CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

Normal Forms in Propositional Logic

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, **Andrei Voronkov** at the University of Manchester, **Emina Torlak** at the University of Washington, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Agenda

- NNF, DNF, CNF (CC Ch. 1.6)

Normal forms

For AR purposes, the language of formulas used to model problems may be too large

AR systems usually transform input formulas to formulas in a more restricted format before reasoning about them

We call these formats *normal forms*

The normal form a formula α is usually logically equivalent to, or at least equisatiable with, α

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Normal forms for propositional logic

These three normal forms are often used:

- Negation normal form (NNF)
- Disjunctive normal form (DNF)
- Conjunctive normal form (CNF)

Every formula of PL can be converted to an equivalent formula in one of these forms

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Negation normal form (NNF)

- Only logical connectives: $\wedge$, $\vee$, and $\neg$
- $\neg$ only appears in literals

Grammar

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\begin{aligned}
\langle \text{Atom} \rangle &:= \top \mid \bot \mid \langle \text{Variable} \rangle \\
\langle \text{Literal} \rangle &:= \langle \text{Atom} \rangle \mid \neg \langle \text{Atom} \rangle \\
\langle \text{Formula} \rangle &:= \langle \text{Literal} \rangle \mid \langle \text{Formula} \rangle \vee \langle \text{Formula} \rangle \mid \langle \text{Formula} \rangle \wedge \langle \text{Formula} \rangle
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NNF transformation

Repeatedly apply the following rewrites ($\longrightarrow$) to the formula and its subformulas, in any order, to *completion*[1]

- Eliminate double implications: $\alpha \Leftrightarrow \beta \longrightarrow (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
- Eliminate implications: $\alpha \Rightarrow \beta \longrightarrow (\neg\alpha \vee \beta)$
- Push negation inside conjunctions: $\neg(\alpha \wedge \beta) \longrightarrow (\neg\alpha \vee \neg\beta)$
- Push negation inside disjunctions: $\neg(\alpha \vee \beta) \longrightarrow (\neg\alpha \wedge \neg\beta)$
- Eliminate double negations: $\neg\neg\alpha \longrightarrow \alpha$
- Eliminate negated bottom: $\neg\bot \longrightarrow \top$
- Eliminate negated top: $\neg\top \longrightarrow \bot$

[1] I.e., until none applies anymore

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NNF transformation properties

Theorem 1

Every wff α not containing double implications ($\Leftrightarrow$) can be transformed into an equivalent NNF α' with a linear increase in the size[2] of the formula

[2] E.g., number of variable occurrences, or number of subformulas

NNF transformation properties

Observe

The NNF of formulas containing $\Leftrightarrow$ can grow exponentially larger in the worst case!

Example

$$(a_1 \Leftrightarrow a_2) \Leftrightarrow (a_3 \Leftrightarrow a_4) \qquad \text{4 vars}$$

$$\downarrow$$

$$(a_1 \Leftrightarrow a_2) \Rightarrow (a_3 \Leftrightarrow a_4) \wedge (a_3 \Leftrightarrow a_4) \Rightarrow (a_1 \Leftrightarrow a_2) \qquad \text{8 vars}$$

$$\downarrow$$

$$\vdots$$

$$\downarrow$$

$$\begin{array}{c} ((a_1 \Rightarrow a_2) \wedge (a_2 \Rightarrow a_1)) \Rightarrow ((a_3 \Rightarrow a_4) \wedge (a_4 \Rightarrow a_3)) \\ \wedge \\ ((a_3 \Rightarrow a_4) \wedge (a_4 \Rightarrow a_3)) \Rightarrow ((a_1 \Rightarrow a_2) \wedge (a_2 \Rightarrow a_1)) \end{array} \qquad \text{16 vars}$$

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Disjunctive normal form (DNF)

- Formula is in NNF
- Formula is a disjunction of conjunctions of literals, i.e., of the form:

$$\bigvee_i \left(\bigwedge_j l_{ij} \right)$$

Grammar

$$\begin{aligned}
\langle \text{Atom} \rangle &:= \top \mid \bot \mid \langle \text{Variable} \rangle \\
\langle \text{Literal} \rangle &:= \langle \text{Atom} \rangle \mid \neg \langle \text{Atom} \rangle \\
\langle \text{Cube} \rangle &:= \langle \text{Literal} \rangle \mid \langle \text{Literal} \rangle \wedge \langle \text{Cube} \rangle \\
\langle \text{Formula} \rangle &:= \langle \text{Cube} \rangle \mid \langle \text{Cube} \rangle \vee \langle \text{Formula} \rangle
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\end{aligned}$$

