Journal of Scientific Computing (2021) 89:8
https://doi.org/10.1007/s10915-021-01614-9

Mixed Finite Element Method for a Hemivariational Inequality of Stationary Navier–Stokes Equations

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Received: 24 April 2021 / Revised: 26 July 2021 / Accepted: 30 July 2021

Abstract
In this paper, we develop and study the mixed finite element method for a hemivariational inequality of the stationary Navier–Stokes equations (NS hemivariational inequality). The NS hemivariational inequality models the motion of a viscous incompressible fluid in a bounded domain, subject to a nonsmooth and nonconvex slip boundary condition. The incompressibility contraint is treated through a mixed formulation. Solution existence and uniqueness are explored. The mixed finite element method is applied to solve the NS hemivariational inequality and error estimates are derived. Numerical results are reported on the use of the P1b/P1 pair, illustrating the optimal convergence order predicted by the error analysis.

Keywords Navier–Stokes equations · Hemivariational inequality · Existence · Uniqueness · Mixed finite element method · Error estimation

Mathematics Subject Classification 65N30 · 76D05 · 47J20

This work is supported by the National Natural Science Foundation of China (No. 12001413) and the Postdoctoral Science Foundation of China (No. 2021M692647).

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Published online: 21 August 2021

1 Introduction

Since Fujita's pioneering work [13,14], there has been steady progress on the modeling, mathematical analysis and numerical approximation of boundary or initial-boundary value problems for flows of viscous incompressible fluid involving nonsmooth slip or leak boundary conditions, cf. e.g., [21,24–27,29,33,44,48] on variational inequalities governed by the Stokes equations, and [9,10,23,28,30,32,43] on variational inequalities governed by the Navier–Stokes equations. In these references, the slip and leak boundary conditions are expressed by monotone relations between physical quantities, and thus the mathematical formulations of the problems are in the form of variational inequalities. When the nonsmooth boundary conditions involve non-monotone relations between physical quantities, the mathematical formulations become hemivariational inequalities. The mixed finite element method for a stationary hemivariational inequality of the Stokes equations with the slip boundary condition has been studied in [11]. In this paper, we consider the mixed finite element method for a hemivariational inequality of the stationary Navier–Stokes equations (NS hemivariational inequality). Navier–Stokes equations have been the research topic of much effort on their mathematical theories, numerical solutions, computer simulations, and applications, cf. [15,46,47] for comprehensive coverages of the subject. A stationary hemivariational inequality for the Navier–Stokes equations is studied in [37], and optimal control problems related to the stationary hemivariational inequality are considered in [36]. The NS hemivariational inequality considered in this paper is different from that found in [36,37]. The NS hemivariational inequality models the motion of a viscous incompressible fluid in a bounded domain, subject to a nonsmooth and nonconvex slip boundary condition. The incompressibility constraint is treated through a mixed formulation. We explore the solution existence and uniqueness, and study the mixed finite element method for the NS hemivariational inequality.

The notion of hemivariational inequalities was first introduced by Panagiotopoulos in early 1980s [41] and it is closely related to the development of the concept of the generalized directional derivative and subdifferential of a locally Lipschitz functional in the sense of Clarke [6,7]. Hemivariational inequalities provide mathematical formulations to treat successfully problems involving nonmonotone, nonsmooth and multivalued constitutive laws, forces, and boundary conditions. During the last four decades, hemivariational inequalities have been shown to be very useful for a large number of problems across a variety of subjects. The mathematical literature dedicated to this field is growing rapidly. The theory, numerical solution and applications of hemivariational inequalities can be found in several monographs [5,38–40,42,45] and in numerous journal articles. Analysis of the finite element method for solving hemivariational inequalities can be found in the monograph [22] where convergence of finite element solutions and solution algorithms are discussed. More recently, optimal order error bounds have been derived for finite element solutions of various kinds of hemivariational inequalities, including elliptic ones [16,18,19], evolutionary ones [3,20], history-dependent ones [49], cf. a summarizing account in [17]. Other numerical methods can be also applied to solve hemivariational inequalities, e.g., the virtual element method is applied and analyzed in [12] for the numerical solution of hemivariational inequalities in contact mechanics, a nonconforming virtual element method is developed and analyzed in [35] for solving a stationary Stokes hemivariational inequality with a slip boundary condition.

The rest of the paper is organized as follows. In Sect. 2 we present some definitions and auxiliary material. In Sect. 3, we introduce the NS hemivariational inequality, and comment on the solution existence and uniqueness. In Sect. 4, we study the mixed finite element method of the NS hemivariational inequality and derive error estimates. In Sect. 5, we report numerical results on some examples using the P1b/P1 pair, illustrating the optimal convergence order predicted by the theoretical error analysis.

2 Preliminaries

All the function spaces in this paper are real. For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. The symbol X_w is used for the space X endowed with the weak topology. Weak convergence will be indicated by the symbol $\rightharpoonup$. The symbol 2^{X^*} represents the set of all subsets of X^*. For simplicity in exposition, in the following we always assume X is a Banach space, unless stated otherwise.

We first recall the definition of generalized directional derivative and subdifferential in the sense of Clarke for a locally Lipschitz function [7].

Definition 2.1 Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of ψ at $x \in X$ in the direction $v \in X$, denoted by $\psi^0(x; v)$, is defined by

$$\psi^0(x;v) = \limsup_{y\to x,\lambda\downarrow 0} \frac{\psi(y+\lambda v) - f(y)}{\lambda}.$$

The generalized gradient or subdifferential of ψ at x, denoted by $\partial\psi(x)$, is a subset of the dual space X^* given by

$$\partial\psi(x) = \{\zeta \in X^* \mid \psi^0(x;v) \geq \langle\zeta, v\rangle_{X^*\times X}\ \forall\, v \in X\}.$$

We list two properties that will be needed later in the paper. The first property is the formula

$$\psi^0(u;v) = \max\{\langle\zeta, v\rangle \mid \zeta \in \partial\psi(u)\} \quad \forall\, u, v \in X. \tag{2.1}$$

This formula provides the generalized directional derivative once the subdifferential is known. The second property is the sub-additivity rule:

$$\psi^0(u; v_1 + v_2) \leq \psi^0(u; v_1) + \psi^0(u; v_2) \quad \forall\, u, v_1, v_2 \in X. \tag{2.2}$$

More properties and detailed discussions of the generalizes directional derivative and subdifferential in the sense of Clarke can be found in [7,38].

We then recall the definition of pseudomonotonicity of a single-valued operator [50].

Definition 2.2 A single-valued operator $F : X \to X^*$ is said to be pseudomonotone, if

(i) F is bounded, i.e., it maps bounded subsets of X into bounded subsets of X^*;
(ii) $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply

$$\langle Fu, u - v\rangle_{X^*\times X} \leq \liminf_{n\to\infty}\langle Fu_n, u_n - v\rangle_{X^*\times X} \quad \forall\, v \in X.$$

It can be proved (see [37], for example) that an operator $F : X \to X^*$ is pseudomonotone iff it is bounded and $u_n \rightharpoonup u$ in X together with $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply $Fu_n \rightharpoonup Fu$ in X^* and $\lim_{n\to\infty}\langle Fu_n, u_n - u\rangle_{X^*\times X} = 0$.

Now consider the case of a multi-valued operator $F\colon X \to 2^{X^*}$. The domain of F is

$$D(F) = \{x \in X \mid F(x) \neq \emptyset\}.$$

Definition 2.3 [38] Let X be a reflexive Banach space. A multi-valued operator $F : X \to 2^{X^*}$ is pseudomonotone if the following conditions hold:

(a) F has values which are nonempty, bounded, closed and convex;
(b) F is upper semicontinuous from each finite dimensional subspace of X into X^*_w;
(c) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \rightharpoonup u$ in X, $u_n^* \in Fu_n$ and $\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0$, we have that for every $v \in X$, there exists $u^*(v) \in Fu$ such that

$$\langle u^*(v), u - v\rangle_{X^*\times X} \le \liminf_{n\to\infty}\langle u_n^*(v), u_n - v\rangle_{X^*\times X}.$$

The following proposition is usually used to check the pseudomonotonicity of an operator.

