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Virtual Element Method for an Elliptic Hemivariational Inequality with Applications to Contact Mechanics

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Abstract
This paper is on the numerical solution of an elliptic hemivariational inequality by the virtual element method. We introduce an abstract framework of the numerical method and provide an error analysis. We then apply the virtual element method to solve two contact problems: a bilateral contact problem with friction and a frictionless normal compliance contact problem. Error estimates of their numerical solutions are derived, which are of optimal order for the linear virtual element method, under appropriate solution regularity assumptions. The discrete problem can be formulated as an optimization problem for a difference of two convex (DC) functions, and a convergent algorithm is introduced to solve it. Numerical examples are reported to show the performance of the proposed methods.

Keywords Virtual element method · Hemivariational inequality · Error estimate · Double bundle method

1 Introduction

The notion of hemivariational inequalities was first introduced by Panagiotopoulos in the early 1980s (cf. [28]) and is closely related to the development of the concept of the generalized gradient of a locally Lipschitz continuous function introduced by Clarke [15]. The theory and applications of hemivariational inequalities can be found in several books (cf. [22,25–27]).

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In recent years, optimal order error estimates have been derived for linear finite element solutions of hemivariational inequalities, e.g. [3,18–21].

The virtual element method (VEM) was first proposed and analyzed in [1,5,7]. The method immediately attracted much attention from the research community due to its advantages in handling problems with complex geometries or problems requiring high-regularity solutions. This method has been applied to a wide variety of scientific and engineering problems; in particular, it has been applied to solve variational inequalities (cf. [17,31,32]). We refer the reader to [8,10,11,14] and the references therein for recent advance of the method.

In this paper, we introduce and analyze VEM for solving elliptic hemivariational inequalities. We provide an error analysis for the numerical method under some standard assumptions. As applications, we employ VEM to solve a bilateral contact problem with friction and a frictionless contact problem with normal compliance. To solve the discrete nonconvex problem, a convexification iterative procedure was applied in [4], where the auxiliary nonsmooth convex problems are solved by classical numerical methods (cf. [24,33]). In this paper, we convert the discrete nonconvex problem into a DC (difference of convex functions) programming, and apply the double bundle method to find the Clarke stationary point (cf. [23]). Numerical results are reported to illustrate computational performance of the VEMs proposed in this paper.

The rest of this paper is organized as follows. In Sect. 2, we recall notions and basic properties of the generalized directional derivative and subdifferential in the sense of Clarke. A general VEM for solving an elliptic hemivariational inequality and its error analysis are given in Sect. 3. In Sect. 4, the VEM is applied to the bilateral contact problem with friction and the frictionless contact problem with normal compliance, and optimal order error estimates are derived. In Sect. 5, we provide a detailed description of an algorithm based on the double bundle method ([23]) to solve the discrete problems discussed in Sect. 4. Two numerical examples are presented in Sect. 6 to illustrate the performance of the VEM studied in this paper.

2 Preliminaries

All linear spaces in this paper are assumed to be real. For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X, and by $X^*_{w^*}$ the space X^* equipped with weak* topology. Weak convergence is indicated by the symbol $\rightharpoonup$. Given two normed space X and Y, $\mathcal{L}(X,Y)$ is the space of all linear continuous operators from X to Y.

In the description of the hemivariational inequality, we need the notion of the generalized (Clarke) directional derivative and the generalized gradient of a locally Lipschitz continuous functional ([15]).

Definition 2.1 Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz continuous functional on a Banach space X. The generalized (Clarke) directional derivative of ψ at $x \in X$ in the direction $v \in X$ is defined by

$$\psi^0(x; v) = \limsup_{y\to x,\,\lambda\downarrow 0} \frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of ψ at x is defined by

$$\partial\psi(x) = \{\zeta \in X^* \mid \psi^0(x; v) \ge \langle\zeta, v\rangle\ \forall v \in X\}.$$

The function $\psi : X \to \mathbb{R}$ is said to be regular in the sense of Clarke at $x \in X$ if for all $v \in X$, the directional derivative $\psi'(x; v)$ exists and $\psi'(x; v) = \psi^0(x; v)$. The function ψ is regular in the sense of Clarke on X if it is regular at every point $x \in X$.

Knowing the generalized subdifferential, we can compute the generalized directional derivative through the formula:

$$\psi^0(x; v) = \max\{\langle \zeta, v\rangle \mid \zeta \in \partial\psi(x)\}. \tag{2.1}$$

The following properties of locally Lipschitz continuous functionals hold (cf. [25, Proposition 3.23]).

Proposition 1 *Suppose $\psi : X \to \mathbb{R}$ is a locally Lipschitz continuous functional on a Banach space X.*
(1) *For every $x \in X$, the function $X \ni v \mapsto \psi^0(x; v) \in \mathbb{R}$ is subadditive:*

$$\psi^0(x; v_1 + v_2) \le \psi^0(x; v_1) + \psi^0(x; v_2) \quad \forall\, v_1, v_2 \in X. \tag{2.2}$$

(2) *The graph of the generalized gradient $\partial\psi$ is closed in $X \times X^*_{w^*}$ topology. It means that if $\{x_n\} \subset X$ and $\{\zeta_n\} \subset X^*$ are sequences such that $\zeta_n \in \partial\psi(x_n)$, $x_n \to x$ in X, and $\zeta_n \to \zeta$ weakly* in X^*, then $\zeta \in \partial\psi(x)$.*

It is convenient to record an elementary result to be used later:

$$a, b, x \ge 0 \text{ and } x^2 \le a\,x + b \quad \Longrightarrow \quad x^2 \le a^2 + 2\,b. \tag{2.3}$$

3 An Elliptic Hemivariational Inequality and Its Numerical Solution

With applications to contact problems in mind, we first introduce an elliptic hemivariational inequality. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$ for applications) be a polyhedral domain, and let $\Gamma_3 \subset \partial\Omega$. Let V be a subspace of $H^1(\Omega; \mathbb{R}^d)$. For some positive integer m, let $\gamma \in \mathcal{L}(V; L^2(\Gamma_3; \mathbb{R}^m))$. Given a bilinear form $a(\cdot, \cdot)\colon V \times V \to \mathbb{R}$, a locally Lipschitz function $j\colon \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$, and a linear bounded functional $f : V \to \mathbb{R}$, we consider the following problem.

Problem (P). Find an element $u \in V$ such that

$$a(u, v) + \int_{\Gamma_3} j^0(\gamma u; \gamma v)\, ds \ge \langle f, v\rangle \quad \forall\, v \in V. \tag{3.1}$$

We allow $j(x, z)$ to depend on the spatial variable x. However, to simplify the notation, we will usually write $j(z)$ with the understanding that it is allowed to depend on the spatial variable. In the study of Problem (P), we need the following assumptions on the data:

(H_a) $a(\cdot, \cdot)\colon V \times V \to \mathbb{R}$ is bilinear, symmetric, continuous and V-elliptic; we will denote the V-ellipticity constant by $m_A > 0$:

$$a(v, v) \ge m_A \|v\|_V^2 \quad \forall\, v \in V. \tag{3.2}$$

(H_j) $j(\cdot, z)$ is measurable on Γ_3 for any $z \in \mathbb{R}^m$ and there exists $z_0 \in L^2(\Gamma_3; \mathbb{R}^m)$ such that $j(\cdot, z_0(\cdot)) \in L^1(\Gamma_3)$. $j(x, \cdot)$ is locally Lipschitz on $\mathbb{R}^m$ for a.e. $x \in \Gamma_3$, and there are constants $c_0, c_1, \alpha_j \ge 0$ such that

$$\|\partial j(z)\|_{\mathbb{R}^m} \le c_0 + c_1 \|z\|_{\mathbb{R}^m} \quad \forall\, z \in \mathbb{R}^m, \tag{3.3}$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \le \alpha_j \|z_1 - z_2\|^2_{\mathbb{R}^m} \quad \forall\, z_1, z_2 \in \mathbb{R}^m. \tag{3.4}$$

Denote by c_γ an upper bound of the norm of the operator $\gamma \in \mathcal{L}(V; L^2(\Gamma_3; \mathbb{R}^m))$:

$$\|\gamma v\|_{L^2(\Gamma_3;\mathbb{R}^m)} \le c_\gamma \|v\|_V \quad \forall\, v \in V. \tag{3.5}$$

The following result can be derived from [20, Theorem 3.1].

Theorem 3.1 *Assume* (H_a), (H_j), *and*

$$\alpha_j c_\gamma^2 < m_A. \tag{3.6}$$

Then for any $f \in V^*$, *Problem* (P) *has a unique solution.*

Proof We sketch a proof of the result. Define a linear operator $A\colon V \to V^*$ by

$$\langle Au, v\rangle = a(u, v), \quad u, v \in V. \tag{3.7}$$

Then $A \in \mathcal{L}(V, V^*)$ and it is strongly monotone with a monotonicity constant m_A. From [34, Proposition 27.6], A is pseudomonotone (a property needed in applying [20, Theorem 3.1]). Define a functional

$$J(z) = \int_{\Gamma_3} j(z)\, ds, \quad z \in L^2(\Gamma_3; \mathbb{R}^m).$$

From [25, Theorem 3.47], we know that $J(\cdot)$ is locally Lipschitz on $L^2(\Gamma_3; \mathbb{R}^m)$ and

$$\partial J(z) \subset \int_{\Gamma_3} \partial j(x, z(x))\, ds \quad \forall\, z \in L^2(\Gamma_3; \mathbb{R}^m), \tag{3.8}$$

$$J^0(z_1; z_2) \le \int_{\Gamma_3} j^0(z_1; z_2)\, ds \quad \forall\, z_1, z_2 \in L^2(\Gamma_3; \mathbb{R}^m), \tag{3.9}$$

$$\|\partial J(z)\|_{L^2(\Gamma_3;\mathbb{R}^m)} \le \sqrt{2\,\mathrm{meas}(\Gamma_3)}\, c_0 + \sqrt{2}\, c_1 \|z\|_{L^2(\Gamma_3;\mathbb{R}^m)} \quad \forall\, z \in L^2(\Gamma_3; \mathbb{R}^m). \tag{3.10}$$

Moreover, (3.8) and (3.9) become equalities when j is regular in the sense of Clarke; however, we do not assume the regularity of j in the study of Problem (P). Here, (3.8) is understood in the sense that for $z^* \in \partial J(z)$, there is a function $\xi(x)$ such that $\xi(x) \in \partial j(x, z(x))$ for a.e. $x \in \Gamma_3$, and

$$\langle z^*, v\rangle_{L^2(\Gamma_3;\mathbb{R}^m)\times L^2(\Gamma_3;\mathbb{R}^m)} = \int_{\Gamma_3} \langle \xi(x), v(x)\rangle_{\mathbb{R}^m\times\mathbb{R}^m} ds \quad \forall\, v \in L^2(\Gamma_3; \mathbb{R}^m). \tag{3.11}$$

Note that

$$\begin{aligned} J^0(z_1; z_2 - z_1) + J^0(z_2; z_1 - z_2) &\le \int_{\Gamma_3} \left[j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2)\right] ds \\ &\le \alpha_j \|z_1 - z_2\|^2_{L^2(\Gamma_3;\mathbb{R}^m)}. \end{aligned}$$

Applying [20, Theorem 3.1], we know that there is a unique solution to the problem

$$u \in V, \quad \langle Au, v\rangle + J^0(\gamma u; \gamma v) \ge \langle f, v\rangle \quad \forall\, v \in V. \tag{3.12}$$

Since

$$\int_{\Gamma_3} j^0(\gamma u; \gamma v)\, ds \ge J^0(\gamma u; \gamma v),$$

it follows from (3.7) that the solution u of (3.12) is also a solution of Problem (P).