DNF transformation

Apply the following rewrites, in any order, to completion

- Apply NNF transformation rewrites
- Distribute $\wedge$ over $\vee$ (another source of exponential increase):
  - $\alpha \wedge (\beta \vee \gamma) \longrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
  - $(\alpha \vee \beta) \wedge \gamma \longrightarrow (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$
- Normalize nested conjunctions and disjunctions
  - $(\alpha \wedge \beta) \wedge \gamma \longrightarrow \alpha \wedge (\beta \wedge \gamma)$
  - $(\alpha \vee \beta) \vee \gamma \longrightarrow \alpha \vee (\beta \vee \gamma)$

Note: Instead of having nested conjunctions or disjunctions, it is convenient to treat $\wedge$ and $\vee$ as n-ary operators for any $n > 1$ (e.g., we treat $\alpha_1 \vee (\alpha_2 \vee (\alpha_3 \vee \alpha_4))$ as $\alpha_1 \vee \alpha_2 \vee \alpha_3 \vee \alpha_4$)

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DNF transformation

Theorem 2

Every wff α can be transformed into a logically equivalent DNF α', with a potentially exponential increase in the size of the formula

Note: The exponential increase can occur even in the absence of $\Leftrightarrow$

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Exercise

Transform each of these formulas (separately) into DNF:

$$\neg((p \vee \neg q) \Rightarrow r) \qquad \neg(a \Rightarrow (\neg b \Rightarrow a))$$

NNF transformation rewrites:

1. $\alpha \Leftrightarrow \beta \longrightarrow (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
2. $\alpha \Rightarrow \beta \longrightarrow \neg\alpha \vee \beta$
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CNF transformation can be exponential

There are formulas whose shortest CNF has an exponential size

Is there any way to avoid exponential blowup? Yes!

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A space-efficient CNF transformation

Using so-called *naming*, *definition introduction*, or *Tseitin's transformation*

1. Take a non-literal subformula α of formula φ
2. Introduce a new *name* n for it, i.e., a fresh propositional variable
3. Add a *definition for* n, i.e., a formula stating that n is equivalent to α

$$\begin{aligned} \varphi \quad &= \quad p_1 \Leftrightarrow (p_2 \Leftrightarrow (p_3 \Leftrightarrow (p_4 \Leftrightarrow \overbrace{(p_5 \Leftrightarrow p_6)}^{\alpha}))) \\ & \quad\;\; n \Leftrightarrow (p_5 \Leftrightarrow p_6) \end{aligned}$$

$$S = \left\{ \begin{array}{l} p_1 \Leftrightarrow (p_2 \Leftrightarrow (p_3 \Leftrightarrow (p_4 \Leftrightarrow n))) \\ n \Leftrightarrow (p_5 \Leftrightarrow p_6) \end{array} \right\}$$

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4. Replace α in φ by its name n:

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The new set S of formulas and the original formula φ are not equivalent but they are *equisatisfiable*:

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After several steps

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The conversion of the original formula to CNF introduces 32 copies of p_6

The conversion of the new set of formulas to CNF introduces 4 copies of p_6

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Clausal Form

Clausal form of a formula α: a set S_α of clauses which is satisfiable iff α is satisfiable

Clausal form of a set S of formulas: a set S' of clauses which is satisfiable iff so is S

Big advantage of clausal normal form over CNF:

we can convert any formula to a set of clauses in almost linear time

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Definitional Clause Form Transformation

How to convert a formula α into a set S of clauses that is a clausal normal form of α:

1. If α has the form $C_1 \wedge \cdots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then

$$S := \{ C_1, \ldots, C_n \}$$

2. Otherwise, introduce a name for each subformula β of α that is not a literal, and use this name instead of β

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Converting a formula to clausal form, Example

	non-literal subformula	definition	clauses
	$\neg((p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r))$		n_1
n_1	$\neg((p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	$n_1 \Leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$ $n_1 \lor n_2$
n_2	$(p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	$n_2 \Leftrightarrow (n_3 \Rightarrow n_7)$	$\neg n_2 \lor \neg n_3 \lor n_7$ $n_3 \lor n_2$ $\neg n_7 \lor n_2$
n_3	$(p \Rightarrow q) \land (p \land q \Rightarrow r)$	$n_3 \Leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$ $\neg n_3 \lor n_5$ $\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \Rightarrow q$	$n_4 \Leftrightarrow (p \Rightarrow q)$	$\neg n_4 \lor \neg p \lor q$ $p \lor n_4$ $\neg q \lor n_4$
n_5	$p \land q \Rightarrow r$	$n_5 \Leftrightarrow (n_6 \Rightarrow r)$	$\neg n_5 \lor \neg n_6 \lor r$ $n_6 \lor n_5$ $\neg r \lor n_5$
n_6	$p \land q$	$n_6 \Leftrightarrow (p \land q)$	$\neg n_6 \lor p$ $\neg n_6 \lor q$ $\neg p \lor \neg q \lor n_6$
n_7	$p \Rightarrow r$	$n_7 \Leftrightarrow (p \Rightarrow r)$	$\neg n_7 \lor \neg p \lor r$ $p \lor n_7$ $\neg r \lor n_7$