Proposition 2.4 [8] *Let X be a real reflexive Banach space. Suppose $F : X \to 2^{X^*}$ is bounded and for each $v \in X$, $F(v)$ is a nonempty, closed and convex subset of X^*. Moreover, if $v_n \rightharpoonup v$ in X, $v_n^* \rightharpoonup v^*$ in X^* with $v_n^* \in F(v_n)$, and $\limsup_{n\to\infty}\langle v_n^*, v_n - v\rangle_{X^*\times X} \le 0$, then $v^* \in F(v)$ and $\langle v_n^*, v_n\rangle_{X^*\times X} \to \langle v^*, v\rangle_{X^*\times X}$. Then the operator F is pseudomonotone.*

We will need the notion of coercivity.

Definition 2.5 An operator $F : X \to 2^{X^*}$ is coercive if either $D(F)$ is bounded or $D(F)$ is unbounded and

$$\lim_{\|u\|_X\to\infty, u\in D(F)} \frac{\inf\{\langle u^*, u\rangle_{X^*\times X} \mid u^* \in Fu\}}{\|u\|_X} = +\infty.$$

The following is the main surjectivity result for pseudomonotone and coercive operators.

Theorem 2.6 [8] *Let X be a reflexive Banach space and $F : X \to 2^{X^*}$ be pseudomonotone and coercive. Then F is surjective, i.e., for any $f \in X^*$, there exists an element $x \in X$ such that $f \in F(x)$.*

3 The NS Hemivariational Inequality

Let Ω be an open bounded connected set in $\mathbb{R}^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary Γ. A generic point in Ω or on Γ is denoted by $\boldsymbol{x}$. Let $\mathbb{S}^d$ be the space of symmetric matrices of order d. In $\mathbb{R}^d$ and $\mathbb{S}^d$, we use the standard inner products

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and below, we adopt the summation convention over a repeated index, e.g., $u_i v_i$ stands for $\sum_{i,j=1}^d u_i v_i$.

Consider the Navier–Stokes equations for steady flows of incompressible viscous fluids:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.1}$$
$$\text{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3.2}$$

where $\boldsymbol{u}(\boldsymbol{x})$ is the flow velocity field, $\nu > 0$ the kinematic viscosity, $p(\boldsymbol{x})$ the pressure, and $\boldsymbol{f}(\boldsymbol{x})$ the density of external forces. The Eq. (3.2) reflects the incompressibility constraint.

The Navier–Stokes equations are to be supplemented by boundary conditions. We assume the boundary $\Gamma = \partial\Omega$ is split into two non-overlapping parts Γ_0 and Γ_1, both with non-zero measures. Let $\boldsymbol{n} = (n_1, \ldots, n_d)^\top$ be the unit outward normal on the boundary Γ. For a vector $\boldsymbol{v}$ defined on Γ, denote by $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_n \boldsymbol{n}$ the normal and tangential component of the vector $\boldsymbol{v}$, respectively. Given the flow velocity $\boldsymbol{u}$ and the pressure p, we let $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$ be the rate of deformation tensor, and let $\boldsymbol{\sigma} = 2\,\nu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\,\boldsymbol{I}$ be the stress tensor, where $\boldsymbol{I}$ is the unit tensor (identity matrix). Restricting $\boldsymbol{\sigma}$ to the boundary Γ, we denote by $\sigma_n = \boldsymbol{n} \cdot \boldsymbol{\sigma} \boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n \boldsymbol{n}$ the normal and tangential components of $\boldsymbol{\sigma}$ on Γ. For the boundary condition, we consider

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{3.3}$$

$$u_n = 0, \ -\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_1, \tag{3.4}$$

where $j(\boldsymbol{u}_\tau)$ is a short-hand notation for $j(\boldsymbol{x}, \boldsymbol{u}_\tau)$. The function $j : \Gamma_1 \times \mathbb{R}^d \to \mathbb{R}$ is called a superpotential; we assume it is locally Lipschitz with respect to its last argument and denote by ∂j the subdifferential of $j(\boldsymbol{x}, \cdot)$ in the sense of Clarke. The condition (3.4) is known as a slip boundary condition. Note that in the case where the function $j(\boldsymbol{x}, u_n)$ is convex with respect to its last argument, the problem (3.1)–(3.4) leads to a variational inequality. Here, we do not assume the convexity of j with respect to its last argument, and then the problem corresponds to a hemivariational inequality.

To present a weak formulation of the problem (3.1)–(3.4), we need some function spaces. For the velocity variable, we will use the space

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0, \ v_n = 0 \text{ on } \Gamma_1 \right\}. \tag{3.5}$$

This is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega; \mathbb{S}^d)}$$

thanks to Korn's inequality and the assumption $|\Gamma_0| > 0$. In the formulation of the reduced problem, we will use the following subspace of V:

$$\tilde{V} = \{ \boldsymbol{v} \in V \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \}. \tag{3.6}$$

Let

$$H = L^2(\Omega; \mathbb{R}^d). \tag{3.7}$$

Then

$$V \subset H = H^* \subset V^*$$

and

$$\tilde{V} \subset H = H^* \subset \tilde{V}^*$$

with all embeddings being dense and compact. We further denote

$$V_0 = H_0^1(\Omega; \mathbb{R}^d), \tag{3.8}$$

$$\tilde{V}_0 = \{ \boldsymbol{v} \in V_0 \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \}. \tag{3.9}$$

For the pressure variable, we will use the space

$$Q = \left\{ q \in L^2(\Omega) \mid (q, 1)_\Omega = 0 \right\}. \tag{3.10}$$

We then introduce two bilinear forms and one trilinear form: for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $q \in Q$,

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega 2\,\nu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \tag{3.11}$$

$$d(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v} \, dx, \tag{3.12}$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{w} \, dx. \tag{3.13}$$

Moreover, for $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^d)$, write

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx, \quad \boldsymbol{v} \in V.$$

Observe that the bilinear form $a(\cdot, \cdot)$ is bounded and elliptic on V. Indeed,

$$|a(\boldsymbol{u}, \boldsymbol{v})| \le 2\,\nu \, \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.14}$$

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2\,\nu \, \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V, \tag{3.15}$$

The bilinear form $d(\cdot, \cdot)$ is bounded in $V \times Q$:

$$|d(\boldsymbol{v}, q)| \le c \, \|\boldsymbol{v}\|_V \|q\|_Q \quad \forall\, \boldsymbol{v} \in V, \; q \in Q. \tag{3.16}$$

The trilinear form $b(\cdot, \cdot, \cdot)$ is bounded over V: for a constant $c_b > 0$,

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le c_b \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{3.17}$$

Moreover,

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.18}$$

Concerning the superpotential j, we assume the following hypothesis:
$H(j)$: $j : \Gamma_1 \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j(\cdot, \boldsymbol{\xi})$ is measurable on Γ_1 for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $j(\cdot, \boldsymbol{0}) \in L^1(\Gamma_1)$;
(ii) $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_1$;
(iii) $|\boldsymbol{\eta}| \le c_0 + c_1 |\boldsymbol{\xi}|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\eta} \in \partial j(\boldsymbol{x}, \boldsymbol{\xi})$, a.e. $\boldsymbol{x} \in \Gamma$ with $c_0, c_1 \ge 0$;
(iv) $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -\alpha \, |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2$ for all $\boldsymbol{\xi}_i \in \mathbb{R}^d$, $\boldsymbol{\eta}_i \in \partial j(\boldsymbol{x}, \boldsymbol{\xi}_i)$, $i = 1, 2$, a.e. $\boldsymbol{x} \in \Gamma$ with $\alpha \ge 0$.

The condition $H(j)$(iv) is known as a relaxed monotonicity condition in the literature [38, Definition 3.49], and it can be written equivalently as

$$j^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j^0(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le \alpha \, |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d.$$

Now we consider the functional $J : L^2(\Gamma; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(\boldsymbol{v}) = \int_{\Gamma_1} j(\boldsymbol{x}, \boldsymbol{v}_\tau(\boldsymbol{x})) \, ds, \quad \boldsymbol{v} \in L^2(\Gamma_1; \mathbb{R}^d). \tag{3.19}$$

The next result is proved in [37] (with some modification).