The uniqueness of a solution to Problem (P) is proved by a standard approach. Assume $\tilde{u} \in V$ is another solution of Problem (P). Then,

$$a(\tilde{u}, v) + \int_{\Gamma_3} j^0(\gamma \tilde{u}; \gamma v)\, ds \geq \langle f, v \rangle \quad \forall\, v \in V. \tag{3.13}$$

Take $v = \tilde{u} - u$ in (3.1), $v = u - \tilde{u}$ in (3.13), and add the two resulting inequalities to get

$$a(\tilde{u} - u, \tilde{u} - u) \leq \int_{\Gamma_3} \left[j^0(\gamma u; \gamma \tilde{u} - \gamma u) + j^0(\gamma \tilde{u}; \gamma u - \gamma \tilde{u}) \right] ds.$$

Applying (H_a) and (H_j), we derive from the above inequality that

$$m_A \|\tilde{u} - u\|_V^2 \leq \alpha_j c_\gamma^2 \|\tilde{u} - u\|_V^2.$$

By the assumption (3.6), we deduce that $\|\tilde{u} - u\|_V = 0$, i.e. $\tilde{u} = u$. □

Remark 3.1 The symmetry assumption on the bilinear form $a(\cdot, \cdot)$ in (H_a) is not needed for the conclusion of Theorem 3.1. However, we will need this symmetry assumption in developing error analysis of the VEM for Problem (P).

From the above proof we can also conclude that under the assumptions in Theorem 3.1, Problem (P) is equivalent to the auxiliary problem (3.12). In fact, let u be the solution of the forgoing problem and $\tilde{u}$ the solution of the latter one. Then, $\tilde{u}$ is also the solution of Problem (P) due to (3.9). Hence, by the unique solvability of Problem (P), $u = \tilde{u}$ as required. Certainly, if the nonlinear functional j is regular in the sense of Clarke, the equality in (3.9) holds, which shows that the previous two problems have the same formulation and are equivalent trivially. We remark that we formulate the problem in the form (3.1) for convenience of application by researchers in the areas of applied sciences, since it only involves the determination of j^0.

In the rest of the section, we assume the conditions stated in Theorem 3.1. Now, we propose a general framework for numerical methods to solve Problem (P). Let $\{\mathcal{T}_h\}_h$, $\mathcal{T}_h := \{K\}_{K \in \mathcal{T}_h}$, be a family of partitions of $\overline{\Omega}$ into polygons, with a generic element denoted by K; $h := \max_{K \in \mathcal{T}_h} h_K$ and $h_K := \mathrm{diam}(K)$. With this mesh, we consider a finite dimensional subspace V_h of V. For a non-negative integer k and an element $K \in \mathcal{T}_h$, denote by $\mathbb{P}_k(K)$ the set of all polynomials on K with the total degree no more than k, and simply write $(\mathbb{P}_k(K))^d$ as $\mathbb{P}_k(K; \mathbb{R}^d)$. Moreover, we assume that the bilinear form $a(\cdot, \cdot)$ can be decomposed as

$$a(v, w) := \sum_{K \in \mathcal{T}_h} a^K(v, w) \quad \forall\, v, w \in V,$$

where $a^K(\cdot, \cdot)$ is a bilinear, symmetric and nonnegative form over $V_K := V_{|K}$. For a function in V, we naturally view it as a function in V_K by its restriction to K. We equip the Hilbert space $V_{|K}$ with a norm or semi-norm $\|\cdot\|_{V,K}$ such that

$$\|v\|_V^2 := \sum_{K \in \mathcal{T}_h} \|v\|_{V,K}^2 \quad \forall\, v \in V, \tag{3.14}$$

and for all $K \in \mathcal{T}_h$, there holds

$$a^K(v, v) \lesssim \|v\|_{V,K}^2 \quad \forall\, v \in V_K, \tag{3.15}$$

where and in what follows, C or c (with or without subscript) denotes a positive constant independent of h_K or h, which may take on different values at different occurrences. For any two quantities a and b, "$a \lesssim b$" stands for "$a \leq Cb$".

In addition, assume that there exists a natural number k such that $\mathbb{P}_k(K;\mathbb{R}^d) \subset V_{h|K}$ for all $K \in \mathcal{T}_h$. With the above preparation, our abstract numerical method for Problem (P) is the following.

Problem (P^h.) Find an element $u_h \in V_h$ such that

$$a_h(u_h, v_h) + \int_{\Gamma_3} j^0(\gamma u_h; \gamma v_h)\, ds \geq \langle f_h, v_h \rangle \quad \forall\, v_h \in V_h, \tag{3.16}$$

where $f_h \in V_h^*$ satisfies the condition

$$\langle f_h, v \rangle \leq c\, \|f\|_{V^*} \|v\|_V \quad \forall\, v \in V_h. \tag{3.17}$$

In addition, the bilinear form is obtained from

$$a_h(u, v) := \sum_{K \in \mathcal{T}_h} a_h^K(u, v),$$

with the symmetric bilinear form $a_h^K(\cdot, \cdot)$ satisfying

- *k-Consistency* For all $p \in \mathbb{P}_k(K;\mathbb{R}^d)$ and for all $v_h \in V_{h|K}$,

$$a_h^K(p, v_h) = a^K(p, v_h). \tag{3.18}$$

- *Stability* There exist two positive constants α_* and α^*, independent of h_K and K, such that

$$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \quad \forall\, v_h \in V_{h|K}. \tag{3.19}$$

By the V-ellipticity of $a(\cdot, \cdot)$ and the stability condition for $a_h^K(\cdot, \cdot)$, a routine computation reveals the V_h-ellipticity of $a_h(\cdot, \cdot)$:

$$a_h(v, v) \geq \alpha_* a(v, v) \geq \tilde{m}_A \|v\|_V^2 \quad \forall\, v \in V_h, \tag{3.20}$$

where

$$\tilde{m}_A = \alpha_* m_A. \tag{3.21}$$

For the study of the discrete problem, we assume further that

$$m_A > \max\left\{1, \frac{1}{\alpha_*}\right\} \alpha_j c_\gamma^2. \tag{3.22}$$

Note that (3.22) implies

$$\tilde{m}_A > \alpha_j c_\gamma^2. \tag{3.23}$$

A discrete analogue of Theorem 3.1 is the following.

Theorem 3.2 *Under the assumptions* (H_a), (H_j), (3.20) *and* (3.22), *Problem* (P^h) *has a unique solution.*

Remark 3.2 The construction of $a_h(\cdot, \cdot)$ is motivated by the ideas of the virtual element method (VEM) introduced in [2,6]. Here, we first propose a general framework of numerical methods for solving Problem (P). Based on this general framework, we devise and analyze virtual element methods for solving elliptic hemivariational inequalities in a unified way.

In the study of the discrete problem, we first show the uniform boundedness of the numerical solutions.

Lemma 3.1 *Assume* (H_a), (H_j), (3.17), (3.20) *and* (3.22). *Then*

$$\|u_h\|_V \lesssim \|f\|_{V^*} + 1.$$

Proof We take $v_h = -u_h$ in (3.16),

$$a_h(u_h, -u_h) + \int_{\Gamma_3} j^0(\gamma u_h; -\gamma u_h)\, ds \geq \langle f_h, -u_h \rangle,$$

which is rewritten as

$$a_h(u_h, u_h) \leq \int_{\Gamma_3} j^0(\gamma u_h; -\gamma u_h)\, ds + \langle f_h, u_h \rangle. \tag{3.24}$$

By (3.20),

$$a_h(u_h, u_h) \geq \tilde{m}_A \|u_h\|_V^2. \tag{3.25}$$

From (3.3) and (3.4),

$$j^0(\gamma u_h; -\gamma u_h) \leq \alpha_j \|\gamma u_h\|_{\mathbb{R}^m}^2 - j^0(0; \gamma u_h) \leq \alpha_j \|\gamma u_h\|_{\mathbb{R}^m}^2 + c_0 \|\gamma u_h\|_{\mathbb{R}^m}.$$

So there is a constant $c > 0$,

$$\int_{\Gamma_3} j^0(\gamma u_h; -\gamma u_h)\, ds \leq \alpha_j c_\gamma^2 \|u_h\|_V^2 + c\, \|u_h\|_V. \tag{3.26}$$

Use (3.25), (3.26) and (3.17) in (3.24) to obtain

$$\tilde{m}_A \|u_h\|_V^2 \leq c\, \|f\|_{V^*} \|u_h\|_V + \alpha_j c_\gamma^2 \|u_h\|_V^2 + c\, \|u_h\|_V.$$

Hence, the combination of (3.23) and the above result gives

$$\|u_h\|_V \lesssim \|f\|_{V^*} + 1,$$

i.e. $u_h \in V_h$ is uniformly bounded independent of h. □

As is customary in the literature on virtual element methods, in order to derive error estimation, we make the following two assumptions for some natural number k.

Assumption B1 For every $v \in H^{k+1}(K; \mathbb{R}^d)$, there exists a function $v_\pi \in \mathbb{P}_k(K; \mathbb{R}^d)$ such that

$$\|v - v_\pi\|_{0,K} + h_K \|v - v_\pi\|_{V,K} \lesssim h_K^{k+1} |v|_{k+1,K} \quad \forall\, v \in H^{k+1}(K; \mathbb{R}^d). \tag{3.27}$$

Assumption B2 There exists an interpolation operator $I_K : H^{k+1}(K; \mathbb{R}^d) \cap V_K \to V_{h|K}$ such that

$$\|v - I_K v\|_{0,K} + h_K \|v - I_K v\|_{V,K} \lesssim h_K^{k+1} |v|_{k+1,K} \quad \forall\, v \in H^{k+1}(K; \mathbb{R}^d) \cap V_K. \tag{3.28}$$

Moreover, if we write v_I as the global interpolant of v, i.e. $v_I(x)$ is equal to $I_K v(x)$ for $x \in K$, we require additionally that $v_I \in V_h$ if $v \in V$.

