CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

Abstract Proof Systems

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Agenda

- Abstract Proof Systems
- Satisfiability Proof Systems
- Soundness, Completeness, Termination, and Progressiveness
- A Decision Procedure for Propositional Logic
- Strategies

Proofs for Automated Reasoning

In AR, representing algorithms as proof systems has several advantages

- They are modularity and composable
- It is easier to prove things about the algorithms
- Can choose which implementation aspects to highlight and which to leave out

Abstract Proof Systems

An *abstract proof system* is a tuple $\mathbb{P} = \langle \mathbb{S}, \mathbb{R} \rangle$

where $\mathbb{S}$ is a set of proof states and $\mathbb{R}$ is a set of proof rules

Proof state: Data structure representing what is known at each stage of the proof

Example: a set of propositional formulas

Proof Rule: A partial function from proof states to sets of proof states

Example: Modus Ponens maps a state $S \supseteq \{\alpha, \alpha \Rightarrow \beta\}$ to the state $S \cup \{\beta\}$

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Proof Rules

- Take an input proof state $\mathcal{S}$
- Are only applicable if $\mathcal{S}$ satisfies some *premises*
- Return one or more *derived* proof states, the *conclusions*

Notation:

$$\textbf{R}\ \frac{P_1 \quad P_2 \quad \cdots \quad P_m}{C_1 \quad C_2 \quad \cdots \quad C_n}$$

- **R** is the rule's name (for reference)
- Each P_i is a premise, each C_i is a conclusion

Note: Intuitively, premises are conjunctive; conclusions are disjunctive

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A Proof System for Propositional Logic

Let $\mathbb{P}_{\mathrm{PL}} = \langle \mathbb{S}_{\mathrm{PL}}, \mathbb{R}_{\mathrm{PL}} \rangle$ where every proof state $\mathcal{S} \in \mathbb{S}_{\mathrm{PL}}$ is a set of wffs of PL

If $\mathbb{R}_{\mathrm{PL}}$ contains the *modus ponens* rule (**MP** for short) we can write **MP** as follows:

$$\textbf{MP}\ \frac{\alpha \in \mathcal{S} \quad \alpha \Rightarrow \beta \in \mathcal{S} \quad \beta \notin \mathcal{S}}{\mathcal{S} \cup \{\beta\}}$$

Technically, **MP** is a proof rule *schema*

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Let a, b, c, d be propositional variables

What is the result of applying **MP** to the following proof states?

1. $\{a, a \Rightarrow b\}$ $\{a, a \Rightarrow b, b\}$
2. $\{\neg d, a \vee \neg c, \neg d \Rightarrow b\}$ $\{a \vee \neg c, \neg d, \neg d \Rightarrow b, b\}$
3. $\{c, d, c \Rightarrow d\}$ does not apply

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A Proof System for Propositional Logic

Let $\mathcal{V}$ be the set of all propositional variables

Consider the following rule for $\mathbb{P}_{\text{PL}}$:

$$\textsc{Split}\ \frac{\alpha \in \mathcal{V} \quad \alpha \text{ occurs in some formula of } \mathcal{S} \quad \alpha \notin \mathcal{S} \quad \neg\alpha \notin \mathcal{S}}{\mathcal{S} \cup \{\alpha\} \qquad\qquad \mathcal{S} \cup \{\neg\alpha\}}$$

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Can we apply **Split** to $\{a \lor (b \land c), \neg d\}$?

Yes, if we choose to instantiate α with a, b, or c but not d

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Then, formally:

$$\{a \vee (b \wedge c), \neg d\} \xrightarrow{\textsc{Split}_b} \{\{a \vee (b \wedge c), \neg d, b\}, \{a \vee (b \wedge c), \neg d, \neg b\}\}$$

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Let $\mathcal{V}$ be the set of all propositional variables and let $\mathcal{L} = \mathcal{V} \cup \{ \neg\alpha \mid \alpha \in \mathcal{V} \}$

$\mathcal{L}$ is the set of all propositional *literals*, variables or negations of variables

Now consider the following rule for $\mathbb{P}_{\mathrm{PL}}$:

$$\textsc{Contr}\ \frac{\alpha \in \mathcal{V} \quad \alpha \in \mathcal{S} \quad \neg\alpha \in \mathcal{S}}{\mathrm{UNSAT}}$$

where UNSAT is a distinguished state

Note: The rule applies only to states with contradictory literals

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  - nodes from $\mathbb{S}$
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Contr **Contr**

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This tree is irreducible

Derivations

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- A *derivation* (in $\mathbb{P}$) from a derivation tree τ_0 is a (possibly infinite) sequence $\tau_0, \tau_1, \ldots$ of derivation trees where each τ_{i+1} is derivable from τ_i by applying a rule from $\mathbb{R}$ to a leaf of τ_i

