\mathbf{u}_h - \mathbf{v}_h|||_1$. Applying Lemma 1, we see that

$$\nu\gamma_0 \,|||\, \mathbf{u}_h - \mathbf{v}_h|||_1^2 \le \nu a_h(\mathbf{u}_h - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) = T_1 + T_2, \tag{32}$$

where

$$T_1 = \nu a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{v}_h), \quad T_2 = \nu a_h(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h).$$

We rewrite T_1 as follows:

$$T_1 = \nu a_h(\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) - \nu a_h(\mathbf{u}, \mathbf{u}_h - \mathbf{v}_h).$$

In (13), choosing $\mathbf{v} = \mathbf{u}_h$ and $\mathbf{v} = 2\mathbf{u} - \mathbf{v}_h$, respectively, we have

$$\nu a_h(\mathbf{u}, \mathbf{u}_h - \mathbf{v}_h) + c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{u}_h - \mathbf{v}_h) + d_h(\mathbf{u}_h - \mathbf{v}_h, p) + j(2\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}}) \\ + j(\mathrm{u}_{h\boldsymbol{\tau}}) - 2j(u_{\boldsymbol{\tau}}) \geq \mathcal{F}(\mathbf{u}_h - \mathbf{v}_h).$$

From (11),

$$\nu a_h(\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) \leq d_h(\mathbf{v}_h - \mathbf{u}_h, p_h) + c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathrm{v}_{h\boldsymbol{\tau}}) - j(\mathrm{u}_{h\boldsymbol{\tau}}) + \mathcal{F}(\mathbf{u}_h - \mathbf{v}_h).$$

Then T_1 is bounded as follows:

$$T_1 \leq d_h(\mathbf{v}_h - \mathbf{u}_h, p_h - p) + j(2\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}}) - 2j(\mathrm{u}_{\boldsymbol{\tau}}) + j(\mathrm{v}_{h\boldsymbol{\tau}}) \\ + c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) - c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{v}_h - \mathbf{u}_h). \tag{33}$$

Considering (12), (14), (30), we have

$$\begin{aligned} d_h(\mathbf{v}_h - \mathbf{u}_h, p_h - p) &= d_h(\mathbf{v}_h - \mathbf{u}_h, p_h - q_h) + d_h(\mathbf{v}_h - \mathbf{u}_h, q_h - p) \\ &= d_h(\mathbf{v}_h - \mathbf{u}, p_h - q_h) + d_h(\mathbf{u} - \mathbf{u}_h, p_h - q_h) + d_h(\mathbf{v}_h - \mathbf{u}_h, q_h - p) \\ &= d_h(\mathbf{v}_h - \mathbf{u}_h, q_h - p). \end{aligned}$$

The terms on the right side of (33) are bounded as follows:

$$|d_h(\mathbf{v}_h - \mathbf{u}_h, q_h - p)| \leq C \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1 \; \|p - q_h\|_0 \leq \varepsilon \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + \frac{C^2}{4\varepsilon} \|p - q_h\|_0^2, \tag{34}$$

$$|j(2\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}}) - 2j(\mathrm{u}_{\boldsymbol{\tau}}) + j(\mathrm{v}_{h\boldsymbol{\tau}})| \leq 4|j(\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}})| \leq C\|g\|_{\infty,\Gamma_S} \|\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}}\|_{0,\Gamma_S}. \tag{35}$$

Using the fact that $\mathbf{u} \in \mathbf{V}$ and $c_h(\mathbf{v}_h; \mathbf{v}_h; \mathbf{w}_h, \mathbf{w}_h) \geq 0$, we can write

$$\begin{aligned} &|c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) - c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{v}_h - \mathbf{u}_h)| \\ &\quad = |c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) - c_h(\mathbf{u}_h; \mathbf{u}; \mathbf{u}, \mathbf{v}_h - \mathbf{u}_h)| \\ &\quad = |c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h - \mathbf{v}_h, \mathbf{v}_h - \mathbf{u}_h) + c_h(\mathbf{u}_h; \mathbf{u}; \mathbf{v}_h - \mathbf{u}, \mathbf{v}_h - \mathbf{u}_h) \\ &\qquad + c_h(\mathbf{u}_h; \mathbf{u}_h - \mathbf{v}_h; \mathbf{v}_h, \mathbf{v}_h - \mathbf{u}_h) + c_h(\mathbf{u}_h; \mathbf{v}_h - \mathbf{u}; \mathbf{v}_h, \mathbf{v}_h - \mathbf{u}_h)| \\ &\quad \leq C_0 \,|||\, \mathbf{v}_h \,|||_1 \; |||\mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + C_0(|||\mathbf{u} \,|||_1 + |||\, \mathbf{v}_h|||_1) \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1 \; |||\mathbf{u} - \mathbf{v}_h \,|||_1 \\ &\quad \leq \frac{\nu\gamma_0}{2} \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + \frac{1}{2}(\nu\gamma_0 + \nu\gamma_0^2) \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1 \; |||\mathbf{u} - \mathbf{v}_h \,|||_1 \\ &\quad \leq \frac{\nu\gamma_0}{2} \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + \varepsilon \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + \frac{\nu^2\gamma_0^2(1 + \gamma_0^2)}{4\varepsilon} \,|||\, \mathbf{u} - \mathbf{v}_h \,|||_1^2 . \end{aligned} \tag{36}$$

