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A new C^0 discontinuous Galerkin method for Kirchhoff plates

Jianguo Huang [a,b,c], Xuehai Huang [a], Weimin Han [d,*]

[a] Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China
[b] Division of Computational Science, E-Institute of Shanghai Universities, Shanghai Normal University, China
[c] Scientific Computing Key Laboratory of Shanghai Universities, Shanghai Normal University, China
[d] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

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ABSTRACT

A general framework of constructing C^0 discontinuous Galerkin (CDG) methods is developed for solving the Kirchhoff plate bending problem, following some ideas in (Castillo et al., 2000) [10] and (Cockburn, 2003) [12]. The numerical traces are determined based on a discrete stability identity, which lead to a class of stable CDG methods. A stable CDG method, called the LCDG method, is particularly interesting in our study. It can be viewed as an extension to fourth-order problems of the LDG method studied in (Castillo et al., 2000) [10] and (Cockburn, 2003) [12]. For this method, optimal order error estimates in certain broken energy norm and H^1-norm are established. Some numerical results are reported, confirming the theoretical convergence orders.

1. Introduction

In the past two decades, discontinuous Galerkin (DG) methods have been widely used for solving many kinds of mathematical and physical problems, including linear and non-linear hyperbolic problems, Navier–Stokes equations, convection-dominated diffusion problems and so on, due to the flexibility of constructing feasible local shape function spaces and the advantage to capture non-smooth or oscillatory solutions effectively. We refer to [13] for an excellent historical survey along this line. In particular, a very extensive and thorough study has been developed for solving second-order equations/systems by DG methods. A unified error analysis of DG methods for second-order elliptic equations was established in [2]; a new framework was proposed in [8] for designing and analyzing DG methods, and the stabilization mechanism frequently used in DG methods was also investigated there. The local discontinuous Galerkin (LDG) methods were introduced and analyzed in [14,10] for time-dependent convection-diffusion systems and second-order elliptic problems, respectively. In [12] a basic procedure was proposed for constructing stable DG methods, which consists of first deriving a discrete formulation involving numerical traces by integration by parts and then determining the traces over edges/faces by means of a discrete stability identity. Some DG methods were presented for solving Friedrichs' systems in [17,18].

To the best of our knowledge, there are few results about DG methods for the biharmonic equation and the Kirchhoff plate bending problems. Because of the fourth-order nature of the partial differential equations, it is more difficult to construct stable DG methods for such problems and the major known methods in the literature are interior penalty (IP) methods (cf. [3,4,7,16, 24–26,28]). In [16], a C^0 IP formulation was presented for Kirchhoff plates and quasi-optimal error estimates were obtained for smooth functions. In [7], rigorous error analysis for the previous method was given under suitable weak regularity assumption on the solution (cf. [15,21]), and a post-processing procedure was formulated that can generate C^1 approximate solutions from the C^0 approximate solutions. A drawback of the forgoing method is the presence of a dimensionless penalty parameter which must be chosen suitably large to guarantee stability, but it can not be precisely quantified *a priori*. Based on this observation, a C^0 DG (CDG) method was introduced in [29] for which the stability condition can be precisely quantified. The fully discontinuous IP method was investigated systematically in [24–26,28] for biharmonic problems, where the subdivision mesh size and the polynomial degree on individual elements can vary arbitrarily, very suitable for the design of *hp*-adaptive algorithms.

In this paper, we intend to develop some new CDG methods for solving the Kirchhoff plate bending problem. We first write the original fourth-order partial differential equation as a second-order system and follow some ideas presented in [10,12] to obtain a framework of constructing CDG methods for solving the original problem. Then, we establish a discrete stability identity, from

* Corresponding author.
E-mail address: whan@math.uiowa.edu (W. Han).

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which we derive feasible choices of numerical traces and get a class of stable CDG methods for Kirchhoff plates. One particular method, called LCDG method, can be viewed as an extension of the LDG method in [10,12] for fourth-order problems. Comparing the formulations of the LCDG method and the method in [29], we find the former is more convenient to implement in actual computation than the latter. Moreover, the LCDG method does not contain any parameter which can not be quantified *a priori*. It only requires to choose $C_{11} = \eta_e h_e^{-1}$ on each edge $e \in \mathscr{E}_h$, with $\{\eta_e\}_{e\in\mathscr{E}_h}$ having a uniform positive bound from above and below, which leads to optimal error estimates (see Section 3 for details). We also show that some existing CDG methods in [7,29] can be obtained from our framework under proper choices of numerical traces. Following some ideas on error analysis of DG methods for second-order elliptic problems (cf. [2]) and detailed technical argument, we derive optimal order error estimates in certain broken energy norm and H^1-norm for the LCDG method. Some numerical results are included to confirm our theoretical convergence orders.

We point out that numerous non-conforming finite elements were developed for solving the Kirchhoff plate bending problem, such as the continuous non-conforming elements of Adini's rectangle (cf. [1]) and Zienkiewicz triangle (cf. [5]), the discontinuous non-conforming elements of Morley's triangle (cf. [23]) and Fraeijs De Veubeke triangle (cf. [20]), and discrete Kirchhoff methods where the Kirchhoff-theory constraint of zero transverse shear strains are required at some discrete points within the plate elements (see, e.g., [22]). However, all these elements are of low order, and in comparison, the DG approach provides an effective way to construct higher order elements for plate bending problems and other fourth-order problems.

The rest of this paper is organized as follows. The basic framework of our CDG methods for Kirchhoff plate bending problems, the determination of numerical traces based on the discrete stability identity, and some other methods are presented in Section 2. Error analysis for the LCDG method is given in Section 3. In Section 4, some numerical results are reported to show the performance of the LCDG method.

2. The CDG method for Kirchhoff plates

2.1. Basic framework for the CDG method

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in L^2(\Omega)$. The mathematical model of a clamped Kirchhoff plate under a vertical load $f \in L^2(\Omega)$ reads (cf. [19,27])

$$\begin{cases} \nabla\cdot(\nabla\cdot\mathscr{M}(u)) + f = 0 & \text{in } \Omega, \\ u = \partial_{\boldsymbol{N}} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\boldsymbol{N}$ is the unit outward normal to $\partial\Omega, \nabla$ is the usual gradient operator, $\nabla\cdot$ stands for the divergence operator acting on tensor-valued or vector-valued functions (cf. [27]), and

$$\mathscr{M}(u) := (1-\nu)\mathscr{K}(u) + \nu\,\mathrm{tr}(\mathscr{K}(u))\mathscr{I},$$

$$\mathscr{K}(u) := (\mathscr{K}_{ij}(u))_{2\times 2}, \quad \mathscr{K}_{ij}(u) := -\partial_{ij}u, \quad 1 \leqslant i,\ j \leqslant 2$$

with $\mathscr{I}$ a second-order identity tensor, tr the trace operator acting on second-order tensors, and $\nu \in (0, 0.5)$ the Poisson ratio of an elastic thin plate occupying the region Ω. Here, we have normalized the rigid flexibility D to simplify the presentation. In fact, all derivations developed in what follows may be extended to the general case after some straightforward modifications.

Introduce an auxiliary tensor-valued function by

$$\boldsymbol{\sigma} := (1-\nu)\mathscr{K}(u) + \nu\,\mathrm{tr}(\mathscr{K}(u))\mathscr{I}. \tag{2.2}$$

Then, problem (2.1) can be reformulated as the following second-order system:

$$\begin{cases} \frac{1}{1-\nu}\boldsymbol{\sigma} - \frac{\nu}{1-\nu^2}(\mathrm{tr}\boldsymbol{\sigma})\mathscr{I} = \mathscr{K}(u) & \text{in } \Omega, \\ \nabla\cdot(\nabla\cdot\boldsymbol{\sigma}) = -f & \text{in } \Omega, \\ u = \partial_{\boldsymbol{N}} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We will define the CDG method for solving problem (2.1) based on the system (2.3). For this, we first introduce some notation frequently used later on. For any Banach space B for components of second-order tensor-valued functions, the subspace of symmetric tensor-valued functions is denoted $(B)^s_{2\times 2}$. Given a bounded domain $G \subset \mathbb{R}^2$ and a non-negative integer m, let $H^m(G)$ be the usual Sobolev space of functions on G. The corresponding norm and semi-norm are denoted respectively by $\|\cdot\|_{m,G}$ and $|\cdot|_{m,G}$. If G is Ω, we abbreviate them by $\|\cdot\|_m$ and $|\cdot|_m$, respectively. Let $H_0^m(G)$ be the closure of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{m,G}$.