Lemma 3.1 *Assume that $j : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis $H(j)$. Then the functional J defined by (3.19) has the following properties.*

(i) *$J(\cdot)$ is locally Lipschitz in $L^2(\Gamma_1; \mathbb{R}^d)$;*
(ii) *$\|\boldsymbol{z}\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le c_0 |\Gamma_1|^{1/2} + c_1 \|\boldsymbol{v}\|_{L^2(\Gamma_1;\mathbb{R}^d)}$ $\forall\, \boldsymbol{v} \in L^2(\Gamma_1; \mathbb{R}^d)$, $\boldsymbol{z} \in \partial J(\boldsymbol{v})$.*
(iii) *$J^0(\boldsymbol{u}; \boldsymbol{v}) \le \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau(\boldsymbol{x}); \boldsymbol{v}_\tau(\boldsymbol{x})) \, ds$ $\forall\, \boldsymbol{u}, \boldsymbol{v} \in L^2(\Gamma_1; \mathbb{R}^d)$.*
(iv) *$(\boldsymbol{z}_1 - \boldsymbol{z}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_{L^2(\Gamma_1;\mathbb{R}^d)} \ge -\alpha \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^2(\Gamma_1)^d}$ $\forall\, \boldsymbol{z}_i \in \partial J(\boldsymbol{u}_i)$, $\boldsymbol{u}_i \in L^2(\Gamma_1; \mathbb{R}^d)$, $i = 1, 2$.*

In the derivation of a weak formulation of the problem (3.1)–(3.4), it is convenient to rewrite (3.1) as

$$-2\,\nu\,\mathrm{div}(\boldsymbol{\varepsilon}(\boldsymbol{u})) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega. \tag{3.20}$$

Assume the problem (3.1)–(3.4) has a smooth solution $(\boldsymbol{u}, p)$ so that the calculations in the following derivation are meaningful. We multiply the Eq. (3.20) by an arbitrary smooth test function $\boldsymbol{v}$, i.e., $\boldsymbol{v} \in V$ is smooth and integrate over Ω,

$$\int_\Omega [-2\,\nu\,\mathrm{div}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} + \nabla p\cdot\boldsymbol{v}]\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx.$$

Perform integration by parts to get

$$\int_\Omega [2\,\nu\,\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - p\,\mathrm{div}\boldsymbol{v}]\,dx - \int_\Gamma \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{v}\,ds = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx.$$

Using the boundary conditions satisfied by $\boldsymbol{v}$, we have

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{v}, p) + \int_{\Gamma_1} (-\boldsymbol{\sigma}_\tau)\cdot\boldsymbol{v}_\tau ds = \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{3.21}$$

By (3.4),

$$-\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_1;$$

then,

$$\int_{\Gamma_1} (-\boldsymbol{\sigma}_\tau)\cdot\boldsymbol{v}_\tau ds \le \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\,ds.$$

So for $\boldsymbol{v} \in V$ smooth, we have

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{v}, p) + \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle.$$

Then we multiply (3.2) by an arbitrary function $q \in Q$ and integrate over Ω to get

$$d(\boldsymbol{u}, q) = 0.$$

Summarizing, we have derived the following hemivariational inequality for the problem (3.1)–(3.4).

Problem 3.2 *Find $\boldsymbol{u} \in V$ and $p \in Q$ such that*

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{v}, p) + \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v} \in V, \tag{3.22}$$

$$d(\boldsymbol{u}, q) = 0 \quad \forall\,q \in Q. \tag{3.23}$$

We can eliminate the unknown variable p to get a reduced hemivariational inequality.

Problem 3.3 *Find $\boldsymbol{u} \in \tilde{V}$ such that*

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v} \in \tilde{V}. \tag{3.24}$$

We first study Problem 3.3. We will need a so-called smallness condition:

$$c_1 < 2\,\nu\,\lambda_0, \tag{3.25}$$

where the constant c_1 is from H(j)(iii) whereas $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda \int_{\Gamma_1} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau ds \quad \forall\, \boldsymbol{v} \in V. \tag{3.26}$$

Note the trace inequality

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le \lambda_0^{-1/2} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{3.27}$$

Theorem 3.4 *Under the assumptions $H(j)$* (i)–(iii) *and* (3.25)*, there exists a solution $\boldsymbol{u} \in \tilde{V}$ to Problem 3.3.*

Proof Define $A\colon V \to V^*$ by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v}), \quad \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.28}$$

and define $B\colon V \times V \to V^*$ by

$$\langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\rangle = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{3.29}$$

Observe that $A \in \mathcal{L}(V; V^*)$ is symmetric and V-elliptic:

$$\langle A\boldsymbol{v}, \boldsymbol{v}\rangle = 2\,\nu\,\|\boldsymbol{v}\|_V^2.$$

The bilinear operator $B\colon V \times V \to V^*$ is continuous, the mapping $\boldsymbol{v} \mapsto B(\boldsymbol{v}, \boldsymbol{v})$ is weakly continuous from V to V^*, and

$$\langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{v}\rangle = 0 \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V.$$

Since $\tilde{V}$ is a subspace of V, by [37, Lemma 9], we know that the mapping $\boldsymbol{v} \mapsto A\boldsymbol{v} + B(\boldsymbol{v}, \boldsymbol{v})$ from $\tilde{V}$ to $\tilde{V}^*$ is coercive and pseudomonotone.

Let $\gamma\colon H^{1/2}(\Gamma; \mathbb{R}^d) \to L^2(\Gamma; \mathbb{R}^d)$ be the embedding operator, and denote by γ^* its adjoint operator. Consider the operator $F\colon \tilde{V} \to 2^{\tilde{V}^*}$ defined by

$$F(\boldsymbol{v}) = A\boldsymbol{v} + B(\boldsymbol{v}, \boldsymbol{v}) + \gamma^*(\partial J(\gamma \boldsymbol{v})).$$

Similar to [37, Proposition 10], we know that F is pseudomonotone and it is coercive under the condition (3.25). The coercivity of F is proved as follows: for all $\boldsymbol{v} \in V$,

$$\begin{aligned}
\langle F(\boldsymbol{v}), \boldsymbol{v}\rangle &\ge \langle A\boldsymbol{v}, \boldsymbol{v}\rangle + \langle B(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{v}\rangle - J^0(\boldsymbol{v}_\tau, \boldsymbol{v}_\tau)\\
&\ge 2\,\nu\,\|\boldsymbol{v}\|_V^2 + 0 - \left(c_0|\Gamma_1|^{1/2} + c_1\|\boldsymbol{v}_\tau\|_{L^2(\Gamma;\mathbb{R}^d)}\right)\|\boldsymbol{v}_\tau\|_{L^2(\Gamma;\mathbb{R}^d)}\\
&\ge \left(2\,\nu - c_1\lambda_0^{-1}\right)\|\boldsymbol{v}\|_V^2 - c_0\lambda_0^{-1/2}\sqrt{|\Gamma_1|}\,\|\boldsymbol{v}\|_V,
\end{aligned}$$

where in the second inequality, we applied Lemma 3.1 (ii) for a lower bound of the term $-J^0(\boldsymbol{v}_\tau, \boldsymbol{v}_\tau)$.

Thus, by Theorem 2.6, the operator $F\colon \tilde{V} \to 2^{\tilde{V}^*}$ is surjective. Then for any $\boldsymbol{f} \in V^* \subset \tilde{V}^*$, there exists $\boldsymbol{u} \in \tilde{V}$ such that

$$\boldsymbol{f} \in F(\boldsymbol{u}),$$

i.e.,

$$\boldsymbol{u} \in \tilde{V}, \quad a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + J^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \tilde{V}. \tag{3.30}$$

Since

$$J^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) \leq \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds,$$

from (3.30) we deduce (3.24). In other words, Problem 3.3 has a solution. □

Let us show that any solution of Problem 3.3 is bounded.

Proposition 3.5 *Keep the assumptions stated in Theorem 3.4. If $\boldsymbol{u} \in \tilde{V}$ is a solution of Problem 3.3, then*

$$\|\boldsymbol{u}\|_V \leq c_2, \tag{3.31}$$

where

$$c_2 = \frac{c_0 \lambda_0^{-1/2} |\Gamma_1|^{1/2} + \|\boldsymbol{f}\|_{V^*}}{2\,\nu - c_1 \lambda_0^{-1}}, \tag{3.32}$$

and the constants c_0, c_1 are from H(j)(iii).