Applying the bounds (34)–(36) in (33), we obtain

$$T_1 \leq \frac{\nu\gamma_0}{2} \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + 2\varepsilon \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + \frac{\nu^2\gamma_0^2(1 + \gamma_0^2)}{4\varepsilon} \,|||\, \mathbf{u} - \mathbf{v}_h \,|||_1^2 + C\|\mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}}\|_{0,\Gamma_S} \\ + \frac{C^2}{4\varepsilon}\|p - q_h\|_0^2. \tag{37}$$

As for T_2, it is easy to see that

$$T_2 \leq C \,|||\, \mathbf{u} - \mathbf{v}_h \,|||_1 \; |||\mathbf{u}_h - \mathbf{v}_h|||_1 \leq \varepsilon \,|||\, \mathbf{u}_h - \mathbf{v}_h \,|||_1^2 + \frac{C^2}{4\varepsilon} \,|||\, \mathbf{u} - \mathbf{v}_h \,|||_1^2 . \tag{38}$$

From (37) and (38), choosing $\varepsilon = \frac{\nu\gamma_0}{8}$ we see that

$$\nu\gamma_0 \, ||| \mathbf{u}_h - \mathbf{v}_h |||_1^2 \le \frac{8\nu^2\gamma_0^2(1+\gamma_0^2)+8C^2}{\nu\gamma_0} ||| \mathbf{u} - \mathbf{v}_h |||_1^2 + \frac{8C^2}{\nu\gamma_0} \| p - q_h \|_0^2 + C \| \mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}} \|_{0,\Gamma_S}. \tag{39}$$

Next, taking $\mathbf{u} \pm \mathbf{w}_h$ as a test function in (13), with an arbitrary $\mathbf{w}_h \in \overset{\circ}{\mathbf{V}}_h$, yields

$$\nu a_h(\mathbf{u}, \mathbf{w}_h) + c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{w}_h) + d_h(\mathbf{w}_h, p) = \mathcal{F}(\mathbf{w}_h).$$

Similarly, we can obtain from (11) that

$$\nu a_h(\mathbf{u}_h, \mathbf{w}_h) + c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{w}_h) + d_h(\mathbf{w}_h, p_h) = \mathcal{F}(\mathbf{w}_h).$$

By subtraction of the above two equations, there holds

$$\nu a_h(\mathbf{u}-\mathbf{u}_h, \mathbf{w}_h) + c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{w}_h) - c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{w}_h) - d_h(\mathbf{w}_h, p - p_h) = 0 \quad \forall\, \mathbf{w}_h \in \overset{\circ}{\mathbf{V}}_h.$$

According to Lemma 5, there holds

$$\begin{aligned}
\beta \| p_h - q_h \|_0 &\le \sup_{\mathbf{w}_h \in \overset{\circ}{\mathbf{V}}_h} \frac{d_h(\mathbf{w}_h, p_h - q_h)}{|||\mathbf{w}_h|||_1} \\
&\le \sup_{\mathbf{w}_h \in \overset{\circ}{\mathbf{V}}_h} \frac{d_h(\mathbf{w}_h, p_h - p) + d_h(\mathbf{w}_h, p - q_h)}{|||\mathbf{w}_h|||_1} \\
&\le \sup_{\mathbf{w}_h \in \overset{\circ}{\mathbf{V}}_h} \frac{c_h(\mathbf{u}_h; \mathbf{u}_h; \mathbf{u}_h, \mathbf{w}_h) - c_h(\mathbf{u}; \mathbf{u}; \mathbf{u}, \mathbf{w}_h) - \nu a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w}_h) + d_h(\mathbf{w}_h, p - q_h)}{|||\mathbf{w}_h|||_1} \\
&\le C(|||\mathbf{u} - \mathbf{u}_h |||_1 + \| p - q_h \|_0).
\end{aligned} \tag{40}$$

The bound (40) follows from (39). Further with the estimate (39), this theorem is completed. □

Remark 2 By the standard finite element approximation theory [7,9,26], if

$$\mathbf{u} \in [H^2(\Omega)]^2, \;\; \mathrm{u}_{\boldsymbol{\tau}}|_{\Gamma_S} \in \tilde{H}^2(\Gamma_S), \;\; p \in H^1(\Omega),$$

then for $k = 1$, there exist $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$ such that

$$|||\mathbf{u} - \mathbf{v}_h |||_1 + \| p - q_h \|_0 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1), \quad \| \mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}} \|_{0,\Gamma_S} \le Ch^2 \| \mathrm{u}_{\boldsymbol{\tau}} \|_{\tilde{H}^2(\Gamma_S)}.$$