Let $\{\mathscr{T}_h\}_{h>0}$ be a regular family of triangulations of Ω (cf. [6,11]); $h := \max_{K\in\mathscr{T}_h} h_K$ and $h_K := \mathrm{diam}(K)$. Let $\mathscr{E}_h$ be the union of all edges of the triangulation $\mathscr{T}_h$ and $\mathscr{E}_h^i$ the union of all interior edges of the triangulation $\mathscr{T}_h$. For any $e \in \mathscr{E}_h$, denote by h_e its length. Based on the triangulation $\mathscr{T}_h$, let

$$\boldsymbol{\Sigma} := \left\{\tau \in (L^2(\Omega))^s_{2\times 2} : \tau_{ij}|_K \in H^1(K)\ \forall\, K \in \mathscr{T}_h,\ i,j = 1,2\right\},$$
$$V := \left\{v \in H_0^1(\Omega) : v|_K \in H^2(K)\ \forall\, K \in \mathscr{T}_h\right\}.$$

The corresponding finite element spaces are given by

$$\boldsymbol{\Sigma}_h := \left\{\tau \in (L^2(\Omega))^s_{2\times 2} : \tau_{ij}|_K \in \mathscr{S}_1(K)\ \forall\, K \in \mathscr{T}_h,\ i,j = 1,2\right\},$$
$$V_h := \left\{v \in H_0^1(\Omega) : v|_K \in \mathscr{S}_2(K)\ \forall\, K \in \mathscr{T}_h\right\},$$

where for a triangle $K \in \mathscr{T}_h, \mathscr{S}_1(K)$ and $\mathscr{S}_2(K)$ are two finite-dimensional spaces of polynomials in K containing $P_l(K)$ and $P_k(K)$, respectively, with $l \geqslant 0$ and $k \geqslant 1$. Here, for a non-negative integer $m, P_m(K)$ stands for the set of all polynomials in K with the total degree no more than m.

To guarantee uniqueness of the solution to the CDG method to be proposed, we always assume that

$$\nabla_h^2 V_h \subset \boldsymbol{\Sigma}_h, \quad \frac{1}{1-\nu}\boldsymbol{\Sigma}_h - \frac{\nu}{1-\nu^2}(\mathrm{tr}\boldsymbol{\Sigma}_h)\mathscr{I} \subset \boldsymbol{\Sigma}_h, \tag{2.4}$$

where $\nabla_h^2 V_h|_K := \nabla^2(V_h|_K)$ for any $K \in \mathscr{T}_h$. For a function $v \in L^2(\Omega)$ with $v|_K \in H^m(K)$ for all $K \in \mathscr{T}_h$, let $\|v\|_{m,h}$ and $|v|_{m,h}$ be the usual broken H^m-type norm and semi-norm of v:

$$\|v\|_{m,h} = \left(\sum_{K\in\mathscr{T}_h}\|v\|_{m,K}^2\right)^{1/2}, \quad |v|_{m,h} = \left(\sum_{K\in\mathscr{T}_h}|v|_{m,K}^2\right)^{1/2}.$$

If v is a vector-valued or tensor-valued function, the above symbols are defined in the similar manners. For a vector or tensor $\boldsymbol{v}$, its length $|\boldsymbol{v}|$ is $(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}$ or $(\boldsymbol{v}:\boldsymbol{v})^{1/2}$. Here, the symbol: denotes the double dot product operation of tensors. Throughout this paper, we also use "$\lesssim \cdots$" to mean that "$\leqslant C\cdots$", where C is a generic positive constant independent of h and other parameters, which may take different values at different appearances.

Consider two adjacent triangles K^+ and K^- sharing an interior edge e. Denote by $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ the unit outward normals to the common edge e of the triangles K^+ and K^-, respectively. For a scalar-valued function v, write $v^+ = v|_{K^+}$ and $v^- = v|_{K^-}$. Similarly, for a second-order tensor-valued function τ, write $\tau^+ = \tau|_{K^+}$ and $\tau^- = \tau|_{K^-}$. Then define averages and jumps on e as follows:

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad [v] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-,$$

$$\{\nabla v\} = \frac{1}{2}(\nabla v^+ + \nabla v^-), \quad [\nabla v] = \nabla v^+\cdot\boldsymbol{n}^+ + \nabla v^-\cdot\boldsymbol{n}^-,$$

$$\{\tau\} = \frac{1}{2}(\tau^+ + \tau^-), \quad [\tau] = \tau^+\boldsymbol{n}^+ + \tau^-\boldsymbol{n}^-.$$

On an edge e lying on the boundary $\partial\Omega$, the above terms are defined by

$$\{v\} = v, \quad [v] = v\boldsymbol{n},$$
$$\{\nabla v\} = \nabla v, \quad [\nabla v] = \nabla v \cdot \boldsymbol{n},$$
$$\{\tau\} = \tau, \quad [\tau] = \tau \cdot \boldsymbol{n},$$

where $\boldsymbol{n} = \boldsymbol{N}$ is the unit outward normal vector on $\partial\Omega$. The jump $[\![\cdot]\!]$ of the vector ∇v is

$$[\![\nabla v]\!] = \frac{1}{2}(\nabla v^+ \otimes \boldsymbol{n}^+ + \boldsymbol{n}^+ \otimes \nabla v^+ + \nabla v^- \otimes \boldsymbol{n}^- + \boldsymbol{n}^- \otimes \nabla v^-),$$

if $e \in \mathscr{E}_h^i$,

$$[\![\nabla v]\!] = \frac{1}{2}(\nabla v \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \nabla v), \quad \text{if } e \in \mathscr{E}_h \cap \partial\Omega.$$

For later uses, we collect some Green's identities in the following lemma, which can be verified by integration by parts readily.

Lemma 2.1. *Assume G is a bounded domain with a Lipschitz boundary ∂G. Let τ be a symmetric second-order tensor-valued function and v a scalar function. Then*

$$\int_G \nabla \cdot (\nabla \cdot \tau) v\, dx = \int_G \nabla^2 v : \tau\, dx - \int_{\partial G} \nabla v \cdot (\tau \boldsymbol{n}) ds + \int_{\partial G} v \boldsymbol{n} \cdot (\nabla \cdot \tau) ds,$$

$$\int_G \nabla^2 v : \tau\, dx = -\int_G \nabla v \cdot (\nabla \cdot \tau) dx + \int_{\partial G} \nabla v \cdot (\tau \boldsymbol{n}) ds,$$

whenever the terms appearing on both sides of the above identities make sense. Here $\boldsymbol{n}$ stands for the unit outward normal to ∂G.

Now, we are ready to introduce a framework to derive CDG methods for problem (2.1). Following the ideas in [10,12], we first derive the variational formulations for the problem (2.1) or equivalently (2.3). Taking a double dot product with a second-order tensor-valued function τ on both sides of the first equation of (2.3) and then integrating over K, we have by Lemma 2.1 that

$$\int_K \left(\frac{1}{1-\nu} \boldsymbol{\sigma} : \tau - \frac{\nu}{1-\nu^2} \mathrm{tr}\boldsymbol{\sigma} \mathrm{tr}\tau \right) dx = \int_K \nabla u \cdot (\nabla \cdot \tau) dx - \int_{\partial K} \nabla u \cdot (\tau \boldsymbol{n}) ds.$$

Multiplying the second equation of (2.3) by a function v and then integrating over K, we have by Lemma 2.1 again that

$$-\int_K f v\, dx = \int_K \nabla^2 v : \boldsymbol{\sigma}\, dx - \int_{\partial K} \nabla v \cdot (\boldsymbol{\sigma} \boldsymbol{n}) ds + \int_{\partial K} \boldsymbol{n} \cdot (\nabla \cdot \boldsymbol{\sigma}) v\, ds. \tag{2.5}$$

Motivated by the above two identities, we may define our CDG method as follows. Find an approximate solution $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ by requiring that

$$\int_K \left(\frac{1}{1-\nu} \boldsymbol{\sigma}_h : \tau - \frac{\nu}{1-\nu^2} \mathrm{tr}\boldsymbol{\sigma}_h \mathrm{tr}\tau \right) dx = \int_K \nabla u_h \cdot (\nabla \cdot \tau) dx - \int_{\partial K} \widehat{\nabla u_h} \cdot (\tau \boldsymbol{n}) ds, \tag{2.6}$$

$$-\int_K f v\, dx = \int_K \nabla^2 v : \boldsymbol{\sigma}_h\, dx - \int_{\partial K} \nabla v \cdot (\hat{\boldsymbol{\sigma}}_h \boldsymbol{n}) ds \tag{2.7}$$

for all $(\tau, v) \in \boldsymbol{\Sigma}_h \times V_h$ and all $K \in \mathscr{T}_h$. Note that any function with the hat superscript is only defined over all edges of the triangulation $\mathscr{T}_h$, which is called a numerical trace in the context of DG methods (cf. [2,12]). Generally, the last term in (2.5) leads to a corresponding term $\int_{\partial K} \boldsymbol{n} \cdot \widehat{\nabla \cdot \boldsymbol{\sigma}_h} v\, ds$ for the right side of the Eq. (2.7). Here we take $\widehat{\nabla \cdot \boldsymbol{\sigma}_h} = \boldsymbol{0}$ for sake of simplicity. The numerical traces $\hat{\boldsymbol{\sigma}}_h$ and $\widehat{\nabla u_h}$ will be selected to guarantee stability of the above method. Since $\boldsymbol{\sigma}_h$ is symmetric, it is natural to choose $\hat{\boldsymbol{\sigma}}_h$ as a symmetric second-order tensor-valued function. Moreover, we only consider the case where the numerical traces are single-valued over all edges (conservation).

