From Counter-Model-based Quantifier Instantiation to Quantifier Elimination in SMT

CADE 27

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The University of Iowa

Acknowledgments

Collaborators:

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(*) past or present CVC4 developer

Outline

Introduction

Satisfiability Modulo Theories (SMT)

- Subfield of automated deduction focussing on specialized reasoning in certain logical theories
- Used in large and diverse number of applications
- Traditionally, strong on quantifier-free reasoning
- However, many applications require a mix of built-in **and** axiomatically defined symbols

Need for Quantifiers in SMT Applications

Automated Theorem Proving

Background axioms: $\forall x.\, g(e,x) = g(x,e) = x,\ \forall x.\, g(x,i(x)) = e$
$\forall x.\, g(x,g(y,z)) = g(g(x,y),x)$

Software Verification

Unfolding: $\forall x.\, \mathit{foo}(x) = \mathit{bar}(x+1)$
Code contracts: $\forall x.\, \mathit{pre}(x) \Rightarrow \mathit{post}(f(x))$
Frame axioms: $\forall x.\, x \neq t \Rightarrow A'(x) = A(x)$

Function Synthesis

Synthesis conjectures: $\forall i{:}\mathit{input}.\, \exists o{:}\mathit{output}.\, R[o,i]$

Planning

Specifications: $\exists p{:}\mathit{plan}\, \forall t{:}\mathit{time}\, F[p,t]$

Grand Challenge in Automated Deduction

Reasoning efficiently about

theory symbols and quantifiers

First-order theorem provers focus mostly on reasoning with quantifiers but some have been extended to theory reasoning:

Vampire, E, SPASS, Beagle

- First-order resolution/superposition [Nieuwenhuis&Rubio 1999, Prevosto&Waldman 2006, Althaus et al. 2009, Baumgartner&Waldman 2013]
- AVATAR [Voronkov 2014, Reger et al. 2015]

iProver

- InstGen calculus [Ganzinger&Korovin 2003]

Princess

- Sequent calculus [Rümmer 2008]

Reasoning with Theories and Quantifiers in FOL — ATP case

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SMT solvers focus mostly on quantifier-free theory reasoning but some have been extended to reasoning with quantifiers:

Alt-Ergo, CVC3, CVC4, veriT, Z3

- Some superposition-based [deMoura et al. 2009]
- Most instantiation-based [Detlefs et al. 2005, deMoura et al. 2007, Ge et al. 2007, ...]

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SMT Solvers using Quantifier Instantiation

Traditionally:

- E-matching [Detlefs et al. 2005, Bjørner et al. 2007, CADE 2007]

More recently:

- Model-Based Instantiation [Ge et al. 2009, CADE 2013]
- Conflict-Based Instantiation [FMCAD 2014, TACAS 2017]
- Theory-specific Approaches
  - Linear arithmetic [Bjørner 2012, CAV 2015, Janota et al. 2015]
  - Bit-Vectors [Wintersteiger et al. 2013, Dutertre 2015]

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CVC4, Z3

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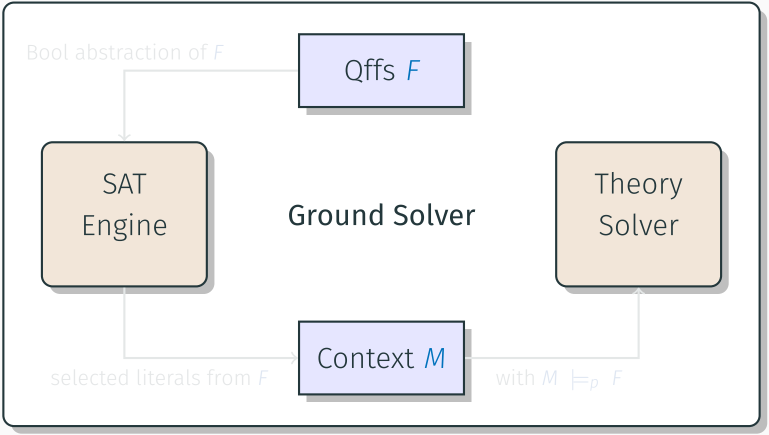
CVC4, Yices, veriT, Z3

SMT Solvers for Ground Formulas



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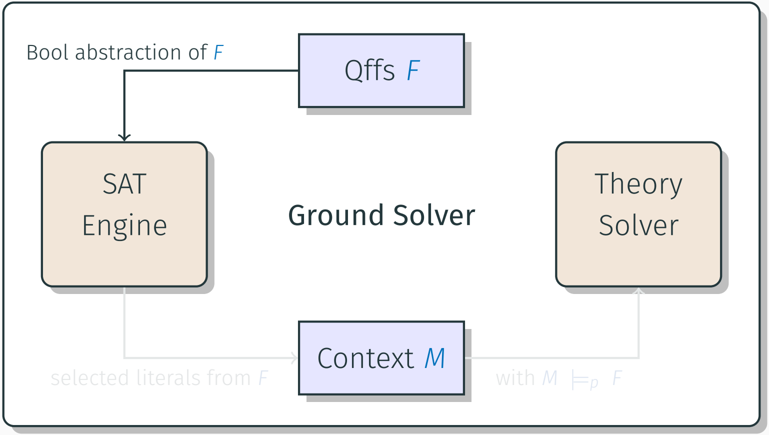

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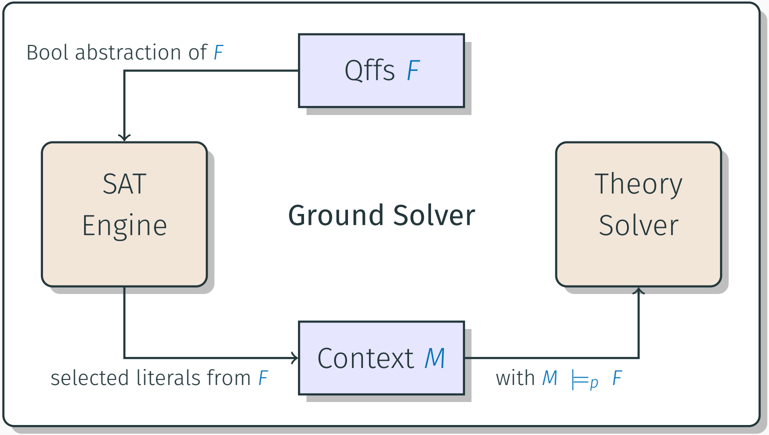

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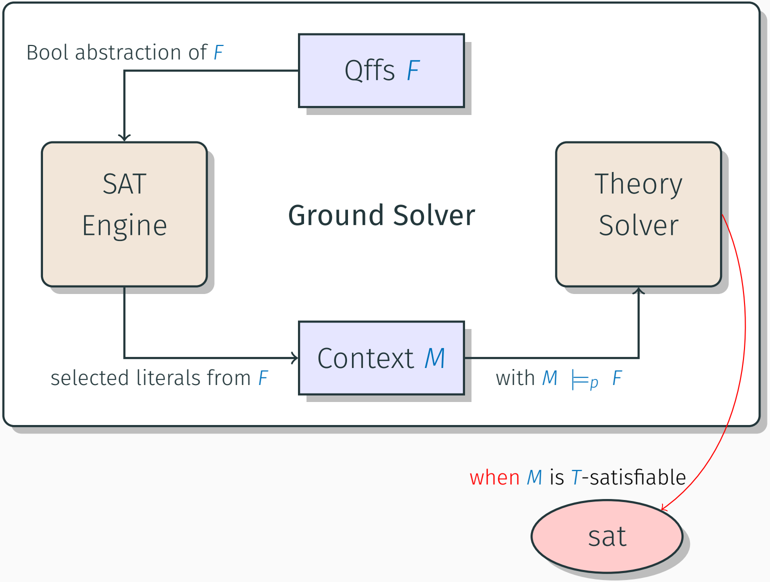