Theorem 3.3 *If Assumptions* B1–B2, (3.17)–(3.19) *hold and the solution u of Problem* (P) *belongs to* $H^{k+1}(\Omega; \mathbb{R}^d)$, *then*

$$\|u - u_h\|_V \lesssim h^k |u|_{k+1,\Omega} + \|f - f_h\|_{V_h^*} + \|\gamma u - \gamma u_I\|_{L^2(\Gamma_3; \mathbb{R}^m)}^{1/2}, \tag{3.29}$$

where

$$\|f - f_h\|_{V_h^*} = \sup_{v_h \in V_h} \frac{\langle f - f_h, v_h \rangle}{\|v_h\|_V}.$$

Proof Let $w = u_I - u_h$. Due to (3.20),

$$\tilde{m}_A \|w\|_V^2 \le a_h(w, w) = a_h(u_I, w) - a_h(u_h, w). \tag{3.30}$$

By the k-consistency (3.18), we have, for $u_\pi \in \mathbb{P}_k(K; \mathbb{R}^d)$,

$$a_h^K(u_\pi, v_h) = a^K(u_\pi, v_h) \quad \forall\, v_h \in V_{h|K}.$$

Therefore,

$$a_h(u_I, w) = \sum_{K \in \mathcal{T}_h} \left(a_h^K(I_K u - u_\pi, w) + a^K(u_\pi - u, w) \right) + a(u, w). \tag{3.31}$$

Take $v = -w$ in (3.1) to obtain

$$a(u, w) \le \int_{\Gamma_3} j^0(\gamma u; \gamma u_h - \gamma u_I)\, ds + \langle f, u_I - u_h \rangle. \tag{3.32}$$

From the discrete hemivariational inequality (3.16),

$$a_h(u_h, w) \ge \langle f_h, u_I - u_h \rangle - \int_{\Gamma_3} j^0(\gamma u_h; \gamma u_I - \gamma u_h)\, ds. \tag{3.33}$$

Use (3.31)–(3.33) in (3.30),

$$\begin{aligned} \tilde{m}_A \|w\|_V^2 \le & \sum_{K \in \mathcal{T}_h} \left(a_h^K(I_K u - u_\pi, w) + a^K(u_\pi - u, w) \right) \\ & + \int_{\Gamma_3} \left[j^0(\gamma u; \gamma u_h - \gamma u_I) + j^0(\gamma u_h; \gamma u_I - \gamma u_h) \right] ds + \langle f - f_h, w \rangle. \end{aligned} \tag{3.34}$$

Note that

$$\langle f - f_h, w \rangle \le c \|f - f_h\|_{V_h^*} \|w\|_V.$$

By the sub-additivity of the generalized directional derivative (see Proposition 1),

$$\begin{aligned} j^0(\gamma u; \gamma u_h - \gamma u_I) &\le j^0(\gamma u; \gamma u_h - \gamma u) + j^0(\gamma u; \gamma u - \gamma u_I), \\ j^0(\gamma u_h; \gamma u_I - \gamma u_h) &\le j^0(\gamma u_h; \gamma u - \gamma u_h) + j^0(\gamma u_h; \gamma u_I - \gamma u). \end{aligned}$$

Thus,

$$\begin{aligned} j^0(\gamma u; \gamma u_h - \gamma u_I) + j^0(\gamma u_h; \gamma u_I - \gamma u_h) \le\; & j^0(\gamma u; \gamma u_h - \gamma u) + j^0(\gamma u_h; \gamma u - \gamma u_h) \\ & + j^0(\gamma u; \gamma u - \gamma u_I) + j^0(\gamma u_h; \gamma u_I - \gamma u). \end{aligned}$$

By (3.4),

$$\begin{aligned} \int_{\Gamma_3} \left[j^0(\gamma u; \gamma u_h - \gamma u) + j^0(\gamma u_h; \gamma u - \gamma u_h) \right] ds &\le \alpha_j \|\gamma u - \gamma u_h\|_{L^2(\Gamma_3; \mathbb{R}^m)}^2 \\ &\le \alpha_j c_\gamma^2 \|u - u_h\|_V^2. \end{aligned}$$

From (3.3),

$$j^0(\gamma u; \gamma u - \gamma u_I) \le (c_0 + c_1 \|\gamma u\|_{\mathbb{R}^m}) \|\gamma u - \gamma u_I\|_{\mathbb{R}^m},$$
$$j^0(\gamma u_h; \gamma u_I - \gamma u) \le (c_0 + c_1 \|\gamma u_h\|_{\mathbb{R}^m}) \|\gamma u_I - \gamma u\|_{\mathbb{R}^m}.$$

Taking the boundedness of $\|\gamma u_h\|_{L^2(\Gamma_3;\mathbb{R}^m)}$ into account, we then get from (3.34) that

$$\tilde{m}_A \|w\|_V^2 \le c\Big[\big(\sum_{K\in\mathcal{T}_h} \|I_K u - u_\pi\|_{V,K}^2\big)^{1/2}\|w\|_V + \big(\sum_{K\in\mathcal{T}_h} \|u - u_\pi\|_{V,K}^2\big)^{1/2}\|w\|_V\Big] + c\|f - f_h\|_{V_h^*}\|w\|_V + \alpha_j c_\gamma^2 \|u - u_h\|_V^2 + c\,\|\gamma u_I - \gamma u\|_{L^2(\Gamma_3;\mathbb{R}^m)}. \tag{3.35}$$

Note that

$$\|u - u_h\|_V^2 \le \|u - u_I\|_V^2 + \|w\|_V^2 + 2\|u - u_I\|_V \|w\|_V. \tag{3.36}$$

Then from (3.35),

$$\left(\tilde{m}_A - \alpha_j c_\gamma^2\right)\|w\|_V^2 \le c\left[\left(\sum_{K\in\mathcal{T}_h} \|I_K u - u_\pi\|_{V,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}_h} \|u - u_\pi\|_{V,K}^2\right)^{1/2}\right]\|w\|_V + c\left(\|f - f_h\|_{V_h^*} + \|u - u_I\|_V\right)\|w\|_V + c\left(\|u - u_I\|_V^2 + \|\gamma u - \gamma u_I\|_{L^2(\Gamma_3;\mathbb{R}^m)}\right).$$

Applying (2.3), we have

$$\|w\|_V^2 \lesssim \sum_{K\in\mathcal{T}_h} \|I_K u - u_\pi\|_{V,K}^2 + \sum_{K\in\mathcal{T}_h} \|u - u_\pi\|_{V,K}^2 + \|u - u_I\|_V^2 + \|f - f_h\|_{V_h^*}^2 + \|\gamma u - \gamma u_I\|_{L^2(\Gamma_3;\mathbb{R}^m)}.$$

Note that

$$\|u - u_h\|_V \le \|u - u_I\|_V + \|w\|_V.$$

Hence,

$$\|u - u_h\|_V \lesssim \left(\sum_{K\in\mathcal{T}_h} \|I_K u - u_\pi\|_{V,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}_h} \|u - u_\pi\|_{V,K}^2\right)^{1/2} + \|u - u_I\|_V + \|f - f_h\|_{V_h^*} + \|\gamma u - \gamma u_I\|_{L^2(\Gamma_3;\mathbb{R}^m)}^{1/2}. \tag{3.37}$$

From the estimate (3.27), we have

$$\big(\sum_{K\in\mathcal{T}_h} \|u - u_\pi\|_{V,K}^2\big)^{1/2} \lesssim h^k |u|_{k+1,\Omega},$$

and from (3.28),

$$\|u - u_I\|_V = \big(\sum_{K\in\mathcal{T}_h} \|u - I_K u\|_{V,K}^2\big)^{1/2} \lesssim h^k |u|_{k+1,\Omega}.$$

Therefore, we have (3.29) from (3.37). □

Remark 3.3 Note that in general we can not expect to achieve optimal error estimates for solving hemivariational inequalities with high order elements. Hence, we restrict our discussion to the lowest order virtual method introduced in [5], i.e. $k = 1$.

4 Numerical Methods for Contact Problems

For definiteness, we let $d = 2$ in the following. Let Ω be the reference configuration of a linear elastic body, assumed to be an open, bounded, connected polygon in $\mathbb{R}^2$. The boundary is made up of three parts: Γ_1, Γ_2 and Γ_3, where meas$(\Gamma_1) > 0$. We assume that the body is clamped on Γ_1, is subject to the action of a surface traction of density $\boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^2)$, and is in contact with a rigid foundation on Γ_3. Volume forces of density $\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^2)$ act in Ω. For a vector $\boldsymbol{v}$, denote on the boundary $\partial\Omega$ by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ its normal component and $\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ the tangential component, respectively. We use $\mathbb{S}^2$ for the space of second order symmetric tensors which is equipped with the canonical inner product ":". For a tensor $\boldsymbol{\sigma} \in \mathbb{S}^2$, define its normal component as $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}$ and tangential component as $\boldsymbol{\sigma}_{\boldsymbol{\tau}} = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. For the contact problems under consideration, we have the linear elastic constitutive law

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{4.1}$$

the equilibrium equation

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{4.2}$$

the displacement boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{4.3}$$

the traction boundary condition

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{4.4}$$

In (4.1), $\mathcal{F}\colon \Omega \times \mathbb{S}^2 \to \mathbb{S}^2$ represents the linear elasticity operator and is assumed to have the following properties (cf. [20]):

$$\begin{cases} (a) \text{ there exists } L_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^2, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \quad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|; \\ (b) \text{ there exists } m_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^2, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \quad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2; \\ (c) \ \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^2. \end{cases} \tag{4.5}$$

Introduce a function space $Q = L^2(\Omega; \mathbb{S}^2)$, which is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})dx;$$

the associated norm is denoted by $\|\cdot\|_Q$. When there is no danger of confusion, we simply write $(\cdot, \cdot)$ for $(\cdot, \cdot)_Q$.

To study the contact problems, the displacement fields will be sought in the following space

$$\boldsymbol{V} := \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^2) \mid \boldsymbol{v}|_{\Gamma_1} = \boldsymbol{0}\},$$

which is equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} := (\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q^{1/2} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.6}$$

Since meas$(\Gamma_1) > 0$, we have by Korn's inequality (see e.g. [9, Remark 1.1]) that

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^2)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V}} \lesssim \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^2)} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.7}$$

We assume the densities of body forces and surface tractions satisfy

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^2), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^2)$$

and define $\boldsymbol{f} \in \boldsymbol{V}^*$ by

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{V}^* \times \boldsymbol{V}} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^2)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^2)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{4.8}$$

Next, we consider two contact problems with two choices of the boundary conditions on the contact boundary Γ_3.