- A derivation is *saturated* if it is finite and ends with an irreducible tree

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Satisfiability Proof Systems

Let $\mathbb{P} = \langle \mathbb{S}, \mathbb{R} \rangle$ be an abstract proof system

$\mathbb{P}$ is a *satisfiability proof system* if $\mathbb{S}$ includes the distinguished states SAT and UNSAT

- A rule of $\mathbb{R}$ is a *refuting* rule if its only conclusion is UNSAT
- A rule of $\mathbb{R}$ is a *corroborating* rule if its only conclusion is SAT
- A *refutation tree* (from S in $\mathbb{P}$) is a derivation tree from S with only UNSAT leaves
- A *refutation* (of S in $\mathbb{P}$) is a derivation from S ending with a refutation tree
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A Satisfiability Proof System for Propositional Logic

Can we extend $\mathbb{P}_{\mathrm{PL}}$ to be a satisfiability proof system?

Yes, simply by adding SAT to $\mathbb{S}_{\mathrm{PL}}$

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Note: We will say just “satisfiability preserving” to mean “strongly satisfiability preserving”

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Theorem 1

$\mathbb{P}$ *is sound if each of its proof rules is satisfiability preserving*

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The proof is by induction on the length of derivations

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Is $\mathbb{P}_{\mathrm{PL}}$ sound wrt $\mathbb{S}^{\mathrm{Sat}}$?

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Soundness Examples

Consider again $\mathbb{P}_{\mathrm{PL}} = \langle \mathbb{S}_{\mathrm{PL}}, \mathbb{R}_{\mathrm{PL}} \rangle$

Let $\mathbb{S}^{\mathrm{Sat}} = \{ \textsf{SAT} \} \cup \{ \mathcal{S} \in \mathbb{S}_{\mathrm{PL}} \mid \mathcal{S} \subseteq \mathcal{W}$ and $\mathcal{S}$ is propositionally satisfiable $\}$

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Exercise. Argue that each of these rules is strongly satisfiability preserving wrt $\mathbb{S}^{\mathrm{Sat}}$

$$\textbf{MP}\ \frac{\alpha \in \mathcal{S} \quad \alpha \Rightarrow \beta \in \mathcal{S} \quad \beta \notin \mathcal{S}}{\mathcal{S} \cup \{\beta\}} \qquad \textbf{Contr}\ \frac{\alpha \in \mathcal{V} \quad \alpha \in \mathcal{S} \quad \neg\alpha \in \mathcal{S}}{\mathsf{UNSAT}}$$

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Exercise

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Which of these new rules is weakly/strongly/non satisfiability preserving wrt $\mathbb{S}^{\mathrm{Sat}}$?

Add-Var1 $\dfrac{\alpha \in \mathcal{V} \quad \alpha \notin \mathcal{S} \quad \neg\alpha \notin \mathcal{S}}{\mathcal{S} \cup \{\alpha\}}$

Add-Var2 $\dfrac{\alpha \in \mathcal{V} \quad \alpha \text{ occurs nowhere in } \mathcal{S}}{\mathcal{S} \cup \{\alpha\}}$

And1 $\dfrac{\alpha \wedge \beta \in \mathcal{S}}{\mathcal{S} \cup \{\alpha\}}$

And2 $\dfrac{\alpha \wedge \beta \in \mathcal{S}}{\mathcal{S} \cup \{\alpha, \beta\}}$

Or-Split $\dfrac{\alpha \vee \beta \in \mathcal{S}}{\mathcal{S} \cup \{\alpha\} \qquad \mathcal{S} \cup \{\beta\}}$

And3 $\dfrac{\mathcal{S} = \mathcal{S}_1 \cup \{\alpha \wedge \beta\}}{\mathcal{S}_1 \cup \{\alpha\}}$

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Unsat $\dfrac{\mathcal{S} = \text{UNSAT}}{\{\alpha\}}$

Completeness and Termination

Let $\mathbb{P}$ be a satisfiability proof system with satisfiability predicate $\mathbb{S}^{\mathsf{Sat}}$

- $\mathbb{P}$ is *complete* (wrt $\mathbb{S}^{\mathsf{Sat}}$) if for every $S \in \mathbb{S}$,
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- $\mathbb{P}$ is *terminating* if every derivation in $\mathbb{P}$ is finite

Recall

$\mathbb{P}$ is sound (wrt $\mathbb{S}^{\mathsf{Sat}}$) if (*i*) no state $S \in \mathbb{S}$ that has a refutation in $\mathbb{P}$ is in $\mathbb{S}^{\mathsf{Sat}}$, and
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How would you prove it?

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Is $\mathbb{P}_{\mathrm{PL}}$ complete? No!

Can you find a satisfiable state other than SAT and UNSAT that is irreducible?