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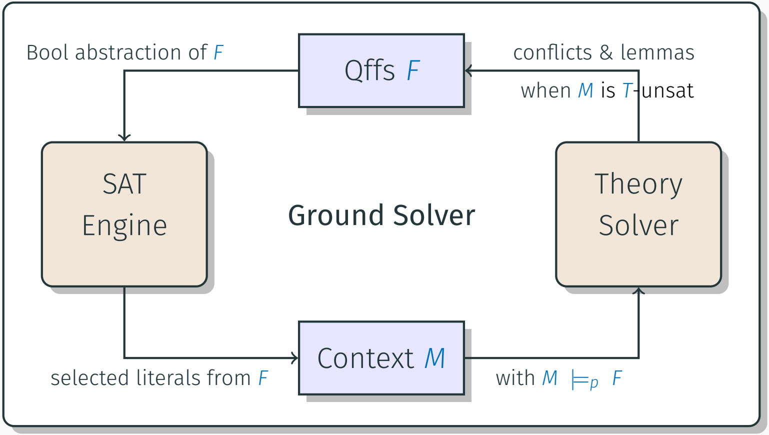

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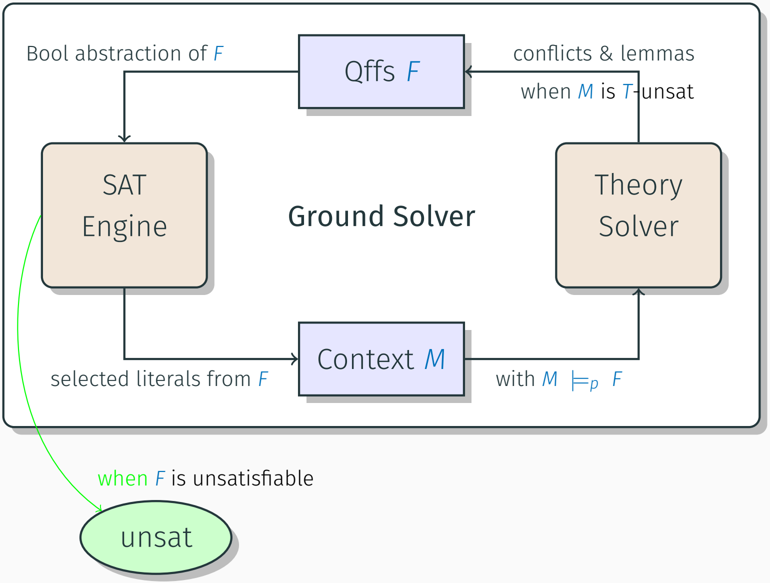

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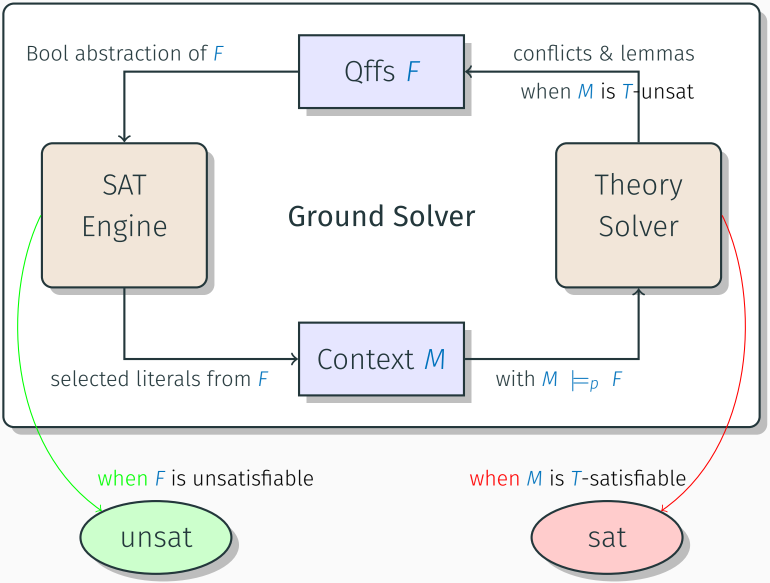
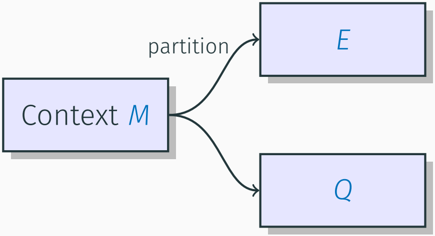


Adding Quantifier Instantiation to SMT Solvers

Formulas F $\quad \{a \approx b \Rightarrow \underbrace{\forall x\, \forall y\, P(x) \vee Q(x,y)}_{q}, \ldots\}$



$\{a \approx b,\ f(a) \neq b,\ P(a) \approx \bot,\ \ldots\}$

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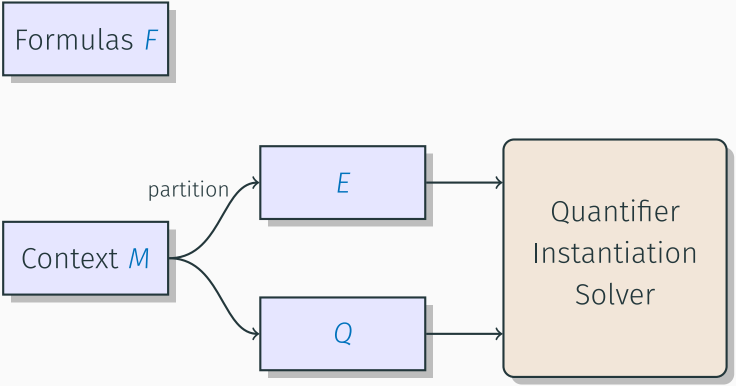

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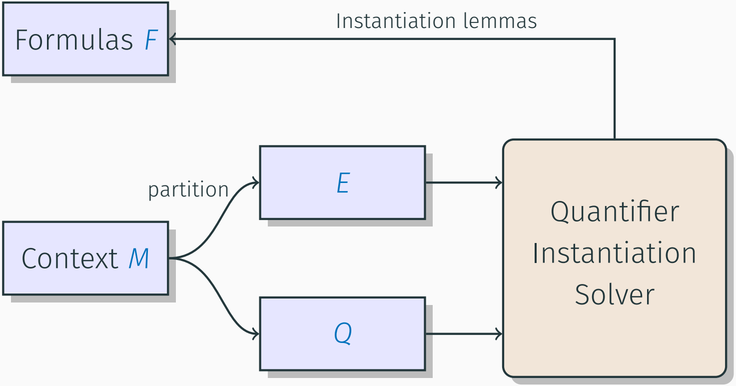

