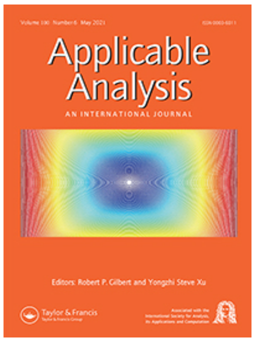

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Unconditional stability and optimal error estimates of discontinuous Galerkin methods for the second-order wave equation

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ABSTRACT

In this paper, we revisit the numerical solution of the scalar second-order wave equation by discontinuous Galerkin methods. The numerical methods are different from the ones found in existing literature. Moreover, we provide a stability analysis and derive optimal order error estimates through a more direct approach. The error estimate in an $H^1(\Omega)$-like norm is derived based on an analysis of the truncation error while that in the $L^2(\Omega)$ norm based on an application of the Aubin-Nitsche technique. Numerical simulation results are reported in support of the theoretical error estimates.

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1. Introduction

The second-order wave equation appears in a wide range of fields, such as electromagnetic, acoustic, elastic, seismic waves, and so on. In view of the important applications of the equation, much effort has been made searching for analytic solution formulas and approximate solutions to the equations by using various numerical methods. In this paper, we consider stability and optimal order error estimation for discontinuous Galerkin (DG) methods of the second-order wave equation. We note that various DG methods have been proposed and studied in the literature, e.g. the penalty DG method (PDG) [1,2], the local DG (LDG) [3–5], the hybrid DG (HDG) [6], and so on. DG methods have been applied to solve a large number of problems from applications, for instance, convection-diffusion equations [5,7], hyperbolic equations [8–11], Navier-Stokes equations [3,12], Hamilton-Jacobi equations [13,14], the radiative transfer equation [15], and variational inequalities [16–20].

A few papers can be found on DG methods for solving the second-order wave equation. In [9], an SIPG (symmetric interior penalty discontinuous Galerkin) method is applied to solve the wave equation and optimal order error estimates are derived for the spatially semi-discrete scheme. In the sequel [10], a fully discrete scheme for the wave equation is studied and an optimal $L^2(\Omega)$ norm error estimate is derived under a CFL (Courant-Friedrichs-Lewy) stability condition. In [21], the spatial

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discretization based on DG methods and the temporal discretization combined with the centered second-order finite difference approximation are applied to the wave equation, spatially semi-discrete schemes and fully discrete schemes are analyzed and optimal order error estimates in the $H^1(\Omega)$ and the $L^2(\Omega)$ norms are derived without restrictive CFL constraints. In this paper, we study DG methods for the second-order wave equation using different initial values as compared to those methods in [21]. Moreover, we provide a stability analysis and derive optimal order error estimates through a more direct approach. We derive optimal order error estimates in an $H^1(\Omega)$-like norm based on a consideration of the truncation error and that in the $L^2(\Omega)$ norm by an application of the Aubin-Nitsche technique.

The organization of this paper is as follows. In the next section, we introduce the initial-boundary value problem for the second-order wave equation and recall a continuous and a discrete Gronwall inequality. In Section 3, we introduce fully discrete schemes based on DG discretization in space and show stability for the fully discrete solutions. In Section 4, through Galerkin projection, we present optimal order error estimates for the numerical solutions in both $H^1(\Omega)$ and $L^2(\Omega)$ norms from the truncation error and by Aubin-Nitsche technique, respectively. Finally, in Section 5, we validate our theory by simulation results on a numerical example.

2. The second-order wave equation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded connected domain with a Lipschitz boundary $\partial\Omega$. For a given $T > 0$, let $[0, T]$ be the time interval of interest, $f \in L^2(0, T; L^2(\Omega))$ represent the external force and $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$ the given initial data. We consider, as in [9,10,21], the following initial-boundary value problem of the scalar wave equation: find $u(\boldsymbol{x}, t)$ such that

$$\partial_t^2 u - \nabla\cdot(b\,\nabla u) = f \quad \text{in } \Omega \times (0, T], \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T], \tag{2.2}$$

$$u = u_0 \quad \text{in } \Omega \times \{t = 0\}, \tag{2.3}$$

$$\partial_t u = v_0 \quad \text{in } \Omega \times \{t = 0\}, \tag{2.4}$$

where u represents the unknown variable of interest, $\partial_t u$ and $\partial_t^2 u$ are its first- and second-order time derivatives. To simplify the notation, we here only concentrate on the case of two dimensional spatial domains, the case of three-dimensional one being similar. In this paper, we assume b is a given smooth function and for two positive constants $b_{\min}$ and $b_{\max}$,

$$b_{\min} \le b(\boldsymbol{x}) \le b_{\max}, \quad \boldsymbol{x} \in \overline{\Omega}, \tag{2.5}$$

Let $V = H_0^1(\Omega)$. The standard variational formulation of the problem (2.1)–(2.4) is as follows.

Problem 2.1: Find $u \in L^2(0, T; V)$ with $\partial_t u \in L^2(0, T; L^2(\Omega))$ and $\partial_t^2 u \in L^2(0, T; H^{-1}(\Omega))$ such that

$$\langle \partial_t^2 u, v\rangle + a(u, v) = (f, v) \quad \forall\, v \in V, \text{ a.e. in } [0, T], \tag{2.6}$$

and

$$u = u_0, \quad \partial_t u = v_0 \quad \text{a.e. in } \Omega \times \{t = 0\}. \tag{2.7}$$

In Problem 2.1, the time derivatives are understood in the distributional sense, $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1}(\Omega)$ and V, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and

$$a(u, v) = \int_\Omega b\,\nabla u\cdot\nabla v\,\mathrm{d}x, \quad u, v \in V. \tag{2.8}$$

It is known, cf. e.g. [22, Chapter 3], that Problem 2.1 has a unique solution and moreover, $u \in C([0, T]; V)$ and $\partial_t u \in C([0, T]; L^2(\Omega))$.

Gronwall's inequalities are important tools for analyzing time-dependent problems. We present both continuous and discrete versions as follows (cf. e.g. [23,24]).