Thus, from Theorem 2, we have the optimal order error bound

$$|||\mathbf{u} - \mathbf{u}_h |||_1 + \| p - p_h \|_0 \le Ch.$$

If

$$\mathbf{u} \in [H^3(\Omega)]^2, \;\; \mathrm{u}_{\boldsymbol{\tau}}|_{\Gamma_S} \in \tilde{H}^3(\Gamma_S), \;\; p \in H^2(\Omega),$$

then for $k = 2$, there exist $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$ such that [7,9,26]

$$|||\mathbf{u} - \mathbf{v}_h |||_1 + \| p - q_h \|_0 \le Ch^2(\|\mathbf{u}\|_3 + \|p\|_2), \quad \| \mathrm{u}_{\boldsymbol{\tau}} - \mathrm{v}_{h\boldsymbol{\tau}} \|_{0,\Gamma_S} \le Ch^3 \| \mathrm{u}_{\boldsymbol{\tau}} \|_{\tilde{H}^3(\Gamma_S)}.$$

Thus, we have the error bound

$$|||\mathbf{u} - \mathbf{u}_h |||_1 + \| p - p_h \|_0 \le Ch^{3/2}.$$

And here, the space $\tilde{H}^s(\Gamma_S)(s = 2, 3)$ is defined as follows: let Γ_S be represented as $\Gamma_S = \cup_{1 \le j \le J} \Gamma_{S,j}$ with each $\Gamma_{S,j}$ a closed subset of an affine hyperplane in $\mathbb{R}^2$.

Then $\tilde{H}^s(\Gamma_S)$ consists of functions $\underline{\mathrm{v}}$ such that $\underline{\mathrm{v}} \in H^s(\Gamma_{S,j})(1 \leq j \leq J)$ with norm $|\underline{\mathrm{v}}|_{\tilde{H}^s(\Gamma_S)} = \left(\sum_{j=1}^{J} |\underline{\mathrm{v}}|^2_{H^s(\Gamma_{S,j})}\right)^{\frac{1}{2}}$.

6 Numerical Simulations

We apply the SIPG method to the following three test problems. The Uzawa iterative algorithm [32,37,44] is employed to solve the variational inequality problem numerically:

Choose an arbitrary $\lambda_h^1 \in \Lambda$, $\Lambda = \{\mu \in L^2(\Gamma_S) : |\mu(x)| \leq 1 \text{ a.e. on } \Gamma_S\}$. Then for $n \geq 1$, with the known λ_h^n, we seek $(\mathbf{u}_h^n, p_h^n)$ and λ_h^{n+1} by

$$\begin{cases} \nu a_h\left(\mathbf{u}_h^n, \mathbf{v}_h\right) + c_h\left(\mathbf{u}_h^{n-1}; \mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h\right) + d_h\left(\mathbf{v}_h, p_h^n\right) \\ \quad = \mathcal{F}(\mathbf{v}_h) - \displaystyle\int_{\Gamma_S} \lambda_h^n \, g \, \mathrm{v}_{h\boldsymbol{\tau}} \, \mathrm{d}s \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ d_h(\mathbf{u}_h^n, q_h) = 0 \quad \forall q_h \in Q_h, \end{cases}$$

and

$$\lambda_h^{n+1} = P_\Lambda\left(\lambda_h^n + \rho g \mathrm{u}_{h\boldsymbol{\tau}}^n\right),$$

where $P_\Lambda(\mu) = \sup\{-1, \inf\{1, \mu\}\}$. If a prescribed error tolerance is reached, stop. Note that the nonlinear term has been linearized by Picard's iteration method and the zero vector has been taken as the iterative initial value of the velocity. In addition, the results served to verify the error bound are exhibited for $k = 1, 2$ (k represents the degree of the polynomial for the velocity function space in Sect. 3), while the others shown in figures are for $k = 1$ since there is no obvious difference between different k.

Example 6.1 Let $\Omega = (0, 1)^2$, and consider a boundary split into the slip boundary $\Gamma_S = (0, 1) \times \{1\}$ and the Dirichlet boundary $\Gamma_D = \partial\Omega \setminus \Gamma_S$. Motivated by the numerical example in [37], let us consider

$$\begin{cases} u_1(x, y) = 20x^2(x-1)^2 y(y-1)(2y-1), \\ u_2(x, y) = -20x(x-1)(2x-1)y^2(y-1)^2, \\ p(x, y) = 20(2x-1)(2y-1), \end{cases} \tag{41}$$

which turns out to be the solution of the Navier–Stokes equations (1) under the adhesive boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$. Here, the external force $\mathbf{f}$ is defined by

$$\begin{cases} f_1(x, y) = -40\nu(6x^2 - 6x + 1)y(y-1)(2y-1) - 120\nu x^2(x-1)^2(2y-1) \\ \qquad\qquad + 40(2y-1) + 400x^3(x-1)^3(2x-1)y^2(y-1)^2(2y^2 - 2y + 1), \\ f_2(x, y) = 120\nu(2x-1)y^2(y-1)^2 + 40\nu x(x-1)(2x-1)(6y^2 - 6y + 1) \\ \qquad\qquad + 40(2x-1) + 400x^2(x-1)^2(2x^2 - 2x + 1)y^3(y-1)^3(2x-1). \end{cases}$$