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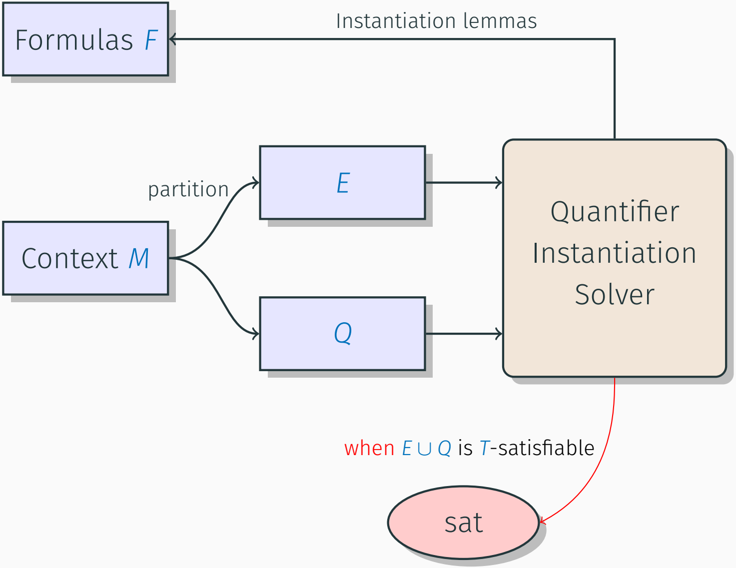

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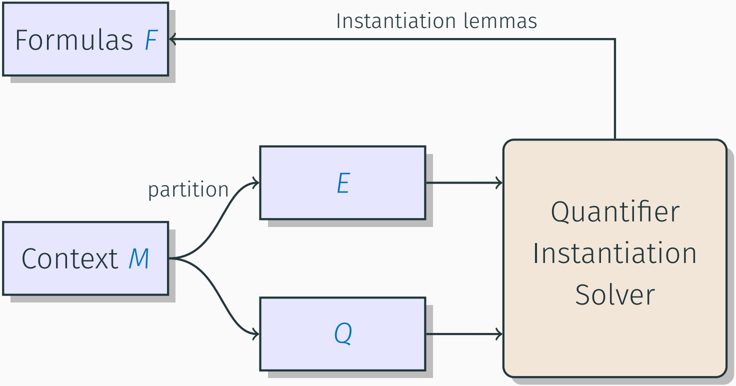

Main Questions:

- Which instantiations likely lead to unsat?
- When can we answer sat?

Quantifier Instantiation

Outline

Instantiations by E-matching

Basic Idea: Choose instances based on pattern matching over *E-graph* of asserted ground (dis-)equalities [Nelson 80]

Most widely used technique for refuting quantified problems in SMT

Exploited in:

- Software Verification
  (Boogie, Dafny, Leon, SPARK, Why3, ...)
- ...
- Automated Theorem Proving
  (Sledgehammer)

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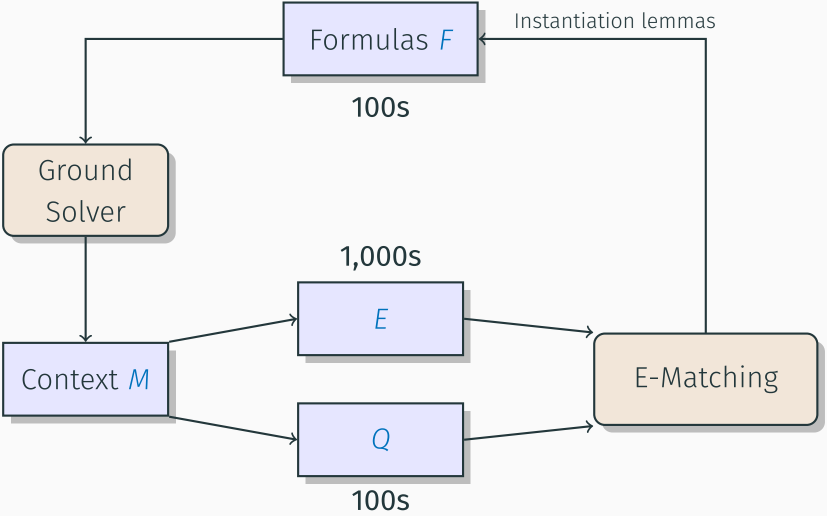
E-matching's Challenge #1 : Too Many Instances



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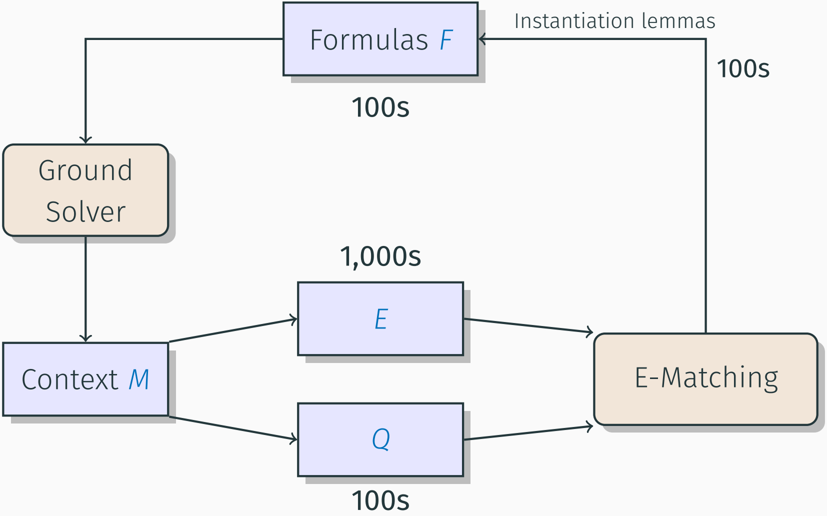

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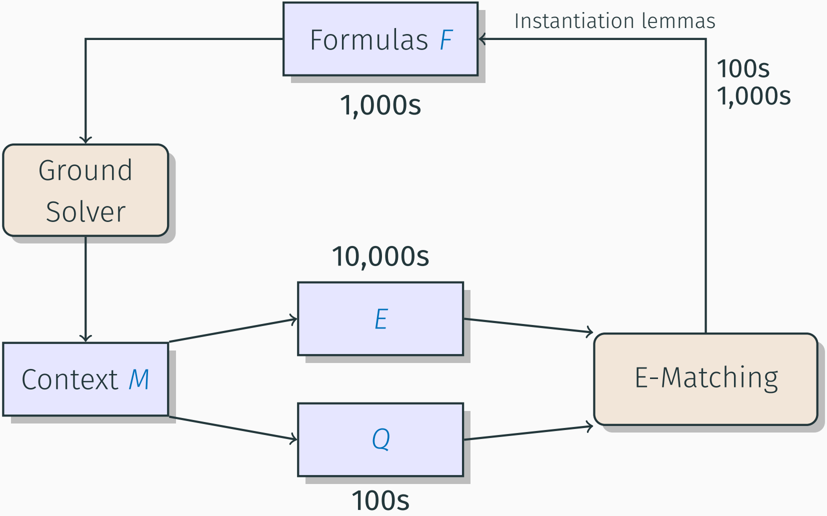

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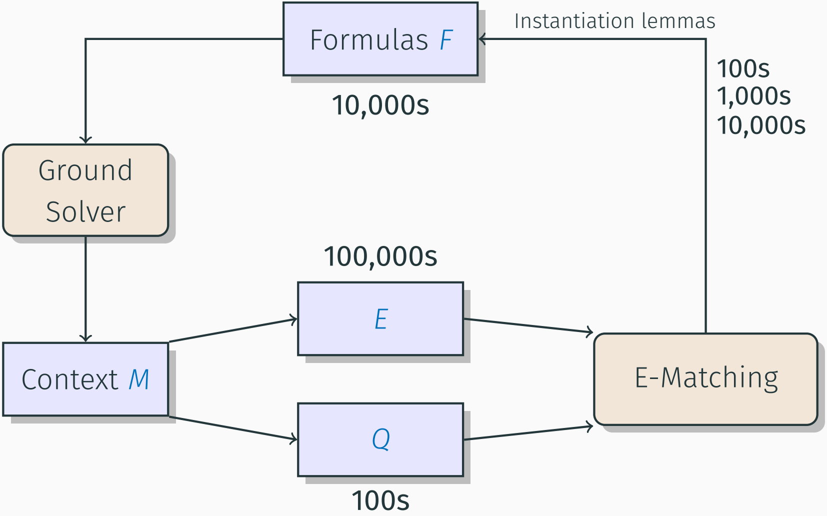