Lemma 2.2: *Let ξ be a non-negative and integrable function on $[0, T]$. Assume for some constants $C_1, C_2 \geq 0$,*

$$\xi(t) \leq C_1 \int_0^t \xi(s)\,\mathrm{d}s + C_2, \quad \text{a.e. } t \in [0, T].$$

Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}), \quad \text{a.e. } t \in [0, T].$$

Lemma 2.3: *Let C,k and a_m, b_m, c_m, d_m, for integers m, be non-negative numbers such that*

$$a_n + k \sum_{m=0}^{n} b_m \leq k \sum_{m=0}^{n-1} d_m a_m + k \sum_{m=0}^{n-1} c_m + C \quad \forall\, m \geq 0.$$

Then

$$a_n + k \sum_{m=0}^{n} b_m \leq \exp\left(k \sum_{m=0}^{n-1} d_m\right)\left(k \sum_{m=0}^{n-1} c_m + C\right) \quad \forall\, m \geq 0.$$

3. Numerical methods

To prepare for the presentation of the discrete schemes for solving Problem 2.1, we assume Ω is a convex polygon as in [10,21]. Let $\{\mathcal{T}_h\}_h$ be a regular family of quasi-uniform finite element triangulations of $\overline{\Omega}$. Corresponding to a finite element mesh $\mathcal{T}_h$ in the family, denote by K a generic element, by $h_K = \mathrm{diam}(K)$ the diameter of K, and by $h = \max\{h_K : K \in \mathcal{T}_h\}$ the finite element mesh-size. Let $\mathcal{E}_h$ be the collection of all the edges of $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of all interior edges, and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ the set of all the edges on the boundary $\partial\Omega$.

As in [10,16,21,25], we use the standard DG notation, for instance, jump $[\![\cdot]\!]$ and average $\{\cdot\}$ and so on. Denote by ∇_h the broken gradient operator defined piecewise by the relation $\nabla_h v = \nabla v$ on any element $K \in \mathcal{T}_h$. Let $\eta : \mathcal{E}_h \to \mathbb{R}$ be the penalty weighting function defined by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h$, $\eta_e > 0$. We introduce the following discontinuous finite element spaces:

$$V^h = \{v^h \in L^2(\Omega) : v^h\,|_K \in P_p(K)\ \forall\, K \in \mathcal{T}_h\},$$
$$\boldsymbol{W}^h = \{\boldsymbol{w}^h \in [L^2(\Omega)]^2 : \boldsymbol{w}^h\,|_K \in [P_p(K)]^2\ \forall\, K \in \mathcal{T}_h\},$$

where $p \geq 1$ is a positive integer and it is used as the local polynomial degree of the DG formulations. Over the space $V(h) = V^h + H^2(\Omega) \cap V$, we define a norm by the relation

$$\|v\|_h^2 = \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![v]\!]\|_{0,e}^2. \tag{3.1}$$

To approximate the bilinear form (2.8), we consider four choices of the DG bilinear form as follows.

$$a_h^{(1)}(u, v) = \int_\Omega b\,\nabla_h u \cdot \nabla_h v\,\mathrm{d}x - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b\,\nabla_h v\}\,\mathrm{d}s - \int_{\mathcal{E}_h} \{b\,\nabla_h u\} \cdot [\![v]\!]\,\mathrm{d}s + \int_{\mathcal{E}_h} b\,\eta [\![u]\!] \cdot [\![v]\!]\,\mathrm{d}s,$$
$$a_h^{(2)}(u, v) = \int_\Omega b\,\nabla_h u \cdot \nabla_h v\,\mathrm{d}x - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b\,\nabla_h v\}\,\mathrm{d}s - \int_{\mathcal{E}_h} \{b\,\nabla_h u\} \cdot [\![v]\!]\,\mathrm{d}s$$

$$+\sum_{e\in\mathcal{E}_h}\int_\Omega b\,\eta_e r_e([\![u]\!])\cdot r_e([\![v]\!])\,\mathrm{d}x,$$

$$a_h^{(3)}(u,v)=\int_\Omega b\,\nabla_h u\cdot\nabla_h v\,\mathrm{d}x-\int_{\mathcal{E}_h}[\![u]\!]\cdot\{b\,\nabla_h v\}\,\mathrm{d}s-\int_{\mathcal{E}_h}\{b\,\nabla_h u\}\cdot[\![v]\!]\,\mathrm{d}s$$
$$+\int_{\mathcal{E}_h} b\,\eta[\![u]\!]\cdot[\![v]\!]\,\mathrm{d}s+\int_\Omega b\,r([\![u]\!])\cdot r([\![v]\!])\,\mathrm{d}x,$$

$$a_h^{(4)}(u,v)=\int_\Omega b\,\nabla_h u\cdot\nabla_h v\,\mathrm{d}x-\int_{\mathcal{E}_h}[\![u]\!]\cdot\{b\,\nabla_h v\}\,\mathrm{d}s-\int_{\mathcal{E}_h}\{b\,\nabla_h u\}\cdot[\![v]\!]\,\mathrm{d}s$$
$$+\sum_{e\in\mathcal{E}_h}\int_\Omega b\,\eta_e r_e([\![u]\!])\cdot r_e([\![v]\!])\,\mathrm{d}x+\int_\Omega b\,r([\![u]\!])\cdot r([\![v]\!])\,\mathrm{d}x.$$

Here, the lifting operators $r_e:[L^2(e)]^2\to \boldsymbol{W}^h$ and $r:[L^2(\mathcal{E}_h)]^2\to \boldsymbol{W}^h$ are defined by relations [25]

$$\int_\Omega r_e(\boldsymbol{q})\cdot\boldsymbol{w}^h\mathrm{d}x=-\int_e \boldsymbol{q}\cdot\{\boldsymbol{w}^h\}\,\mathrm{d}s\quad\forall\,\boldsymbol{w}^h\in\boldsymbol{W}^h,$$
$$\int_\Omega r(\boldsymbol{q})\cdot\boldsymbol{w}^h\mathrm{d}x=-\int_{\mathcal{E}_h} \boldsymbol{q}\cdot\{\boldsymbol{w}^h\}\,\mathrm{d}s\quad\forall\,\boldsymbol{w}^h\in\boldsymbol{W}^h.$$

The first DG bilinear form corresponds to the interior penalty (IP) method ([1,2]) and is used in [10]. The other three DG bilinear forms correspond to the method in [26], the simplified local DG (SLDG) method in [4], and the method in [27], respectively.