2.2. Numerical traces through stability identity, the LCDG method

We begin by deriving a stability identity for the continuous problem (2.1) or equivalently (2.3), which is crucial in constructing feasible numerical traces to get a stable CDG method from (2.6) and (2.7) (cf. [10,12]). To do so, taking a double dot product with $\boldsymbol{\sigma}$ on both sides of the first equation of (2.3) and then integrating over Ω, we have

$$\int_\Omega \left(\frac{1}{1-\nu} |\boldsymbol{\sigma}|^2 - \frac{\nu}{1-\nu^2} (\mathrm{tr}\boldsymbol{\sigma})^2 \right) dx = -\int_\Omega \nabla^2 u : \boldsymbol{\sigma}\, dx.$$

Multiplying the second equation of (2.3) by u and then integrating over Ω, we find from Lemma 2.1 and the homogeneous boundary conditions of u that

$$-\int_\Omega f u\, dx = \int_\Omega \nabla^2 u : \boldsymbol{\sigma}\, dx.$$

Adding the last two equations leads to

$$\int_\Omega \left(\frac{1}{1-\nu} |\boldsymbol{\sigma}|^2 - \frac{\nu}{1-\nu^2} (\mathrm{tr}\boldsymbol{\sigma})^2 \right) dx = \int_\Omega f u\, dx, \tag{2.8}$$

i.e.,

$$\int_\Omega \left(\frac{\sigma_{11}^2 + \sigma_{22}^2}{1+\nu} + \frac{\sigma_{12}^2 + \sigma_{21}^2}{1-\nu} + \frac{\nu(\sigma_{11} - \sigma_{22})^2}{1-\nu^2} \right) dx = \int_\Omega f u\, dx.$$

This is the stability identity we sought for the solution of the original equation.

Next, we mimic the above derivation to get a discrete analogue of the stability identity (2.8) for the CDG method (2.6) and (2.7). Taking $\tau = \boldsymbol{\sigma}_h$ in (2.6), using Lemma 2.1, and then summing over all $K \in \mathscr{T}_h$, we know

$$\int_\Omega \left(\frac{1}{1-\nu} |\boldsymbol{\sigma}_h|^2 - \frac{\nu}{1-\nu^2} (\mathrm{tr}\boldsymbol{\sigma}_h)^2 \right) dx = -\int_\Omega \nabla_h^2 u_h : \boldsymbol{\sigma}_h\, dx + \sum_{K \in \mathscr{T}_h} \int_{\partial K} (\nabla u_h - \widehat{\nabla u_h}) \cdot (\boldsymbol{\sigma}_h \boldsymbol{n}) ds.$$

Taking $v = u_h$ in (2.7) and summing over all $K \in \mathscr{T}_h$ again, we have

$$-\int_\Omega f u_h\, dx = \int_\Omega \nabla_h^2 u_h : \boldsymbol{\sigma}_h\, dx - \sum_{K \in \mathscr{T}_h} \int_{\partial K} \nabla u_h \cdot (\hat{\boldsymbol{\sigma}}_h \boldsymbol{n}) ds.$$

Adding the last two equations we obtain the desired discrete stability identity, described as follows:

$$\int_\Omega \left(\frac{1}{1-\nu} |\boldsymbol{\sigma}_h|^2 - \frac{\nu}{1-\nu^2} (\mathrm{tr}\boldsymbol{\sigma}_h)^2 \right) dx + \Theta_h = \int_\Omega f u_h\, dx, \tag{2.9}$$

where

$$\Theta_h := \sum_{K \in \mathscr{T}_h} \left(\int_{\partial K} (\widehat{\nabla u_h} - \nabla u_h) \cdot (\boldsymbol{\sigma}_h \boldsymbol{n}) ds + \int_{\partial K} \nabla u_h \cdot (\hat{\boldsymbol{\sigma}}_h \boldsymbol{n}) ds \right). \tag{2.10}$$

Observing that $\boldsymbol{\sigma}_h$ and $\hat{\boldsymbol{\sigma}}_h$ are symmetric, and for a symmetric second-order tensor $\boldsymbol{A}$ and an anti-symmetric second-order tensor $\boldsymbol{B}, \boldsymbol{A} : \boldsymbol{B} = 0$, we have after a direct manipulation that

$$\begin{aligned}
\sum_{K \in \mathscr{T}_h} \int_{\partial K \setminus \partial\Omega} \widehat{\nabla u_h} \cdot (\boldsymbol{\sigma}_h \boldsymbol{n}) ds &= \sum_{e \in \mathscr{E}_h^i} \int_e [\boldsymbol{\sigma}_h] \cdot \widehat{\nabla u_h} ds, \\
\sum_{K \in \mathscr{T}_h} \int_{\partial K \setminus \partial\Omega} \nabla u_h \cdot (\widehat{\sigma_h} \boldsymbol{n}) ds &= \sum_{e \in \mathscr{E}_h^i} \int_e \widehat{\sigma_h} : [\![\nabla u_h]\!] ds, \\
\sum_{K \in \mathscr{T}_h} \int_{\partial K \setminus \partial\Omega} \nabla u_h \cdot (\boldsymbol{\sigma}_h \boldsymbol{n}) ds &= \sum_{e \in \mathscr{E}_h^i} \int_e ([\boldsymbol{\sigma}_h] \cdot \{\nabla u_h\} + \{\boldsymbol{\sigma}_h\} : [\![\nabla u_h]\!]) ds.
\end{aligned} \tag{2.11}$$

Therefore, we can rewrite Θ_h as

$$\Theta_h = \sum_{e\in\mathscr{E}_h^i}\int_e\Big([\boldsymbol{\sigma}_h]\cdot\Big(\widehat{\nabla u}_h - \{\nabla u_h\}\Big) + (\widehat{\boldsymbol{\sigma}}_h - \{\boldsymbol{\sigma}_h\}) : [\![\nabla u_h]\!]\Big)ds$$
$$+\int_{\partial\Omega}\nabla u_h\cdot((\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h)\boldsymbol{n})ds + \int_{\partial\Omega}\widehat{\nabla u}_h\cdot(\boldsymbol{\sigma}_h\boldsymbol{n})ds.$$

If Θ_h is non-negative, we can derive from (2.9) the discrete stable estimate:

$$\|\boldsymbol{\sigma}_h\|_0^2 \lesssim \int_\Omega f u_h\,dx,$$

which is essential in constructing a reliable DG method (cf. [12]). Because of this, we will define consistent (cf. [2]) numerical traces $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{\nabla u}_h$ so that Θ_h is non-negative. Thus, if $e\in\mathscr{E}_h^i$, we take

$$\widehat{\boldsymbol{\sigma}}_h = \{\boldsymbol{\sigma}_h\} + C_{11}[\![\nabla u_h]\!],$$
$$\widehat{\nabla u}_h = \{\nabla u_h\} + C_{22}[\boldsymbol{\sigma}_h] \tag{2.12}$$

and if $e\in\mathscr{E}_h\cap\partial\Omega$, we take

$$\widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h + C_{11}[\![\nabla u_h]\!],$$
$$\widehat{\nabla u}_h = \mathbf{0}, \tag{2.13}$$

where C_{11} and C_{22} are two non-negative continuous functions on e. Note that when $C_{22}=0$, the corresponding method for second-order elliptic problems is called the LDG method in [10,12]; we therefore call the above method in this case the LCDG method. In the next section, we will perform systematic error analysis for this method.

For the above choice of numerical traces, we have by some direct manipulation that

$$\Theta_h = \sum_{e\in\mathscr{E}_h^i}\int_e\Big(C_{22}|[\boldsymbol{\sigma}_h]|^2 + C_{11}|[\![\nabla u_h]\!]|^2\Big)ds + \int_{\partial\Omega}C_{11}|[\![\nabla u_h]\!]|^2 ds \geqslant 0, \tag{2.14}$$

so the required condition given before is satisfied. Let us further show the unique solvability of problem (2.6) and (2.7) with the numerical traces given by (2.12) and (2.13), whenever $C_{11}>0$ and the finite element spaces $\boldsymbol{\Sigma}_h$ and V_h satisfy conditions (2.4). In fact, it suffices to verify that this CDG method only has zero solution when $f=0$. We have by (2.9), (2.10) and (2.14) that

$$\int_\Omega\left(\frac{1}{1-\nu}|\boldsymbol{\sigma}_h|^2 - \frac{\nu}{1-\nu^2}(\mathrm{tr}\boldsymbol{\sigma}_h)^2\right)dx + \int_{\mathscr{E}_h^i}C_{22}|[\boldsymbol{\sigma}_h]|^2 ds$$
$$+\int_{\mathscr{E}_h}C_{11}|[\![\nabla u_h]\!]|^2 ds = 0,$$

which implies that $\boldsymbol{\sigma}_h=\mathbf{0}$ in Ω, $[\![\nabla u_h]\!]=\mathbf{0}$ on $\mathscr{E}_h^i$ and $\nabla u_h=\mathbf{0}$ on $\partial\Omega$, owing to $C_{11}>0$. Moreover, from the condition that $[\![\nabla u_h]\!]=\mathbf{0}$ on $\mathscr{E}_h^i$ and a direct manipulation, we have $\nabla u_h^+ = \nabla u_h^-$ on each interior edge $e\in\mathscr{E}_h^i$. Therefore, it follows from (2.6), the definition of $\widehat{\nabla u}_h$ (see (2.12) and (2.13)), and Lemma 2.1 that

$$\int_K \boldsymbol{\tau}:\nabla^2 u_h\,dx = 0 \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h,$$

which together with the condition (2.4) yields $\nabla^2 u_h = \mathbf{0}$. Combining this with $\nabla u_h=\mathbf{0}$ on $\partial\Omega$ and $u_h\in H_0^1(\Omega)$, we conclude that $u_h=0$ in Ω, as required.