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$$\textbf{Split}\ \frac{\alpha \in \mathcal{V} \quad \alpha \text{ occurs in some formula in } \mathcal{S} \quad \alpha \notin \mathcal{S} \quad \neg\alpha \notin \mathcal{S}}{\mathcal{S} \cup \{\alpha\} \qquad \mathcal{S} \cup \{\neg\alpha\}}$$

Is $\mathbb{P}_{\text{PL}}$ complete? No!

Can you find a satisfiable state other than SAT and UNSAT that is irreducible?

How about $\{b\}$?

Proof Systems and Decision Procedures

If $\mathbb{P}$ is sound and complete wrt $\mathbb{S}^{\text{Sat}}$ and terminating,
it induces a decision procedure for checking whether a $\mathcal{S}$ is in $\mathbb{S}^{\text{Sat}}$:

- Simply start with $\mathcal{S}$ and produce any derivation
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A Decision Procedure for Propositional Logic

Recall: A variable assignment v is a partial mapping from $\mathcal{V}$ to $\{\text{true}, \text{false}\}$, and $v \models \mathcal{S}$ means that each formula in $\mathcal{S}$ evaluates to true under v

Let $\mathcal{S}$ be a set of propositional formulas

The *variable assignment v induced by $\mathcal{S}$* is defined as follows:

$$v(p) = \begin{cases} \text{true} & \text{if } p \in \mathcal{S} \\ \text{false} & \text{if } \neg p \in \mathcal{S} \\ \textit{undefined} & \text{otherwise} \end{cases}$$

$\mathcal{S}$ fully defines v if

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Let $\mathbb{P}_{\mathrm{E}} = \langle \mathbb{S}_{\mathrm{E}}, \mathbb{R}_{\mathrm{E}} \rangle$ where

- $\mathbb{S}_{\mathrm{E}}$ consists of all sets of wffs plus the distinguished states SAT and UNSAT
- $\mathbb{R}_{\mathrm{E}}$ consists of the following proof rules:

$$\textsc{Split}\ \frac{p \in \mathcal{V} \qquad p \text{ occurs in some formula in } \mathcal{S} \qquad p \notin \mathcal{S} \qquad \neg p \notin \mathcal{S}}{\mathcal{S} \cup \{p\} \qquad\qquad \mathcal{S} \cup \{\neg p\}}$$

$$\textsc{Sat}\ \frac{\mathcal{S} \text{ fully defines } v \qquad v \models \mathcal{S}}{\text{SAT}}$$

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A Decision Procedure for Propositional Logic

Let $\mathbb{S}^{\mathrm{Sat}}$ consist of SAT and all satisfiable sets of wffs

Theorem 1

Each rule in $\mathbb{P}_{\mathrm{E}}$ is satisfiability preserving wrt $\mathbb{S}^{\mathrm{Sat}}$

Corollary 2

$\mathbb{P}_{\mathrm{E}}$ *is sound wrt* $\mathbb{S}^{\mathrm{Sat}}$

Theorem 3

$\mathbb{P}_{\mathrm{E}}$ *is terminating*

Theorem 4

$\mathbb{P}_{\mathrm{E}}$ *is complete*

Therefore, $\mathbb{P}_{\mathrm{E}}$ can be used as a decision procedure for the SAT problem

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Derivation Strategies

Let $\mathbb{P} = \langle \mathbb{S}, \mathbb{R} \rangle$ be a proof system

- A *(derivation) strategy* for $\mathbb{P}$ is a partial function that, when defined, takes a derivation tree τ in $\mathbb{P}$ and returns a new derivation tree τ' such that (τ, τ') is a derivation in $\mathbb{P}$

- A derivation D in $\mathbb{P}$ *follows* a strategy π for $\mathbb{P}$
  1. if each non-initial derivation tree in D is the result of applying π to the previous derivation tree, and
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Derivation Strategy Example

Let $\prec$ be a total order on literals in $\mathcal{L}$ defined as alphabetical by variable name, with variables smaller than their negations (e.g., $a \prec \neg a \prec b \prec \neg b \prec \cdots$)

Consider the following strategy π_{PL} for $\mathbb{P}_{\mathrm{PL}}$, usable when every formula is either a literal or an implication between literal:

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1. Find the first reducible leaf in a left-to-right depth-first traversal of the tree; if none, then stop (π_{PL} is undefined in that case)
2. if **MP** applies, apply it to the formulas l_1 and $l_1 \Rightarrow l_2$ where l_1 is minimal according to $\prec$, breaking ties by choosing a minimal l_2
3. Otherwise, if **Split** applies, apply it to the smallest variable p among those occurring in the state
4. Otherwise, apply **Contr** if possible

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Exercise

Apply π_{PL} to

$$\mathcal{S} = \{a \Rightarrow c, a \Rightarrow \neg b, \neg b \Rightarrow \neg a\}$$

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