It can be shown that all the four DG methods $a_h(u,v)=a_h^{(j)}(u,v)$, $1\le j\le 4$ are consistent, i.e. if the solution of Problem 2.1 has the regularity property $u\in L^2(0,T;H^2(\Omega))$, then for a.e. $t\in[0,T]$,

$$(\partial_t^2 u,v^h)+a_h(u,v^h)=(f,v^h)\quad\forall\,v^h\in V^h.\tag{3.2}$$

We also have the boundedness, i.e. there exists a constant c_b such that for $a_h=a_h^{(j)}$, $1\le j\le 4$,

$$|a_h(u,v)|\le c_b\,\|u\|_h\|v\|_h\quad\forall\,u,v\in V(h).$$

For stability, if $\eta_0=b_{\min}\inf_e\eta_e$ is sufficiently large for $j=1,2$ and $\eta_0>0$ for $j=3,4$, then there exists a constant c_s such that for $a_h=a_h^{(j)}$, $1\le j\le 4$,

$$a_h(v,v)\ge c_s\|v\|_h^2\quad\forall\,v\in V^h.$$

Details can be found in [16,21,25]. Thanks to the boundedness and stability of the four bilinear forms $a_h(u,v)$, we have

$$c_2\|v^h\|_h^2\le a_h(v^h,v^h)\le c_1\|v^h\|_h^2.$$

Thus $\|v^h\|_{a_h}=a_h(v^h,v^h)^{1/2}$ defines a norm for $v^h\in V^h$, and the norm is equivalent to the norm $\|v^h\|_h$. In addition, we notice that

$$\|w\|_h\le c\,\|w\|_2\quad\forall\,w\in H^2(\Omega).\tag{3.3}$$

For simplicity in notation, we only focus on the case of evenly spaced nodes $t_n=nk$ $(0\le n\le N)$, where $0=t_0<t_1<\cdots<t_N=T$ form a uniform partition of $I=[0,T]$ into subintervals

$I_n = (t_n, t_{n+1})$, $n = 0, 1, \ldots, N-1$, with a uniform time step $k = t_{n+1} - t_n = T/N$. For a generic continuous function u of time, set $u_n = u(\cdot, t_n)$. The symbols $\bar{\delta}_k$, δ_k, and d_k are defined by

$$\bar{\delta}_k u_n = \frac{u_{n+1} + u_{n-1}}{2},$$
$$\delta_k u_n = \frac{u_{n+1} - u_{n-1}}{2k},$$
$$d_k u_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{k^2}.$$

Let Π^h be the Galerkin projection onto the space V^h, i.e. for $w \in V$,

$$\Pi^h w \in V^h, \quad a_h(\Pi^h w, v^h) = a_h(w, v^h) \quad \forall\, v^h \in V^h.$$

The following error bounds hold [10, Lemma 4.1]:

$$\|w - \Pi^h w\|_0 + h\|w - \Pi^h w\|_h \le c\, h^{\min\{p+1,m\}} \|w\|_m \quad \forall\, w \in H^m(\Omega),\ m \ge 1. \tag{3.4}$$

Let $a_h(\cdot,\cdot)$ be one of the bilinear forms $a_h^{(j)}(\cdot,\cdot)$ with $1 \le j \le 4$. Assume $v_0 \in V$. Then the fully discrete approximation of Problem 2.1 is as follows.

Problem 3.1: Find $\{u_n^{hk}\}_{n=0}^N \subset V^h$ such that for $1 \le n \le N-1$,

$$(d_k u_n^{hk}, v^h) + a_h(\bar{\delta}_k u_n^{hk}, v^h) = (f_n, v^h) \quad \forall\, v^h \in V^h, \tag{3.5}$$

and

$$u_0^{hk} = \Pi^h u_0, \tag{3.6}$$

$$u_1^{hk} = u_0^{hk} + k\,\Pi^h v_0 + \frac{k^2}{2}\,\tilde{u}_0^h, \tag{3.7}$$

where

$$\tilde{u}_0^h \in V^h, \quad (\tilde{u}_0^h, v^h) = (f_0, v^h) - a_h(u_0, v^h) \quad \forall\, v^h \in V^h. \tag{3.8}$$

We have the next result.

Lemma 3.2 (Stability): *Problem* 3.1 *has a unique solution* $\{u_n^{hk}\}_{n=0}^N \subset V^h$ *and the following stability estimate holds*: *for* $0 \le n \le N-1$,

$$\left\|\frac{u_{n+1}^{hk} - u_n^{hk}}{k}\right\|_0^2 \le C\left(\left\|\frac{u_1^{hk} - u_0^{hk}}{k}\right\|_0^2 + \|u_1^{hk}\|_{a_h}^2 + \|u_0^{hk}\|_{a_h}^2 + k\sum_{i=1}^n \|f_i\|_0^2\right), \tag{3.9}$$

and

$$a_h(u_n^{hk}, u_n^{hk}) \le C\left(\left\|\frac{u_1^{hk} - u_0^{hk}}{k}\right\|_0^2 + \|u_1^{hk}\|_{a_h}^2 + \|u_0^{hk}\|_{a_h}^2 + k\sum_{i=1}^n \|f_i\|_0^2\right). \tag{3.10}$$

Proof: It is easy to show that Problem 3.1 has a unique solution. Take $v^h = u^{hk}_{n+1} - u^{hk}_{n-1} \in V^h$ in (3.5),

$$\frac{1}{k^2}\left(u^{hk}_{n+1} - 2\,u^{hk}_n + u^{hk}_{n-1}, \quad u^{hk}_{n+1} - u^{hk}_{n-1}\right) + a_h\left(\frac{u^{hk}_{n+1} + u^{hk}_{n-1}}{2}, \quad u^{hk}_{n+1} - u^{hk}_{n-1}\right) = (f_n, u^{hk}_{n+1} - u^{hk}_{n-1}),$$

which can be rewritten as

$$\left\|\frac{u^{hk}_{n+1} - u^{hk}_n}{k}\right\|_0^2 - \left\|\frac{u^{hk}_n - u^{hk}_{n-1}}{k}\right\|_0^2 + \frac{1}{2}\left(a_h(u^{hk}_{n+1}, \quad u^{hk}_{n+1}) - a_h(u^{hk}_{n-1}, u^{hk}_{n-1})\right) = (f_n, u^{hk}_{n+1} - u^{hk}_{n-1}).$$

We change n to i and sum on the relation for $1 \le i \le n$,

$$\begin{aligned}&\left\|\frac{u^{hk}_{n+1} - u^{hk}_n}{k}\right\|_0^2 + \frac{1}{2}\left(a_h(u^{hk}_{n+1}, u^{hk}_{n+1}) + a_h(u^{hk}_n, u^{hk}_n)\right)\\ &= \left\|\frac{u^{hk}_1 - u^{hk}_0}{k}\right\|_0^2 + \frac{1}{2}(\|u^{hk}_1\|^2_{a_h} + \|u^{hk}_0\|^2_{a_h}) + k\sum_{i=1}^{n}\left(f_i, \frac{u^{hk}_{i+1} - u^{hk}_i}{k} + \frac{u^{hk}_i - u^{hk}_{i-1}}{k}\right).\end{aligned}$$