4.1 A Bilateral Contact Problem with Friction

The contact boundary condition is

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_{\boldsymbol{\tau}} \in \partial j_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}}) \quad \text{on } \Gamma_3. \tag{4.9}$$

For the potential function $j_{\boldsymbol{\tau}} : \Gamma_3 \times \mathbb{R}^2 \to \mathbb{R}$ (cf. [20]), we assume

$$\begin{cases} (a)\ j_{\boldsymbol{\tau}}(\cdot, \boldsymbol{z}) \text{ is measurable on } \Gamma_3 \text{ for all } \boldsymbol{z} \in \mathbb{R}^2 \text{ and } j_{\boldsymbol{\tau}}(\cdot, \boldsymbol{z}_0(\cdot)) \in L^1(\Gamma_3) \\ \quad \text{for some } \boldsymbol{z}_0 \in L^2(\Gamma_3, \mathbb{R}^2); \\ (b)\ j_{\boldsymbol{\tau}}(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^2 \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ (c)\ |\partial j_{\boldsymbol{\tau}}(\boldsymbol{x}, \boldsymbol{z})| \le \bar{c}_0 + \bar{c}_1 \|\boldsymbol{z}\| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for all } \boldsymbol{z} \in \mathbb{R}^2 \text{ with } \bar{c}_0, \bar{c}_1 \ge 0; \\ (d)\ j_{\boldsymbol{\tau}}^0(\boldsymbol{x}, \boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) + j_{\boldsymbol{\tau}}^0(\boldsymbol{x}, \boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2) \le \alpha_{j_\tau} \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|^2 \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \quad \text{for all } \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^2 \text{ with } \alpha_{j_\tau} \ge 0. \end{cases} \tag{4.10}$$

Let $m = 2$ and

$$\boldsymbol{V}_1 := \{\boldsymbol{v} \in \boldsymbol{V} \mid v_\nu|_{\Gamma_3} = 0\}, \quad \gamma \boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{\tau}} \ \text{ for } \ \boldsymbol{v} \in \boldsymbol{V},$$

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})), \quad j(\cdot, \boldsymbol{z}) = j_{\boldsymbol{\tau}}(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathbb{R}^2.$$

From (4.5)(b) and the definition (4.6), the assumption (H_a) is satisfied with $m_A = m_{\mathcal{F}}$, and (H_j) is satisfied with $\alpha_j = \alpha_{j_\tau}$ from (4.10) (cf. [20]). The inequality (3.5) holds for any $c_\gamma \ge \lambda_{1,V}^{-1/2}$, $\lambda_{1,V} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in \boldsymbol{V}_1, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} ds \quad \forall \boldsymbol{v} \in \boldsymbol{V}_1.$$

We assume additionally

$$\alpha_{j_\tau} < \lambda_{1,V} m_{\mathcal{F}}.$$

For the first contact problem described by (4.1)–(4.4) and (4.9), proceeding in a standard way, we can obtain the following weak formulation:

Problem (P_1). Find an element $\boldsymbol{u} \in \boldsymbol{V}_1$ and $\boldsymbol{\xi}_{\boldsymbol{\tau}} \in L^2(\Gamma_3; \mathbb{R}^2)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_{\Gamma_3} \boldsymbol{\xi}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} ds \quad \forall \boldsymbol{v} \in \boldsymbol{V}_1 \tag{4.11}$$

with $-\boldsymbol{\xi}_{\boldsymbol{\tau}} \in \partial j_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}})$ a.e. on Γ_3.

According to Theorem 3.1, there is a unique solution to the problem

$$\boldsymbol{u} \in \boldsymbol{V}_1, \quad a(\boldsymbol{u}, \boldsymbol{v}) + \int_{\Gamma_3} j^0_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}})\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_1, \tag{4.12}$$

which is also the unique solution of the auxiliary problem (cf. Remark 3.1)

$$\boldsymbol{u} \in \boldsymbol{V}_1, \quad a(\boldsymbol{u}, \boldsymbol{v}) + J^0(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}}) \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_1, \tag{4.13}$$

where

$$J(\boldsymbol{z}) := \int_{\Gamma_3} j_{\boldsymbol{\tau}}(\boldsymbol{z})\, ds, \quad \boldsymbol{z} \in L^2(\Gamma_3; \mathbb{R}^2).$$

We next show the unique solvability of Problem (P_1). Let $\boldsymbol{u} \in \boldsymbol{V}_1$ be the solution of (4.13). Then by (2.1), there is a function $\boldsymbol{\eta} \in \partial J(\boldsymbol{u}_{\boldsymbol{\tau}})$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + \langle \boldsymbol{\eta}, \boldsymbol{v}_{\boldsymbol{\tau}} \rangle \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_1.$$

For all $\boldsymbol{v} \in \boldsymbol{V}_1$, the above inequality still holds if $\boldsymbol{v}$ is replaced by $-\boldsymbol{v}$. These two inequalities together readily imply

$$a(\boldsymbol{u}, \boldsymbol{v}) + \langle \boldsymbol{\eta}, \boldsymbol{v}_{\boldsymbol{\tau}} \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_1. \tag{4.14}$$

On the other hand, owing to (3.11), for $\boldsymbol{\eta} \in \partial J(\boldsymbol{u}_{\boldsymbol{\tau}})$, there is a function $-\boldsymbol{\xi}_{\boldsymbol{\tau}} \in \partial j_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}})$ such that

$$\langle \boldsymbol{\eta}, \boldsymbol{v}_{\boldsymbol{\tau}} \rangle = -\int_{\Gamma_3} \boldsymbol{\xi}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}\, ds. \tag{4.15}$$

Inserting (4.15) into (4.14) gives (4.11), that means, the solution of (4.13) is a solution of Problem (P_1). It is evident from Remark 3.1 that the solution of (4.12) is also a solution of Problem (P_1). The uniqueness of a solution to Problem (P_1) can be shown similarly as in the uniqueness part of the proof of Theorem 3.1.

Now, we introduce a virtual element method to solve Problem (P_1). As in [13,14], we make the following assumption on the family of meshes $\{\mathcal{T}_h\}_h$.

Assumption B3 For each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K such that $\mathcal{T}_K$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_K$ is bounded from below by a constant multiple of h_K. Each edge of K is a side of certain triangle in $\mathcal{T}_K$.

It is evident to check that the above assumption covers the usual conditions satisfied by $K \in \mathcal{T}_h$, given as follows (cf. [1,5,7]).

C1. There exists a real number $\gamma > 0$ such that each element $K \in \mathcal{T}_h$ is star-shaped with respect to a disk of radius $\rho_K \geq \gamma h_K$.

C2. There exists a real number $\gamma_1 > 0$ such that for each element $K \in \mathcal{T}_h$, the distance between any two vertices of K is $\geq \gamma_1 h_K$.

From now on, we always assume that the family of meshes $\{\mathcal{T}_h\}_h$ satisfies the assumption **B3**. Furthermore, we express the three parts of the boundary Γ as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \leq k \leq 3.$$

Then, we construct virtual linear element spaces corresponding to $\mathcal{T}_h$. Let

$$V_1(K) := \{v \in H^1(K) \mid \Delta v = 0 \text{ in } K,\ v|_{\partial K} \in C(\partial K),\ v|_e \in \mathbb{P}_1(e) \text{ for each edge } e \subset \partial K\},$$

$W_h := \{v \in C(\bar{\Omega}) \mid v|_K \in V_1(K) \text{ for all } K \in \mathcal{T}_h\}.$

The displacement fields will be sought in the space

$$\boldsymbol{V}_h := (W_h)^2 \cap \boldsymbol{V}_1.$$

Furthermore, we briefly describe how to construct the bilinear form $a_h^K(\cdot,\cdot)$. Let $\boldsymbol{\Pi}_K$ be defined as a projection operator from $\boldsymbol{V}_h(K)$ into $\mathbb{P}_0(K)^{2\times 2}_{sym}$ such that for any given $\boldsymbol{v}_h \in \boldsymbol{V}_h(K) := (V_1(K))^2$,

$$\int_K \boldsymbol{\Pi}_K(\boldsymbol{v}_h) : \boldsymbol{\varepsilon}^P dx = \int_K \boldsymbol{\varepsilon}(\boldsymbol{v}_h) : \boldsymbol{\varepsilon}^P dx \quad \forall \boldsymbol{\varepsilon}^P \in \mathbb{P}_0(K)^{2\times 2}_{sym},$$

where $\mathbb{P}_0(K)^{2\times 2}_{sym}$ stands for the set of all second order symmetric tensor fields with each entry being constant. Intuitively, $\boldsymbol{\Pi}_K(\boldsymbol{v}_h)$ is a constant projection of the strain field $\boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ over K. Then, following [2, Eqn. (12)], define

$$a_h^K(\boldsymbol{v}_h, \boldsymbol{w}_h) = \int_K \mathcal{F}\boldsymbol{\Pi}_K(\boldsymbol{v}_h) : \boldsymbol{\Pi}_K(\boldsymbol{w}_h)\, dx + b_h^K(\boldsymbol{v}_h, \boldsymbol{w}_h) \quad \forall \boldsymbol{v}_h,\ \boldsymbol{w}_h \in \boldsymbol{V}_h(K), \tag{4.16}$$

where the second term plays a stabilization role. We mention that the first term on the right of (4.16) is essentially equivalent to the first term given in equation (4.1) of the paper [6]. However, the construction of $b_h^K(\cdot,\cdot)$ is rather involved, requiring that $b_h^K(\cdot,\cdot)$ be a symmetric and positive semidefinite bilinear form whose kernel is exactly $(\mathbb{P}_1(K))^2$ (cf. [6, pp. 808–809]). To simplify the presentation, we refer to [2,6] for details along this line.

Then we introduce a local projection $\Pi_1^\nabla : H^1(K) \to \mathbb{P}_1(K)$ as follows. For all $v \in H^1(K)$,

$$\begin{cases} (\nabla \Pi_1^\nabla v, \nabla p)_K = (\nabla v, \nabla p)_K & \forall p \in \mathbb{P}_1(K), \\ \overline{\Pi_1^\nabla v} = \overline{v}, \end{cases}$$

where $(\cdot,\cdot)_K$ stands for the $L^2(K)$ inner product, and $\overline{v}$ is the integral average of v on the boundary ∂K of K. To simplify the presentation, we also use Π_1^∇ to represent the related element-wise defined global operator.

For the right-hand side $\boldsymbol{f}$, we define the approximation $\boldsymbol{f}_h$ such that

$$\langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle := \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{f}_0 \cdot \boldsymbol{\Pi}_1^\nabla \boldsymbol{v}_h dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}_h dx \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.17}$$

where $\boldsymbol{\Pi}_1^\nabla$ is the vectorized analog of Π_1^∇, i.e. for all $\boldsymbol{v} = (v_1, v_2)^T$, $\boldsymbol{\Pi}_1^\nabla \boldsymbol{v} := (\Pi_1^\nabla v_1, \Pi_1^\nabla v_2)^T$.

According to [14, Corollary 3.8] and (4.7), we have by the Cauchy-Schwarz inequality that, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{f}_0 \cdot \boldsymbol{\Pi}_1^\nabla \boldsymbol{v}_h dx &\le \Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{f}_0\|^2_{L^2(K,\mathbb{R}^2)}\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\Pi}_1^\nabla \boldsymbol{v}_h\|^2_{L^2(K,\mathbb{R}^2)}\Big)^{1/2} \\ &\lesssim \Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{f}_0\|^2_{L^2(K,\mathbb{R}^2)}\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{v}_h\|^2_{L^2(K,\mathbb{R}^2)}\Big)^{1/2} \\ &\lesssim \|\boldsymbol{f}_0\|_{L^2(\Omega,\mathbb{R}^2)} \|\boldsymbol{v}_h\|_{L^2(\Omega,\mathbb{R}^2)} \\ &\lesssim \|\boldsymbol{f}_0\|_{L^2(\Omega,\mathbb{R}^2)} \|\boldsymbol{v}_h\|_V. \end{aligned} \tag{4.18}$$

Using the Cauchy–Schwarz inequality again to the second term in (4.17) and then combining the estimate with (4.18), we find

$$\langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \lesssim (\|\boldsymbol{f}_0\|^2_{L^2(\Omega,\mathbb{R}^2)} + \|\boldsymbol{f}_2\|^2_{L^2(\Gamma_2,\mathbb{R}^2)})^{1/2} \|\boldsymbol{v}_h\|_{\boldsymbol{V}}.$$

Thus, we have verified the condition (3.17).