As in [10,12], the CDG method (2.6) and (2.7) with the numerical traces (2.12) and (2.13) can also be written in a mixed formulation (cf. [9]). After some direct manipulation, the approximate solution $(\boldsymbol{\sigma}_h, u_h)$ can be characterized as the unique solution of the following variational problem: find $(\boldsymbol{\sigma}_h,u_h)\in\boldsymbol{\Sigma}_h\times V_h$ such that

$$a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + b(u_h,\boldsymbol{\tau}) = 0, \tag{2.15}$$
$$-b(v,\boldsymbol{\sigma}_h) + c(u_h,v) = F(v), \tag{2.16}$$

for all $(\boldsymbol{\tau},v)\in\boldsymbol{\Sigma}_h\times V_h$, where

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) := \int_\Omega\left(\frac{1}{1-\nu}\boldsymbol{\sigma}:\boldsymbol{\tau} - \frac{\nu}{1-\nu^2}\mathrm{tr}\boldsymbol{\sigma}\mathrm{tr}\boldsymbol{\tau}\right)dx + \int_{\mathscr{E}_h^i}C_{22}[\boldsymbol{\sigma}]\cdot[\boldsymbol{\tau}]ds,$$
$$b(v,\boldsymbol{\tau}) := \sum_{K\in\mathscr{T}_h}\int_K\boldsymbol{\tau}:\nabla^2 v\,dx - \int_{\mathscr{E}_h}\{\boldsymbol{\tau}\}:[\![\nabla v]\!]ds,$$
$$c(u,v) := \int_{\mathscr{E}_h}C_{11}[\![\nabla u]\!]:[\![\nabla v]\!]ds,$$
$$F(v) := \int_\Omega f v\,dx.$$

2.3. Some other choices of numerical traces

In this subsection, we derive some existing CDG methods within our basic framework of CDG methods (2.6) and (2.7) with certain choices of numerical traces $\hat{\boldsymbol{\sigma}}_h$ and $\widehat{\nabla u}_h$. Analogous to [2], we can eliminate the dual variable $\boldsymbol{\sigma}_h$ to recast the original method (2.6) and (2.7) in the following primal formulation:

$$\int_\Omega(1-\nu)\nabla_h^2 u_h:\nabla_h^2 v\,dx + \int_\Omega \nu\,\mathrm{tr}\Big(\nabla_h^2 u_h\Big)\mathrm{tr}\Big(\nabla_h^2 v\Big)dx$$
$$+\int_{\mathscr{E}_h^i}\{\widehat{\nabla u}_h - \nabla u_h\}\cdot\Big((1-\nu)[\nabla_h^2 v] + \nu[\mathrm{tr}\Big(\nabla_h^2 v\Big)]\Big)ds$$
$$+\int_{\mathscr{E}_h}[\![\widehat{\nabla u}_h - \nabla u_h]\!]:\Big((1-\nu)\{\nabla_h^2 v\} + \nu\,\mathrm{tr}\Big(\{\nabla_h^2 v\}\Big)\mathscr{I}\Big)ds$$
$$+\int_{\mathscr{E}_h^i}\{\nabla v\}\cdot[\hat{\boldsymbol{\sigma}}_h]ds + \int_{\mathscr{E}_h}[\![\nabla v]\!]:\{\hat{\boldsymbol{\sigma}}_h\}ds = \int_\Omega f v\,dx. \tag{2.17}$$

Some CDG methods found in the literature can be obtained from (2.17) by proper choices of numerical traces $\hat{\boldsymbol{\sigma}}_h$ and $\widehat{\nabla u}_h$. For example, taking

$$\begin{cases}\widehat{\nabla u}_h = \nabla u_h + \frac{1}{2}[\nabla u_h]\boldsymbol{n},\\ \hat{\boldsymbol{\sigma}}_h = \left(\left\{\frac{\partial^2 u_h}{\partial\boldsymbol{n}^2}\right\} + \frac{\eta}{h_e}[\nabla u_h]\right)\mathscr{I},\end{cases}$$

we obtain from (2.17) that

$$\int_\Omega(1-\nu)\nabla_h^2 u_h:\nabla_h^2 v\,dx + \int_\Omega\nu\,\mathrm{tr}\Big(\nabla_h^2 u_h\Big)\mathrm{tr}\Big(\nabla_h^2 v\Big)dx$$
$$+\int_{\mathscr{E}_h}[\nabla u_h]\left((1-\nu)\left\{\frac{\partial^2 v}{\partial\boldsymbol{n}^2}\right\} + \nu\,\mathrm{tr}\Big(\{\nabla_h^2 v\}\Big)\right)ds$$
$$+\int_{\mathscr{E}_h}\left\{\frac{\partial^2 u_h}{\partial\boldsymbol{n}^2}\right\}[\nabla v]ds + \int_{\mathscr{E}_h}\frac{\eta}{h_e}[\nabla u_h][\nabla v]ds = \int_\Omega f v\,dx.$$

This is the formulation studied in [7]. For the second example, we introduce a global lifting operator $\boldsymbol{r}:(L^2(\mathscr{E}_h))^s_{2\times2}\to\boldsymbol{\Sigma}_h$ defined by

$$\int_\Omega\boldsymbol{r}(\boldsymbol{\phi}):\boldsymbol{\tau}\,dx = -\int_{\mathscr{E}_h}\boldsymbol{\phi}:\{\boldsymbol{\tau}\}ds \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h,\ \boldsymbol{\phi}\in\Big(L^2(\mathscr{E}_h)\Big)^s_{2\times2}. \tag{2.18}$$

Moreover, for each $e\in\mathscr{E}_h$, introduce a local lifting operator $\boldsymbol{r}_e:(L^2(e))^s_{2\times2}\to\boldsymbol{\Sigma}_h$ by

$$\int_\Omega\boldsymbol{r}_e(\boldsymbol{\phi}):\boldsymbol{\tau}\,dx = -\int_e\boldsymbol{\phi}:\{\boldsymbol{\tau}\}ds \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h,\ \boldsymbol{\phi}\in\Big(L^2(e)\Big)^s_{2\times2}. \tag{2.19}$$

Then, it is easy to check that $\boldsymbol{r}_e(\boldsymbol{\phi})$ may only be non-zero in the triangles with e as one edge, and there holds the identity

$$\boldsymbol{r}(\boldsymbol{\phi}) = \sum_{e\in\mathscr{E}_h}\boldsymbol{r}_e(\boldsymbol{\phi}|_e) \quad \forall\,\boldsymbol{\phi}\in\Big(L^2(\mathscr{E}_h)\Big)^s_{2\times2}.$$

For $e\in\mathscr{E}_h^i$ we take

$$\hat{\boldsymbol{\sigma}}_h = -(1-\nu)\{\nabla_h^2 u_h\} - \nu\,\mathrm{tr}(\{\nabla_h^2 u_h\})\mathscr{I} - (1-\nu)\{\boldsymbol{r}([\![\nabla u_h]\!])\}$$
$$-\nu\{\boldsymbol{r}(\mathrm{tr}([\![\nabla u_h]\!])\mathscr{I})\} - (1-\nu)\{\eta\boldsymbol{r}_e([\![\nabla u_h]\!])\}$$
$$-\nu\{\eta\,\mathrm{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\}\mathscr{I},$$

$$\widehat{\nabla u}_h = \{\nabla u_h\},$$

and for $e \in \mathscr{E}_h \cap \partial\Omega$ we take

$$\hat{\boldsymbol{\sigma}}_h = -(1-\nu)\nabla_h^2 u_h - \nu\,\mathrm{tr}(\nabla_h^2 u_h)\mathscr{I} - (1-\nu)\boldsymbol{r}([\![\nabla u_h]\!]) - \nu\boldsymbol{r}(\mathrm{tr}([\![\nabla u_h]\!])\mathscr{I}) - (1-\nu)\eta\boldsymbol{r}_e([\![\nabla u_h]\!]) - \nu\eta\,\mathrm{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\mathscr{I},$$

$$\widehat{\nabla u}_h = \mathbf{0}.$$

With these choices of numerical traces, (2.17) becomes

$$\int_\Omega (1-\nu)\Big(\nabla_h^2 u_h + \boldsymbol{r}([\![\nabla u_h]\!])\Big) : \Big(\nabla_h^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big)dx + \int_\Omega \nu\,\mathrm{tr}\Big(\nabla_h^2 u_h + \boldsymbol{r}([\![\nabla u_h]\!])\Big)\mathrm{tr}\Big(\nabla_h^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big)dx + \sum_{e\in\mathscr{E}_h}\int_\Omega \eta((1-\nu)\boldsymbol{r}_e([\![\nabla u_h]\!])\boldsymbol{r}_e([\![\nabla v]\!]) + \nu\,\mathrm{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\mathrm{tr}(\boldsymbol{r}_e([\![\nabla v]\!])))dx = \int_\Omega f v\,dx, \quad (2.20)$$

which is the CDG method proposed in [29].