Applying Cauchy–Schwarz inequality and the inequality $(a+b)^2 \le 2\,(a^2 + b^2)$ to the right summing term in the above equality, we have

$$\begin{aligned}&\left\|\frac{u^{hk}_{n+1} - u^{hk}_n}{k}\right\|_0^2 + \frac{1}{2}\left(a_h(u^{hk}_{n+1}, u^{hk}_{n+1}) + a_h(u^{hk}_n, u^{hk}_n)\right)\\ &\le \left\|\frac{u^{hk}_1 - u^{hk}_0}{k}\right\|_0^2 + \frac{1}{2}(\|u^{hk}_1\|^2_{a_h} + \|u^{hk}_0\|^2_{a_h}) + k\sum_{i=1}^{n}\|f_i\|_0\left(\left\|\frac{u^{hk}_{i+1} - u^{hk}_i}{k}\right\|_0 + \left\|\frac{u^{hk}_i - u^{hk}_{i-1}}{k}\right\|_0\right)\\ &\le \left\|\frac{u^{hk}_1 - u^{hk}_0}{k}\right\|_0^2 + \frac{1}{2}(\|u^{hk}_1\|^2_{a_h} + \|u^{hk}_0\|^2_{a_h}) + k\sum_{i=1}^{n}\frac{1}{2}\|f_i\|_0^2\\ &\quad + k\sum_{i=1}^{n}\left(\left\|\frac{u^{hk}_{i+1} - u^{hk}_i}{k}\right\|_0^2 + \left\|\frac{u^{hk}_i - u^{hk}_{i-1}}{k}\right\|_0^2\right).\end{aligned}$$

For $k \le 1/2$, we can apply Lemma 2.3 to the above inequality to get

$$\left\|\frac{u^{hk}_{n+1} - u^{hk}_n}{k}\right\|_0^2 + a_h(u^{hk}_n, u^{hk}_n) \le C\left(\left\|\frac{u^{hk}_1 - u^{hk}_0}{k}\right\|_0^2 + \|u^{hk}_1\|^2_{a_h} + \|u^{hk}_0\|^2_{a_h} + k\sum_{i=1}^{n}\|f_i\|_0^2\right).$$

So (3.9) and (3.10) hold. ■

4. Error estimates

In this section, we shall present optimal $H^1(\Omega)$-like norm error estimates from the truncation errors and the $L^2(\Omega)$ norm optimal order error estimates by Aubin-Nitsche technique for the fully discrete schemes. This technique used here is a kind of improvement and development for the existing paper (cf. [9,10,21]).

Theorem 4.1: *Let u and u^{hk} be the solutions of Problem* 2.1 *and Problem* 3.1, *respectively. Assume* $u \in C^2([0,T];H^{p+1}(\Omega)), \partial_t^3 u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \partial_t^4 u \in L^2(0,T;L^2(\Omega))$. *Then the following error bound holds*

$$\max_{0\le n\le N-1} k^{-1}\|(u_{n+1}-u_{n+1}^{hk})-(u_n-u_n^{hk})\|_0 + \max_{0\le n\le N-1}\|u_n-u_n^{hk}\|_h \le c\,h^p\|u\|_{C^2([0,T];H^{p+1}(\Omega))}$$
$$+ c\,k^2\left(\|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))} + \|\partial_t^3 u\|_{C([0,T];L^2(\Omega))} + \|\partial_t^3 u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))}\right) \tag{4.1}$$

where the constant $c > 0$ is independent of the mesh size h and the time step k.

Proof: Write the error $e_n = u_n - u_n^{hk}$ at time t_n as $e_n = e_{1,n} + e_{2,n}$ with $e_{1,n} = u_n - \Pi^h u_n$ and $e_{2,n} = \Pi^h u_n - u_n^{hk}$. As in [21], for $0 \le n \le N-1$, we have the error decomposition

$$k^{-1}\|(u_{n+1}-u_{n+1}^{hk})-(u_n-u_n^{hk})\|_0 \le k^{-1}\|e_{1,n+1}-e_{1,n}\|_0 + k^{-1}\|e_{2,n+1}-e_{2,n}\|_0, \tag{4.2}$$

$$\|u_n-u_n^{hk}\|_h \le \|e_{1,n}\|_h + \|e_{2,n}\|_h. \tag{4.3}$$

First, we bound the first term in (4.2) and (4.3), respectively. Note that

$$e_{1,n+1}-e_{1,n} = \left(I-\Pi^h\right)u_{n+1} - \left(I-\Pi^h\right)u_n = \int_{t_n}^{t_{n+1}}\left(I-\Pi^h\right)\partial_t u(\cdot,s)\,\mathrm{d}s$$

and by (3.4),

$$k^{-1}\|e_{1,n+1}-e_{1,n}\|_0 \le k^{-1}\int_{t_n}^{t_{n+1}}\|\left(I-\Pi^h\right)\partial_t u(\cdot,s)\|_0\,\mathrm{d}s \le c\,h^{p+1}\|\partial_t u\|_{C([0,T];H^{p+1}(\Omega))}, \tag{4.4}$$

$$\|e_{1,n}\|_h = \|u_n - \Pi^h u_n\|_h \le c\,h^p\|u_n\|_{p+1} \le c\,h^p\|u\|_{C([0,T];H^{p+1}(\Omega))}. \tag{4.5}$$

Next, we estimate the combined error bound $k^{-1}\|e_{2,n+1}-e_{2,n}\|_0 + \|e_{2,n}\|_h \le c\,(h^{p+1}+k^2)$ in (4.2) and (4.3). Denote

$$r_n = d_k u_n - \partial_t^2 u_n - \frac{k^2}{2}\nabla\cdot(b\,\nabla d_k u_n), \quad n = 1,2,\dots,N-1,$$

where r_n is the truncation errors due to the time discretization. With the help of Taylor's formula with an integral remainder (cf. [10])

$$d_k u_n = \frac{1}{k^2}\int_{t_{n-1}}^{t_{n+1}}(k-|s-t_n|)\,\partial_t^2 u(\cdot,s)\,\mathrm{d}s, \tag{4.6}$$

and

$$d_k u_n - \partial_t^2 u_n = \frac{1}{6k^2}\int_{t_{n-1}}^{t_{n+1}}(k-|t_n-s|)^3\partial_t^4 u(\cdot,s)\,\mathrm{d}s.$$