By a direct computation, we find

$$\max_{\overline{\Gamma_S}} |\sigma_{\boldsymbol{\tau}}| = \max_{0 \leq x \leq 1} |20\nu x^2(x-1)^2| = 1.25\nu.$$

Now, instead of the adhesive boundary condition, we impose the slip boundary condition on Γ_S, for a fixed function g. Then it can be seen that

$$\begin{cases} g(\mathbf{x}) > \sigma_{\boldsymbol{\tau}}(\mathbf{x}) \text{ for all } \mathbf{x} \in \Gamma_S \Rightarrow (41) \text{ remains the solution} \Rightarrow \text{No-slip occurs.} \\ g(\mathbf{x}_0) = \sigma_{\boldsymbol{\tau}}(\mathbf{x}_0) \text{ for some } \mathbf{x}_0 \in \Gamma_S \Rightarrow (41) \text{ is no longer a solution} \Rightarrow \text{Slip occurs.} \end{cases}$$

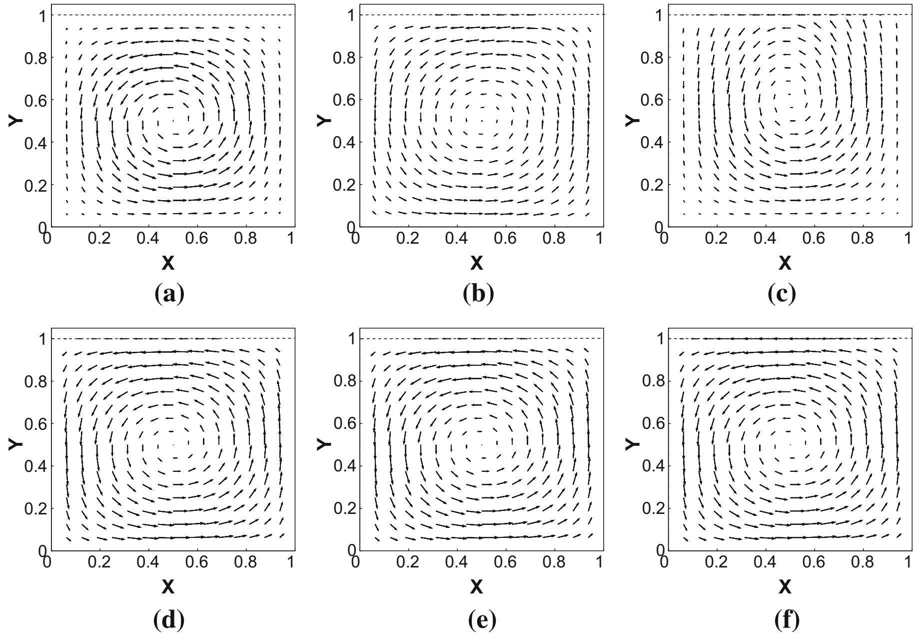

Fig. 1 SIPG method ($\epsilon = -1$): velocity field in Ω with different constant and function g under $k = 1$. **a** g = 2.0. **b** g = 0.8. **c** g = 0.2. **d** $g = x$. **e** $g = \frac{\sin x+1}{2}$. **f** $g = 20x^2(x-1)^2$

In particularly, for a constant g, it can be intuitively observed:

$$\begin{cases} g > 1.25\nu \Rightarrow \text{(41) remains the solution} \Rightarrow \text{No-slip occurs.} \\ g \leq 1.25\nu \Rightarrow \text{(41) is no longer a solution} \Rightarrow \text{Slip occurs.} \end{cases}$$

The slip and non-slip phenomena are clearly observed in Fig. 1 for different values of g on a uniform 16×16 grid. In fact, slip phenomena ($\mathrm{u}_{h\boldsymbol{\tau}} \neq 0$) take place on Γ_S for $g = 0.2, 0.8$, whereas no slip is observed for $g = 2.0$ ($\nu = 1.0$). When g is a fixed function, if the values of $g(\mathbf{x}_0)$ is bigger than $\sigma_{\boldsymbol{\tau}}(\mathbf{x}_0)$, no slip occurs along the top boundary of the computational domain, while slip phenomena appear at the positions where the values of $g(\mathbf{x}_0)$ are less than $\sigma_{\boldsymbol{\tau}}(\mathbf{x}_0)$, and the degree of slip is closely related to the value of the friction function g (see Fig. 1).

Figures 2 and 3 display the tangential velocities $\mathrm{u}_{h\boldsymbol{\tau}}$ along the slip boundary for the SIPG method and finite element method, corresponding to Fig. 1, respectively. We see that the locations, where slip and non-slip switch, are captured by the DG methods through discontinuous velocities, while these discontinuous points are connected in the finite element method since continuous function spaces are used (lowest order finite element pair [36,42]). These comparisons suggest that the DG methods are superior than continuous finite element method on capturing the discontinuity phenomena. In addition, we fix $\gamma = 10$ for better accuracy for Figs. 1, 2, and 3 and $\nu = 1.0$, which is omitted in g for simplicity.