Now, we define a VEM for solving Problem (P_1) as follows.

Problem (P_1^h). Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $\boldsymbol{\xi}_{\boldsymbol{\tau}}^h \in L^2(\Gamma_3; \mathbb{R}^2)$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle = \int_{\Gamma_3} \boldsymbol{\xi}_{\boldsymbol{\tau}}^h \cdot \boldsymbol{v}_{\boldsymbol{\tau}}^h ds \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{4.19}$$

with $-\boldsymbol{\xi}_{\boldsymbol{\tau}}^h \in \partial j_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}}^h)$ a.e. on Γ_3.

For an error analysis, we first note that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned}
|\langle \boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v}_h \rangle| &= \Big| \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{f}_0 \cdot (\boldsymbol{v}_h - \boldsymbol{\Pi}_1^{\nabla} \boldsymbol{v}_h) dx \Big| \\
&\leq \sum_{K \in \mathcal{T}_h} \|\boldsymbol{f}_0\|_{L^2(K,\mathbb{R}^2)} \|\boldsymbol{v}_h - \boldsymbol{\Pi}_1^{\nabla} \boldsymbol{v}_h\|_{L^2(K,\mathbb{R}^2)} \\
&\lesssim \sum_{K \in \mathcal{T}_h} \|\boldsymbol{f}_0\|_{L^2(K,\mathbb{R}^2)} h_K |\boldsymbol{v}_h|_{H^1(K,\mathbb{R}^2)} \\
&\lesssim h \|\boldsymbol{f}_0\|_{L^2(\Omega,\mathbb{R}^2)} \|\boldsymbol{v}_h\|_{\boldsymbol{V}},
\end{aligned}$$

i.e.

$$\|\boldsymbol{f} - \boldsymbol{f}_h\|_{\boldsymbol{V}_h^*} \lesssim h. \tag{4.20}$$

We comment that under Assumption B3, it is easy to derive the estimate (3.27) using the classical Scott-Dupont theory in the case $V = H^1(\Omega; \mathbb{R}^d)$ (cf. [12]). Moreover, according to [10,14], there exists a nodal interpolation operator $I_K : H^2(K) \to V_1(K)$ such that

$$\|v - I_K v\|_{0,K} + h_K |v - I_K v|_{1,K} \lesssim h_K^2 \|v\|_{2,K} \quad \forall v \in H^2(K).$$

When $d = 2$, we write $\boldsymbol{I}_K$ as the vectorized analog of I_K defined above. Moreover, for $\boldsymbol{v} \in H^2(\Omega; \mathbb{R}^2)$ write its global interpolant as $\boldsymbol{v}_I$. It is easy to check that if $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{v}_I \in \boldsymbol{V}_h$. Hence, Assumption B2 holds for $k = 1$ by using the interpolation operator $\boldsymbol{I}_K$.

Moreover, it can be proved using the arguments in [2,6,9,14] that the bilinear form $a_h^K(\cdot, \cdot)$ from (4.16) satisfies conditions (3.18) and (3.19) under Assumption B3. Hence, applying Theorem 3.3 and the finite element interpolation error estimates (cf. [10]), we conclude the optimal order error bound

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}} \lesssim h \tag{4.21}$$

under the regularity assumptions

$$\boldsymbol{u} \in H^2(\Omega; \mathbb{R}^2), \quad \boldsymbol{u}_{\boldsymbol{\tau}}|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i}; \mathbb{R}^2), \ 1 \leq i \leq i_3. \tag{4.22}$$

Now, we consider a concrete example of $j_{\boldsymbol{\tau}}$,

$$j_{\boldsymbol{\tau}}(\boldsymbol{z}) = \int_0^{\|\boldsymbol{z}\|} \mu(t)\, dt. \tag{4.23}$$

Here $\mu(t)$ can be interpreted as a friction bound function. This function is assumed to be measurable from $[0,\infty)$ to $\mathbb{R}$, $\mu(0+)>0$, and with two positive constant c_1, c_2,

$$0 \le \mu(t) \le c_1(1+t) \quad \forall\, t \ge 0, \tag{4.24}$$

$$\mu(t_2)-\mu(t_1) \ge -c_2(t_2-t_1) \;\; \forall\, t_2 > t_1 \ge 0. \tag{4.25}$$

In this situation, the function $j_{\boldsymbol{\tau}}$ defined by (4.23) is regular in the sense of Clarke (cf. [3, Lemma 3.2]).

4.2 A Frictionless Normal Compliance Contact Problem

The contact boundary condition is

$$-\sigma_\nu \in \partial j_\nu(u_\nu), \quad \boldsymbol{\sigma_\tau} = \mathbf{0} \quad \text{on } \Gamma_3. \tag{4.26}$$

The first relation in (4.26) is a normal compliance contact condition, whereas the second relation indicates that the contact is frictionless. Here, we assume the following properties (cf. [20]) on the potential function $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$:

$$\begin{cases} (a)\ j_\nu(\cdot, z) \text{ is measurable on } \Gamma_3 \text{ for all } z \in \mathbb{R} \text{ and } j_\nu(\cdot, z_0(\cdot)) \in L^1(\Gamma_3) \\ \quad \text{for some } z_0 \in L^2(\Gamma_3); \\ (b)\ j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ (c)\ |\partial j_\nu(\boldsymbol{x}, z)| \le \bar{c}_0 + \bar{c}_1 \|z\| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for all } z \in \mathbb{R} \text{ with } \bar{c}_0, \bar{c}_1 \ge 0; \\ (d)\ j_\nu^0(\boldsymbol{x}, z_1; z_2 - z_1) + j_\nu^0(\boldsymbol{x}, z_2; z_1 - z_2) \le \alpha_{j_\nu} |z_1 - z_2|^2 \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \quad \text{for all } z_1, z_2 \in \mathbb{R} \text{ with } \alpha_{j_\nu} \ge 0. \end{cases} \tag{4.27}$$

For the contact problem described by (4.1)–(4.4) and (4.26), proceeding in a standard way, we can obtain the following weak formulation.

Problem (P_2). Find an element $\boldsymbol{u} \in \boldsymbol{V}$ and $\sigma_\nu \in L^2(\Gamma_3)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_{\Gamma_3} \sigma_\nu \cdot v_\nu ds \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{4.28}$$

with $-\sigma_\nu \in \partial j_\nu(u_\nu)$ a.e. on Γ_3.

The displacement fields will be sought in the space

$$\boldsymbol{V}_h := (W_h)^2 \cap \boldsymbol{V}.$$

We follow the discussion in Sect. 4.1, with the following modifications:

$$m = 1, \quad \gamma \boldsymbol{v} = v_\nu \ \text{ for } \ \boldsymbol{v} \in \boldsymbol{V},$$

$$j(\cdot, z) = j_\nu(z), \quad z \in \mathbb{R},$$

$\alpha_j = \alpha_{j_\nu}$, and $c_\gamma \ge \lambda_{2,V}^{-1/2}$, $\lambda_{2,V} > 0$ being the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in \boldsymbol{V}, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} u_\nu \cdot v_\nu ds \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

We assume additionally that

$$\alpha_{j_\nu} < \lambda_{2,V} m_{\mathcal{F}}.$$

Similar to Problem (P_1), Problem (P_2) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$.

Now, we introduce the following approximation of Problem (P_2):

Problem (P_2^h). Find an element $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $\sigma_\nu^h \in L^2(\Gamma_3)$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle = \int_{\Gamma_3} \sigma_\nu^h \cdot v_\nu^h ds \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.29}$$

with $-\sigma_\nu^h \in \partial j_\nu(u_\nu^h)$ a.e. on Γ_3.

Similar to the derivation of the error estimate (4.21), we can conclude the optimal order error bound

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}} \lesssim h \tag{4.30}$$

under the regularity assumptions

$$\boldsymbol{u} \in H^2(\Omega; \mathbb{R}^2), \quad u_\nu|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i}),\ 1 \le i \le i_3. \tag{4.31}$$

5 An Algorithm for Solving Discrete Problems

We now present an efficient algorithm to solve the discrete problems. To this end, we require to formulate the discrete hemivariational inequality under study as a minimization problem following some ideas in [4]. For definiteness, we only consider Problem (P_1^h) in detail. Let N_0 be the number of nodal points, and let $\{\boldsymbol{\phi}_i\}_{i=1}^{2N_0}$ be the shape basis functions of $\boldsymbol{V}_h$. For $\boldsymbol{v} \in \boldsymbol{V}_h$, define a function $\ell : \mathbb{R}^{2N_0} \to \mathbb{R}$ by

$$\ell(\boldsymbol{\alpha}) = \int_{\Gamma_3} j(\boldsymbol{v}) ds = \int_{\Gamma_3} j\Big(\sum_{k=1}^{2N_0} \alpha_k \boldsymbol{\phi}_k(\boldsymbol{x})\Big)\, ds \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{2N_0}, \tag{5.1}$$

where $\boldsymbol{\alpha} := [\alpha_k]_{k=1}^{2N_0}$ and $\boldsymbol{v} := \sum_{k=1}^{2N_0} \alpha_k \boldsymbol{\phi}_k(\boldsymbol{x})$. Recalling the relation (3.8), we know that for any $\boldsymbol{\eta} \in \partial \ell(\boldsymbol{\alpha})$, there exists $-\boldsymbol{\xi}_{\boldsymbol{\tau}}^h(\boldsymbol{x}) \in \partial j(\boldsymbol{v})$ a.e. on Γ_3 such that

$$\langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle_{\mathbb{R}^{2N_0} \times \mathbb{R}^{2N_0}} = \int_{\Gamma_3} -\boldsymbol{\xi}_{\boldsymbol{\tau}}^h(\boldsymbol{x}) \cdot \boldsymbol{v}_1\, ds \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^{2N_0}, \tag{5.2}$$

where $\boldsymbol{v}_1 := \sum_{i=1}^{2N_0} \zeta_i \boldsymbol{\phi}_i(\boldsymbol{x})$ and $\boldsymbol{\zeta} := [\zeta_i]_{i=1}^{2N_0}$. Next, define

$$\boldsymbol{b} = [b_i]_{i=1}^{2N_0}, \quad b_i = \langle \boldsymbol{f}_h, \boldsymbol{\phi}_i \rangle; \quad \boldsymbol{A} = [A_{ik}]_{i,k=1}^{2N_0}, \quad A_{ik} = a_h(\boldsymbol{\phi}_i, \boldsymbol{\phi}_k).$$

If a vector $\boldsymbol{\alpha}^* = [\alpha_k^*]_{k=1}^{2N_0}$ is a solution of the inclusion

$$\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\alpha} \in \partial \ell(\boldsymbol{\alpha}), \tag{5.3}$$

then we take $\boldsymbol{\eta} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{\alpha}^*$ in (5.2), from which we immediately know $\boldsymbol{u}_h = \sum_{k=1}^{2N_0} \alpha_k^* \boldsymbol{\phi}_k(\boldsymbol{x})$ is exactly the solution of Problem (P_1^h). So we turn to solve the problem (5.3).