Therefore, under the stated regularity condition, we have

$$\begin{aligned}\|r_n\|_0 &\le \|d_k u_n - \partial_t^2 u_n\|_0 + \frac{k^2}{2}\|\nabla\cdot(b\,\nabla d_k u_n)\|_0\\ &\le \frac{k}{6}\int_{t_{n-1}}^{t_{n+1}}\|\partial_t^4 u(\cdot,s)\|_0\,\mathrm{d}s + \frac{k}{2}\int_{t_{n-1}}^{t_{n+1}}\|\nabla\cdot(b\,\nabla\partial_t^2 u)\|_0\,\mathrm{d}s\\ &\le c\,k\int_{t_{n-1}}^{t_{n+1}}\|\partial_t^4 u(\cdot,s)\|_0\,\mathrm{d}s + c\,k\int_{t_{n-1}}^{t_{n+1}}\|\partial_t^2 u(\cdot,s)\|_{H^2}\,\mathrm{d}s\\ &\le c\,k\int_{t_{n-1}}^{t_{n+1}}\left(\|\partial_t^4 u(\cdot,s)\|_0 + \|\partial_t^2 u(\cdot,s)\|_{H^2}\right)\,\mathrm{d}s.\end{aligned} \tag{4.7}$$

According to the truncation errors r_n,

$$(d_k u_n, v^h) + a_h(\bar{\delta}_k u_n, v^h) = (f_n + r_n, v^h) \quad \forall\, v^h \in V^h,\ n = 1, 2, \ldots, N-1. \tag{4.8}$$

Subtracting (3.5) from (4.8), we have

$$(d_k u_n - d_k u_n^{hk}, v^h) + a_h(\bar{\delta}_k u_n - \bar{\delta}_k u_n^{hk}, v^h) = (r_n, v^h) \quad \forall\, v^h \in V^h.$$

Since $a_h(u_n - \Pi^h u_n, v^h) = 0$, by the definition of the Galerkin projection, we obtain

$$(d_k e_{2,n}, v^h) + a_h(\bar{\delta}_k e_{2,n}, v^h) = (r_n - d_k e_{1,n}, v^h) \quad \forall\, v^h \in V^h,\ n = 1, 2, \ldots, N-1. \tag{4.9}$$

Taking $v^h = \delta_k e_{2,n}$ in (4.9) and by the definition of symbols $\bar{\delta}_k$, δ_k, d_k, then

$$\begin{aligned}
&\left(\frac{e_{2,n+1} - 2e_{2,n} + e_{2,n-1}}{k^2}, \frac{e_{2,n+1} - e_{2,n-1}}{2k}\right) + a_h\left(\frac{e_{2,n+1} + e_{2,n-1}}{2}, \frac{e_{2,n+1} - e_{2,n-1}}{2k}\right)\\
&\quad = \frac{1}{2}\left(r_n - d_k e_{1,n}, \frac{e_{2,n+1} - e_{2,n-1}}{k}\right).
\end{aligned}$$

By adding and subtracting the factor $e_{2,n}$ and multiplying both side $2k$,

$$\begin{aligned}
&\left\|\frac{e_{2,n+1} - e_{2,n}}{k}\right\|_0^2 - \left\|\frac{e_{2,n} - e_{2,n-1}}{k}\right\|_0^2 + \frac{1}{2}\left(\|e_{2,n+1}\|_{a_h}^2 - \|e_{2,n-1}\|_{a_h}^2\right)\\
&\quad = k\left(r_n - d_k e_{1,n}, \frac{e_{2,n+1} - e_{2,n-1}}{k}\right).
\end{aligned}$$

Change n to j and make a summation of the relation for $j = 1, 2, \ldots, n-1$,

$$\left\|\frac{e_{2,n} - e_{2,n-1}}{k}\right\|_0^2 - \left\|\frac{e_{2,1} - e_{2,0}}{k}\right\|_0^2 + \frac{1}{2}(\|e_{2,n}\|_{a_h}^2 + \|e_{2,n-1}\|_{a_h}^2 - \|e_{2,1}\|_{a_h}^2 - \|e_{2,0}\|_{a_h}^2) = \text{RHS}, \tag{4.10}$$

where

$$\begin{aligned}
\text{RHS} &= \sum_{j=1}^{n-1} k\left(r_j - d_k e_{1,j}, \frac{e_{2,j+1} - e_{2,j}}{k} + \frac{e_{2,j} - e_{2,j-1}}{k}\right)\\
&\le k\sum_{j=1}^{n-1}\|r_j\|_0^2 + k\sum_{j=1}^{n-1}\|d_k e_{1,j}\|_0^2 + k\sum_{j=1}^{n-1}\left(\left\|\frac{e_{2,j+1} - e_{2,j}}{k}\right\|_0^2 + \left\|\frac{e_{2,j} - e_{2,j-1}}{k}\right\|_0^2\right).
\end{aligned}$$

By applying Cauchy–Schwartz inequality, $ab \le \frac{1}{2}(a^2 + b^2)$, and $(a+b)^2 \le 2(a^2 + b^2)$ to RHS and from (4.10), we know that

$$\begin{aligned}
&\left\|\frac{e_{2,n} - e_{2,n-1}}{k}\right\|_0^2 + \frac{1}{2}\|e_{2,n}\|_{a_h}^2 + \frac{1}{2}\|e_{2,n-1}\|_{a_h}^2\\
&\quad \le \left\|\frac{e_{2,1} - e_{2,0}}{k}\right\|_0^2 + \frac{1}{2}\|e_{2,1}\|_{a_h}^2 + \frac{1}{2}\|e_{2,0}\|_{a_h}^2 + \sum_{j=1}^{n-1} k\|r_j\|_0^2 + \sum_{j=1}^{n-1} k\|d_k e_{1,j}\|_0^2\\
&\qquad + k\sum_{j=1}^{n-1}\left(\left\|\frac{e_{2,j+1} - e_{2,j}}{k}\right\|_0^2 + \left\|\frac{e_{2,j} - e_{2,j-1}}{k}\right\|_0^2\right).
\end{aligned} \tag{4.11}$$

Apply Lemma 2.3 to (4.11),

$$\max_n \left\| \frac{e_{2,n} - e_{2,n-1}}{k} \right\|_0 + \max_n \|e_{2,n}\|_{a_h}$$
$$\leq c \left(\|e_{2,0}\|_{a_h} + \|e_{2,1}\|_{a_h} + \left\| \frac{e_{2,1} - e_{2,0}}{k} \right\|_0 + \left(k \sum_{j=1}^{N-1} \|d_k e_{1,j}\|_0^2 \right)^{1/2} + \left(k \sum_{j=1}^{N-1} \|r_j\|_0^2 \right)^{1/2} \right). \tag{4.12}$$