3. Error analysis for the LCDG method

In this section, we provide error analysis for the CDG method (2.15) and (2.16) in the case $C_{22} = 0$, which is referred to as the LCDG method in the previous section. For this purpose, we first derive a primal formulation for the approximate method. Since $C_{22} = 0$, with the help of the global lifting operator $\boldsymbol{r}$ (see (2.18)), (2.15) can be expressed as

$$\int_\Omega \left(\frac{1}{1-\nu}\boldsymbol{\sigma}_h - \frac{\nu}{1-\nu^2}(\mathrm{tr}\boldsymbol{\sigma}_h)\mathscr{I}\right) : \tau\,dx = -\int_\Omega \nabla_h^2 u_h : \tau\,dx - \int_\Omega \boldsymbol{r}([\![\nabla u_h]\!]) : \tau\,dx,$$

for all $\tau \in \boldsymbol{\Sigma}_h$. On the other hand, it follows from (2.4) that

$$\frac{1}{1-\nu}\boldsymbol{\sigma}_h - \frac{\nu}{1-\nu^2}(\mathrm{tr}\boldsymbol{\sigma}_h)\mathscr{I} \in \boldsymbol{\Sigma}_h, \quad \nabla_h^2 u_h \in \boldsymbol{\Sigma}_h,$$

and hence the above equation implies that

$$\frac{1}{1-\nu}\boldsymbol{\sigma}_h - \frac{\nu}{1-\nu^2}(\mathrm{tr}\boldsymbol{\sigma}_h)\mathscr{I} = -\nabla_h^2 u_h - \boldsymbol{r}([\![\nabla u_h]\!]),$$

i.e.,

$$\boldsymbol{\sigma}_h = -(1-\nu)(\nabla_h^2 u_h + \boldsymbol{r}([\![\nabla u_h]\!])) - \nu\,\mathrm{tr}(\nabla_h^2 u_h + \boldsymbol{r}([\![\nabla u_h]\!]))\mathscr{I}.$$

Substituting $\boldsymbol{\sigma}_h$ from the last equation into (2.16), we get a primal formulation for the LCDG method as follows:

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v\,dx \quad \forall\ v \in V_h, \quad (3.1)$$

where

$$a_h(w, v) := \int_\Omega (1-\nu)\Big(\nabla_h^2 w + \boldsymbol{r}([\![\nabla w]\!])\Big) : \Big(\nabla_h^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big)dx + \int_\Omega \nu\,\mathrm{tr}\Big(\nabla_h^2 w + \boldsymbol{r}([\![\nabla w]\!])\Big)\mathrm{tr}\Big(\nabla_h^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big)dx + \int_{\mathscr{E}_h} C_{11}[\![\nabla w]\!] : [\![\nabla v]\!]ds.$$

Let us compare it with the method (2.20) (cf. [29]). For the two methods, the first two terms on the left and the term on the right are the same, but the third term on the left for the LCDG method is much simpler than that of the method (2.20). Therefore, the LCDG method is more convenient to implement in actual computation than the method in [29]. Moreover, the LCDG method does not contain any parameter which can not be quantified *a priori*. It only requires to choose $C_{11} = \eta_e h_e^{-1}$ on each $e \in \mathscr{E}_h$, with $\{\eta_e\}_{e\in\mathscr{E}_h}$ having a uniform positive bound from above and below, which leads to optimal error estimates shown later on. In what follows, we always assume that C_{11} is chosen as above.

Now, we are in a position to give error analysis for the LCDG method (3.1). The main idea of our derivation is based on the framework on error analysis of DG methods for second-order elliptic problems (cf. [2]). Introduce a finite element space $W_h \subset V_h$ by

$$W_h := \Big\{ v \in H_0^1(\Omega) : v|_K \in P_k(K)\ \forall\ K \in \mathscr{T}_h \Big\}.$$

Let Q_h be the usual Lagrange interpolation operator from V onto W_h (cf. [6,11]). For sake of clarity, we simply write Q_h for the restriction of Q_h to any $K \in \mathscr{T}_h$. Let $\boldsymbol{P}_h$ be the usual L^2-orthogonal projection operator onto $\boldsymbol{\Sigma}_h$. Using the scaling argument and the trace theorem, we can easily obtain the following result.

Lemma 3.1. *For all $v \in H^{m+3}(K)$, $\boldsymbol{\sigma} \in (H^{m+1}(\Omega))^s_{2\times2}$ with m a nonnegative integer and all $K \in \mathscr{T}_h$ with e as one edge, we have the estimates*

$$|v - Q_h v|_{1,K} + h_K|v - Q_h v|_{2,K} \lesssim h_K^{\min\{m+1,k-1\}+1}\|v\|_{m+3,K},$$
$$\|\nabla(v - Q_h v)\|_{0,\partial K} \lesssim h_K^{\min\{m+1,k-1\}+1/2}\|v\|_{m+3,K},$$
$$\|\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}\|_{0,K} \lesssim h_K^{\min\{m+1,l+1\}}\|\boldsymbol{\sigma}\|_{m+1,K},$$
$$h_e^{1/2}\|\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}\|_{0,e} \lesssim h_K^{\min\{m+1,l+1\}}\|\boldsymbol{\sigma}\|_{m+1,K}.$$

Next, we consider the consistency of the method (3.1). Assume that the exact solution u of problem (2.1) lies in $H^{m+3}(\Omega)$ for some nonnegative integer m. From the definition (2.2) and the second equation of (2.3), we know that $\boldsymbol{\sigma} \in (H^{m+1}(\Omega))^s_{2\times2} \subset (H^1(\Omega))^s_{2\times2}$ and $\nabla\cdot(\nabla\cdot\boldsymbol{\sigma}) \in L^2(\Omega)$. Since $u \in H^{m+3}(\Omega)$, $[\![\nabla u]\!] = \mathbf{0}$ on $\mathscr{E}_h$. Therefore, for all $v \in V_h \subset H_0^1(\Omega)$, we have by (2.2) that

$$a_h(u, v) = \int_\Omega \Big((1-\nu)\nabla^2 u : \Big(\nabla_h^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big) + \nu\,\mathrm{tr}\Big(\nabla^2 u\Big)\mathrm{tr}\Big(\nabla_h^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big)\Big)dx = -\sum_{K\in\mathscr{T}_h}\int_K \boldsymbol{\sigma} : \Big(\nabla^2 v + \boldsymbol{r}([\![\nabla v]\!])\Big)dx,$$

from which, the definition of lifting operator $\boldsymbol{r}$ (see (2.18)), and the fact that $\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}$, we further have

$$a_h(u, v) = \tilde{a}_h(u, v) - \int_{\mathscr{E}_h}[\![\nabla v]\!] : \{\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}\}ds - \sum_{K\in\mathscr{T}_h}\int_K(\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}) : \boldsymbol{r}([\![\nabla v]\!])dx,$$

where

$$\tilde{a}_h(u, v) := -\sum_{K\in\mathscr{T}_h}\int_K \boldsymbol{\sigma} : \nabla^2 v dx + \int_{\mathscr{E}_h}[\![\nabla v]\!] : \boldsymbol{\sigma} ds.$$

In addition, it follows from Lemma 2.1 that

$$\tilde{a}_h(u, v) = \sum_{K\in\mathscr{T}_h}\int_K \nabla v\cdot(\nabla\cdot\boldsymbol{\sigma})dx + \int_{\mathscr{E}_h}[\![\nabla v]\!] : \boldsymbol{\sigma} ds - \sum_{K\in\mathscr{T}_h}\int_{\partial K}\nabla v\cdot(\boldsymbol{\sigma}\boldsymbol{n})ds.$$

Observing that $[\boldsymbol{\sigma}] = 0$ and using the same technique for deriving identities (2.11), we find

$$\tilde{a}_h(u, v) = \int_\Omega \nabla v\cdot(\nabla\cdot\boldsymbol{\sigma})dx.$$

Finally, since $v = 0$ on $\partial\Omega$, we deduce from Lemma 2.1 and (2.3) that

$$\tilde{a}_h(u, v) = -\int_\Omega v\nabla \cdot (\nabla \cdot \boldsymbol{\sigma})dx + \int_{\partial\Omega} v\boldsymbol{n} \cdot (\nabla \cdot \boldsymbol{\sigma})ds$$
$$= -\sum_{K\in\mathcal{T}_h} \int_K v\nabla \cdot (\nabla \cdot \boldsymbol{\sigma})dx = \int_\Omega f v dx.$$

Hence, we know the LCDG method (3.1) is not consistent with respect to the bilinear form $a_h(\cdot,\cdot)$, but it admits the following identity:

$$a_h(u-u_h, v) = -\int_{\mathcal{E}_h} [\![\nabla v]\!] : \{\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}\}ds - \sum_{K\in\mathcal{T}_h}\int_K (\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}) : \boldsymbol{r}([\![\nabla v]\!])dx \quad \forall v \in V_h. \tag{3.2}$$