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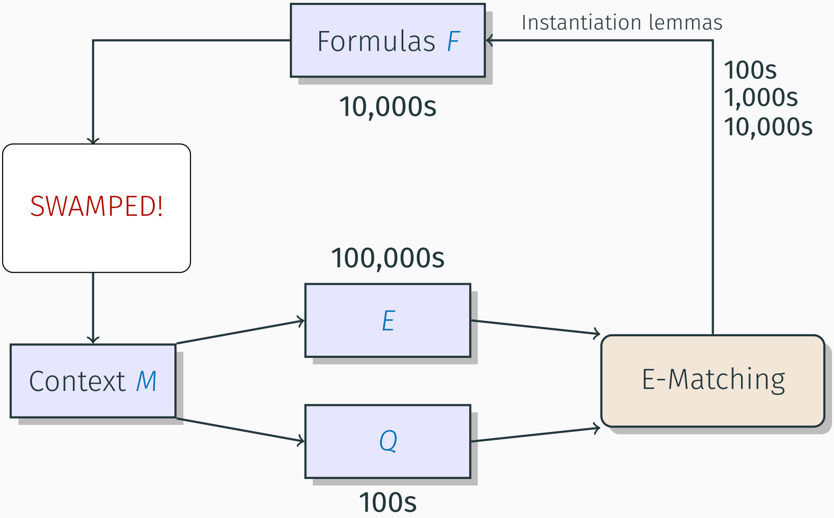
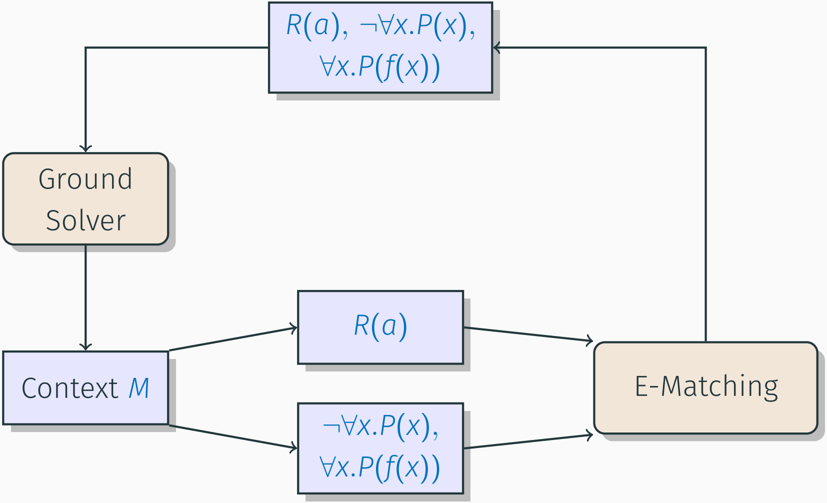


Ground solver gets overloaded and times out

E-matching's Challenge #2 : Incompleteness



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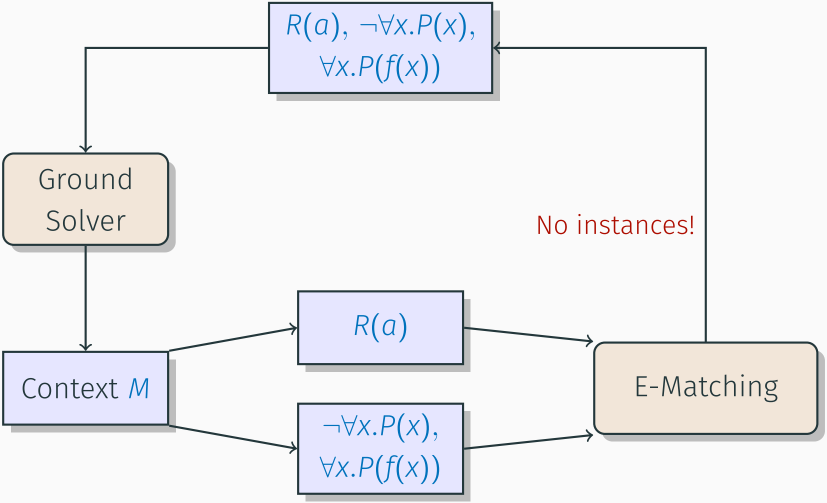

Unsatisfiability goes undetected

Addressing E-matching's Challenges

Too many instances?

- ▷ Try *conflict-based instantiation* first [FMCAD 2014]

Apply E-matching

No instances and input may be satisfiable?

- ▷ Try *model-based instantiation* next [Ge&deMoura 2009]

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Conflict-Based QI

Basic idea: Given $E \cup \{\forall \mathbf{x}.\, \varphi[\mathbf{x}], \ldots\}$,

- Try to find one *conflict instance* $\forall \mathbf{x}.\, \varphi[\mathbf{x}] \Rightarrow \varphi[\mathbf{t}]$ such that
$$E, \varphi[\mathbf{t}] \models_T \bot$$
- If this is possible, E-matching is not needed

Leads to fewer instances, improving ability to answer **unsat**

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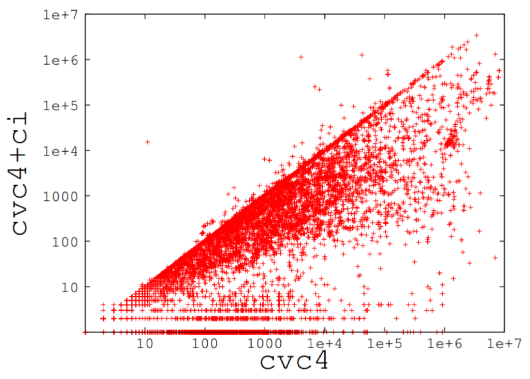
Impact of Conflict-Based QI in CVC4

CBQI (cvc4+ci) needs 10^{-1}x instances to show **unsat** vs. E-matching alone



(evaluation on SMT-LIB, TPTP, and Isabelle benchmarks [FMCAD 2014])

CBQI's Challenge #1: Finding Conflicting Instances

Our solution: Construct instances via a stronger version of matching [FMCAD 2014]

Intuition: with $\forall \mathbf{x}.\, P[\mathbf{x}] \vee Q[\mathbf{x}]$ only match on $P[\mathbf{t}]$ where

$$P[\mathbf{t}] \equiv_{\mathrm{EUF}} \perp$$

Formalized as calculus based on ground E-(dis)unification [TACAS 2017]

CBQI's Challenge #2: Theory Symbols

Difficulty of finding conflicting instances in the presence of theory symbols:

$$E = \{f(1) \approx 5, \ldots\} \quad Q = \{\underbrace{\forall x, y. f(x+y) > x + 2 \cdot y}_{q}, \ldots\}$$

Generally, use fast and incomplete procedure for quantifiers + theories

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$$q \Rightarrow f(1) > 5$$

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Model-based Quantifier Instantiation

$E = \{\text{ground literals}\} \qquad Q = \{\text{quantified formulas}\}$

Basic idea:

If E-matching saturates, build a *candidate model* $\mathcal{I}$ satisfying E

1. Check if $\mathcal{I}$ also satisfies Q
   (using a ground satisfiability query)
2. If not, add instance of formula in Q falsified by $\mathcal{I}$
3. Repeat

Gives ability to answer **sat**

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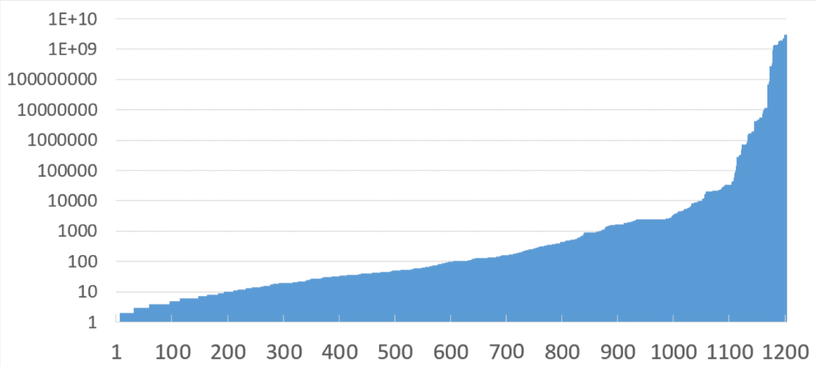

x = 1,203 satisfiable TPTP benchmarks

y = # of instances potentially generated by exhaustive instantiation

E.g. $4^3 = 64$ instances for $\forall x, y, z : \mathbf{A}. P(x, y, z)$ when $|A| = 4$

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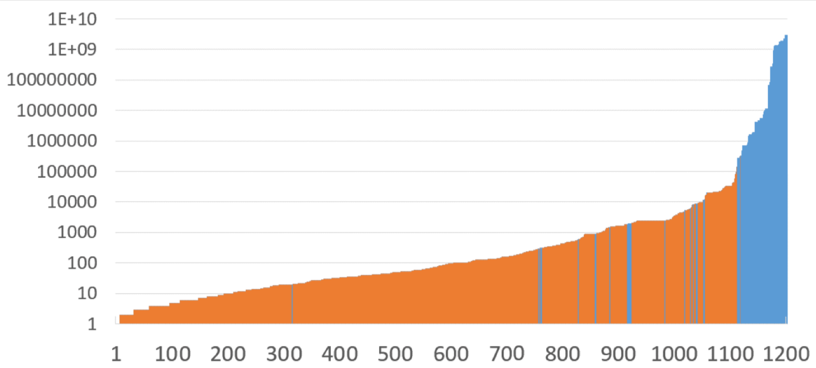

CVC4 Finite Model Finding + Model-Based instantiation [CADE 2013]

Scales only up to $\sim$150K instances with a 30s timeout

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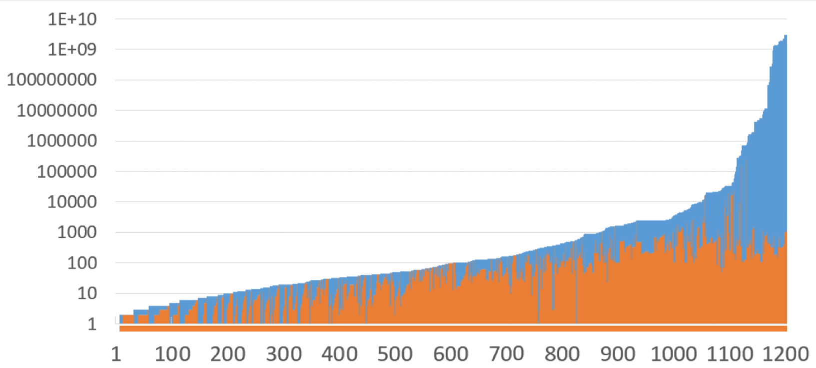

CVC4 Finite Model Finding + Model-Based instantiation [CADE 2013]

Scales to >2B instances with a 30s timeout,
generates only a fraction of possible instances

Model-Based QI: Challenges

How do we build interpretations $\mathcal{I}$?

Typically, build $\mathcal{I}$ where every function is *almost constant*:

$$f^{\mathcal{I}} := \lambda x.\, \mathsf{ite}(x = t_1, v_1, \mathsf{ite}(x = t_2, v_2, \ldots, \mathsf{ite}(x = t_n, v_n, v_{\mathrm{def}}) \ldots))$$

This works well in EUF

Model-Based QI: Challenges

How do we build interpretations $\mathcal{I}$?

However, more sophisticated models are needed when other theories are involved:

$\forall x, y : \mathbf{Int}.\,(f(x, y) \geq x \wedge f(x, y) \geq y)$ $\qquad$ $f^{\mathcal{I}} := \lambda x, y : \mathbf{Int}.\,\mathbf{ite}(x \geq y, x, y)$

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$\forall x : \textbf{Int}.\, 3 \cdot g(x) + 5 \cdot h(x) = x$

$g^{\mathcal{I}} := \lambda x.\, x - 3 \cdot x$
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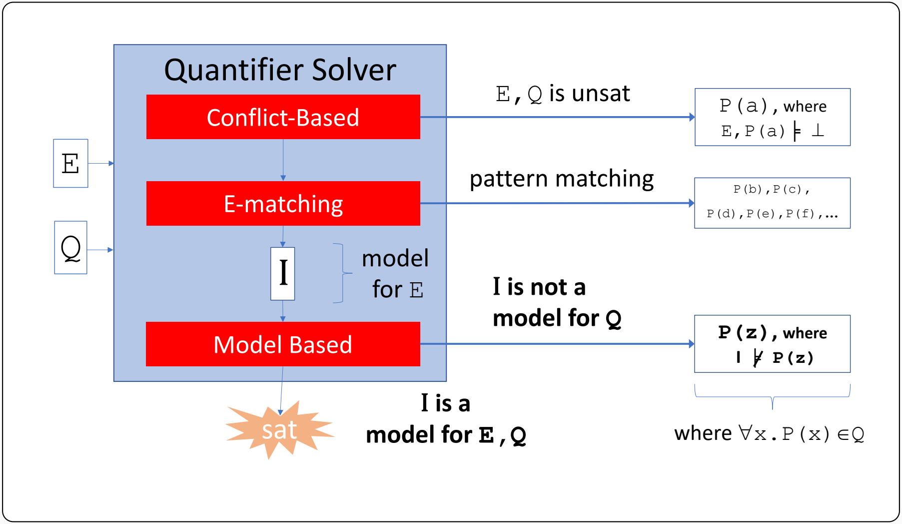
More research is needed!

(may leverage recent advantages in syntax-guided synthesis?)

Putting It All Together in CVC4



General Challenge

Reasoning efficiently about quantifiers + EUF + other theories is still hard!

E-matching: Pattern selection, matching modulo theories

Conflict-based: Matching is incomplete, entailment tests are expensive

Model-based: Models are complex, interpreted domains may be infinite

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Reasoning efficiently about quantifiers + EUF + other theories is not as bad

- Classic QE algorithms are decision procedures for LRA [Ferrante&Rackoff 79, Loos&Wiespfenning 93], LIA [Cooper 72], datatypes [Maher 1988], ...
- Some have been leveraged successfully in SMT applications [Monniaux 2010, Bjorner 2012, Reynolds et al. 2015, Bjorner&Janota 2016, ...]