Proof Note that $b(\boldsymbol{u}, \boldsymbol{u}, -\boldsymbol{u}) = 0$ from (3.18). We take $\boldsymbol{v} = -\boldsymbol{u}$ in (3.24) to obtain

$$a(\boldsymbol{u}, \boldsymbol{u}) \leq \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; -\boldsymbol{u}_\tau)\, ds + \langle \boldsymbol{f}, \boldsymbol{u} \rangle.$$

Then, recalling (3.15), $H(j)$(iii) and (3.27),

$$\begin{aligned} 2\,\nu\, \|\boldsymbol{u}\|_V^2 &\leq \left(c_0 |\Gamma_1|^{1/2} + c_1 \|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1)}\right) \|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1)} + \|\boldsymbol{f}\|_{V^*} \|\boldsymbol{u}\|_V \\ &\leq \left(c_0 \lambda_0^{-1/2} |\Gamma_1|^{1/2} + \|\boldsymbol{f}\|_{V^*}\right) \|\boldsymbol{u}\|_V + c_1 \lambda_0^{-1} \|\boldsymbol{u}\|_V^2. \end{aligned}$$

Rearranging the terms in the above inequality, we obtain (3.31). □

Now we turn to the solution uniqueness.

Theorem 3.6 *Keep the assumptions of Theorem 3.4. Assume in addition that*

$$\alpha + c_b c_2 \lambda_0 < 2\,\nu\, \lambda_0. \tag{3.33}$$

Then Problem 3.3 has a unique solution.

Proof By Theorem 3.4, under the stated assumptions, Problem 3.3 has a solution. So we only need to prove the solution uniqueness.

Assume Problem 3.3 has two solutions $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \tilde{V}$. Take $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$ in the defining relation (3.24) for the solution $\boldsymbol{u}_1$ to obtain

$$a(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + b(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + \int_{\Gamma_1} j^0(\boldsymbol{u}_{1\tau}; \boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau})\, ds \geq \langle \boldsymbol{f}, \boldsymbol{u}_2 - \boldsymbol{u}_1 \rangle.$$

Similarly, take $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ in (3.24) for the solution $\boldsymbol{u}_2$,

$$a(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + b(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + \int_{\Gamma_1} j^0(\boldsymbol{u}_{2\tau}; \boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau})\, ds \geq \langle \boldsymbol{f}, \boldsymbol{u}_1 - \boldsymbol{u}_2 \rangle.$$

Add the above two inequalities,

$$a(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2)\le b(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_2-\boldsymbol{u}_1)+b(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2)$$
$$+\int_{\Gamma_1}\left[j^0(\boldsymbol{u}_{1\tau};\boldsymbol{u}_{2\tau}-\boldsymbol{u}_{1\tau})+j^0(\boldsymbol{u}_{2\tau};\boldsymbol{u}_{1\tau}-\boldsymbol{u}_{2\tau})\right]ds. \tag{3.34}$$

Write

$$b(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_2-\boldsymbol{u}_1)+b(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2)$$
$$=b(\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_2-\boldsymbol{u}_1)+b(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_2-\boldsymbol{u}_1),$$

and note that

$$b(\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_2-\boldsymbol{u}_1)=0,$$
$$b(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_2-\boldsymbol{u}_1)\le c_b\|\boldsymbol{u}_2\|_V\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2.$$

The norm $\|\boldsymbol{u}_2\|_V$ is bounded by (3.31). Then from (3.34),

$$2\,\nu\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2\le c_bc_2\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2+\alpha\,\|\boldsymbol{u}_{1\tau}-\boldsymbol{u}_{2\tau}\|_{L^2(\Gamma_1)^d}^2$$
$$\le\left(c_bc_2+\alpha\lambda_0^{-1}\right)\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2.$$

Thus, because of (3.33), we conclude that $\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V=0$, i.e., $\boldsymbol{u}_1=\boldsymbol{u}_2$. □

We turn to a study of Problem 3.2. Recall the inf-sup condition [46]

$$\beta_0\|q\|_Q\le\sup_{\boldsymbol{v}\in V_0}\frac{d(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_V}\quad\forall\,q\in Q. \tag{3.35}$$

Theorem 3.7 *Under the assumptions of Theorem 3.6, Problem 3.2 has a unique solution.*

Proof Let $\boldsymbol{u}\in\tilde{V}$ be the unique solution of Problem 3.3. Then,

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{f},\boldsymbol{v}\rangle\quad\forall\,\boldsymbol{v}\in\tilde{V}_0.$$

Thanks to the inf-sup condition (3.35), by a classical result from functional analysis [4, Chapter 4], there is a function $p\in Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})+d(\boldsymbol{v},p)=\langle\boldsymbol{f},\boldsymbol{v}\rangle\quad\forall\,\boldsymbol{v}\in V_0. \tag{3.36}$$

Now for an arbitrary $\boldsymbol{v}\in V$, due to the inf-sup condition (3.35), we can find $\boldsymbol{v}_1\in V_0$ such that

$$d(\boldsymbol{v}_1,q)=d(\boldsymbol{v},q)\quad\forall\,q\in Q. \tag{3.37}$$

Define $\boldsymbol{v}_2=\boldsymbol{v}-\boldsymbol{v}_1$. Then $\boldsymbol{v}_2\in V$ and

$$d(\boldsymbol{v}_2,q)=0\quad\forall\,q\in Q.$$

Hence, $\boldsymbol{v}_2\in\tilde{V}$. Use this element $\boldsymbol{v}_2$ as the test function in (3.24),

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{v}_1)+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}-\boldsymbol{v}_1)+\int_{\Gamma_1}j^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau-\boldsymbol{v}_{1\tau})\,ds\ge\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{v}_1\rangle.$$

Note that $\boldsymbol{v}_1=\boldsymbol{0}$ on Γ_1. So we derive from the above inequality that, with the use of (3.36) and (3.37) in turn,

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq a(\boldsymbol{u}, \boldsymbol{v}_1) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_1) + \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{v}_1 \rangle$$
$$= -d(\boldsymbol{v}_1, p) + \langle \boldsymbol{f}, \boldsymbol{v} \rangle$$
$$= -d(\boldsymbol{v}, p) + \langle \boldsymbol{f}, \boldsymbol{v} \rangle,$$

i.e., (3.22) holds. Observe that (3.23) follows from $\boldsymbol{u} \in \tilde{V}$.

Finally, we prove the uniqueness of a solution $(\boldsymbol{u}, p)$ of Problem 3.2. Since the solution component $\boldsymbol{u}$ satisfies (3.24) and Problem 3.3 has a unique solution, we conclude that $\boldsymbol{u}$ is unique. Now assume $(\boldsymbol{u}, p_1)$ and $(\boldsymbol{u}, p_2)$ are two solutions of Problem 3.2. Then by (3.36),

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, p_1) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V_0,$$
$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, p_2) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V_0.$$

Hence,

$$d(\boldsymbol{v}, p_1 - p_2) = 0 \quad \forall\, \boldsymbol{v} \in V_0.$$

Applying the inf-sup condition (3.35) to $q = p_1 - p_2$, we find that $\|p_1 - p_2\|_Q = 0$, i.e., $p_1 = p_2$. □

From the proof of Theorem 3.7, we see that Problems 3.2 and 3.3 are equivalent, i.e., if $(\boldsymbol{u}, p) \in V \times Q$ is the solution of Problem 3.2, then $\boldsymbol{u} \in \tilde{V}$ and $\boldsymbol{u}$ is the solution of Problem 3.3; conversely, if $\boldsymbol{u} \in \tilde{V}$ is the solution of Problem 3.3, then there exists a unique $p \in Q$ such that $(\boldsymbol{u}, p) \in V \times Q$ is the solution of Problem 3.2.

4 Mixed Finite Element Method for NS Hemivariational Inequality

We keep the assumptions stated in Theorem 3.6 so that Problems 3.2 and 3.3 have unique solutions. In this section, we focus on the mixed finite element method to solve Problem 3.2.