We next present a useful result on the lifting operators. For this, let $V(h) := V_h + H_0^2(\Omega)$ and define a mesh-dependent energy norm (broken energy norm) for $v \in V(h)$ by

$$|||v|||^2 = |v|_{2,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2.$$

Lemma 3.2. *For any $v \in V(h)$ and $e \in \mathcal{E}_h$,*

$$\|\boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2 \lesssim h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2. \tag{3.3}$$

Consequently,

$$\|\boldsymbol{r}([\![\nabla v]\!])\|_{0,h}^2 \lesssim \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2 \quad \forall v \in V(h). \tag{3.4}$$

Proof. Since $v \in H_0^2(\Omega)$ implies that $[\![\nabla v]\!] = \boldsymbol{0}$, it suffices to verify the result for $v \in V_h$. Taking $\boldsymbol{\phi} = [\![\nabla v]\!]$ and $\boldsymbol{\tau} = \boldsymbol{r}_e([\![\nabla v]\!])$ in (2.19), we see that

$$\|\boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2 = -\int_e [\![\nabla v]\!] : \{\boldsymbol{r}_e([\![\nabla v]\!])\}ds \leqslant \|[\![\nabla v]\!]\|_{0,e}\|\{\boldsymbol{r}_e([\![\nabla v]\!])\}\|_{0,e}. \tag{3.5}$$

By the scaling argument, the trace theorem, and the local inverse inequality for finite elements, it follows that

$$\|\{\boldsymbol{r}_e([\![\nabla v]\!])\}\|_{0,e}^2 \lesssim h_e^{-1}\|\boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2 + h_e|\boldsymbol{r}_e([\![\nabla v]\!])|_{1,h}^2 \lesssim h_e^{-1}\|\boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2. \tag{3.6}$$

Therefore, (3.3) is a direct consequence of (3.5) and (3.6). Note that

$$\|\boldsymbol{r}([\![\nabla v]\!])\|_{0,h}^2 \lesssim \sum_{e\in\mathcal{E}_h} \|\boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2.$$

The inequality (3.4) follows from (3.3). □

Lemma 3.3 (*boundedness*).

$$a_h(w, v) \lesssim |||w|||\ |||v||| \quad \forall (w, v) \in V(h) \times V(h). \tag{3.7}$$

Proof. Applying the Cauchy–Schwarz inequality, we have

$$a_h(w, v) \lesssim \sum_{K\in\mathcal{T}_h}\left(\int_K \left|\nabla^2 w + \boldsymbol{r}([\![\nabla w]\!])\right|^2 dx \cdot \int_K \left|\nabla^2 v + \boldsymbol{r}([\![\nabla v]\!])\right|^2 dx\right)^{1/2} + \sum_{e\in\mathcal{E}_h}\left(\int_e h_e^{-1}|[\![\nabla w]\!]|^2 ds \cdot \int_e h_e^{-1}|[\![\nabla v]\!]|^2 ds\right)^{1/2}$$
$$\lesssim \left(|w|_{2,h}^2 + \|\boldsymbol{r}([\![\nabla w]\!])\|_{0,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla w]\!]\|_{0,e}^2\right)^{1/2} \cdot \left(|v|_{2,h}^2 + \|\boldsymbol{r}([\![\nabla v]\!])\|_{0,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2\right)^{1/2}.$$

The result (3.7) then follows from the above inequality and (3.4). □

Denote $\eta_0 := \min_{e\in\mathcal{E}_h}\eta_e$. By our assumption, $\eta_0 > 0$.

Lemma 3.4 (*stability*).

$$a_h(v, v) \gtrsim |||v|||^2 \quad \forall v \in V_h. \tag{3.8}$$

Proof. Using the Cauchy–Schwarz inequality and (3.3) we find

$$a_h(v, v) \geqslant (1-\nu)\left\|\nabla^2 v + \boldsymbol{r}([\![\nabla v]\!])\right\|_{0,h}^2 + \sum_{e\in\mathcal{E}_h}\int_e \eta_e h_e^{-1}|[\![\nabla v]\!]|^2 ds$$
$$\geqslant (1-\nu)\left(|v|_{2,h}^2 + \|\boldsymbol{r}([\![\nabla v]\!])\|_{0,h}^2 + 2\sum_{K\in\mathcal{T}_h}\int_K \nabla^2 v : \boldsymbol{r}([\![\nabla v]\!])\,dx\right) + \eta_0\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2$$
$$\geqslant (1-\nu)\left((1-\epsilon)|v|_{2,h}^2 + \left(1-\frac{1}{\epsilon}\right)\|\boldsymbol{r}([\![\nabla v]\!])\|_{0,h}^2\right) + \eta_0\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2 \geqslant (1-\nu)(1-\epsilon)|v|_{2,h}^2$$
$$+ \left(\eta_0 - (1-\nu)\left(\frac{1}{\epsilon} - 1\right)C_1\right)\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2,$$

where $\epsilon \in (0, 1)$ is arbitrary and C_1 is the generic positive constant in (3.4). Since $\eta_0 > 0$, we can choose $\epsilon \in (0, 1)$ such that

$$\eta_0 - (1-\nu)\left(\frac{1}{\epsilon} - 1\right)C_1 > 0.$$

Therefore, (3.8) holds. □

Now we are ready to prove an optimal order error estimate in the mesh-dependent energy norm $|||\cdot|||$.

Theorem 3.5. *Assume the solution of (2.1) satisfies $u \in H^{m+3}(\Omega)$ for some non-negative integer m, and let $u_h \in V_h$ be the solution of (3.1). Then*

$$|||u - u_h||| \lesssim h^{\min\{m+1,k-1,l+1\}}\|u\|_{m+3,\Omega}.$$

Proof. By the stability (3.8), the identity (3.2), the boundedness (3.7) and Lemmas 3.1, 3.2,

$$|||Q_hu - u_h|||^2 \lesssim a_h(Q_hu - u_h, Q_hu - u_h)$$
$$= a_h(Q_hu - u, Q_hu - u_h) + a_h(u - u_h, Q_hu - u_h)$$
$$= a_h(Q_hu - u, Q_hu - u_h) - \int_{\mathcal{E}_h} [\![\nabla(Q_hu - u_h)]\!] : \{\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}\}ds - \sum_{K\in\mathcal{T}_h}\int_K (\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}) : \boldsymbol{r}([\![\nabla(Q_hu - u_h)]\!])dx$$
$$\lesssim |||Q_hu - u|||\ |||Q_hu - u_h||| + h^{\min\{m+1,l+1\}}\|u\|_{m+3}|||Q_hu - u_h|||.$$

Thus,

$$|||Q_hu - u_h||| \lesssim |||Q_hu - u||| + h^{\min\{m+1,l+1\}}\|u\|_{m+3}.$$

Together with the triangle inequality and Lemma 3.1, we get

$$|||u - u_h||| \leqslant |||u - Q_hu||| + |||Q_hu - u_h|||$$
$$\lesssim |||u - Q_hu||| + h^{\min\{m+1,l+1\}}\|u\|_{m+3}$$
$$\lesssim h^{\min\{m+1,k-1,l+1\}}\|u\|_{m+3,\Omega},$$

as required. □

Table 1
Errors $|u-u_h|_1$ and $|||u-u_h|||$ for Example 4.1.

k,l	h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
$k=2, l=1$	$\|u-u_h\|_1$	0.51744	0.19173	0.05847	0.01617	0.00449
	$\|\|\|u-u_h\|\|\|$	4.89493	2.62657	1.29978	0.63654	0.31434
$k=2, l=0$	$\|u-u_h\|_1$	0.41739	0.15846	0.04943	0.01384	0.00391
	$\|\|\|u-u_h\|\|\|$	5.75214	2.90550	1.37018	0.65405	0.31871
$k=1, l=0$	$\|u-u_h\|_1$	1.20110	1.16104	1.12206	1.09690	1.08326
	$\|\|\|u-u_h\|\|\|$	8.44959	8.20780	8.15277	8.14459	8.14539

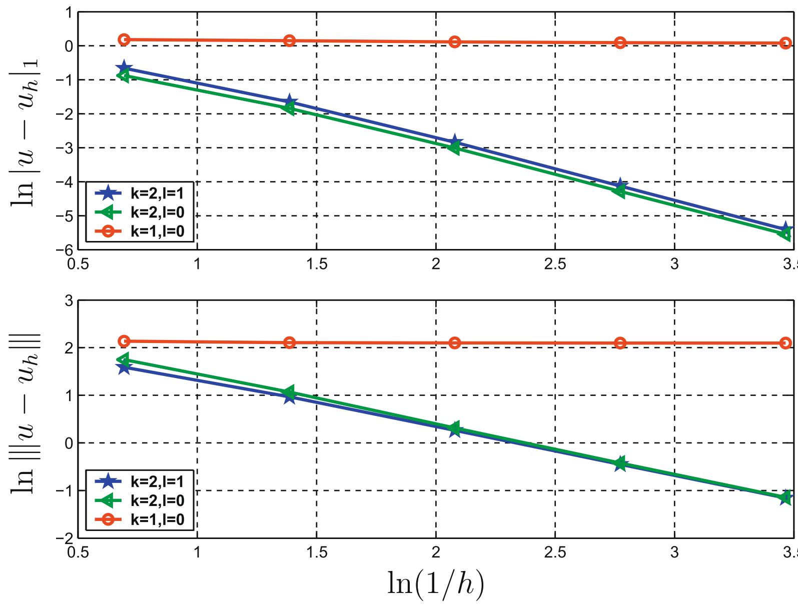

Fig. 1. Errors $|u-u_h|_1$ and $|||u-u_h|||$ vs $1/h$ in ln – ln scale for Example 4.1.

