CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

Combining Theories and Their Solvers

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Need for Combining Theories and Solvers

Recall: Many applications give rise to formulas like

$$a = b + 2 \wedge A \doteq \mathrm{write}(B, a, 4) \wedge (\mathrm{read}(A, b + 3) \doteq b - 2 \vee f(a - b) \not\doteq f(b + 1))$$

Solving that formula requires reasoning over

- the theory of integer arithmetic ($\mathcal{T}_{\mathrm{LIA}}$)
- the theory of arrays ($\mathcal{T}_{\mathrm{A}}$)
- the theory of uninterpreted functions ($\mathcal{T}_{\mathrm{EUF}}$)

Given solvers for each theory, can we combine them modularly into one for a theory that combines $\mathcal{T}_{\mathrm{LIA}}$, $\mathcal{T}_{\mathrm{A}}$ and $\mathcal{T}_{\mathrm{EUF}}$?

The answer is **yes**, under certain conditions

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First-order theories and their combination

Recall: A *theory* $\mathcal{T}$ is a pair (Σ, S), where:

- Σ is a signature, consisting of a set Σ^S of *sort symbols* and a set Σ^F of function symbols
- S is a class of Σ-interpretations closed under variable re-assignment

We limit interpretations of Σ-formulas to those in S

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$$\Sigma_1 \oplus \Sigma_2 = (\Sigma_1^S \cup \Sigma_2^S, \Sigma_1^F \cup \Sigma_2^F)$$

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Note: Signatures with no shared function symbols are trivially compatible

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$$\mathcal{T}_1 \oplus \mathcal{T}_2 = (\Sigma, S)$$

where $\Sigma = \Sigma_1 \oplus \Sigma_2$ and $S = \{ \mathcal{I} \mid \mathcal{I}^{\Sigma_1} \in S_1 \text{ and } \mathcal{I}^{\Sigma_2} \in S_2 \}$

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Recall: the reduct $\mathcal{I}^{\Omega}$ of a Σ-interpretation $\mathcal{I}$ to a subsignature Ω of Σ is an Ω-interpretation defined exactly as $\mathcal{I}$ over the symbols in Ω

Convex Theories

We want to build theory solvers for combined theory by modularly combining theory solvers for the individual theories

This is easier to do when individual theories are convex

A $\mathcal{T}$-theory $\mathcal{T}$ is convex if for all sets Γ of $\mathcal{T}$-literals over the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $n > 0$

$$\Gamma \models_{\mathcal{T}} x_1 \doteq y_1 \vee \cdots \vee x_n \doteq y_n \quad \text{iff} \quad \Gamma \models_{\mathcal{T}} x_k \doteq y_k \quad \text{for some } k \in 1, \ldots, n$$

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Convex Theories: Examples

Linear real arithmetic is convex

This is a consequence of the fact that sets of literals in this theory define convex polytopes (recall the linear programming slides)

Linear integer arithmetic is non-convex, for instance

$x \doteq 1,\ y \doteq 2,\ 1 \leq z,\ z \leq 2 \ \models_{\mathrm{LIA}}\ z \doteq x \vee z \doteq y$ holds, while neither

$x \doteq 1,\ y \doteq 2,\ 1 \leq z,\ z \leq 2 \ \models_{\mathrm{LIA}}\ z \doteq x$ nor

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Combining Theory Solvers

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two theory solvers deciding the satisfiability of sets of literals in theories $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively

We are interested in constructing a theory solver deciding the satisfiability of sets L of literals in $\mathcal{T}_1 \oplus \mathcal{T}_2$ by modularly combining $\mathbf{S}_1$ and $\mathbf{S}_2$

A popular procedure that achieves this combination consists of four main steps:

1. **Purification.** Purify L into a set L_1 of Σ_1-literals and a set L_2 of Σ_2-literals
2. **Propagation.** Exchange entailed equalities between variables shared by L_1 and L_2
3. **Decision.** If either $\mathcal{T}_1$ or $\mathcal{T}_2$ is non-convex, guess non-entailed equalities and disequalities between the shared variables. Go to 2
4. **Check.** Check the satisfiability of L_i locally in $\mathcal{T}_i$ for $i = 1, 2$