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Counter-Example-Guided Quantifier Instantiation

Counterexample-Guided QI

Counterexample-Guided Quantifier Instantiation

Variants implemented in number of tools:

- Z3 [Bjorner 2012, Bjorner&Janota 2016]
- SPACER [Komuravelli et al. 2014][1]
- Yices [Dutertre 2015]
- CVC4 [CAV 2015, CAV 2018]
- UFO [Fedyukovich et al. 2016][2]
- Boolector [Preiner et al. 2017]

[1] Originally using Z3 as backend, now integrated in Z3
[2] Using Z3 as backend

Counterexample-Guided Instantiation

Basic idea: Derived from quantifier elimination (e.g., for LIA):

$$\exists x.\, \psi[x, \mathbf{y}] \equiv_T \psi[t_1, \mathbf{y}] \vee \cdots \vee \psi[t_n, \mathbf{y}] \text{ for some } t_1, \ldots, t_n$$

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$$\forall x.\, \neg\psi[x, \mathbf{y}] \equiv_T \neg\psi[t_1, \mathbf{y}] \wedge \cdots \wedge \neg\psi[t_n, \mathbf{y}] \quad \text{for some } t_1, \ldots, t_n$$

Counterexample-Guided Instantiation

Basic idea: Derived from quantifier elimination (e.g., for LIA):

$$\forall x.\, \neg\psi[x, \mathbf{y}] \equiv_T \neg\psi[t_1, \mathbf{y}] \wedge \cdots \wedge \neg\psi[t_n, \mathbf{y}] \text{ for some } t_1, \ldots, t_n$$

Enumerate instances via a counterexample-guided loop that is

1. terminating: generate a finite set $S \supseteq \{t_1, \ldots, t_n\}$
2. efficient in practice: typically terminates after $<< n$ instances

basic-CEGQI($\forall \mathbf{x}.\, \psi[\mathbf{x}, \mathbf{y}]$)

    $G := \emptyset$ (instances of $\forall \mathbf{x}.\, \psi$)

    repeat

        if G is T-unsatisfiable

            return **unsat** (because $\forall \mathbf{x}.\, \psi \models_T G$)

        else

            let $G' = G \cup \{\neg \psi\}$

            if is G' is T-unsatisfiable

                return **sat** (because $G \models_T \forall \mathbf{x}.\, \psi$)

            else

                let $\mathcal{I}$ be a T-model of G'

                let $\mathbf{t}[\mathbf{y}] = \mathrm{Sel}(\mathbf{x}, \psi, \mathcal{I}, G)$

                $G := G \cup \{\psi[\mathbf{t}, \mathbf{y}]\}$

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High-level View of Basic Procedure

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Relies on *selection function* Sel

Selection Function

Right selection functions make CEGQI a decision procedure for various theories T

Termination Requirements:

1. Quantifier-free fragment of T is decidable
2. For all qffs $\psi[\mathbf{x}, \mathbf{y}]$, selection function Sel is
   2.1 *finite*:
   there is a finite set $S_{\psi,\mathbf{x}}$ s.t. $\mathrm{Sel}(\mathbf{x}, \psi, \mathcal{I}, G) \in S_{\psi,\mathbf{x}}$ for all legal $\mathcal{I}, G$
   2.2 *monotonic*:
   if $G \models_T \psi[\mathbf{t}, \mathbf{y}]$ then $\mathrm{Sel}(\mathbf{x}, \psi, \mathcal{I}, G) \neq \mathbf{t}$ for all legal $\mathcal{I}, G$

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Theorem. Under (1), procedure **basic-CEGQI** always terminates if *sel* is finite and monotonic

From CEGQI ...

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$G := G \cup \{\psi[t, y]\}$

From CEGQI to Quantifier Elimination

project($x, \psi[x, y]$)
  $G := \emptyset$
  repeat
    if G is T-unsatisfiable
      return $\perp$
    else
      let $G' = G \cup \{\neg\psi\}$
      if is G' is T-unsatisfiable
        return $\bigwedge G$
      else
        let $\mathcal{I}$ be a T-model of G'
        let $t[y] = \mathrm{Sel}(x, \psi, \mathcal{I}, G)$
        $G := G \cup \{\psi[t, y]\}$

From CEGQI to Quantifier Elimination

$\textbf{project}(x, \psi[x, y])$

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Note:

Let $\varphi[y] = \textbf{project}(x, \psi[x, y])$

Then $\varphi \equiv_T \forall \boldsymbol{x}.\, \psi$

From CEGQI to QE: General Case

Assumption: Consider only NNF formulas φ containing a subformula $\forall \mathbf{x}.\, \varphi_1 \vee \varphi_2$ (resp. $\exists \mathbf{x}.\, \varphi_1 \wedge \varphi_2$) only if $\varphi_1 \vee \varphi_2$ (resp. $\varphi_1 \wedge \varphi_2$) is quantifier-free[3]

[3] For simplicity and wlog by (lazy) PNF transformation

From CEGQI to QE: General Case

$$
\begin{aligned}
\mathsf{qe}(x, \varphi) \quad &:= \quad \text{if } \varphi \text{ is quantifier-free then} \\
&\qquad\quad \mathsf{project}(x, \varphi) \\
&\qquad \text{else} \\
&\qquad\quad \text{match } \varphi \text{ with} \\
&\qquad\qquad\qquad \varphi_1 \wedge \varphi_2 : \quad \mathsf{qe}(x, \varphi_1) \wedge \mathsf{qe}(x, \varphi_2) \\
&\qquad\qquad\qquad \exists z.\, \psi : \quad \neg\, \mathsf{qe}(x, \forall z.\, \mathsf{nnf}(\neg\psi)) \\
&\qquad\qquad\qquad \forall z.\, \psi : \quad \mathsf{qe}(x, \mathsf{qe}(z, \psi)) \\
\mathsf{nnf}(\varphi) \quad &:= \quad \text{negation normal form of } \varphi
\end{aligned}
$$

From CEGQI to QE: General Case

$\mathbf{qe}(\boldsymbol{x}, \varphi)$:= if φ is quantifier-free then

$\quad \mathbf{project}(\boldsymbol{x}, \varphi)$

else

$\quad$ match φ with

Note:

1. Avoiding full prenex normal form transformation increases scalability in practice
2. Implementation of general CEBQI in CVC4 is similar in spirit to **qe** but is fully integrated into SMT loop [FMSD 2017]

Selection Functions

Linear real arithmetic (LRA) [FMSD 2017]

- Maximal lower (minimal upper) bounds [Loos+Wiespfenning 1993]

$$l_1 < k, \ldots, l_n < k \quad \Longrightarrow \quad \{x \mapsto l_{\max} + d\}$$

  (may involve virtual terms δ, ∞)

- Interior point method [Ferrante&Rackoff 1979]

$$l_{\max} < k < u_{\min} \quad \Longrightarrow \quad \{x \mapsto (l_{\max} + u_{\min})/2\}$$

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- In

Common termination argument:
a finite number of instances cover all cases

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Selection Functions

Finite domains

- Model-based value instantiations [Wintersteiger et al. 2013]

$$D = \{d_1, \ldots, d_n\} \quad \Longrightarrow \quad \{x \mapsto d_i\}$$

Fixed-size Bit vectors

- Value instantiations [Neimetz et al. 2016]

$$0 \leq i < w \quad \Longrightarrow \quad \{x \mapsto 2^i\}$$

- Invertibility conditions [CAV 2018]

(next slides)

Datatypes

- Stay tuned ...