On the other hand, by [15, Proposition 2.3.3], the inclusion (5.3) is equivalent to the following:

$$\boldsymbol{0} \in \partial H(\boldsymbol{\alpha}), \tag{5.4}$$

where

$$H(\boldsymbol{\alpha}) = \frac{1}{2}\boldsymbol{\alpha}^T \boldsymbol{A}\boldsymbol{\alpha} - \boldsymbol{b}^T \boldsymbol{\alpha} + \ell(\boldsymbol{\alpha}). \tag{5.5}$$

By [4, Lemma 4], we know that $H(\boldsymbol{\alpha})$ defined in (5.5) attains a global minimum. Note that Problem (P_1^h) has a unique solution, so it is equivalent to the minimization problem:

$$H(\boldsymbol{\alpha}^*) = \inf\{H(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in \mathbb{R}^{2N_0}\}, \tag{5.6}$$

where $H(\boldsymbol{\alpha})$ is given by (5.5).

Furthermore, we approximate $\ell(\boldsymbol{\alpha})$ with the trapezoidal rule to get

$$\int_{\Gamma_3} j\left(\sum_{k=1}^{2N_0} \alpha_k \boldsymbol{\phi}_k(\boldsymbol{x})\right) ds \approx \sum_{i\in I} w_i\, j\left(\sum_{k=1}^{2N_0} \alpha_k \boldsymbol{\phi}_k(\boldsymbol{x}_i)\right) =: \boldsymbol{w}^T \boldsymbol{j}(\boldsymbol{\alpha}), \tag{5.7}$$

where $\{\boldsymbol{x}_i\}_{i\in I}$ is the set of the nodal points of $\mathcal{T}_h$ on Γ_3 and $\{w_i\}_{i\in I}$ are the weights of the integration formula. Assuming that N_1 components corresponding to the index set I are listed first, we write the vector $\boldsymbol{\alpha}$ in block form as $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1^T, \boldsymbol{\alpha}_2^T)^T$ with $\boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1}$. Similarly,

$$\boldsymbol{A} := \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T & \boldsymbol{A}_{22} \end{pmatrix}, \quad \boldsymbol{b} := \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}, \quad \boldsymbol{w} := \begin{pmatrix} \boldsymbol{w}_1 \\ \boldsymbol{0} \end{pmatrix}. \tag{5.8}$$

Then the problem (5.6) can be recast approximately as follows: find $\boldsymbol{\alpha}_1^* \in \mathbb{R}^{N_1}$ and $\boldsymbol{\alpha}_2^* \in \mathbb{R}^{N_2}$ such that

$$F(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = \inf\{F(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \mid \boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1}, \boldsymbol{\alpha}_2 \in \mathbb{R}^{N_2}\}, \tag{5.9}$$

where $N_2 = 2N_0 - N_1$, and

$$F(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) := \frac{1}{2}(\boldsymbol{\alpha}_1^T \boldsymbol{A}_{11}\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_1^T \boldsymbol{A}_{12}\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_2^T \boldsymbol{A}_{22}\boldsymbol{\alpha}_2) - \boldsymbol{b}_1^T \boldsymbol{\alpha}_1 - \boldsymbol{b}_2^T \boldsymbol{\alpha}_2 + \boldsymbol{w}_1^T \boldsymbol{j}(\boldsymbol{\alpha}_1).$$

It is evident that the first order conditions of the optimization problem (5.9) are

$$\boldsymbol{0} \in \boldsymbol{A}_{11}\boldsymbol{\alpha}_1 + \boldsymbol{A}_{12}\boldsymbol{\alpha}_2 - \boldsymbol{b}_1 + \partial(\boldsymbol{w}_1^T \boldsymbol{j}(\boldsymbol{\alpha}_1)), \tag{5.10}$$

$$\boldsymbol{0} = \boldsymbol{A}_{12}^T\boldsymbol{\alpha}_1 + \boldsymbol{A}_{22}\boldsymbol{\alpha}_2 - \boldsymbol{b}_2, \tag{5.11}$$

where ∂ denotes the generalized gradient of a Lipschitz function (cf. [26,27]). We have by (5.11) that

$$\boldsymbol{\alpha}_2 = \boldsymbol{A}_{22}^{-1}(\boldsymbol{b}_2 - \boldsymbol{A}_{12}^T\boldsymbol{\alpha}_1). \tag{5.12}$$

Using this formula in (5.9) to eliminate $\boldsymbol{\alpha}_2$, we find the following reduced minimization problem from (5.9):

Find $\boldsymbol{\alpha}_1^* \in \mathbb{R}^{N_1}$ such that

$$\hat{F}(\boldsymbol{\alpha}_1^*) = \inf\{\hat{F}(\boldsymbol{\alpha}_1) \mid \boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1}\}, \tag{5.13}$$

where

$$\hat{F}(\boldsymbol{\alpha}_1) := \frac{1}{2}\boldsymbol{\alpha}_1^T \widetilde{\boldsymbol{A}_1}\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{b}_1}^T \boldsymbol{\alpha}_1 + \boldsymbol{w}_1^T \boldsymbol{j}(\boldsymbol{\alpha}_1),$$

with

$$\widetilde{\boldsymbol{A}_1} := \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{12}^T, \quad \widetilde{\boldsymbol{b}_1} := \boldsymbol{b}_1 - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{b}_2.$$

In our numerical simulation given in the next section, we will use the double bundle method (cf. [23]) to solve problem (5.13).

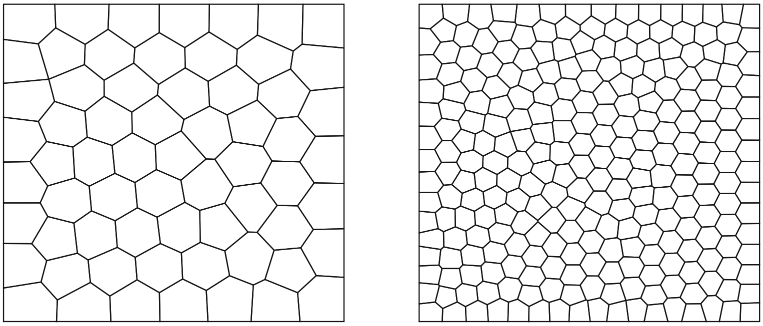

Fig. 1 Polygonal mesh for $N = 64$ (left) and $N = 256$ (right)

6 Numerical Experiments

In this section, we report numerical results to illustrate performance of the numerical method. In our numerical simulation, the polygonal meshes are produced by an algorithm presented in [30] and the codes obtained are written based on the program in [29]. Meshes with element numbers $N = 64$ and $N = 256$ for the unit square are displayed in Fig. 1.

Example 6.1 First, we consider an example of Problem (P_1). The contact is bilateral and frictional. The domain $\Omega = (0, 4) \times (0, 4)$ is the cross section of a three-dimensional linearly elastic body and the plane stress condition is imposed. The boundary $\partial\Omega$ is decomposed into three parts: $\Gamma_D = \{4\} \times (0, 4)$ where the body is clamped, $\Gamma_C = (0, 4) \times \{0\}$ where frictional contact takes place, and the remaining part $\Gamma_N = (\{0\} \times (0, 4)) \cup ((0, 4) \times \{4\})$ is for traction boundary condition. The elasticity tensor $\mathcal{F}$ is given by

$$(\mathcal{F}\boldsymbol{\varepsilon})_{ij} = \frac{E\nu}{1-\nu^2}(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}, \quad 1 \le i, j \le 2,$$

where E is the Young modulus, ν is the Poisson ratio of the material and δ_{ij} is the Kronecker delta. We use the following data:

$$\begin{aligned} E &= 2000\,\mathrm{daN/mm^2}, \quad \nu = 0.4, \\ \boldsymbol{f}_0 &= (0, 0)^T\ \mathrm{daN/mm^2}, \\ \boldsymbol{f}_2(x_1, x_2) &= \begin{cases} (200(5 - x_2), -200)^T\ \mathrm{daN/mm} & \text{on } \{0\} \times (0, 4), \\ \boldsymbol{0} \quad \text{on } (0, 4) \times \{4\}. \end{cases} \end{aligned}$$

The contact condition is determined by (4.9) and (4.23), where

$$\mu(t) = (a - b)\, e^{-\alpha t} + b.$$

Hence,

$$j_\tau(\boldsymbol{u_\tau}^h) = \int_0^{\|\boldsymbol{u_\tau}^h\|} \left[(a - b)e^{-\alpha t} + b\right] dt = f_0(\boldsymbol{u_\tau}^h) - \tilde{f}_0(\boldsymbol{u_\tau}^h),$$

with

$$f_0(\boldsymbol{u_\tau}^h) := a\,\|\boldsymbol{u_\tau}^h\| + \frac{a-b}{\alpha}, \quad \tilde{f}_0(\boldsymbol{u_\tau}^h) := \frac{a-b}{\alpha}(e^{-\alpha\|\boldsymbol{u_\tau}^h\|} + \alpha\,\|\boldsymbol{u}_\tau^h\|).$$

Since $a > b > 0$ and $\alpha > 0$, after using numerical integration to $\int_{\Gamma_3} j_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}}^h)\,ds$, we can similarly get

$$\int_{\Gamma_3} j_{\boldsymbol{\tau}}(\boldsymbol{u}_{\boldsymbol{\tau}}^h)\,ds \approx \boldsymbol{w}_1^T \boldsymbol{j}(\boldsymbol{\alpha}_1) =: j(\boldsymbol{\alpha}_1). \tag{6.1}$$

In this situation, the function $j(\boldsymbol{\alpha}_1)$ is regular (cf. [3]). We observe that the Lipschitz function $j(\boldsymbol{\alpha}_1)$ is a DC function, i.e. it is a difference of two convex functions. Obviously, it is satisfied with the necessities of the double bundle method (cf. [23]). We choose $a = 900$, $b = 450$ and $\alpha = 2000$. Combining with (5.7), we obtain that

$$j(\boldsymbol{\alpha}_1) = f_0(\boldsymbol{\alpha}_1) - \tilde{f}_0(\boldsymbol{\alpha}_1),$$

where

$$f_0(\boldsymbol{\alpha}_1) := 900\boldsymbol{w}_1^T|\boldsymbol{\alpha}_1| + \frac{9}{40}\text{meas}(\Gamma_3), \tag{6.2}$$

$$\tilde{f}_0(\boldsymbol{\alpha}_1) := \frac{9}{40}\boldsymbol{w}_1^T(e^{-2000|\boldsymbol{\alpha}_1|} + 2000|\boldsymbol{\alpha}_1|), \tag{6.3}$$

and $|\boldsymbol{\alpha}_1|$ denotes a new vector formed by taking the absolute value for each entry of $\boldsymbol{\alpha}_1$, whereas $\boldsymbol{w}_1$ denotes a vector formed by the coefficients of the integration formula. The problem (5.13) is equivalent to the following DC minimization problem:

$$\min\left\{f(\boldsymbol{\alpha}_1) = f_1(\boldsymbol{\alpha}_1) - f_2(\boldsymbol{\alpha}_1) \mid \boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1}\right\},$$

where

$$f_1(\boldsymbol{\alpha}_1) := \frac{1}{2}\boldsymbol{\alpha}_1^T\widetilde{\boldsymbol{A}_1}\boldsymbol{\alpha}_1 + f_0(\boldsymbol{\alpha}_1), \quad f_2(\boldsymbol{\alpha}_1) := \widetilde{\boldsymbol{b}_1}^T\boldsymbol{\alpha}_1 + \tilde{f}_0(\boldsymbol{\alpha}_1). \tag{6.4}$$

It is verified that the method terminates after a finite number of steps and the solution is approximately Clarke stationary (cf. [23, Section 5.4]). The associated Fortran code can be obtained from http://napsu.karmitsa.fi/nsosoftware/.