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Combining Theory Solvers: Step 1 Example

Let $\mathcal{T}_1 = \mathcal{T}_{\mathsf{EUF}}$ and $\mathcal{T}_2 = \mathcal{T}_{\mathsf{LRA}}$

1. Purify and partition input set

$$L = \begin{cases} f(f(x) - f(y)) \doteq a \\ f(0) > a + 2 \\ x \doteq y \end{cases} \longrightarrow \begin{cases} f(v_1 - v_2) \doteq a,\ v_1 \doteq f(x),\ v_2 \doteq f(y) \\ f(v_3) > a + 2,\ v_3 \doteq 0 \\ x \doteq y \end{cases}$$

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An *i-term* is a non-variable term of signature Σ_i for $i = 1$ or $i = 2$

Purification: Given a set L of $\Sigma_1 \oplus \Sigma_2$-literals:

1. Find an i-term t that is a subterm of a non-Σ_i-literal $l \in L$
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3. Repeat Steps 1 and 2 until every literal is *pure* (i.e, is either a Σ_1- or a Σ_2-literal)
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Note: L is equisatisfiable with $L_1 \cup L_2$ in $\mathcal{T}_1 \oplus \mathcal{T}_2$

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Combining Theory Solvers: Step 2-4 Example

Let $\mathcal{T}_1 = \mathcal{T}_{\text{EUF}}$ and $\mathcal{T}_2 = \mathcal{T}_{\text{LRA}}$

2. Propagate entailed equalities between the shared variables $v_1, v_2, v_3, v_4, v_5, a$

L_1	L_2
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$f(v_4) \doteq a$	$v_4 \doteq v_1 - v_2$
$v_1 \doteq f(x)$	$v_5 > a + 2$
$v_2 \doteq f(y)$	$v_3 \doteq 0$
$v_5 \doteq f(v_3)$	$v_1 \doteq v_2$
$x \doteq y$	$a \doteq v_5$
$v_3 \doteq v_4$	

3. If either $\mathcal{T}_1$ or $\mathcal{T}_2$ is non-convex, …

   No action because both theories are convex

Combining Theory Solvers: Step 2-4 Example

Let $\mathcal{T}_1 = \mathcal{T}_{\text{EUF}}$ and $\mathcal{T}_2 = \mathcal{T}_{\text{LRA}}$

2. Propagate entailed equalities between the shared variables $v_1, v_2, v_3, v_4, v_5, a$

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$x \doteq y$	$a \doteq v_5$
$v_3 \doteq v_4$	

4. Check for satisfiability of L_1 and of L_2 locally

$L_1 \not\models_{\text{EUF}} \perp$ and $L_2 \models_{\text{LRA}} \perp$

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Combining Theory Solvers: Step 3 Example (non-convex case)

Let $\mathcal{T}_1 = \mathcal{T}_{\text{EUF}}$ and $\mathcal{T}_2 = \mathcal{T}_{\text{LIA}}$

3. Since $\mathcal{T}_2$ is non-convex, guess non-entailed equalities and disequalities between the shared variables

L_1	L_2
$f(v_1) \doteq a$	$1 \leq x$
$f(x) \doteq b$	$x \leq 2$
$f(v_2) \doteq v_3$	$v_1 \doteq 1$
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Note: No entailed equalities, but $L_2 \models_{\mathrm{LIA}} x \doteq v_1 \vee x \doteq v_2$

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Consider each case of $x \doteq v_1 \vee x \doteq v_2$ separately

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Case 1) $x \doteq v_1$

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Case 2) $x = v_2$

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The Combination Method

Bare-bones, non-deterministic, non-incremental version:

Input: $L_1 \cup L_2$ with L_i finite set of $\mathcal{T}_i$-literals
Output: SAT or UNSAT

1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over the variables V shared by L_1 and L_2 such that

$$u \doteq v \in A \text{ or } u \neq v \in A \text{ for all } u, v \in V$$

2. If $L_i \cup A$ is unsatisfiable in $\mathcal{T}_i$ for $i = 1$ or $i = 2$, return UNSAT
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Correctness of the Combination Method

Theorem 1 (Refutation Soundness)

If the method returns UNSAT *for every arrangement, the input is unsatisfiable in* $T_1 \oplus T_2$.