Using the usual duality argument, we can additionally derive an optimal order error estimate in the $H^1(\Omega)$-norm.

Theorem 3.6. *Let Ω be a convex bounded polygonal domain. Assume the solution of (2.1) satisfies $u \in H^{m+3}(\Omega)$ for some non-negative integer m, and let $u_h \in V_h$ be the solution of (3.1). Then*

$$|u-u_h|_1 \lesssim h^{\min\{1,k-1\}+\min\{m+1,k-1,l+1\}}\|u\|_{m+3,\Omega}.$$

Proof. Let $(\tilde{\boldsymbol{\sigma}}, \tilde{u})$ be the solution of the auxiliary problem:

$$\begin{cases} \frac{1}{1-\nu}\tilde{\boldsymbol{\sigma}} - \frac{\nu}{1-\nu^2}(\mathrm{tr}\tilde{\boldsymbol{\sigma}})\mathscr{I} = \mathscr{K}(\tilde{u}) & \text{in } \Omega, \\ \nabla\cdot(\nabla\cdot\tilde{\boldsymbol{\sigma}}) = \Delta(u-u_h) & \text{in } \Omega, \\ \tilde{u} = \partial_{\boldsymbol{N}}\tilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Formally, (3.9) is problem (2.3) with f replaced by $-\Delta(u-u_h)$, where $\tilde{\boldsymbol{\sigma}}$ is obtained from (2.2) with u replaced by $\tilde{u}$. Since $\triangle(u-u_h) \notin L^2(\Omega)$, the second equation of (3.9) is interpreted by the following relation

$$\int_\Omega (\nabla\cdot\tilde{\boldsymbol{\sigma}})\cdot\nabla v\,dx = \int_\Omega \nabla(u-u_h)\cdot\nabla v\,dx \quad \forall\, v \in H_0^1(\Omega). \tag{3.10}$$

Since Ω is a convex bounded polygonal domain, we know $\tilde{u} \in H^3(\Omega)\cap H_0^1(\Omega)$ with the bound (cf. [15,21])

$$\|\tilde{u}\|_{3,\Omega} \lesssim \|\Delta(u-u_h)\|_{-1,\Omega}. \tag{3.11}$$

Note that $\nabla^2\tilde{u} \in (H^1(\Omega))^s_{2\times 2}$ and $u-u_h \in H_0^1(\Omega)$. Hence, using the definitions of $\boldsymbol{r}$ and $a_h(\cdot,\cdot)$, (3.9), (3.10) with $v = u-u_h$, Lemma 2.1, the identity (3.2), and the technique for deriving identities (2.11), we have

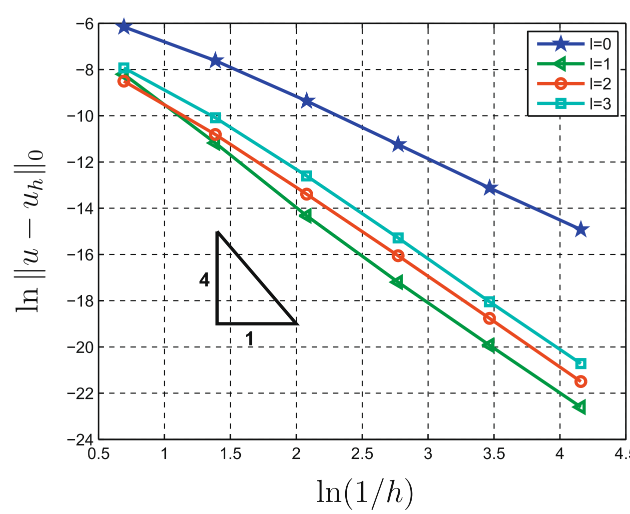

Fig. 2. Error $\|u-u_h\|_0$ vs $1/h$ in ln – ln scale for Example 4.2.

Table 2
Error $\|u-u_h\|_0$ for Example 4.2.

h	l			
	0	1	2	3
2^{-1}	2.1197E–03	2.7407E–04	2.0288E–04	3.5907E–04
2^{-2}	4.9406E–04	1.4006E–05	2.0260E–05	4.1786E–05
2^{-3}	8.5966E–05	6.0178E–07	1.5281E–06	3.3586E–06
2^{-4}	1.3122E–05	3.3942E–08	1.0675E–07	2.2988E–07
2^{-5}	2.0047E–06	2.2153E–09	7.0898E–09	1.4516E–08
2^{-6}	3.3140E–07	1.5255E–10	4.6244E–10	1.0042E–09

Table 3
Error $|u-u_h|_1$ for Example 4.2.

h	l			
	0	1	2	3
2^{-1}	1.0247E–02	2.4728E–03	1.7610E–03	2.2239E–03
2^{-2}	2.6650E–03	3.1270E–04	2.4622E–04	3.1858E–04
2^{-3}	5.5946E–04	3.8016E–05	3.0225E–05	3.7722E–05
2^{-4}	1.0482E–04	4.6806E–06	3.6865E–06	4.4299E–06
2^{-5}	1.9090E–05	5.7719E–07	4.5507E–07	5.3818E–07
2^{-6}	3.5221E–06	7.1548E–08	5.6585E–08	6.6626E–08

$$
\begin{aligned}
|u-u_h|_1^2 &= \int_\Omega \nabla(u-u_h)\cdot(\nabla\cdot\tilde{\sigma})dx = \sum_{K\in\mathcal{T}_h}\int_K \nabla(u-u_h)\cdot(\nabla\cdot\tilde{\sigma})dx\\
&= -\sum_{K\in\mathcal{T}_h}\int_K \tilde{\sigma}:\nabla^2(u-u_h)dx + \int_{\mathcal{E}_h}\tilde{\sigma}:[\![\nabla(u-u_h)]\!]ds\\
&= \sum_{K\in\mathcal{T}_h}\int_K (1-\nu)\nabla^2\tilde{u}:\nabla^2(u-u_h)dx\\
&\quad+\sum_{K\in\mathcal{T}_h}\int_K \nu\,\mathrm{tr}(\nabla^2\tilde{u})\mathrm{tr}(\nabla^2(u-u_h))dx\\
&\quad-\int_{\mathcal{E}_h}(1-\nu)\nabla^2\tilde{u}:[\![\nabla(u-u_h)]\!]ds\\
&\quad-\int_{\mathcal{E}_h}\nu\,\mathrm{tr}(\nabla^2\tilde{u})\mathrm{tr}([\![\nabla(u-u_h)]\!])ds = a_h(u-u_h,\tilde{u}-Q_h\tilde{u})\\
&\quad+a_h(u-u_h,Q_h\tilde{u}) = a_h(u-u_h,\tilde{u}-Q_h\tilde{u})\\
&\quad+\int_{\mathcal{E}_h}[\![\nabla(\tilde{u}-Q_h\tilde{u})]\!]:\{\sigma-\boldsymbol{P}_h\sigma\}ds\\
&\quad+\sum_{K\in\mathcal{T}_h}\int_K(\sigma-\boldsymbol{P}_h\sigma):\boldsymbol{r}([\![\nabla(\tilde{u}-Q_h\tilde{u})]\!])dx.
\end{aligned}
$$

Therefore, it follows from (3.7), (3.11), and Lemmas 3.1, 3.2 that

$$
\begin{aligned}
|u-u_h|_1^2 &\lesssim |||u-u_h|||\ |||\tilde{u}-Q_h\tilde{u}|||+h^{\min\{m+1,l+1\}}\|u\|_{m+3}\\
&\quad\times\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\![\nabla(\tilde{u}-Q_h\tilde{u})]\!]\|_{0,e}^2\right)^{1/2}\\
&\lesssim h^{\min\{1,k-1\}}(|||u-u_h|||+h^{\min\{m+1,l+1\}}\|u\|_{m+3})\|\tilde{u}\|_3\\
&\lesssim h^{\min\{1,k-1\}}(|||u-u_h|||+h^{\min\{m+1,l+1\}}\|u\|_{m+3})\|\triangle(u-u_h)\|_{-1}\\
&\lesssim h^{\min\{1,k-1\}}(|||u-u_h|||+h^{\min\{m+1,l+1\}}\|u\|_{m+3})|u-u_h|_1.
\end{aligned}
$$

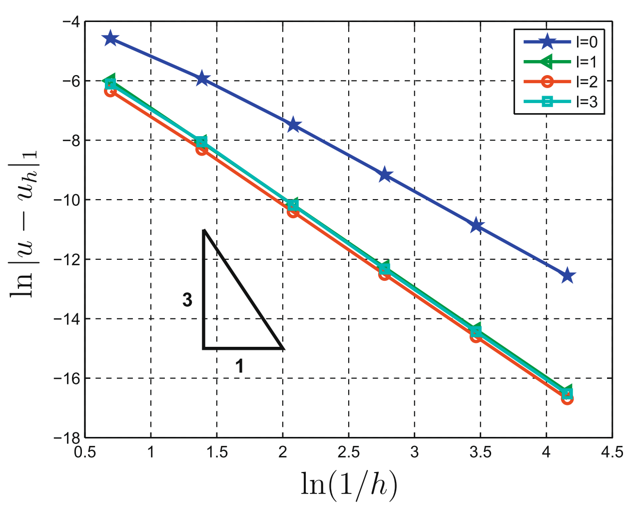

Fig. 3. Error $|u-u_h|_1$ vs $1/h$ in ln – ln for Example 4.2.