In Table 1, we report the numerical errors of the velocity and pressure with different friction constant g, respectively. Since the explicit solution is unknown when $g = 0.2$, we regard the approximate solution on grid of 128×128 as a reference solution ($\mathbf{u}_{ref}$, p_{ref}) for $k = 1$, and on a 64×64 grid as a reference solution ($\mathbf{u}_{ref}$, p_{ref}) for $k = 2$ in this example.

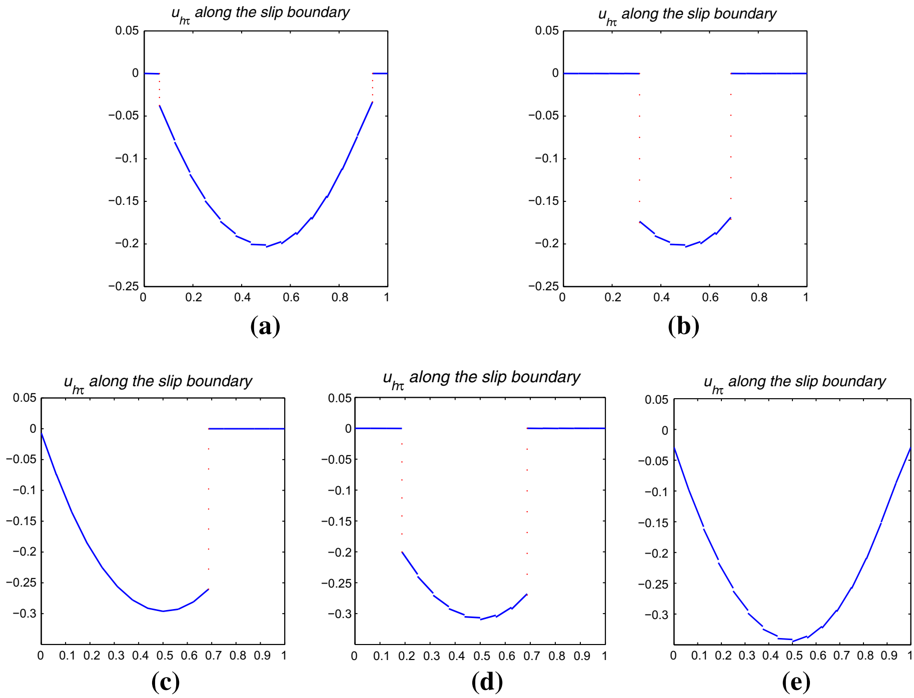

Fig. 2 SIPG method ($\epsilon = -1$): values of $u_{h\tau}$ along the slip boundary under different constant and function g ($k = 1$). **a** g = 0.2. **b** g = 0.8. **c** $g = x$. **d** $g = \frac{\sin x + 1}{2}$. **e** $g = 20x^2(x-1)^2$

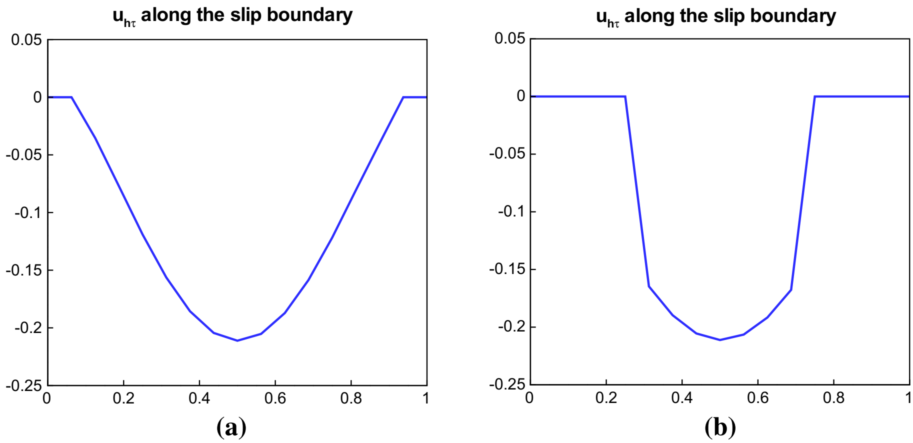

Fig. 3 Finite element method: values of $u_{h\tau}$ along the slip boundary under different constant g (stabilized lowest order finite element pair). **a** g = 0.2. **b** g = 0.8.