For simplicity, assume Ω is a polygonal/polyhedral domain and let $\{\mathcal{T}^h\}$ be a regular family of finite element partitions of $\overline{\Omega}$. Corresponding to a partition $\mathcal{T}^h$, let V^h and Q^h be finite element spaces approximating V and Q. Denote $V_0^h = V^h \cap H_0^1(\Omega)$. We assume the discrete inf-sup condition is valid: for a constant $\beta > 0$ independent of $h > 0$,

$$\beta\, \|q^h\|_Q \leq \sup_{\boldsymbol{v}^h \in V_0^h} \frac{d(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_V} \quad \forall\, q^h \in Q^h. \tag{4.1}$$

The finite element approximation of Problem 3.2 is the following.

Problem 4.1 *Find $\boldsymbol{u}^h \in V^h$ and $p^h \in Q^h$ such that*

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{v}^h, p^h) + \int_{\Gamma_1} j^0(\boldsymbol{u}^h_\tau; \boldsymbol{v}^h_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in V^h, \tag{4.2}$$
$$d(\boldsymbol{u}^h, q^h) = 0 \quad \forall\, q^h \in Q^h. \tag{4.3}$$

Similar to the case of continuous problems, we define a subspace of V^h:

$$\tilde{V}^h = \left\{ \boldsymbol{v}^h \in V^h \mid d(\boldsymbol{v}^h, q^h) = 0\, \forall\, q^h \in Q^h \right\},$$

and introduce a reduced version of Theorem 4.1.

Problem 4.2 *Find $\boldsymbol{u}^h \in \tilde{V}^h$ such that*

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) + \int_{\Gamma_1} j^0(\boldsymbol{u}^h_\tau; \boldsymbol{v}^h_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in \tilde{V}^h. \tag{4.4}$$

Under the assumptions stated in Theorem 3.6 and (4.1), similar to Theorems 3.6 and 3.7, we can show that Problems 4.1 and 4.2 have unique solutions and the two problems are equivalent. Moreover, the discrete analogue of Proposition 3.5 is

$$\|\boldsymbol{u}^h\|_V \leq c_2, \tag{4.5}$$

where c_2 is defined by (3.32).

Let us bound the error. Recall (3.36),

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V_0. \tag{4.6}$$

The discrete analogue of (4.6) is derived from (4.2),

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{v}^h, p^h) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in V_0^h. \tag{4.7}$$

Let $q^h \in Q^h$ be arbitrary. Write

$$\|p - p^h\|_Q \leq \|p - q^h\|_Q + \|p^h - q^h\|_Q.$$

By (4.1),

$$\beta\, \|p^h - q^h\|_Q \leq \sup_{\boldsymbol{v}^h \in V_0^h} \frac{d(\boldsymbol{v}^h, p^h - q^h)}{\|\boldsymbol{v}^h\|_V}. \tag{4.8}$$

Now

$$d(\boldsymbol{v}^h, p^h - q^h) = d(\boldsymbol{v}^h, p^h - p) + d(\boldsymbol{v}^h, p - q^h),$$

and

$$d(\boldsymbol{v}^h, p^h - p) = d(\boldsymbol{v}^h, p^h) - d(\boldsymbol{v}^h, p).$$

By (4.7) and (4.6),

$$d(\boldsymbol{v}^h, p^h) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle - a(\boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h),$$
$$d(\boldsymbol{v}^h, p) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle - a(\boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h).$$

So

$$\begin{aligned} d(\boldsymbol{v}^h, p^h - p) &= a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) \\ &= a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h). \end{aligned}$$

Hence, from (4.8), we have

$$\beta\, \|p^h - q^h\|_Q \leq c \left(1 + \|\boldsymbol{u}\|_V + \|\boldsymbol{u}^h\|_V\right) \|\boldsymbol{u} - \boldsymbol{u}^h\|_V + c\, \|p - q^h\|_Q.$$

Since $\|\boldsymbol{u}\|_V$ and $\|\boldsymbol{u}^h\|_V$ are bounded by the constant c_2,

$$\|p^h - q^h\|_Q \leq c \left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|p - q^h\|_Q\right).$$

Using the triangle inequality

$$\|p - p^h\|_Q \leq \|p - q^h\|_Q + \|p^h - q^h\|_Q,$$

we then derive the inequality

$$\|p - p^h\|_Q \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|p - q^h\|_Q\right). \tag{4.9}$$

On the other hand, for any $\boldsymbol{v}^h \in V^h$,

$$\begin{aligned}
2\,\nu\,\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 &\le a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h)\\
&= a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h) - a(\boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h)\\
&= a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h) - a(\boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + a(\boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h).
\end{aligned} \tag{4.10}$$

By (3.22) with $\boldsymbol{v} = \boldsymbol{u}^h - \boldsymbol{u}$,

$$a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h) \le b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{u}) + d(\boldsymbol{u}^h - \boldsymbol{u}, p) + \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{u}_\tau)\,ds - \langle \boldsymbol{f}, \boldsymbol{u}^h - \boldsymbol{u}\rangle.$$

By (4.2) with $\boldsymbol{v}^h$ replaced by $\boldsymbol{v}^h - \boldsymbol{u}^h$,

$$\begin{aligned}
a(\boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) &\le b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + d(\boldsymbol{v}^h - \boldsymbol{u}^h, p^h)\\
&+ \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\,ds - \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h\rangle.
\end{aligned}$$

Also,

$$-a(\boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) = a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + a(\boldsymbol{u}, \boldsymbol{v}^h - \boldsymbol{u}),$$

and by (3.22) with $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{v}^h$,

$$a(\boldsymbol{u}, \boldsymbol{v}^h - \boldsymbol{u}) \le b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}^h) + d(\boldsymbol{u} - \boldsymbol{v}^h, p) + \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h)\,ds - \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{v}^h\rangle.$$

Using these inequalities in (4.10),

$$2\,\nu\,\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 \le a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + I_b + I_d + I_j, \tag{4.11}$$

where

$$\begin{aligned}
I_b &= b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{u}) + b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}^h),\\
I_d &= d(\boldsymbol{u}^h - \boldsymbol{u}, p) + d(\boldsymbol{v}^h - \boldsymbol{u}^h, p^h) + d(\boldsymbol{u} - \boldsymbol{v}^h, p),\\
I_j &= \int_{\Gamma_1} \left[j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{u}_\tau) + j^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h) + j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h)\right] ds.
\end{aligned}$$

Write

$$\begin{aligned}
I_b &= b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) + b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h)\\
&= b(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) + b(\boldsymbol{u}, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) + b(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h)\\
&= b(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) + b(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h)\\
&= b(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + b(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h),
\end{aligned}$$

where in the last step, we used the fact that $b(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{u}) = 0$. Then we bound I_b by

$$I_b \le c\,\|\boldsymbol{u}\|_V\|\boldsymbol{u} - \boldsymbol{u}^h\|_V\|\boldsymbol{u} - \boldsymbol{v}^h\|_V + c_b c_2\|\boldsymbol{u} - \boldsymbol{u}^h\|_V\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V.$$

Using (3.23) and (4.3) repeatedly, we have

$$
\begin{aligned}
I_d &= d(\boldsymbol{u}^h - \boldsymbol{u}, p - q^h) + d(\boldsymbol{v}^h, p^h) - d(\boldsymbol{v}^h, p) \\
&= d(\boldsymbol{u}^h - \boldsymbol{u}, p - q^h) + d(\boldsymbol{v}^h, p^h - p) \\
&= d(\boldsymbol{u}^h - \boldsymbol{u}, p - q^h) + d(\boldsymbol{v}^h - \boldsymbol{u}, p^h - p).
\end{aligned}
$$

Thus,

$$
I_d \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|p - q^h\|_Q + \|\boldsymbol{u} - \boldsymbol{v}^h\|_V \|p - p^h\|_Q\right).
$$

By the sub-additivity (2.2),

$$
j^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h) \le j^0(\boldsymbol{u}_\tau^h; \boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h) + j^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau).
$$