The numerical solutions corresponding to several meshes with $N = 200$, $N = 800$, $N = 3200$, $N = 12{,}800$ are displayed in Fig. 2, respectively. A convergence trend is evident for the numerical solutions as N increases.

In Table 1 and Fig. 3, we report relative errors $\|\boldsymbol{u}_{\text{ref}} - \boldsymbol{u}_h\|_E/\|\boldsymbol{u}_{\text{ref}}\|_E$ of the numerical solutions in the energy norm on square meshes, where the energy norm is given by

$$\|\boldsymbol{v}\|_E := \frac{1}{\sqrt{2}}(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{v})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q^{1/2}.$$

Note that the error bound (4.21) predicts an optimal first order convergence of the numerical solutions measured in the energy norm, under the regularity assumptions (4.22). Since the true solution $\boldsymbol{u}$ is not available, we use the numerical solution with a fine mesh as the "reference" solution $\boldsymbol{u}_{\text{ref}}$ in computing the solution errors. Specifically, the "reference" solution $\boldsymbol{u}_{\text{ref}}$ is set as the numerical solution with $h = 1/32$.

The relative errors in energy norm are shown in Fig. 3. □

Example 6.2 Now, we consider an example of Problem (P_2). The first relation in (4.26) is a normal compliance contact condition, whereas the second relation indicates that the contact is frictionless. The domain $\Omega = (0, 1) \times (0, 1)$ is the cross section of a three-dimensional linearly elastic body and plane strain condition is assumed. On the part $\Gamma_1 = (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])$ the body is clamped. Vertical tractions act on Γ_2, where

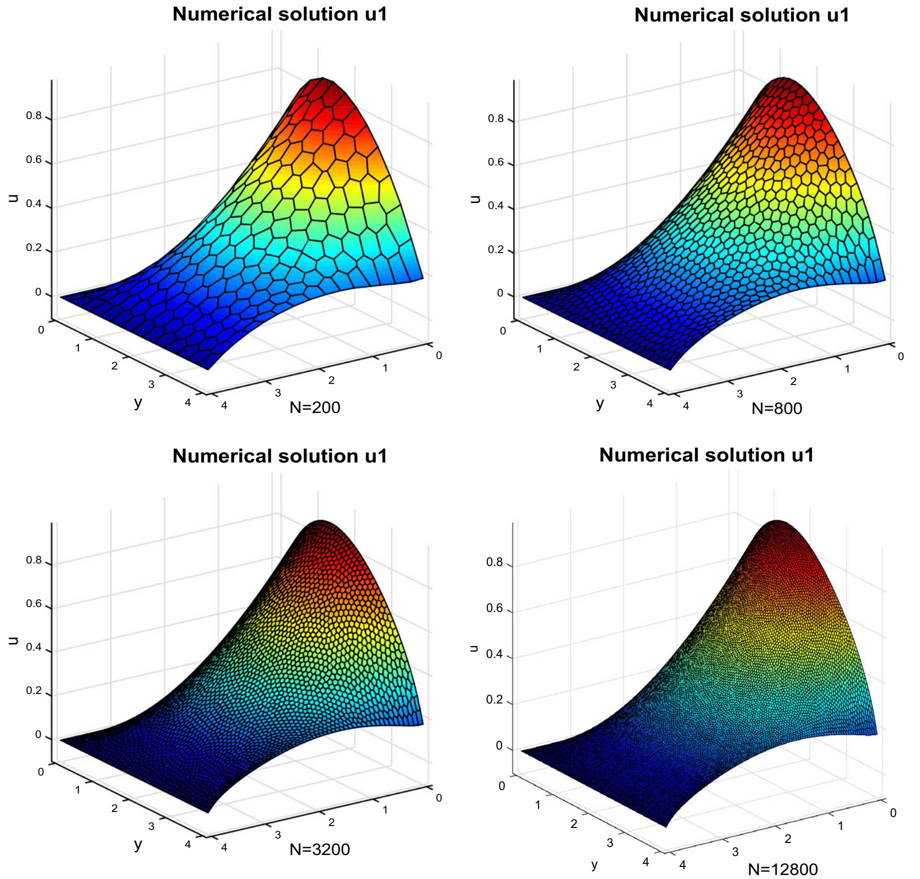

Fig. 2 The numerical solution related to different numbers of elements: $N = 200$ (upper left), $N = 800$ (upper right), $N = 3200$ (bottom left) and $N = 12{,}800$ (bottom right)

Table 1 Numerical errors on square meshes for lowest-order VEM

h	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Error	31.914%	14.319%	6.353%	2.713%	1.129%

$\Gamma_2 = [0, 1] \times \{1\}$. The contact part of the boundary is $\Gamma_3 = [0, 1] \times \{0\}$. The elasticity tensor $\mathcal{F}$ satisfies

$$(\mathcal{F}\boldsymbol{\varepsilon})_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}, \quad 1 \leqslant i, j \leqslant 2,$$

where E is the Young's modulus, ν is the Poisson's ratio of the material and δ_{ij} is the Kronecker symbol. For the computation below, we use the following data

$$E = 70\,\text{GPa}, \quad \nu = 0.3.$$

No body forces are assumed to act on the body during the process, and

$$\boldsymbol{f}_0 = (0, 0)\,\text{GPa}, \quad \boldsymbol{f}_2 = (0, -52)\,\text{GPa} \quad \text{on } \Gamma_2.$$

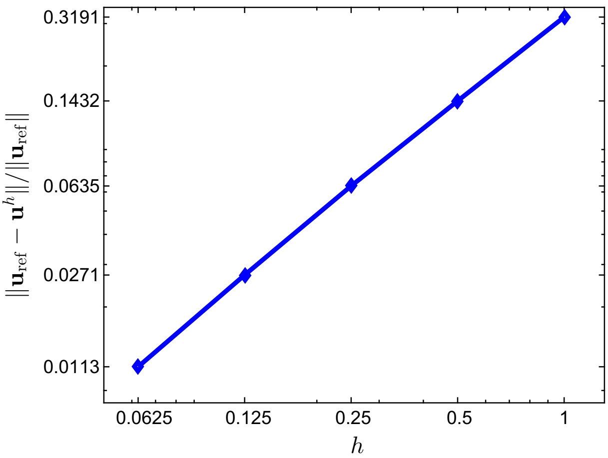

Fig. 3 Relative errors in energy norm

For Problem (P_2) with boundary (4.26), we choose

$$-\sigma_\nu = \begin{cases} 0 \ \text{ if } u_\nu \le 0, \\ 100u_\nu \ \text{ if } u_\nu \in (0, 0.1], \\ 20 - 100u_\nu \ \text{ if } \ u_\nu \in (0.1, 0.15), \\ 400u_\nu - 55 \ \text{ if } \ u_\nu \ge 0.15. \end{cases} \tag{6.5}$$

We have the following formula by (4.26),

$$j_\nu(u_\nu) = \begin{cases} 0 \ \text{ if } u_\nu \le 0, \\ 50u_\nu^2 \ \text{ if } u_\nu \in (0, 0.1], \\ 20u_\nu - 50u_\nu^2 - 1 \ \text{ if } u_\nu \in (0.1, 0.15), \\ 200u_\nu^2 - 55u_\nu + 4.625 \ \text{ if } u_\nu \ge 0.15. \end{cases} \tag{6.6}$$

Let $j_\nu(u_\nu) = f_0(u_\nu) - \tilde{f}_0(u_\nu)$, where $f_0(u_\nu)$ and $\tilde{f}_0(u_\nu)$ defined as follows:

$$f_0(u_\nu) = \begin{cases} 50u_\nu^2 \ \text{ if } u_\nu \le 0, \\ 100u_\nu^2 \ \text{ if } u_\nu \in (0, 0.1], \\ 20u_\nu - 1 \ \text{ if } u_\nu \in (0.1, 0.15), \\ 250u_\nu^2 - 55u_\nu + 4.625 \ \text{ if } u_\nu \ge 0.15, \end{cases} \qquad \text{and} \quad \tilde{f}_0(u_\nu) = 50u_\nu^2.$$

For the numerical solution u_ν^h, we choose

$$j_\nu(u_\nu^h) = f_0(u_\nu^h) - \tilde{f}_0(u_\nu^h),$$

as in the case of (6.1). Then

$$\int_{\Gamma_3} j_\nu(u_\nu^h)\, ds = \int_{\Gamma_3} (f_0(u_\nu^h) - \tilde{f}_0(u_\nu^h))ds \approx \boldsymbol{w}_1^T \boldsymbol{j}(\boldsymbol{\alpha}_1) =: j(\boldsymbol{\alpha}_1),$$

with

$$j(\boldsymbol{\alpha}_1) := f_0(\boldsymbol{\alpha}_1) - \tilde{f}_0(\boldsymbol{\alpha}_1),$$

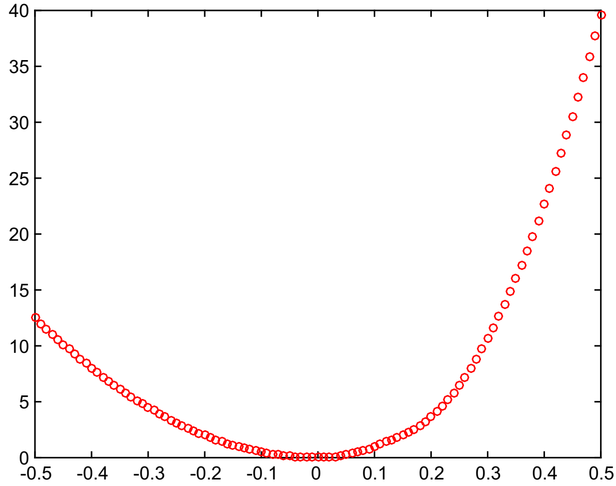

Fig. 4 The function $f_0(u_\nu)$

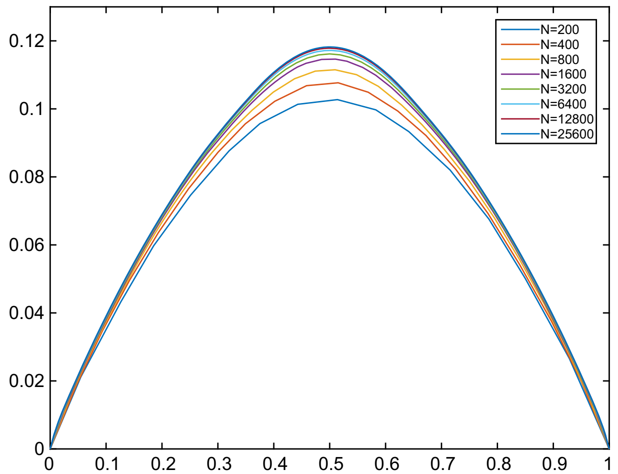

Fig. 5 The numerical solution of normal direction for different meshes

where $\boldsymbol{w}_1$ denotes a vector formed by the coefficients of an numerical integration formula [cf. (5.7)]. It is noted the function $f_0(u_\nu)$ is convex, which is sketched in Fig. 4.