Table 1 SIPG method ($\epsilon = -1$): numerical errors for $g = 0.2$ and $g = 2.0$

$\frac{1}{h}$	$g = 0.2$				$g = 2.0$			
	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_1$	Order	$\|p - p_h\|_0$	Order	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_1$	Order	$\|p - p_h\|_0$	Order
$k = 1$		$\gamma = 4$				$\gamma = 3$		
8	1.5384e+00	–	9.5763e−01	–	5.4842e−01	–	9.9278e−01	–
16	7.2479e−01	1.09	4.8711e−01	0.98	2.3638e−01	1.21	4.9287e−01	1.01
32	3.2932e−01	1.14	2.4211e−01	1.00	1.0693e−02	1.14	2.4483e−01	1.00
64	1.3553e−01	1.13	1.0968e−01	1.14	5.0372e−02	1.09	1.2189e−01	1.00
$k = 2$		$\gamma = 10$				$\gamma = 8$		
8	2.8419e−01	–	1.3589e−01	–	5.9491e−02	–	5.7145e−02	–
16	1.0907e−01	1.38	5.2631e−01	1.37	1.2786e−03	2.10	11.4165e−03	2.01
32	3.7839e−02	1.53	1.9626e−02	1.42	2.8938e−03	2.06	3.5287e−03	2.01

However, we know the exact solution (41) when $g = 2.0$ and thus we take $\mathbf{u}_{ref} = \mathbf{u}$, $p_{ref} = p$. The specific penalty parameters γ given in Table 1 are the smallest integer values which guarantee the provided methods are stable, besides, the viscosity coefficient remain to be 1.0, which is also left out here in g as Figs. 1, 2 and 3.

Moreover, convergence behaviors for different ν (1.0, 0.025, 0.01) and different constant g are exhibited in Fig. 4, at this time $\gamma = 10, 200, 2000$, respectively. From Table 1 and Fig. 4, we see that when $k = 1$, the expected first order convergence is observed in both the broken H^1-norm for velocity and L^2-norm for pressure; when $k = 2$, an error of size $\mathcal{O}(h^2)$ is obtained when no slip occurs, while the convergence order reaches $\frac{3}{2}$ as slip occurs, these results are consistent with the theoretical analysis.

Example 6.2 A wall-driven semi-circular cavity flow is simulated. We investigate properties of the numerical method: its stability, and its ability in handling the boundary layers and the effects associated with the nonlinear advection. The geometric region is

$$\Omega = \left\{\mathbf{x} = (x, y) \in R^2 \mid y < 0, x^2 + y^2 < 1/4\right\}.$$

On the straight part of the boundary Γ_D, we specify a velocity condition: $\mathbf{u} = (1, 0)$. The curvilinear part Γ_S is chosen as the slip boundary, cf. Fig. 5a.

Non-uniform grid is obtained by the Delaunay mesh generation with 5258 triangles (Fig. 5b). Different values of the viscosity coefficient ν are used combined with constant or variable g. When $\nu = 0.001$ the penalty parameter $\gamma = 200$, and otherwise, $\gamma = 10$. From Fig. 6 we see that when slip occurs, the DG method has the ability to handle the slip layers, while the slip phenomenon disappears as the viscosity coefficient becomes small. The capability of the DG method in dealing with the advection-dominated cases is illustrated in Fig. 6c, f, which is consistent with the known results in [31]. When g are the set functions, whether the slip phenomenon occurs depends on the values of g on Γ_S (see Fig. 6a, d), which is reasonable according to the analysis in Example 6.1.

Example 6.3 This example provides an application of the interior penalty DG method on general polygonal meshes [46] for solving the variational inequality problem. Let $\Omega = (0, 1) \times (0, 1)$, the two slip boundaries are $\Gamma_S = \{x = 1, 0 < y \leq 1\} \cup \{y = 1, 0 < x \leq 1\}$, and the remain boundaries naturally become the Dirichlet boundaries Γ_D (see Fig.

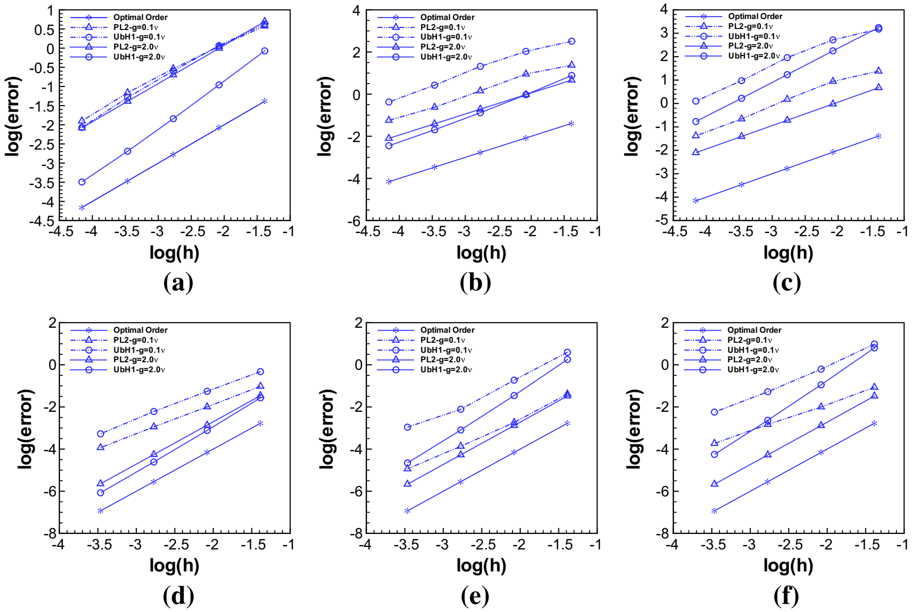

Fig. 4 Convergence behavior for different viscosity coefficient ν and different constant g for the SIPG method (top: $k = 1$, bottom: $k = 2$). **a** $\nu = 1.0$. **b** $\nu = 0.025$. **c** $\nu = 0.01$. **d** $\nu = 1.0$. **e** $\nu = 0.025$. **f** $\nu = 0.01$