Proof.

Because unsatisfiability in $T_1 \oplus T_2$ is preserved. □

Theorem 2 (Solution Soundness)

If $\Sigma_1^F \cap \Sigma_2^F = \emptyset$ *and* T_1 *and* T_2 *are stably infinite over* $\Sigma_1^S \cap \Sigma_2^S$*, when the method returns* SAT *for some arrangement, the input is satisfiable in* $T_1 \oplus T_2$.

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Because satisfiability in $T_1 \oplus T_2$ is preserved for stably infinite theories. □

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The method is terminating.

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Because there is only a finite number of arrangements to guess. □

Theorem 4 (Decidability)

If $\Sigma_1^F \cap \Sigma_2^F = \emptyset$, T_1 and T_2 are stably infinite over $\Sigma_1^S \cap \Sigma_2^S$, and the satisfiability of quantifier-free formulas in T_i is decidable for $i = 1, 2$, then the satisfiability of quantifier-free formulas in $T_1 \oplus T_2$ is decidable.

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If $\Sigma_1^F \cap \Sigma_2^F = \emptyset$, T_1 and T_2 are stably infinite over $\Sigma_1^S \cap \Sigma_2^S$, and the satisfiability of quantifier-free formulas in $\mathcal{T}_i$ is decidable for $i = 1, 2$, then the satisfiability of quantifier-free formulas in $\mathcal{T}_1 \oplus \mathcal{T}_2$ is decidable.

Stably Infinite Theories

Let $\mathcal{T}$ be a theory or signature Σ, let $S \subset \Sigma^S$

$\mathcal{T}$ is *stably-infinite with respect to* S if every quantifier-free formula satisfiable in $\mathcal{T}$ is satisfiable in $\mathcal{T}$-interpretation $\mathcal{I}$ such that $\sigma^{\mathcal{I}}$ is infinite for all σ on S.

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Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer/real arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF with uninterpreted sorts, linear real arithmetic)

  Recall: With convex theories, arrangements do not need to be guessed as they can be computed by (theory) propagation

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The combination method has been extended to over the years to various classes of non-stably infinite theories

Why the combination method needs stably infiniteness

The theory of fixed-size bit-vectors contains sorts whose domains are all finite. Hence, this theory cannot be stably-infinite.

Example: Consider T_{array} where both indices and elements are of the same sort $\mathtt{bv}$, so that the sorts of T_{array} are $\{\mathtt{array}, \mathtt{bv}\}$, and a theory T_{bv} that requires the sort $\mathtt{bv}$ to be interpreted as bit-vectors of size 1.

- Both theories are decidable and we would like to decide the combination theory in a Nelson-Oppen-like framework.
- Let $a_1, \ldots, a_5$ be array variables and consider the following constraints: $a_i \neq a_j$, for $1 \leq i < j \leq 5$.
- These constraints are entirely within T_{array}. Array theory solver is given all constraints and the bit-vector theory solver is given none.
- **Problem:** Array solver tells us these constraints are SAT, but there are only four possible different arrays with elements and indices over bit-vectors of size 1.

SMT Solving with *Multiple* Theories

Let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be theories with respective solvers $S_1, \ldots, S_n$

How can we integrate all of them cooperatively into a single SMT solver for $\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_n$?