So I_j can be bounded by

$$
\begin{aligned}
I_j &\le \int_{\Gamma_1} \left[j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{u}_\tau) + j^0(\boldsymbol{u}_\tau^h; \boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h) + j^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau) + j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h)\right] ds \\
&\le \int_{\Gamma_1} \left[\alpha\, |\boldsymbol{u}_\tau - \boldsymbol{u}_\tau^h|^2 + c\left(1 + |\boldsymbol{u}_\tau| + |\boldsymbol{u}_\tau^h|\right) |\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h|\right] ds \\
&\le \alpha\, \lambda_0^{-1} \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + c\left(1 + \|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)} + \|\boldsymbol{u}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}\right) \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}.
\end{aligned}
$$

Note that $\|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le c\,\|\boldsymbol{u}\|_V$ and $\|\boldsymbol{u}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le c\,\|\boldsymbol{u}^h\|_V$ are both bounded by a constant independent of h. Hence, from (4.11),

$$
\begin{aligned}
2\,\nu\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 \le{}& c\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{u} - \boldsymbol{v}^h\|_V + c_b c_2 \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V \\
&+ c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|p - q^h\|_Q + \|\boldsymbol{u} - \boldsymbol{v}^h\|_V \|p - p^h\|_Q\right) \\
&+ \alpha\, \lambda_0^{-1} \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + c\, \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}.
\end{aligned}
$$

By using the triangle inequality

$$
\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_V \le \|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|\boldsymbol{u} - \boldsymbol{v}^h\|_V
$$

and the modified Cauchy inequality

$$
a\, b \le \epsilon\, a^2 + b^2/(4\,\epsilon), \quad a, b \in \mathbb{R}
$$

for $\epsilon > 0$ arbitrarily small, we get

$$
\begin{aligned}
&\left(2\,\nu - \alpha\, \lambda_0^{-1} - c_b c_2 - \epsilon\right) \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 \\
&\quad \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|p - q^h\|_Q^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}\right) + \epsilon\, \|p - p^h\|_Q^2. \qquad (4.12)
\end{aligned}
$$

By choosing $\epsilon > 0$ sufficiently small, we can combine (4.12) and (4.9) to get

$$
\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + \|p - p^h\|_Q^2 \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|p - q^h\|_Q^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}\right). \qquad (4.13)
$$

As examples, consider P1b/P1 finite elements [2]

$$
V_h = \{\boldsymbol{v}_h \in V \cap C^0(\overline{\Omega})^d : \boldsymbol{v}_h|_T \in [P_1(T)]^d \oplus B(T)\ \forall\, T \in \mathcal{T}^h\}, \qquad (4.14)
$$

$$
Q_h = \{q_h \in Q \cap C^0(\overline{\Omega}) : q_h|_T \in P_1(T)\ \forall\, T \in \mathcal{T}^h\}, \qquad (4.15)
$$

or P2/P1 finite elements [15, Chapter II, Corollary 4.1]

$$V_h = \{\boldsymbol{v}_h \in V \cap C^0(\overline{\Omega})^d : \boldsymbol{v}_h|_T \in [P_2(T)]^d \ \forall\, T \in \mathcal{T}^h\}, \tag{4.16}$$

$$Q_h = \{q_h \in Q \cap C^0(\overline{\Omega}) : q_h|_T \in P_1(T) \ \forall\, T \in \mathcal{T}^h\}, \tag{4.17}$$

where $P_k(T)$ represents the space of polynomials of a total degree less than or equal to k in T, and $B(T)$ is the space of bubble functions on T. For these choices, the discrete inf-sup condition (4.1) is satisfied. We can then derive an optimal order error estimate for the P1b/P1 element solution from (4.13) and standard finite element interpolation error bounds, under certain solution regularity assumptions. We write Γ_1 as the union of a finite number of plat components:

$$\Gamma_1 = \cup_{i=1}^{i_0} \Gamma_{1,i},$$

where each $\Gamma_{1,i}$ is a line segment in 2D or a polygon in 3D.

Theorem 4.3 *Keep the assumptions of Theorem 3.6. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of Problems 3.2 and 4.1 with the P1b/P1 elements (4.14)–(4.15). Assume the solution regularities $\boldsymbol{u} \in H^2(\Omega)^d$, $\boldsymbol{u}_\tau|_{\Gamma_{1,i}} \in H^2(\Gamma_{1,i})^d$, $1 \le i \le i_0$, and $p \in H^1(\Omega)$. Then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_{L^2(\Omega)} \le c\, h. \tag{4.18}$$

5 Numerical Results

In the numerical experiments, we take the function j in the form

$$j(z) = \int_0^{|z|} \mu(t)\, dt,$$

where $\mu : [0, \infty) \to \mathbb{R}$ is continuous, $\mu(0) > 0$. Then the slip boundary condition $(-\boldsymbol{\sigma}_\tau) \in \partial j(\boldsymbol{u}_\tau)$ is equivalent to

$$|\boldsymbol{\sigma}_\tau| \le \mu(0) \text{ if } \boldsymbol{u}_\tau = \boldsymbol{0}, \quad -\boldsymbol{\sigma}_\tau = \mu(|\boldsymbol{u}_\tau|)\, \frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}.$$

Introduce a Lagrange multiplier

$$\boldsymbol{\lambda} = \frac{-\boldsymbol{\sigma}_\tau}{\mu(|\boldsymbol{u}_\tau|)}$$

and define a set

$$\Lambda = \left\{ \boldsymbol{\lambda} \in L^2(\Gamma_1; \mathbb{R}^d) \mid |\boldsymbol{\lambda}| \le 1 \text{ a.e. on } \Gamma_1 \right\}.$$

Then the weak formulation for the problem (3.1)–(3.4) can be stated as follows.

Problem 5.1 *Find $\boldsymbol{u} \in V$, $p \in Q$, and $\boldsymbol{\lambda} \in \Lambda$ such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, p) + \int_{\Gamma_1} \mu(|\boldsymbol{u}_\tau|)\, \boldsymbol{\lambda} \cdot \boldsymbol{v}_\tau\, ds = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V, \tag{5.1a}$$

$$d(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q, \tag{5.1b}$$

$$\boldsymbol{\lambda} \cdot \boldsymbol{u}_\tau = |\boldsymbol{u}_\tau| \quad \text{a.e. on } \Gamma_1. \tag{5.1c}$$

Table 1 Numerical convergence orders for Example 1

h	$\left\|\boldsymbol{u}^h - \boldsymbol{u}^*\right\|_{L^2}$	Order(h)	$\left\|\boldsymbol{u}^h - \boldsymbol{u}^*\right\|_{H^1}$	Order(h)	$\left\|p^h - p^*\right\|_{L^2}$	Order(h)
$\frac{1}{5}$	1.2985e−02	–	1.5361e−01	–	1.3323e-01	–
$\frac{1}{10}$	3.8085e−03	1.7696	7.9351e−02	0.9530	5.9016e−02	1.1748
$\frac{1}{15}$	1.7086e−03	1.9768	5.4006e−02	0.9490	3.2931e−02	1.4389
$\frac{1}{20}$	9.4859e−04	2.0456	3.9288e−02	1.1061	2.1644e−02	1.4588
$\frac{1}{25}$	5.9909e−04	2.0595	3.1681e−02	0.9644	1.5618e−02	1.4623
$\frac{1}{30}$	4.1358e−04	2.0325	2.7595e−02	0.7573	1.1786e−02	1.5441
$\frac{1}{35}$	3.1081e−04	1.8532	2.2418e−02	1.3478	9.3060e−03	1.5326
$\frac{1}{40}$	2.4896e−04	1.6617	1.9616e−02	0.9998	7.6436e−03	1.4737
$\frac{1}{45}$	1.9174e−04	2.2170	1.7604e−02	0.9190	6.3363e−03	1.5925
$\frac{1}{50}$	1.5219e−04	2.1931	1.5563e−02	1.1695	5.4798e−03	1.3785