Clearly, the function $j(\boldsymbol{\alpha}_1)$ is regular (cf. [16, Proposition 5.6.15(b)]) and directionally differentiable. The problem (5.13) is equivalent to the following DC minimization problem of the form:

$$\min\left\{ f(\boldsymbol{\alpha}_1) = f_1(\boldsymbol{\alpha}_1) - f_2(\boldsymbol{\alpha}_1) \mid \boldsymbol{\alpha}_1 \in \mathbb{R}^{N_1} \right\},$$

where

$$f_1(\boldsymbol{\alpha}_1) := \frac{1}{2}\boldsymbol{\alpha}_1^T \widetilde{\boldsymbol{A}_1}\boldsymbol{\alpha}_1 + f_0(\boldsymbol{\alpha}_1), \quad f_2(\boldsymbol{\alpha}_1) := -\widetilde{\boldsymbol{b}_1}^T \boldsymbol{\alpha}_1 + \tilde{f}_0(\boldsymbol{\alpha}_1). \tag{6.7}$$

According to the numerical results of the numerical solution of normal direction on the boundary $[0, 1] \times \{0\}$, a similar convergence trend is clearly observed (cf. Fig. 5).

Table 2 Numerical errors on square meshes for lowest-order VEM

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
error	36.877%	24.739%	14.944%	8.679%	4.575%

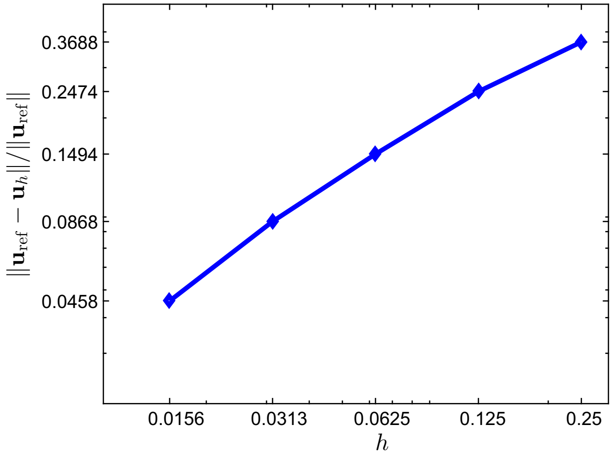

Fig. 6 Relative errors in energy norm

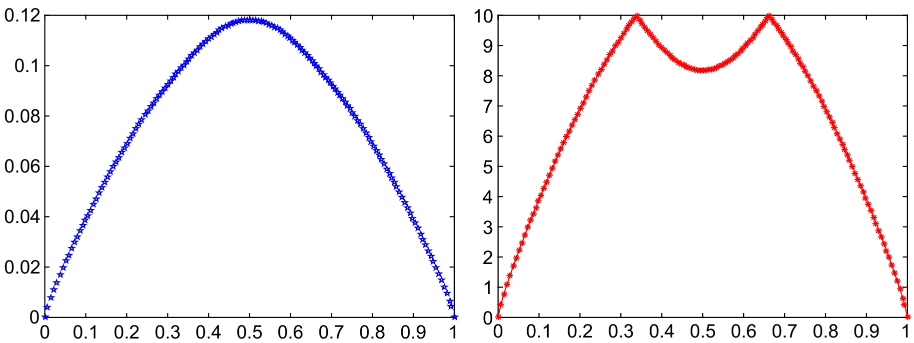

Fig. 7 The normal displacement (left) and normal stress (right) for a mesh with $N = 25{,}600$

In this situation, the numerical solution with $h = 1/256$ is taken to be the "reference" solution $\boldsymbol{u}_{\text{ref}}$ (Table 2).

The relative errors in energy norm are shown in Fig. 6.

On the other hand, we fix the number of elements in the mesh $N = 25600$, and plot the displacement u_ν and force $-\sigma_\nu$ in Fig. 7.

All nodes are in status of normal compliance, i.e. $0 \leq u_\nu < 0.15$. Nevertheless for part of the nodes we have $0 \leq u_\nu < 0.1$ and for the other part we have $0.1 \leq u_\nu < 0.15$. We note that for $0 \leq u_\nu < 0.1$ the normal forces increase with respect to the penetration and for $0.1 \leq u_\nu < 0.15$ they decrease. It arises since there $-\sigma_\nu = k_\nu(u_\nu)$ and k_ν is an increasing function on $[0, 0.1]$, and it is decreasing on $[0.1, 0.15]$. □

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References

1. Ahmad, B., Alsaedi, A., Brezzi, F., Marini, L.D., Russo, A.: Equivalent projectors for virtual element methods. Comput. Math. Appl. **66**(3), 376–391 (2013)
2. Artioli, E., Beirao Da Veiga, L., Lovadina, C., et al.: Arbitrary order 2D virtual elements for polygonal meshes: part I, elastic problem. Comput. Mech. **60**(3), 355–377 (2017)
3. Barboteu, M., Bartosz, K., Han, W., Janiczko, T.: Numerical analysis of a hyperbolic hemivariational inequality arising in dynamic contact. SIAM J. Numer. Anal. **53**(1), 527–550 (2015)
4. Barboteu, M., Bartosz, K., Kalita, P.: An analytical and numerical approach to a bilateral contact problem with nonmonotone friction. Int. J. Appl. Math. Comput. Sci. **23**(2), 263–276 (2013)
5. Beirao Da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., et al.: Basic principles of virtual element methods. Math. Models Methods Appl. Sci. **23**(1), 199–214 (2013)
6. Beirao Da Veiga, L., Brezzi, F., Marini, L.D.: Virtual elements for linear elasticity problems. SIAM J. Numer. Anal. **51**(2), 794–812 (2013)
7. Beirao Da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: The Hitchhiker's guide to the virtual element method. Math. Models Methods Appl. Sci. **24**(8), 1541–1573 (2014)
8. Beirao Da Veiga, L., Lovadina, C., Russo, A.: Stability analysis for the virtual element method. Math. Models Methods Appl. Sci. **27**(13), 2557–2594 (2017)
9. Brenner, S.C.: Korn's inequalities for piecewise H^1 vector fields. Math. Comput. **73**(247), 1067–1087 (2004)
10. Brenner, S.C., Guan, Q., Sung, L.: Some estimates for virtual element methods. Comput. Methods Appl. Math. **17**(4), 553–574 (2017)
11. Brenner, S.C., Sung, L.: Virtual element methods on meshes with small edges or faces. Math. Models Methods Appl. Sci. **28**(7), 1291–1336 (2018)
12. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 3rd edn. Springer, New York (2008)
13. Brezzi, F., Buffa, A., Lipnikov, K.: Mimetic finite differences for elliptic problems. ESAIM: Math. Model. Numer. Anal. **43**, 277–295 (2009)
14. Chen, L., Huang, J.: Some error analysis on virtual element methods. Calcolo **55**(1), 5 (2018)
15. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
16. Denkowski, Z., Migorski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Theory. Kluwer Academic Publishers, Boston (2003)
17. Feng, F., Han, W., Huang, J.: Virtual element methods for elliptic variational inequalities of the second kind. J. Sci. Comput. **80**, 60–80 (2019)
18. Han, W.: Numerical analysis of stationary variational–hemivariational inequalities with applications in contact mechanics. Math. Mech. Solids **23**(3), 279–293 (2018)
19. Han, W., Migorski, S., Sofonea, M.: A class of variational-hemivariational inequalities with applications to frictional contact problems. SIAM J. Math. Anal. **46**(6), 3891–3912 (2014)
20. Han, W., Sofonea, M., Barboteu, M.: Numerical analysis of elliptic hemivariational inequalities. SIAM J. Numer. Anal. **55**(2), 640–663 (2017)
21. Han, W., Sofonea, M., Danan, D.: Numerical analysis of stationary variational-hemivariational inequalities. Numer. Math. **139**, 563–592 (2018)
22. Haslinger, J., Miettinen, M., Panagiotopoulos, P.D.: Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications. Kluwer Academic, Dordrecht (1999)
23. Joki, K., Bagirov, A.M., Karmitsa, N., et al.: Double bundle method for finding Clarke stationary points in nonsmooth DC programming. SIAM J. Optim. **28**(2), 1892–1919 (2018)
24. Laursen, T.: Computational Contact and Impact Mechanics. Springer, New York (2002)
25. Migorski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. Springer, New York (2013)
26. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker, New York (1995)
27. Panagiotopoulos, P.D.: Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer, Berlin (1993)
28. Panagiotopoulos, P.D.: Nonconvex energy functions, hemivariational inequalities and substationarity principles. Acta Mech. **48**(3–4), 111–130 (1983)

29. Sutton, O.J.: The virtual element method in 50 lines of MATLAB. Numer. Algorithms **75**, 1141–1159 (2017)
30. Talischi, C., Paulino, G.H., Pereira, A., Menezes, I.F.M.: PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab. Struct. Multidiscip. Optim. **45**(3), 309–328 (2012)
31. Wang, F., Wei, H.: Virtual element method for simplified friction problem. Appl. Math. Lett. **85**, 125–131 (2018)
32. Wang, F., Wei, H.: Virtual element methods for the obstacle problem. IMA J. Numer. Anal. (2018). https://doi.org/10.1093/imanum/dry055
33. Wriggers, P.: Computational Contact Mechanics, 2nd edn. Springer, Berlin (2006)
34. Zeidler, E.: Nonlinear Functional Analysis and Its Applications. II/B: Nonlinear Monotone Operators. Springer, New York (1990)