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Quick Solution:

1. Combine $S_1, \ldots, S_n$ into a theory solver for $\mathcal{T}$
2. Build a CDCL($\mathcal{T}$) solver as usual

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Better Solution:

1. Extend CDCL($\mathcal{T}$) to CDCL($\mathcal{T}_1, \ldots, \mathcal{T}_n$)
2. Lift combination method to the CDCL($X_1, \ldots, X_n$) level
3. Build a CDCL($\mathcal{T}_1, \ldots, \mathcal{T}_n$) solver

Modeling CDCL($\mathcal{T}_1, \ldots, \mathcal{T}_n$) Abstractly

- Let $n = 2$, for simplicity
- Let $\mathcal{T}_i$ be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let C be a set of fresh constants
- Assume wlog that each input literal has signature $(\mathcal{T}_1 \cup C)$ or $(\mathcal{T}_2 \cup C)$ (no *mixed* literals)
- Let $\mathrm{M}|_i \stackrel{\text{def}}{=} \{\Sigma_{i \cup C}\text{-literals of M and their complement}\}$
- Let $\mathrm{I}(\mathrm{M}) \stackrel{\text{def}}{=} \{c = d \mid c, d \text{ occur in } C, \mathrm{M}|_1 \text{ and } \mathrm{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } C, \mathrm{M}|_1 \text{ and } \mathrm{M}|_2\}$

  (*interface literals*)

Abstract CDCL Modulo Multiple Theories

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

Decide $\dfrac{l \in \texttt{Lits}(F) \cup \mathrm{I}(M) \quad l, \overline{l} \notin M}{M := M \bullet l}$

Only change: decide on interface equalities as well

$\mathcal{T}$-Propagate $\dfrac{l \in \texttt{Lits}(F) \cup \mathrm{I}(M) \quad i \in \{1, 2\} \quad M \models_{\mathcal{T}_i} l \quad l, \overline{l} \notin M}{M := M\, l}$

Only change: propagate interface equalities as well, but reason locally in each $\mathcal{T}_i$

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Abstract CDCL Modulo Multiple Theories

$\mathcal{T}$-Conflict

$$\frac{\mathsf{C} = \mathtt{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{\mathcal{T}_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \overline{l}_1 \vee \cdots \vee \overline{l}_n}$$

$\mathcal{T}$-Explain

$$\frac{\mathsf{C} = l \vee D \quad \overline{l}_1, \ldots, \overline{l}_n \models_{\mathcal{T}_i} \overline{l} \quad i \in \{1, 2\} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_{\mathsf{M}} \overline{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each $\mathcal{T}_i$

$\mathcal{T}$-Learn

$$\frac{\models_{\mathcal{T}_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

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$\mathcal{T}$-Explain

$$\frac{\mathsf{C} = l \vee D \quad \overline{l}_1, \ldots, \overline{l}_n \models_{\mathcal{T}_i} \overline{l} \quad i \in \{1, 2\} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_{\mathsf{M}} \overline{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each $\mathcal{T}_i$

I-Learn

$$\frac{\models_{\mathcal{T}_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

Example — Convex Theories

$$\Delta := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = v_2}_{1} \wedge \underbrace{f(y) = v_3}_{2} \wedge \underbrace{f(v_4) = v_5}_{3} \wedge \underbrace{x = y}_{4} \wedge \underbrace{v_2 - v_3 = v_1}_{5} \wedge \underbrace{v_4 = 0}_{6} \wedge \underbrace{v_5 > a + 2}_{7}$$

$$\underbrace{v_2 = v_3}_{8} \quad \underbrace{v_1 = v_4}_{9} \quad \underbrace{a = v_5}_{10}$$

M	Δ	C	rule
	Δ_0	no	
0 1 2 3 4 5 6 7	Δ_0	no	by Propagate$^+$
0 1 2 3 4 5 6 7 8	Δ_0	no	by $\mathcal{T}$-Propagate $(1, 2, 4 \models_{\text{EUF}} 8)$
0 1 2 3 4 5 6 7 8 9	Δ_0	no	by $\mathcal{T}$-Propagate $(5, 6, 8 \models_{\text{LRA}} 9)$
0 1 2 3 4 5 6 7 8 9 10	Δ_0	no	by $\mathcal{T}$-Propagate $(0, 3, 9 \models_{\text{EUF}} 10)$
0 1 2 3 4 5 6 7 8 9 10	Δ_0	$\overline{7} \vee \overline{10}$	by $\mathcal{T}$-Conflict $(7, 10 \models_{\text{LRA}} \bot)$
UNSAT			by Fail