Note that $e_{2,0} = \Pi^h u_0 - u_0^{hk} = 0$, thus,

$$\|e_{2,0}\|_{a_h} = 0. \tag{4.13}$$

Also note that $e_{2,1} = \Pi^h u_1 - u_1^{hk}$ and $u(x,0) = u_0, \partial_t u(x,0) = v_0$. From Taylor's expansion, we have

$$u_1 = u_0 + kv_0 + \frac{k^2}{2}\partial_t^2 u(x,0) + \frac{1}{2}\int_0^{t_1} (t_1 - s)^2 \partial_t^3 u(\cdot, s)\, \mathrm{d}s.$$

Thus, by a_h-consistency (3.2) and (3.8) and noting (3.3), we have

$$\|e_{2,1}\|_{a_h} \leq c\, k^2 \left(\|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))} + \|\partial_t^3 u\|_{L^2(0,T;H^2(\Omega))} \right). \tag{4.14}$$

Consequently,

$$\left\| \frac{e_{2,1} - e_{2,0}}{k} \right\|_0 = \frac{1}{k}\|e_{2,1}\|_0 \leq c\, h^{p+1} \|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))} + c\, k^2 \|\partial_t^3 u\|_{C([0,T];L^2(\Omega))}. \tag{4.15}$$

By (4.6)

$$\|d_k e_{1,j}\|_0 = \|d_k (I - \Pi^h) u_j\|_0 \leq \frac{1}{k}\int_{t_{j-1}}^{t_{j+1}} \| \left(I - \Pi^h \right) \partial_t^2 u(\cdot, s)\|_0 \,\mathrm{d}s \leq \frac{c\, h^{p+1}}{k} \int_{t_{j-1}}^{t_{j+1}} \|\partial_t^2 u(\cdot, s)\|_{p+1} \mathrm{d}s.$$

So

$$\left(k \sum_{j=1}^{n-1} \|d_k e_{1,j}\|_0^2 \right)^{1/2} \leq c\, h^{p+1} \|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))}. \tag{4.16}$$

By (4.7)

$$\left(k \sum_{n=1}^{N-1} \|r_n\|_0^2 \right)^{1/2} \leq c\, k^2 \left(\|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t^2 u\|_{L^2(0,T;H^2(\Omega))} \right), \tag{4.17}$$

Collecting (4.12)–(4.17), we have

$$\max_n \left\| \frac{e_{2,n} - e_{2,n-1}}{k} \right\|_0 + \max_n \|e_{2,n}\|_{a_h} \leq c\, h^{p+1} \|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))}$$
$$+ c\, k^2 \left(\|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))} + \|\partial_t^3 u\|_{C([0,T];L^2(\Omega))} + \|\partial_t^3 u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))} \right). \tag{4.18}$$

Using (4.2)–(4.5) and (4.18), we arrive at the error bound (4.1). ∎

Next, we present the optimal L^2 error estimate for the fully discrete scheme by Aubin-Nitsche technique.

Theorem 4.2: *Let u and u^{hk} be the solutions of Problem 2.1 and Problem 3.1, respectively. Assume $u \in C^2([0,T];H^{p+1}(\Omega))$, $\partial_t^3 u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, and $\partial_t^4 u \in L^2(0,T;L^2(\Omega))$. Then, we have the following error bound*

$$\max_{0\le n\le N-1} \|u_n - u_n^{hk}\|_0 \le c\,h^{p+1}\|u\|_{C^2([0,T];H^{p+1}(\Omega))} + c\,k^2\left(\|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))} + \|\partial_t^3 u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))}\right), \tag{4.19}$$

where the constant $c > 0$ is independent of the mesh size h and the time step k.

Proof: We consider the dual problem: find $\psi \in H^2(\Omega) \cap V$ solution of

$$-\nabla\cdot(b\,\nabla\psi) = u_n - u_n^{hk} \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega.$$

From the assunptions on the domain and b is smooth, the elliptic regularity theory gives the inequality $\|\psi\|_2 \le c\|u_n - u_n^{hk}\|_0$. Moreover, the four DG bilinear forms considered in this paper are adjoint

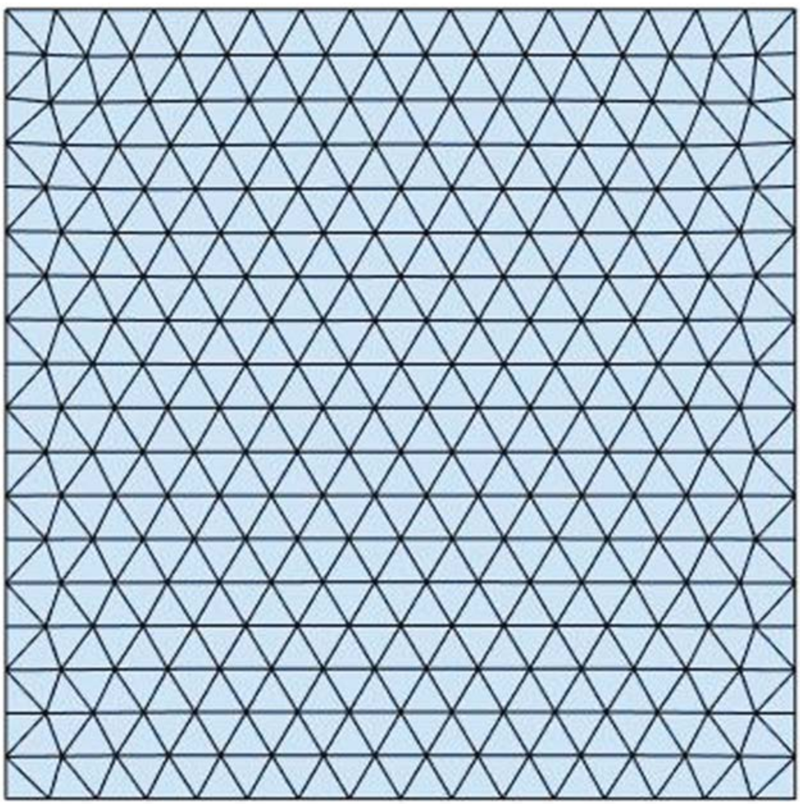

Figure 1. A quasi-uniform triangulation partition of the domain with $h = 1/16$.