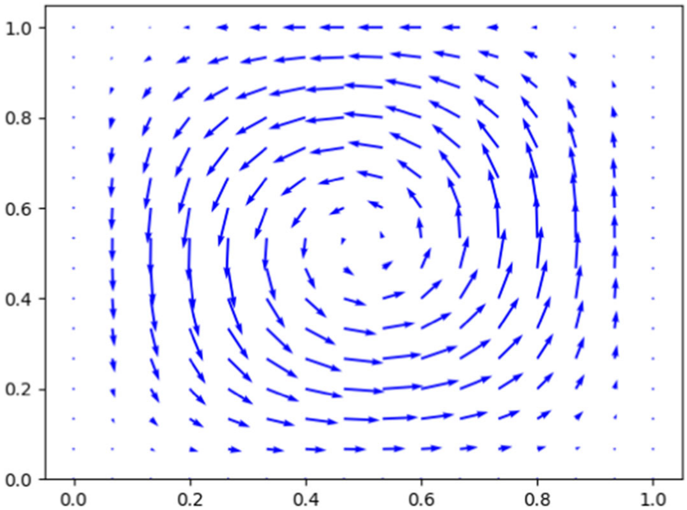

Fig. 1 Plot of velocity

An Uzawa algorithm for the mixed finite element solution of Problem 5.1 is the following [34], with a constant parameter $\rho > 0$. Choose an initial guess $\boldsymbol{u}_0^h$. Then for $n = 1, 2, \ldots,$ find $(\boldsymbol{u}_n^h, p_n^h) \in V^h \times Q^h$ such that

$$a(\boldsymbol{u}_n^h, \boldsymbol{v}^h) + b(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{v}^h, p_n^h) = \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle - \int_{\Gamma_1} \mu(|\boldsymbol{u}_{\tau,n-1}^h|)\, \boldsymbol{\lambda}_n^h \cdot \boldsymbol{v}_\tau^h \, ds \quad \forall\, \boldsymbol{v}^h \in V^h, \tag{5.2}$$

$$d(\boldsymbol{u}_n^h, q^h) = 0 \quad \forall\, q^h \in Q^h, \tag{5.3}$$

and

$$\boldsymbol{\lambda}_{n+1}^h = P(\boldsymbol{\lambda}_n^h + \rho\, \boldsymbol{u}_{\tau,n}^h), \tag{5.4}$$

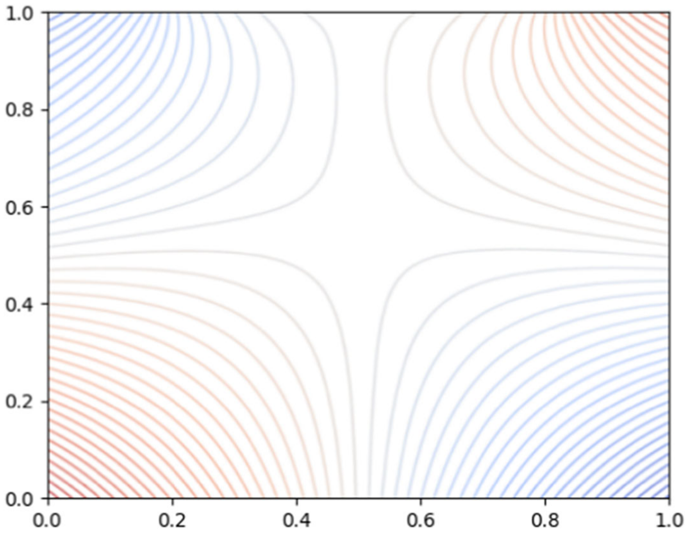

Fig. 2 Plot of pressure

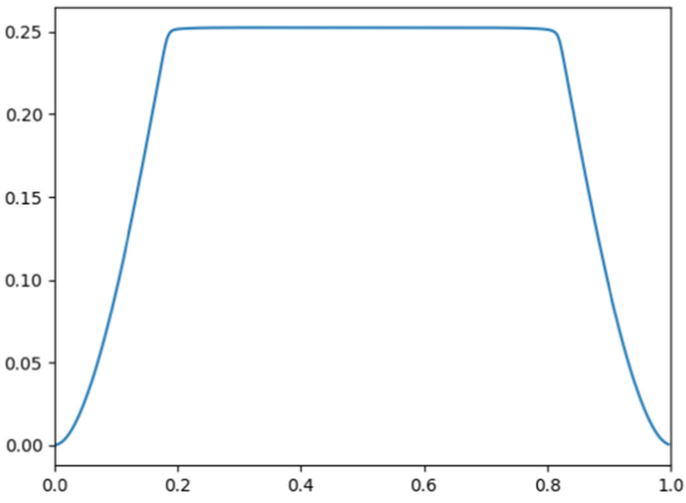

Fig. 3 Plot of σ_t along the slip boundary

where P is the orthogonal projection operator from $\mathbb{R}^d$ to the unit closed ball in $\mathbb{R}^d$. This problem is solved as follows: Choose an initial guess $\boldsymbol{\lambda}^h_{n,0}$, then for $l = 0, 1, \ldots,$

1. solve (5.2)–(5.3) for $(\boldsymbol{u}^h_{n,l}, p^h_{n,l})$;
2. $\boldsymbol{\lambda}^h_{n,l+1} = P\left(\boldsymbol{\lambda}^h_{n,l} + \rho\, \boldsymbol{u}^h_{\tau,n,l}\right)$;
3. iterate Steps 1 and 2 until $||\boldsymbol{u}^h_{n,l} - \boldsymbol{u}^h_{n,l-1}||_{L^2(\Omega)} < \epsilon_2$ and at that time, let $(\boldsymbol{u}^h_n, p^h_n, \boldsymbol{\lambda}^h_{n+1})$ be the most recent iterates.

Repeat the above procedure until $||\boldsymbol{u}^h_n - \boldsymbol{u}^h_{n-1}||_{L^2(\Omega)} < \epsilon_1$.

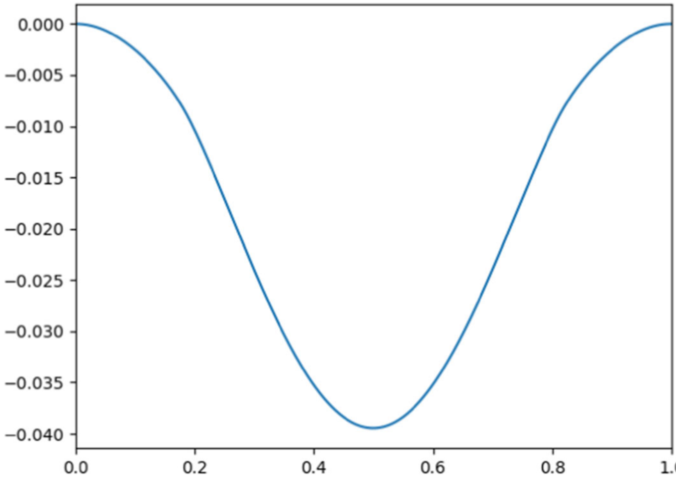

Fig. 4 Plot of u_t along the slip boundary

Table 2 Numerical convergence orders for Example 2

h	$\left\Vert \boldsymbol{u}^h - \boldsymbol{u}^* \right\Vert_{L^2}$	Order(h)	$\left\Vert \boldsymbol{u}^h - \boldsymbol{u}^* \right\Vert_{H^1}$	Order(h)	$\left\Vert p^h - p^* \right\Vert_{L^2}$	Order(h)
$\frac{1}{5}$	3.7852e−01	–	4.3703	–	3.5743	–
$\frac{1}{10}$	1.1298e−01	1.7443	2.2649	0.9483	1.3354	1.4204
$\frac{1}{15}$	5.2071e−02	1.9104	1.5462	0.9416	7.1503e−01	1.5407
$\frac{1}{20}$	2.9645e−02	1.9581	1.1256	1.1034	4.6414e−01	1.5021
$\frac{1}{25}$	1.9016e−02	1.9898	9.0808e−01	0.9625	3.2808e−01	1.5547
$\frac{1}{30}$	1.3091e−02	2.0476	7.9176e−01	0.7518	2.4673e−01	1.5631
$\frac{1}{35}$	9.6460e−03	1.9813	6.4434e−01	1.3366	1.9618e−01	1.4870
$\frac{1}{40}$	7.3037e−03	2.0831	5.6343e−01	1.0049	1.5847e−01	1.5990
$\frac{1}{45}$	5.7437e−03	2.0400	5.0624e−01	0.9088	1.3201e−01	1.5510
$\frac{1}{50}$	4.6182e−03	2.0700	4.4730e−01	1.1749	1.1463e−01	1.3392

In the examples below we use

$$\mu(t) = (a - b)\, e^{-\beta t} + b \tag{5.5}$$

with $a > b$. Each solve of (5.2)–(5.3) is computed using Python with the finite element software library FEniCS [1]. In all examples we use the following parameters for the iteration: For the outer iteration we use $\boldsymbol{u}_0^h = \mathbf{0}$ as an initial guess with stopping tolerance $\epsilon_1 = 10^{-6}$ or a maximum of fifty iterations. For the Uzawa iteration, we use the initial guess $\boldsymbol{\lambda}_{0,0}^h = \mathbf{0}$ the first time through and subsequently use the previous iteration's solution $\boldsymbol{\lambda}_{n-1}^h$ as an initial guess, i.e. $\boldsymbol{\lambda}_{n,0}^h = \boldsymbol{\lambda}_{n-1}^h$. For the remaining parameters we use $\rho = 1$ and $\epsilon_2 = 10^{-6}$ or ten iterations.