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0 1 2 3 4 5 6 7 8 9 10	Δ_0	`no`	by $\mathcal{T}$-**Propagate** $(0, 3, 9 \models_{\mathrm{EUF}} 10)$
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0 1 2 3 4 5 6 7 8 9 10	Δ_0	$\overline{7} \vee \overline{10}$	by $\mathcal{T}$-**Conflict** $(7, 10 \models_{LRA} \perp)$
UNSAT			by **Fail**

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

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M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by PROPAGATE$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-PROPAGATE ($0, 3 \models_{\text{EUF}} 10$)
$0 \cdots 9\ 10$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-LEARN ($\models_{\text{LIA}} \bar{4} \vee \bar{5} \vee 11 \vee 12$)
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE ($0, 1, 11 \models_{\text{EUF}} 13$)
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \bar{13}$	by $\mathcal{T}$-CONFLICT ($7, 13 \models_{\text{EUF}} \bot$)
$0 \cdots 9\ 10\ \bar{13}$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE ($0, 1, \bar{13} \models_{\text{EUF}} \bar{11}$)
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}\ 12$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by PROPAGATE
$\ldots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by FAIL

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$0 \cdots 9\ 10$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by I-LEARN $(\models_{\mathsf{LIA}} \bar{4} \vee \bar{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, 11 \models_{\mathsf{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \bar{13}$	by $\mathcal{T}$-CONFLICT $(7, 13 \models_{\mathsf{EUF}} \bot)$
$0 \cdots 9\ 10\ \bar{13}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, \bar{13} \models_{\mathsf{EUF}} \bar{11})$
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}\ 12$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by PROPAGATE
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UNSAT	$\cdots$	$\cdots$	by FAIL

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$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

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$0 \cdots 9\ 10$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-LEARN $(\models_{\text{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, 11 \models_{\text{EUF}} 13)$
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$0 \cdots 9\ 10\ \overline{13}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, \overline{13} \models_{\text{EUF}} \overline{11})$
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$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \bar{13}$	by $\mathcal{T}$-CONFLICT $(7, 13 \models_{\mathsf{EUF}} \bot)$
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$0 \cdots 9\ 10\ \bar{13}\ \bar{11}$	$\Delta_0, \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, \bar{13} \models_{\mathsf{EUF}} \bar{11})$
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$0 \cdots 9$	Δ_0	no	by PROPAGATE$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-PROPAGATE $(0, 3 \models_{\text{EUF}} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-LEARN $(\models_{\text{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, 11 \models_{\text{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by $\mathcal{T}$-CONFLICT $(7, 13 \models_{\text{EUF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0, 1, \overline{13} \models_{\text{EUF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by PROPAGATE
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by FAIL

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-**Propagate** $(0,\ 3 \models_{\mathrm{EUF}} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \bar{4} \vee \bar{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \bar{13}$	by $\mathcal{T}$-**Conflict** $(7,\ 13 \models_{\mathrm{EUF}} \bot)$
$0 \cdots 9\ 10\ \bar{13}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-**Propagate** $(0,\ 1,\ \bar{13} \models_{\mathrm{EUF}} \bar{11})$
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}\ 12$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \le x}_{4} \wedge \underbrace{x \le 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by PROPAGATE$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-PROPAGATE $(0,\ 3 \models_{\mathrm{EUF}} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by I-LEARN $(\models_{\mathrm{LIA}} \bar{4} \vee \bar{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0,\ 1,\ 11 \models_{\mathrm{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \bar{13}$	by $\mathcal{T}$-CONFLICT $(7,\ 13 \models_{\mathrm{EUF}} \bot)$
$0 \cdots 9\ 10\ \bar{13}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0,\ 1,\ \bar{13} \models_{\mathrm{EUF}} \bar{11})$
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}\ 12$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by PROPAGATE
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by FAIL

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by PROPAGATE$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-PROPAGATE $(0,\ 3 \models_{\text{EUF}} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-LEARN $(\models_{\text{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0,\ 1,\ 11 \models_{\text{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by $\mathcal{T}$-CONFLICT $(7,\ 13 \models_{\text{EUF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0,\ 1,\ \overline{13} \models_{\text{EUF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by PROPAGATE
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by FAIL