consistent. This implies that for any $\psi^h \in V^h$,

$$\begin{aligned}\|u_n - u_n^{hk}\|_0^2 &= a_h(u_n - u_n^{hk}, \psi) \\ &= a_h\left(u_n - \Pi^h u_n, \psi - \psi^h\right) + a_h\left(\Pi^h u_n - u_n^{hk}, \psi\right) \\ &\leq \|u_n - \Pi^h u_n\|_{a_h} \|\psi - \psi^h\|_{a_h} + \|\Pi^h u_n - u_n^{hk}\|_{a_h} \|\psi\|_{a_h}. \end{aligned} \tag{4.20}$$

Choosing ψ^h to be the L^2-projection in (4.20) and noting (3.3) and (3.4),

$$\begin{aligned}\|u_n - u_n^{hk}\|_0^2 &\leq c\, h^{p+1} \|u_n\|_{p+1} \|\psi\|_2 + \|e_{2,n}\|_{a_h} \|\psi\|_2 \\ &\leq c\, h^{p+1} (\|u_n\|_{p+1} + \|e_{2,n}\|_{a_h}) \|u_n - u_n^{hk}\|_0. \end{aligned} \tag{4.21}$$

Table 1. Numerical convergence orders in H^1 norm at $t = 1$ for $p = 1,2,3$.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
2^{-3}	2.5142	–	3.7461e−01	–	4.6749e−02	–
2^{-4}	7.0173e−01	1.8411	9.3429e−02	2.0034	4.9951e−03	3.2264
2^{-5}	2.8794e−01	1.2851	2.2917e−02	2.0275	5.6842e−04	3.1355
2^{-6}	1.3810e−01	1.0601	5.6047e−03	2.0317	6.7242e−05	3.0795
2^{-7}	7.2788e−02	0.9239	1.4156e−03	1.9852	8.3929e−06	3.0021

Table 2. Numerical convergence orders in L^2 norm at $t = 1$ for $p = 1,2,3$.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
2^{-3}	1.9391e−01	–	6.0501e-03	–	7.8546e-04	–
2^{-4}	4.2349e−02	2.1950	6.5649e−04	3.2041	4.4895e−05	4.1289
2^{-5}	9.1337e−03	2.2131	7.8248e−05	3.0686	2.6909e−06	4.0604
2^{-6}	2.1348e−03	2.0971	9.1213e−06	3.1007	1.5777e−07	4.0922
2^{-7}	5.2088e−04	2.0351	9.6664e−07	3.2382	9.6193e−09	4.0357

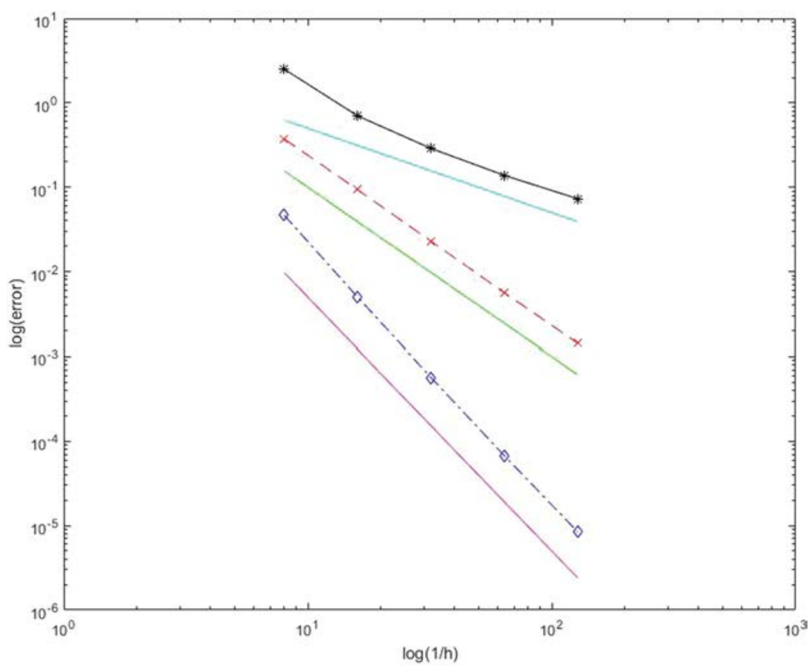

Figure 2. Numerical convergence orders in H^1 norm at $t = 1$ for $p = 1,2,3$.

Combining with (4.18), we yield

$$\|u_n - u_n^{hk}\|_0 \le c\,h^{p+1}(\|u\|_{C([0,T];H^{p+1}(\Omega))} + \|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))}) \\ + c\,k^2\left(\|\partial_t^2 u\|_{C([0,T];H^{p+1}(\Omega))} + \|\partial_t^3 u\|_{C([0,T];L^2(\Omega))} + \|\partial_t^3 u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))}\right).$$

Then, we finish the proof of Theorem 4.2. ■

5. Numerical results

In this section, we present numerical results to illustrate the efficiency and accuracy of the proposed numerical schemes for the polynomial degrees $p = 1,2,3$. Here, we consider the initial-boundary value Problem 2.1 with a spatial domain $\Omega := [0,1]^2$. Let $b = 1$ and choose the exact solution

$$u(x,y,t) = e^{-t/2}\sin(2\pi x)\sin(4\pi y)$$

with

$$u_0(x,y) = \sin(2\pi x)\sin(4\pi y),$$
$$v_0(x,y) = -\tfrac{1}{2}\sin(2\pi x)\sin(4\pi y).$$

Determining the source function $f(x,y,t) = \left(\frac{1}{4} + 20\pi^2\right)e^{-t/2}\sin(2\pi x)\sin(4\pi y)$ from equation (2.1) and taking fully discrete initial values defined by (3.6)–(3.8).

We make use of distmesh software in MATLAB to get a sequence of quasi-uniform triangulations $\mathcal{T}_h$ shown in Figure 1 to partition $\overline{\Omega}$. The second-order wave equation is discretized by the fully discrete scheme (3.5)–(3.7) with the IPDG method and the penalty parameter $\eta_e = 300\,(p+1)^2$.