In the examples $\Omega = (0, 1) \times (0, 1)$ is the unit square and we take the parameter $\nu = 1$. The impermeability and slip boundary conditions are imposed along the top of the domain $\Gamma_1 = (0, 1) \times \{1\}$ and a homogeneous Dirichlet boundary condition along the remaining

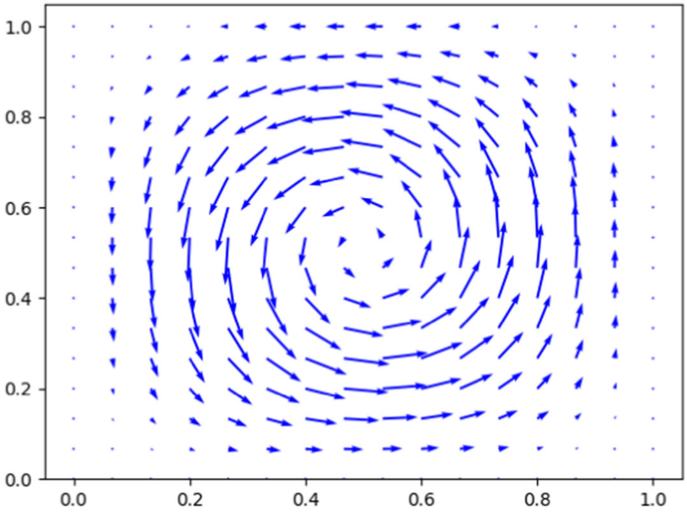

Fig. 5 Plot of velocity

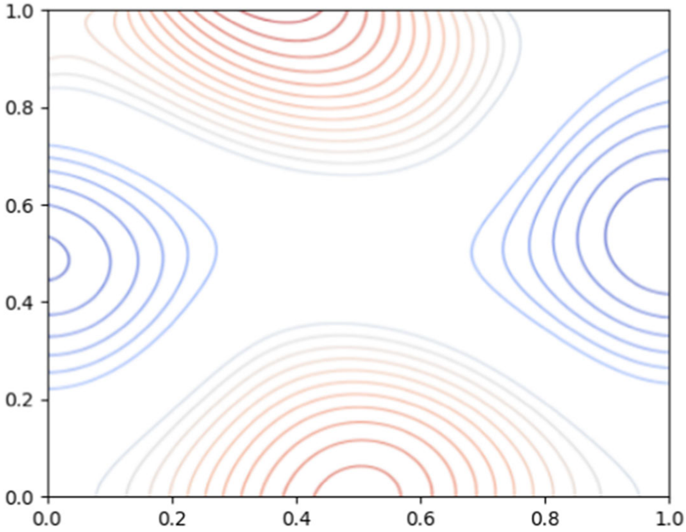

Fig. 6 Plot of pressure

portion of the boundary. We use a sequence of uniform triangular meshes with the interval $[0, 1]$ being split into h^{-1} equal sub-intervals for $h = 1/5, 1/10, \ldots$. In computing the numerical solution errors, we use $\boldsymbol{u}^* = \boldsymbol{u}^{1/250}$ and $p^* = p^{1/250}$ as the reference solution.

5.1 Example 1

The source function is defined by $\boldsymbol{f}_0 = -\nu\Delta\boldsymbol{u}_0 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla p_0$ with

$$\boldsymbol{u}_0(x, y) = \begin{pmatrix} 10x^2(1-x)^2y(1-y)(1-2y) \\ -10x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix}, \quad p_0(x, y) = (2x-1)(2y-1).$$

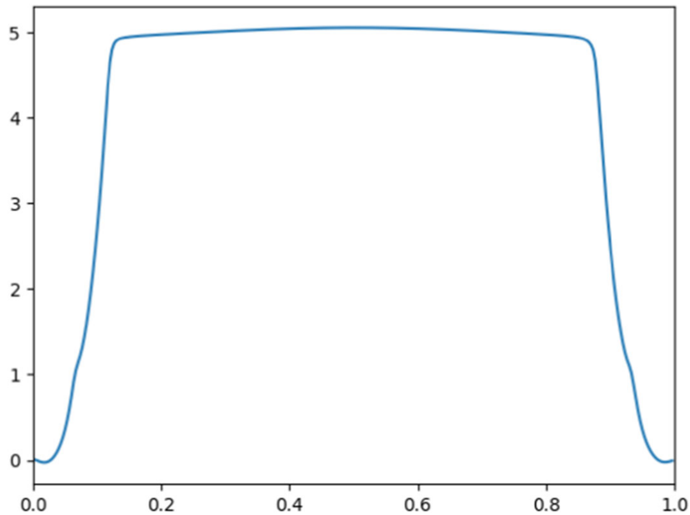

Fig. 7 Plot of σ_t along the slip boundary

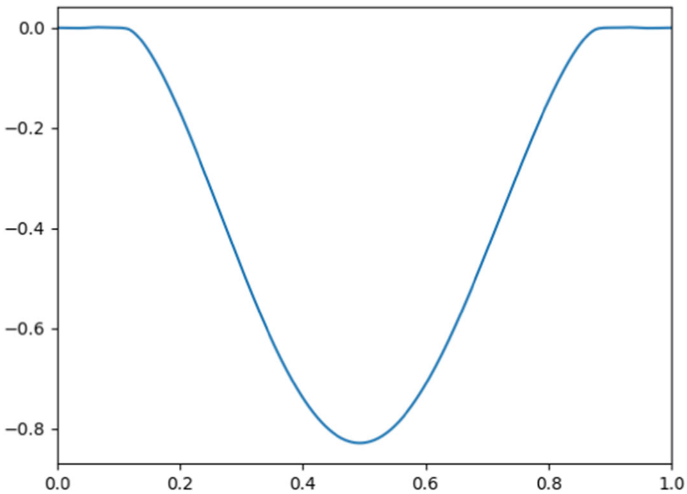

Fig. 8 Plot of u_t along the slip boundary

Similar examples have been used in the context of inequality problems for the Navier–Stokes equation in [31] and for Stokes equations with leak boundary conditions in [26].

For the function μ of (5.5), we take $a = 0.255$, $b = 0.25$, $\beta = 10$.

The numerical results agree with the solution's desired behavior as indicated in Figs. 1, 2, 3 and 4 and Table 1 shows good agreement with the estimates derived in Sect. 4.

5.2 Example 2

In the second example, we use

$$\boldsymbol{f} = -\Delta \boldsymbol{u}_0 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla p_0$$

where

$$\boldsymbol{u}_0(x,y)=\begin{pmatrix}-\cos(2\pi x)\sin(2\pi y)+\sin(2\pi y)\\ \sin(2\pi x)\cos(2\pi y)-\sin(2\pi x)\end{pmatrix},\quad p_0(x,y)=2\pi(\cos(2\pi y)-\cos(2\pi x)).$$

For the function μ of (5.5), we take $a = 5.01$, $b = 5.0$, $\beta = 10$. The numerical results are shown in Figs. 5, 6, 7 and 8 and Table 2. Again we observe a good agreement with the error estimates derived in Sect. 4.

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