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by Propagate$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-Propagate $(0,\ 3 \models_{\text{EUF}} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by I-Learn $(\models_{\text{LIA}} \bar{4} \vee \bar{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by Decide
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-Propagate $(0,\ 1,\ 11 \models_{\text{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \overline{13}$	by $\mathcal{T}$-Conflict $(7,\ 13 \models_{\text{EUF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by Backjump
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-Propagate $(0,\ 1,\ \overline{13} \models_{\text{EUF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by Propagate
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by Fail

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by Propagate$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-Propagate $(0, 3 \models_{EUF} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-Learn $(\models_{LIA} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by Decide
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-Propagate $(0, 1, 11 \models_{EUF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by $\mathcal{T}$-Conflict $(7, 13 \models_{EUF} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by Backjump
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-Propagate $(0, 1, \overline{13} \models_{EUF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$\Delta_0,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by Propagate
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by Fail

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \land \underbrace{f(x) = b}_{1} \land \underbrace{f(v_2) = v_3}_{2} \land \underbrace{f(v_1) = v_4}_{3} \land \underbrace{1 \leq x}_{4} \land \underbrace{x \leq 2}_{5} \land \underbrace{v_1 = 1}_{6} \land \underbrace{a = b + 2}_{7} \land \underbrace{v_2 = 2}_{8} \land \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by **Propagate**$^{+}$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-**Propagate** $(0,\ 3 \models_{\text{EUF}} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	no	by **I-Learn** $(\models_{\text{LIA}} \bar{4} \lor \bar{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	no	by $\mathcal{T}$-**Propagate** $(0,\ 1,\ 11 \models_{\text{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	$\bar{7} \lor \bar{13}$	by $\mathcal{T}$-**Conflict** $(7,\ 13 \models_{\text{EUF}} \perp)$
$0 \cdots 9\ 10\ \bar{13}$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	no	by $\mathcal{T}$-**Propagate** $(0,\ 1,\ \bar{13} \models_{\text{EUF}} \bar{11})$
$0 \cdots 9\ 10\ \bar{13}\ \bar{11}\ 12$	$\Delta_0,\ \bar{4} \lor \bar{5} \lor 11 \lor 12$	no	by **Propagate**
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$\Delta_0 := \underbrace{f(v_1) = a}_{0} \wedge \underbrace{f(x) = b}_{1} \wedge \underbrace{f(v_2) = v_3}_{2} \wedge \underbrace{f(v_1) = v_4}_{3} \wedge \underbrace{1 \le x}_{4} \wedge \underbrace{x \le 2}_{5} \wedge \underbrace{v_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{v_2 = 2}_{8} \wedge \underbrace{v_3 = v_4 + 3}_{9}$$

$$\underbrace{a = v_4}_{10} \quad \underbrace{x = v_1}_{11} \quad \underbrace{x = v_2}_{12} \quad \underbrace{a = b}_{13}$$

M	Δ	C	rule
	Δ_0	no	
$0 \cdots 9$	Δ_0	no	by PROPAGATE$^+$
$0 \cdots 9\ 10$	Δ_0	no	by $\mathcal{T}$-PROPAGATE $(0,\ 3 \models_{EUF} 10)$
$0 \cdots 9\ 10$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by I-LEARN $(\models_{LIA} \bar{4} \vee \bar{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by DECIDE
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0,\ 1,\ 11 \models_{EUF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	$\bar{7} \vee \overline{13}$	by $\mathcal{T}$-CONFLICT $(7,\ 13 \models_{EUF} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by BACKJUMP
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by $\mathcal{T}$-PROPAGATE $(0,\ 1,\ \overline{13} \models_{EUF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$\Delta_0,\ \bar{4} \vee \bar{5} \vee 11 \vee 12$	no	by PROPAGATE
$\cdots$			(exercise)
UNSAT	$\cdots$	$\cdots$	by FAIL