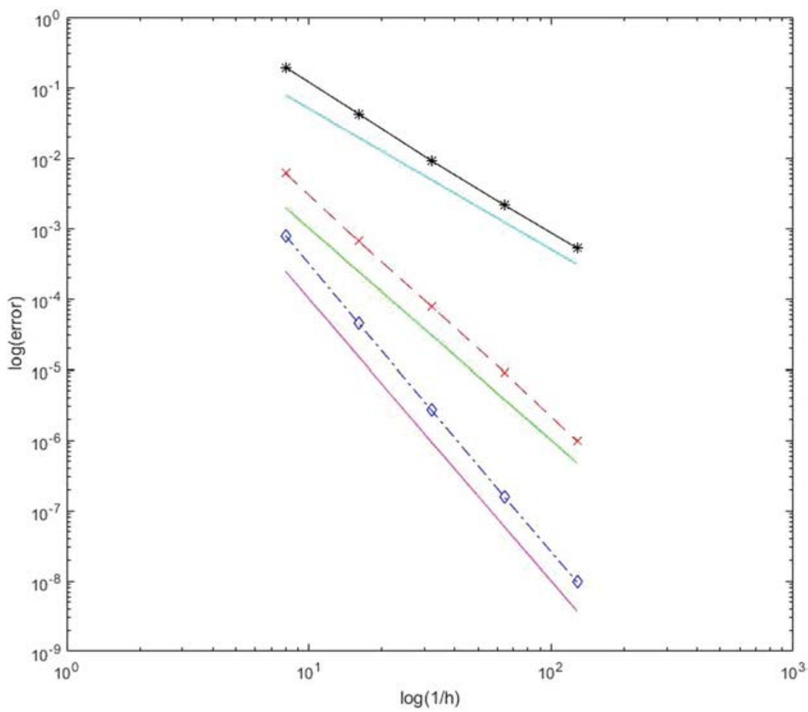

Figure 3. Numerical convergence orders in L^2 norm at $t = 1$ for $p = 1,2,3$.

To illustrate the dependence of the numerical errors on the mesh size h, taking the time step $k = 1 \times 10^{-3}$ for $p = 1$, $k = \frac{1}{2} \times 10^{-3}$ for $p = 2$ and $k = 1 \times 10^{-4}$ for $p = 3$, we list $H^1(\Omega)$ and $L^2(\Omega)$ numerical errors and convergence orders of space in Tables 1 and 2 with varying spacing $h = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$ at $t = 1$. The numerical convergence orders are also shown in Figures 2 and 3. We observe that the numerical convergence orders for H^1 norm and L^2 norm are around p and $p+1$, respectively, which confirm well with the theoretical results. Moreover, we provide plots of the exact solution u at $t = 1$ and the numerical solution at $t = 1$ for $p = 3$ and $h = 1/128$ in Figures 4 and 5. It implies that the numerical solutions approximate the exact solution well.

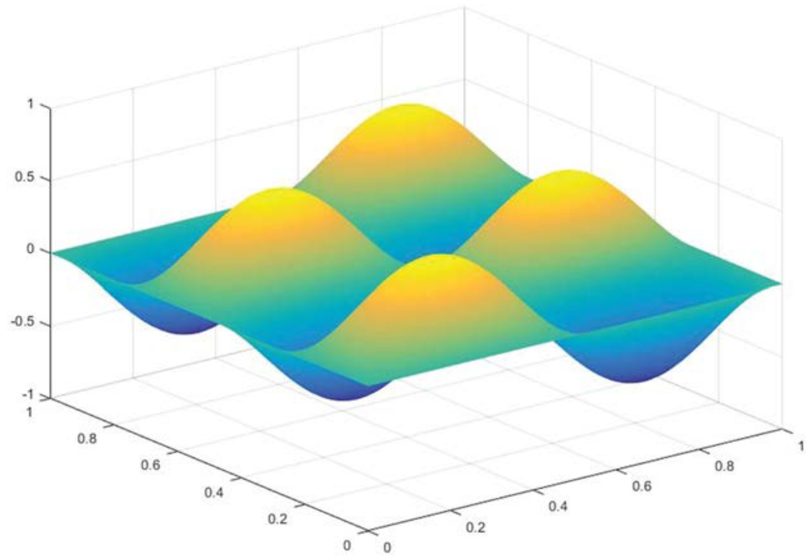

Figure 4. The exact solution at $t = 1$.

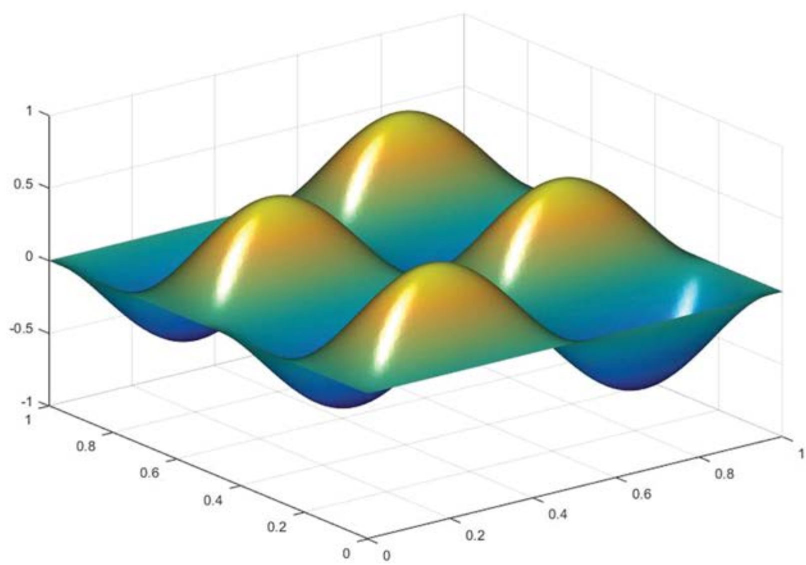

Figure 5. The numerical solution at $t = 1$ for $p = 3$ and $h = 1/128$.

Table 3. Numerical convergence orders at $t = 1.0$ with respect to the time step k for $p = 1$.

(h, k)	L^2 errors	Order	(h, k)	H^1 errors	Order
$(1/64, 1/64)$	1.9318e−03	–	$(1/100, 1/10)$	8.6080e−02	–
$(1/128, 1/128)$	5.0956e−04	1.9226	$(1/144, 1/12)$	6.3890e−02	1.6351
$(1/256, 1/256)$	1.2864e−04	1.9859	$(1/256, 1/16)$	3.6228e−02	1.9721

Next, we examine the orders of convergence with respect to the time step size k for linear element ($p=1$). We take $h = O(k)$ for $L^2(\Omega)$ norm and $h = O(k^2)$ for $H^1(\Omega)$ norm, the results are depicted in Table 3. We observe that the numerical convergence orders are nearly 2 with respect to k, which supports the theoretical results in Theorems 4.1 and 4.2.

Disclosure statement

No potential conflict of interest was reported by the authors.

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