Converting a formula to clausal form, Example

	non-literal subformula	definition	clauses
	$\neg((p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r))$		n_1
n_1	$\neg((p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	$n_1 \Leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$ $n_1 \lor n_2$
n_2	$(p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	$n_2 \Leftrightarrow (n_3 \Rightarrow n_7)$	$\neg n_2 \lor \neg n_3 \lor n_7$ $n_3 \lor n_2$ $\neg n_7 \lor n_2$
n_3	$(p \Rightarrow q) \land (p \land q \Rightarrow r)$	$n_3 \Leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$ $\neg n_3 \lor n_5$ $\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \Rightarrow q$	$n_4 \Leftrightarrow (p \Rightarrow q)$	$\neg n_4 \lor \neg p \lor q$ $p \lor n_4$ $\neg q \lor n_4$
n_5	$p \land q \Rightarrow r$	$n_5 \Leftrightarrow (n_6 \Rightarrow r)$	$\neg n_5 \lor \neg n_6 \lor r$ $n_6 \lor n_5$ $\neg r \lor n_5$
n_6	$p \land q$	$n_6 \Leftrightarrow (p \land q)$	$\neg n_6 \lor p$ $\neg n_6 \lor q$ $\neg p \lor \neg q \lor n_6$
n_7	$p \Rightarrow r$	$n_7 \Leftrightarrow (p \Rightarrow r)$	$\neg n_7 \lor \neg p \lor r$ $p \lor n_7$ $\neg r \lor n_7$

Consider all subformulas that are not literals

Converting a formula to clausal form, Example

	non-literal subformula	definition	clauses
	$\neg((p \Rightarrow q) \wedge (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow r))$		n_1
n_1	$\neg((p \Rightarrow q) \wedge (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	$n_1 \Leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \Rightarrow q) \wedge (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	$n_2 \Leftrightarrow (n_3 \Rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \Rightarrow q) \wedge (p \wedge q \Rightarrow r)$	$n_3 \Leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \Rightarrow q$	$n_4 \Leftrightarrow (p \Rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \Rightarrow r$	$n_5 \Leftrightarrow (n_6 \Rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \Leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \Rightarrow r$	$n_7 \Leftrightarrow (p \Rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Introduce names for these formulas

Converting a formula to clausal form, Example

	non-literal subformula	definition	clauses
	$\neg((p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r))$		n_1
n_1	$\neg((p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	$n_1 \Leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$ $n_1 \lor n_2$
n_2	$(p \Rightarrow q) \land (p \land q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	$n_2 \Leftrightarrow (n_3 \Rightarrow n_7)$	$\neg n_2 \lor \neg n_3 \lor n_7$ $n_3 \lor n_2$ $\neg n_7 \lor n_2$
n_3	$(p \Rightarrow q) \land (p \land q \Rightarrow r)$	$n_3 \Leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$ $\neg n_3 \lor n_5$ $\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \Rightarrow q$	$n_4 \Leftrightarrow (p \Rightarrow q)$	$\neg n_4 \lor \neg p \lor q$ $p \lor n_4$ $\neg q \lor n_4$
n_5	$p \land q \Rightarrow r$	$n_5 \Leftrightarrow (n_6 \Rightarrow r)$	$\neg n_5 \lor \neg n_6 \lor r$ $n_6 \lor n_5$ $\neg r \lor n_5$
n_6	$p \land q$	$n_6 \Leftrightarrow (p \land q)$	$\neg n_6 \lor p$ $\neg n_6 \lor q$ $\neg p \lor \neg q \lor n_6$
n_7	$p \Rightarrow r$	$n_7 \Leftrightarrow (p \Rightarrow r)$	$\neg n_7 \lor \neg p \lor r$ $p \lor n_7$ $\neg r \lor n_7$

Introduce definitions

Converting a formula to clausal form, Example

	non-literal subformula	definition	clauses
	$\neg((p \Rightarrow q) \wedge (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow r))$		n_1
n_1	$\neg((p \Rightarrow q) \wedge (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	$n_1 \Leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \Rightarrow q) \wedge (p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	$n_2 \Leftrightarrow (n_3 \Rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \Rightarrow q) \wedge (p \wedge q \Rightarrow r)$	$n_3 \Leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \Rightarrow q$	$n_4 \Leftrightarrow (p \Rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \Rightarrow r$	$n_5 \Leftrightarrow (n_6 \Rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \Leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \Rightarrow r$	$n_7 \Leftrightarrow (p \Rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Convert the definition formulas to CNF in the standard way

DNF vs. CNF for satisfiability checking

DNF

- Satisfiability is decidable in linear time, with one traversal of the cubes
  - The DNF is unsatisfiable iff every cube contains both a literal and its complement
- However, converting to an equivalent DNF may result in exponential size increase

CNF

- Deciding satisfiability is hard (NP-hard)
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- However, converting into an *equisatisfiable* CNF can be done with only a linear size increase

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They choose to tackle the hardness of the satisfiability problem at runtime rather than at transformation time

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