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Outline

Motivation

Example: Prove unsatisfiability of

$$\psi = \forall x.\, x + s \not\approx t$$

with x, s, t bit vectors of size n

It is crucial to find good set of instantiation candidates for x

Motivation

Example: Prove unsatisfiability of

$$\psi = \forall x.\, x + s \not\approx t$$

with x, s, t bit vectors of size n

Naive approach: Enumerate 2^n possible values for x

Example: Prove unsatisfiability of

$$\psi = \forall x.\, x + s \not\approx t$$

with x, s, t bit vectors of size n

Better approach:

1. Try to solve $\neg(x + s \not\approx t)$ for x (yielding $x = t - s$)
2. Instantiate ψ with computed symbolic solution

$$\underbrace{\overbrace{t - s}^{x} + s \not\approx t}_{\text{UNSAT}}$$

Quantifier Instantiation for Bit Vectors

Idea: Compute symbolic solutions of bit vector constraints

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Problem: hard or impossible in general

Quantifier Instantiation for Bit Vectors

Idea: Compute symbolic solutions of bit vector constraints

Problem: hard or impossible in general

- **Example:** $2 \cdot x \approx 3$ is unsolvable

Quantifier Instantiation for Bit Vectors

Idea: Compute symbolic solutions of bit vector constraints

Problem: hard or impossible in general

Our Answer:

1. Consider restricted case where φ has the form

   $$x \diamond s \bowtie t \quad \text{or} \quad s \diamond x \bowtie t$$

   with $\bowtie$ relational operator and x not in s or t

2. Consider *conditional* symbolic solutions

   (e.g., identify conditions under which $s \cdot x \approx t$ is solvable)

Invertibility Condition

Exact condition under which a bit vector operation is solvable for some x

Example: $x \cdot s \approx t$

- Invertibility condition: $(-s \mid s) \mathbin{\&} t \approx t$
- $(-s \mid s) \mathbin{\&} t \approx t \;\equiv_{\mathrm{BV}}\; \exists x.\, x \cdot s \approx t$

Invertibility Conditions

- 162 IC's for: $\{\approx, \not\approx, <_u, \leq_u, >_u, \geq_u, <_s, \leq_s, >_s, \geq_s\} \;\times$ $\{\sim, \&, |, \ll, \gg, \gg_a, -, +, \cdot, \text{mod}, \div, \circ, [:]\}$
- 83 crafted manually
- 79 generated automatically with syntax-guided synthesizer

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- $(-s \mid s) \mathbin{\&} t \approx t \;\; \equiv_{\text{BV}} \;\; \exists x.\, x \cdot s \approx t$

$x \cdot s = t$ is solvable for x

iff

s has fewer trailing zeroes than t

Invertibility

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A Few Invertibility Conditions

$\ell[x]$	$\approx$	$\not\approx$
$x \cdot s \bowtie t$	$(-s \mid s)\ \&\ t \approx t$	$s \not\approx 0 \lor t \not\approx 0$
$x \bmod s \bowtie t$	$\sim(-s) \geq_u t$	$s \not\approx 1 \lor t \not\approx 0$
$s \bmod x \bowtie t$	$(t + t - s)\ \&\ s \geq_u t$	$s \not\approx 0 \lor t \not\approx 0$
$x \div s \bowtie t$	$(s \cdot t) \div s \approx t$	$s \not\approx 0 \lor t \not\approx \sim 0$
$s \div x \bowtie t$	$s \div (s \div t) \approx t$	$\begin{cases} s\ \&\ t \approx 0 & \text{for } \kappa(s) = 1 \\ \top & \text{otherwise} \end{cases}$
$x\ \&\ s \bowtie t$	$t\ \&\ s \approx t$	$s \not\approx 0 \lor t \not\approx 0$
$x \mid s \bowtie t$	$t \mid s \approx t$	$s \not\approx \sim 0 \lor t \not\approx \sim 0$
$x \gg s \bowtie t$	$(t \ll s) \gg s \approx t$	$t \not\approx 0 \lor s <_u \kappa(s)$
$s \gg x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \gg i \approx t$	$s \not\approx 0 \lor t \not\approx 0$
$x \gg_a s \bowtie t$	$(s <_u \kappa(s) \Rightarrow (t \ll s) \gg_a s \approx t) \land$ $(s \geq_u \kappa(s) \Rightarrow (t \approx \sim 0 \lor t \approx 0))$	$\top$
$s \gg_a x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \gg_a i \approx t$	$(t \not\approx 0 \lor s \not\approx 0) \land$ $(t \not\approx \sim 0 \lor s \not\approx \sim 0)$

A Few More Invertibility Conditions

$\ell[x]$	$<_s$	$>_s$
$x \cdot s \bowtie t$	$\sim(-t)\ \&\ (-s \mid s) <_s t$	$t <_s t - ((s \mid t) \mid -s)$
$x \bmod s \bowtie t$	$\sim t <_s (-s \mid -t)$	$(s >_s 0 \Rightarrow t <_s \sim(-s)) \wedge$ $(s \leq_s 0 \Rightarrow t \not\approx \max_s) \wedge$ $(t \not\approx 0 \vee s \not\approx 1)$
$s \bmod x \bowtie t$	$s <_s t \vee 0 <_s t$	$(s \geq_s 0 \Rightarrow s >_s t) \wedge$ $(s <_s 0 \Rightarrow ((s-1) \gg 1) >_s t)$
$x \div s \bowtie t$	$t \leq_s 0 \Rightarrow \min_s \div s <_s t$	$\sim 0 \div s >_s t \vee \max_s \div s >_s t$
$s \div x \bowtie t$	$s <_s t \vee t \geq_s 0$	$\begin{cases} s >_s t & \text{for } \kappa(s) = 1 \\ (s \geq_s 0 \Rightarrow s >_s t) \wedge & \text{otherwise} \\ (s <_s 0 \Rightarrow s \gg 1 >_s t) & \end{cases}$
$x\ \&\ s \bowtie t$	$\sim(-t)\ \&\ s <_s t$	$t <_s s\ \&\ \max_s$
$x \mid s \bowtie t$ $s \mid x \bowtie t$	$\sim(s-t) \mid s <_s t$	$t <_s (s \mid \max_s)$
$x \gg s \bowtie t$	$\sim(-t) \gg s <_s t$	$t <_s (\max_s \ll s) \gg s$
$s \gg x \bowtie t$	$s <_s t \vee 0 <_s t$	$(s <_s 0 \Rightarrow s \gg 1 >_s t) \wedge$ $(s \geq_s 0 \Rightarrow s >_s t)$

From Invertibility Conditions to Symbolic Instantiations

Hilbert choice functions $\varepsilon x.\, \varphi$

- Represents a solution for φ if there is one
- Represents arbitrary value, otherwise

Embed invertibility conditions into choice functions

BV literal:	$l[x] = x \diamond s \bowtie t$
Inv. condition:	IC_x
Symbolic solution:	$\varepsilon y.\,(IC_x \Rightarrow l[y])$

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Note 1: Choice function expresses all conditional solutions with a single term

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Embed invertibility conditions into choice functions

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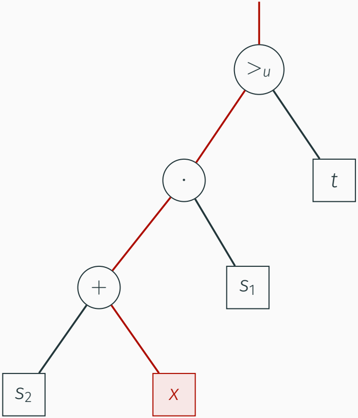
Symbolic solution: $\varepsilon y.\, (IC_x \;\Rightarrow\; l[y])$