Table 4
Error $|u-u_h|_{2,h}$ for Example 4.2.

h	l			
	0	1)	2	3
2^{-1}	5.3236E−02	3.9586E−02	2.9261E−02	2.8676E−02
2^{-2}	2.8347E−02	1.1407E−02	9.1097E−03	8.8621E−03
2^{-3}	1.2726E−02	2.9990E−03	2.4285E−03	2.3507E−03
2^{-4}	4.9306E−03	7.5159E−04	6.1440E−04	5.9585E−04
2^{-5}	1.8685E−03	1.8631E−04	1.5388E−04	1.4940E−04
2^{-6}	7.3637E−04	4.6273E−05	3.8473E−05	3.7365E−05

This with Theorem 3.5 immediately leads to

$$
\begin{aligned}
|u-u_h|_1 &\lesssim h^{\min\{1,k-1\}}(|||u-u_h|||+h^{\min\{m+1,l+1\}}\|u\|_{m+3})\\
&\lesssim h^{\min\{1,k-1\}+\min\{m+1,k-1,l+1\}}\|u\|_{m+3}.
\end{aligned}
$$

Finally, note that $u-u_h\in H_0^1(\Omega)$ and over $H_0^1(\Omega)$, the semi-norm $|\cdot|_1$ is equivalent to the $H^1(\Omega)$-norm. □

It is better to include some results to show the optimality of our estimates derived. For example, if we take $S_1(K) = P_l(K)$ and $S_2(K) = P_k(K)$, a choice mostly suitable for practical applications, it is easy

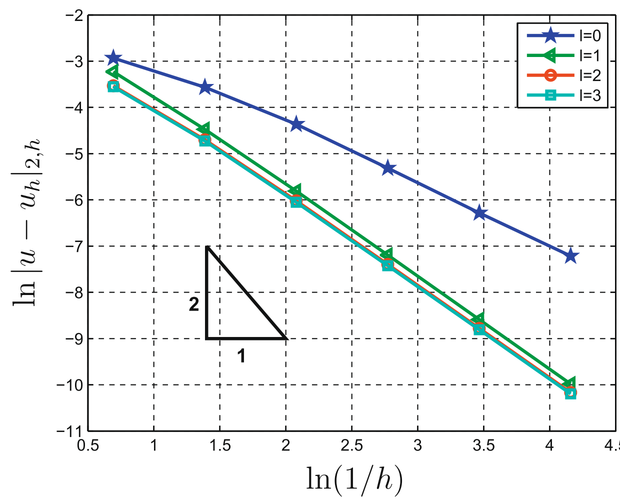

Fig. 4. Error $\|u-u_h\|_{2,h}$ vs $1/h$ in ln – ln scale for Example 4.2.

Table 5
Error $|||u-u_h|||$ for Example 4.2.

h	l			
	0	1	2	3
2^{-1}	1.3639E−01	6.2654E−02	3.2563E−02	3.0111E−02
2^{-2}	6.9217E−02	1.8041E−02	1.1177E−02	9.6508E−03
2^{-3}	2.9829E−02	4.7445E−03	3.1305E−03	2.6043E−03
2^{-4}	1.1344E−02	1.2027E−03	8.0420E−04	6.6835E−04
2^{-5}	4.1340E−03	3.0108E−04	2.0253E−04	1.6845E−04
2^{-6}	1.5114E−03	7.5241E−05	5.0753E−05	4.2215E−05

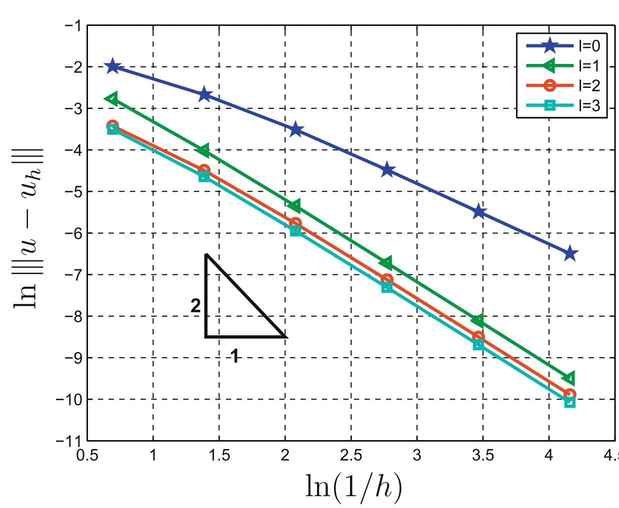

Fig. 5. Error $|||u-u_h|||$ vs $1/h$ in ln – ln scale for Example 4.2.

to check that the first condition of (2.8) implies $k-2 \leq l$. Therefore, we have the following result from the previous theorems directly.

Corollary 3.7. *Assume the solution of* (2.1) *satisfies* $u \in H^{m+3}(\Omega)$, *for some nonnegative integer m, and let* $u_h \in V_h$ *be the solution of* (3.1) *with* $S_1(K) = P_l(K)$ *and* $S_2(K) = P_k(K)$ *for all* $K \in \mathscr{T}_h$. *Then we have the following optimal error estimates*

$$|||u-u_h||| \lesssim h^{\min\{m+1,k-1\}}\|u\|_{m+3,\Omega},$$
$$\|u-u_h\|_1 \lesssim h^{\min\{1,k-1\}+\min\{m+1,k-1\}}\|u\|_{m+3,\Omega}.$$

4. Numerical results

In this section, we present numerical results to show the computational performance of our LCDG method. Let $\Omega = (-1,1)\times(-1,1)$, $\nu = 0.3$, and

$$f(x_1,x_2) = 24(1-x_1^2)^2 + 24(1-x_2^2)^2 + 32(3x_1^2-1)(3x_2^2-1).$$

It can be verified that the exact solution of (2.1) is $u(x_1,x_2) = (1-x_1^2)^2(1-x_2^2)^2$.We use the uniform triangulation $\mathscr{T}_h$ of Ω. For any $K \in \mathscr{T}_h$, we take $S_1(K) = P_l(K)$ and $\mathscr{S}_2(K) = P_k(K)$, with $k \geqslant 1,\ l \geqslant 0$.

Example 4.1. In this example, we consider the numerical results of our LCDG method in the lower order case, that is, $k = 1,2$ and $l = 0,1$. From Table 1 and Fig. 1, we observe that the numerical convergence rates of $|u-u_h|_1$ and $|||u-u_h|||$ are $O(h^2)$ and $O(h)$, respectively, when $k = 2$, but there is no convergence for $k = 1$. These phenomena agree with the theoretical predictions given in Theorems 3.5 and 3.6. When $k = 2$, accuracies of the numerical results are nearly the same for $l = 0$ and $l = 1$. Certainly, it is more convenient to simulate for $l = 0$ than for $l = 1$.

Example 4.2. Now we consider the higher order case for our LCDG method, where $k = 3$ and $l = 0,1,2,3$. Note that the condition (2.4) requires $l \geqslant 1$. The numerical results of L^2 norm of error $\|u-u_h\|_0$ for $l = 0,1,2,3$ are given in Table 2 and Fig. 2, from which we can see that $\|u-u_h\|_0 = O(h^4)$ for $l = 1,2,3$. But the convergence rate for $l = 0$ does not reach $O(h^4)$, only between $O(h^2)$ and $O(h^3)$. Then we examine the numerical values of $|u-u_h|_1$ with respect to h. It is shown in Table 3 and Fig. 3 that $|u-u_h|_1 = O(h^3)$ for $l = 1,2,3$, and the convergence rate for $l = 0$ is less than $O(h^{2.5})$. We also investigate the convergence rates for $|u-u_h|_{2,h}$ and $|||u-u_h|||$. The corresponding results are given in Table 4 and Fig. 4 for $|u-u_h|_{2,h}$ and in Table 5 and Fig. 5 for $|||u-u_h|||$, respectively. We may find from these numerical results that both $|u-u_h|_{2,h}$ and $|||u-u_h|||$ are of the size $O(h^2)$ for $l = 1,2,3$. Again, we can see that the convergence rates of $|u-u_h|_{2,h}$ and $|||u-u_h|||$ are no more than $O(h^{1.5})$ for $l = 0$. From the above data analysis we may conclude that condition (2.4) is crucial for achieving the optimal convergence rates. In our example discussed here, $l \geqslant 1$ implies this condition.

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