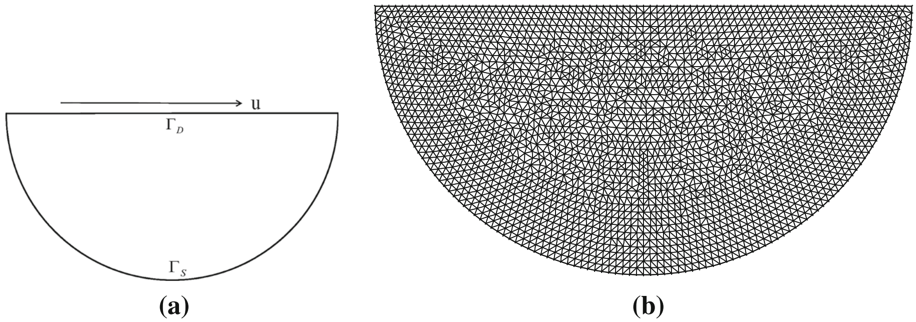

Fig. 5 Wall-driven flow in a semi-circular cavity and the triangulation of this domain. **a** A semi-circular cavity. **b** Triangulation of $\mathscr{T}_h$

7a). The general polygonal mesh generation of Ω is shown in Fig. 7b. The exact solution $(\mathbf{u}, p)$ of the Navier–Stokes equations (1)–(2) is [42,44]:

$$\mathbf{u}(x, y) = \begin{pmatrix} -x^2 y(x-1)(3y-2) \\ xy^2(y-1)(3x-2) \end{pmatrix}, \quad p(x, y) = (2x-1)(2y-1).$$

Then the body force $\mathbf{f}$ can be calculated by (1), and it is easy to verify that the $\mathbf{u}$ satisfies the boundary conditions (3)–(4) on Γ_D and Γ_S, respectively. We can specify $\sigma_{\boldsymbol{\tau}}$ as follows:

$$\begin{cases} \sigma_{\boldsymbol{\tau}} = 4\nu y^2(y-1) \quad \text{on} \quad S_1, \\ \sigma_{\boldsymbol{\tau}} = 4\nu x^2(x-1) \quad \text{on} \quad S_2. \end{cases}$$

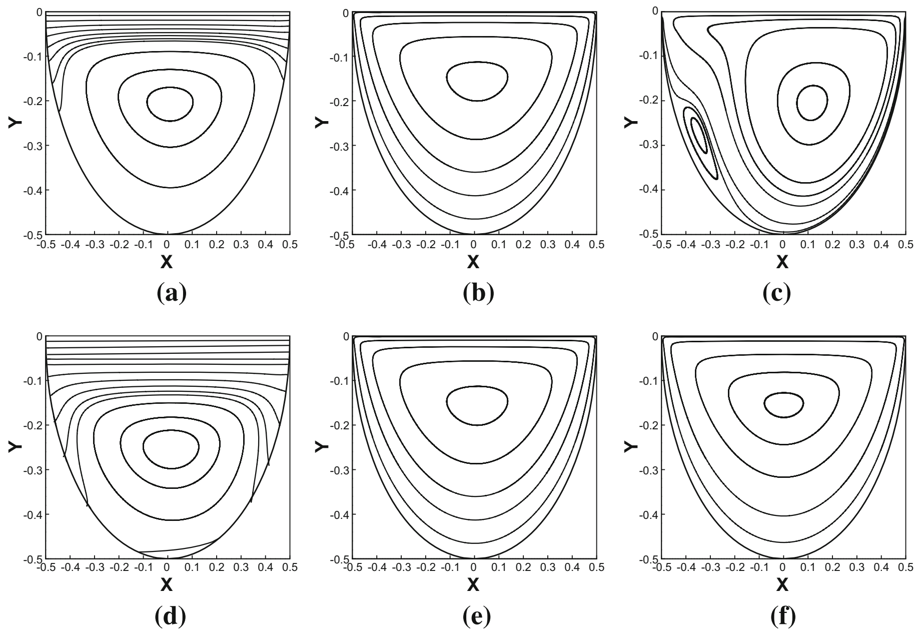

Fig. 6 Streamlines of SIPG method ($\epsilon = -1$) for different viscosity coefficient ν and different function g under $k = 1$. **a** $g = 1.0, \nu = 1.0$. **b** $g = 1.0, \nu = 0.1$. **c** $g = 1.0, \nu = 0.001$. **d** $g = 4x^2 + y^2,\ \nu = 1.0$. **e** $g = 4x^2 + y^2,\ \nu = 0.1$. **f** $g = 10(\cos(\pi x) + 1),\ \nu = 1.0$