Note 2: The ε binder can be later eliminated from instances by Skolemization:

$$\varphi[\varepsilon y.\, (IC_x \;\Rightarrow\; l[y])] \;\longrightarrow\; \varphi[k] \land (IC_x \;\Rightarrow\; l[k])$$

More General Case by Example: $\forall x.\, (s_2 + x) \cdot s_1 \leq_u t$



1. Pick variable to solve for (x)
2. Compute inverse/IC's along path to x
3. Solve $z \cdot s_1 >_u t$ for z

$$\begin{aligned} IC_z &= t <_u -s \mid s \\ z &= \varepsilon y.\, IC_z \Rightarrow y \cdot s_1 >_u t \end{aligned}$$

4. Solve $s_2 + x \approx z$ for x

$$\begin{aligned} IC_x &= \top \\ x &= z - s_2 \end{aligned}$$

Instantiation for x: $\varepsilon y.\, (t <_u -s \mid s \Rightarrow s_1 \cdot y >_u t) - s_2$

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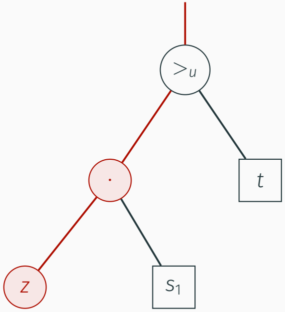

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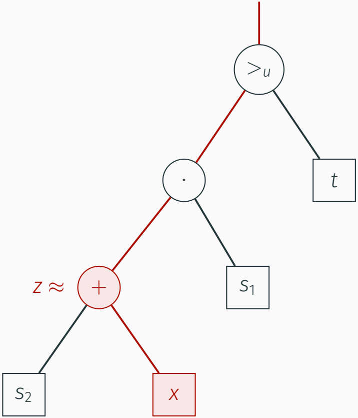

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2. Compute inverse/IC's along path to x
3. Solve $z \cdot s_1 >_u t$ for z

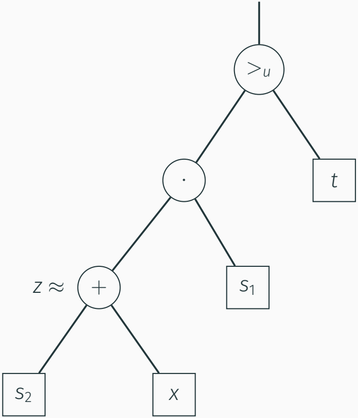
$$\begin{aligned} IC_z &= t <_u -s \mid s \\ z &= \varepsilon y.\, IC_z \Rightarrow y \cdot s_1 >_u t \end{aligned}$$

4. Solve $s_2 + x \approx z$ for x

$$\begin{aligned} IC_x &= \top \\ x &= z - s_2 \end{aligned}$$

Instantiation for x: $\varepsilon y. (t <_u -s \mid s \Rightarrow s_1 \cdot y >_u t) - s_2$

More General Case by Example: $\forall x. (s_2 + x) \cdot s_1 \leq_u t$



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Multiple Variable Occurrences

Non-linear constraints (multiple occurrences of a variable)

- Try to linearize with rewriting/normalization
  e.g., $x + x + s \approx t \longrightarrow 2 \cdot x + s \approx t$
- Otherwise, replace extra occurrences of x with value in current model $\mathcal{I}$
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► **Future work:** Use SyGuS to synthesize IC's for non-linear cases

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Experimental Results

	$\text{CVC4}_{\text{base}}$	Q3B	Boolector	Z3	CVC4_{ic}
keymaera (4035)	3823	3805	4025	**4031**	3993
psyco (194)	**194**	99	193	193	190
scholl (374)	239	214	**289**	271	246
tptp (73)	**73**	**73**	72	**73**	**73**
uauto (284)	112	256	180	190	**274**
wintersteiger (191)	168	**184**	154	162	168
Total (5151)	4609	4631	4913	4920	**4944**

Limits: 300 seconds CPU time limit, 100G memory limit

CVC4_{ic} won division BV at SMT-COMP 2018

Outline

Invertibility Conditions for Floating Point Arithmetic

Similar approach can be applied to Floating Points [CAV 2019]

However, invertibility conditions are much more complex

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Similar approach can be applied to Floating Points [CAV 2019]

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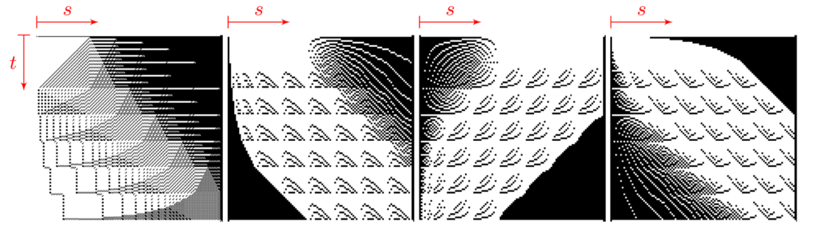


(a) $x + s \approx t$ (b) $x \cdot s \approx t$ (c) $s \div x \approx t$ (d) $x \div s \approx t$

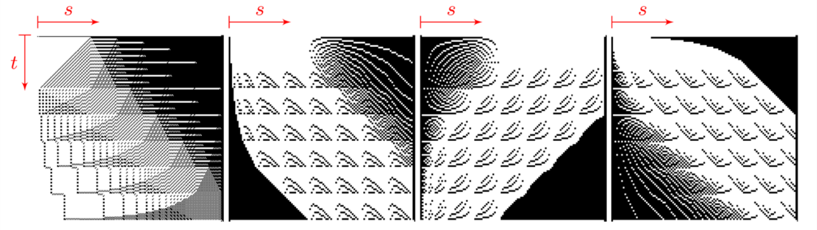
(Shown for FP[3,5])

(white dot = IC is **sat**, black dot = IC is **unsat**)

Invertibility Conditions for Floating Point Arithmetic

Similar approach can be applied to Floating Points [CAV 2019]

However, invertibility conditions are much more complex



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(a) $(t \overset{RTN}{-} s) \overset{R}{+} s \approx t \ \vee \ t \approx s \ \vee \ (t \overset{RTP}{-} s) \overset{R}{+} s \approx t$

(b) $t \approx (t \overset{RTP}{\div} s) \overset{R}{\cdot} s \vee t \approx (t \overset{RTN}{\div} s) \overset{R}{\cdot} s \vee (s \approx \pm\infty \wedge t \approx \pm\infty) \vee (s \approx \pm 0 \wedge t \approx \pm 0)$

...

Invertibility Conditions for Floating Point Arithmetic

$$\exists x.\, x \overset{R}{+} s \approx t$$

$$\equiv_{\mathrm{BV}}$$

$$(t \overset{RTN}{-} s) \overset{R}{+} s \approx t \quad \vee \quad t \approx s \quad \vee \quad (t \overset{RTP}{-} s) \overset{R}{+} s \approx t$$

rounding towards negative	corner case (zero)	rounding towards positive
$x = t \overset{RTP}{-} s$	$x = \pm 0$	$x = t \overset{RTN}{-} s$

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Conclusion

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- SMT solvers do not operate just on ground formulas
- There has been considerable progress on reasoning with quantified formulas in SMT
- Still, a lot more needs to be done
- The quest of combining theory and quantifier reasoning efficiently is still on
- The CVC4 team is at the forefront of this quest
- CVC4 is available at `http://cvc4.cs.stanford.edu/`
- Join our quest!
- We are hiring PhD students and postdocs and welcome collaborations with other groups

Related and Future Work

- Symmetry elimination
- Proofs
- Synthesis
- Abduction

Thank you