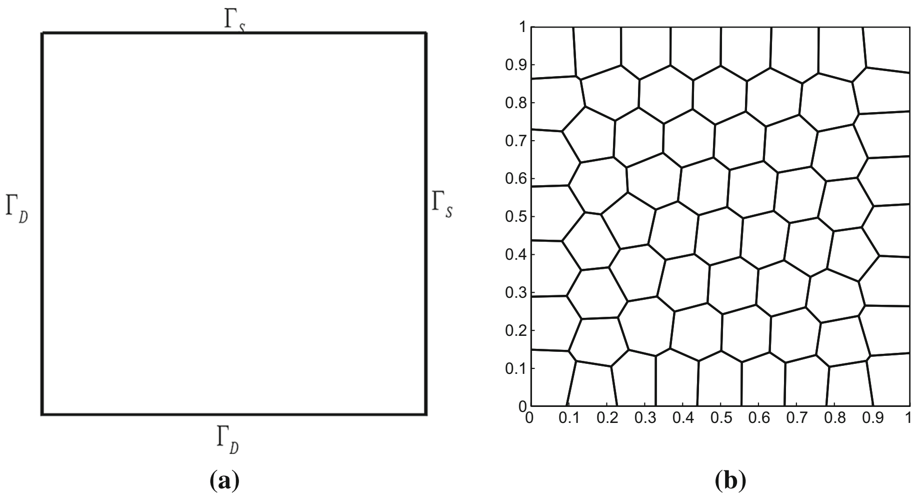

Fig. 7 The computational domain and general polygonal meshes. **a** Ω with Γ_D and $\Gamma_S = S_1 \cup S_2$. **b** Polygonal meshes

Moreover, the position friction function g can be chosen as $-\sigma_{\boldsymbol{\tau}}$ on each slip boundary Γ_S by (4). In Tables 2 and 3, errors and convergence orders of velocity and pressure are displayed for the SIPG and NIPG methods, where h represents the average value of the radius of all the polygons, N is the total number of the polygons, and $\nu = 1$. We see that the numerical results

Table 2 SIPG method ($\epsilon = -1$): numerical errors and convergence behaviors

h	N	$k = 1$				$k = 2$			
		$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_1$	Order	$\|p - p_h\|_0$	Order	$\|\mathbf{u} - \mathbf{u}_h\|_1$	Order	$\|p - p_h\|_0$	Order
2.88e−01	10	1.6628e−01	–	4.4608e−01	–	6.3820e−02	–	9.6259e−01	–
9.61e−02	64	7.1598e−02	0.77	1.2553e−01	1.16	1.33012e−02	1.43	1.0487e−01	2.02
4.61e−02	256	3.7001e−02	0.90	4.4286e−02	1.42	3.1469e−03	1.96	2.0947e−02	2.19
2.26e−02	1024	1.8140e−02	1.00	1.7784e−02	1.28	8.1071e−04	1.90	5.0710e−03	1.99

Table 3 NIPG method ($\epsilon = 1$): numerical errors and convergence behaviors

h	N	$k = 1$				$k = 2$			
		$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_1$	Order	$\|p - p_h\|_0$	Order	$\|\mathbf{u} - \mathbf{u}_h\|_1$	Order	$\|p - p_h\|_0$	Order
2.88e−01	10	1.6644e−01	–	2.2243e−01	–	6.4343e−02	–	1.2722e+00	–
9.61e−02	64	7.2582e−02	0.77	8.0056e−02	0.93	1.4053e−02	1.37	1.3743e−01	2.03
4.61e−02	256	3.8826e−02	0.85	3.6553e−02	1.07	3.6459e−03	1.84	2.7910e−02	2.17
2.26e−02	1024	2.0037e−02	0.93	1.7207e−02	1.05	8.5856e−04	2.02	6.7345e−03	1.99

match our theoretical analysis and show some superconvergence. This example shows the potential of extending DG methods to arbitrary polygonal meshes.

7 Conclusion and Future Work

Several discontinuous Galerkin methods are employed to solve the steady Navier–Stokes equations with a nonlinear slip boundary condition of friction type. We establish the stability of the DG scheme, existence and uniqueness of the numerical solution. We prove the optimal order error bound $\mathcal{O}(h)$ when piecewise linear functions are used for the velocity and piecewise constant functions for the pressure. We provide numerical simulation results to illustrate the slip and non-slip phenomena, convergence behaviors, the capability of the DG methods to capture the discontinuity of the velocity, the ability of handling the boundary layers when slip phenomenon appears, the capacity of the proposed methods in dealing with the advection-dominated cases, and extension of the methods to general polygonal meshes.

In future studies, a p-adaptive technique would be introduced to improve the error bound when the quadratic or higher order polynomial velocity subspaces are used, extension to a 3D domain, the theoretical analysis for the polymesh and other high-precision numerical method, e.g. hybrid discontinuous Galerkin method, will be also considered.

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